Memorandum COSOR 97-14

Subset selection for logistic populations using loss functions:
Some basic probabilities

P. van der Laan
C. van Eeden

Eindhoven, June 1997
The Netherlands
1 Introduction

Given are \( k \geq 2 \) random variables \( X_1, \ldots, X_k \) associated with \( k \) populations \( \pi_1, \ldots, \pi_k \), respectively. We assume that these random variables have logistic distributions differing only in their location parameters and we are interested in selecting a subset of them which contains the best population, that is the population with the largest value of the location parameter. We use the following selection rule

\[
\text{include } \pi_i \text{ in the subset } \iff X_i \geq \max_{1 \leq j \leq k} X_j - c,
\]

where \( c \) is a positive constant.

Results for Gupta’s subset selection procedure for this logistic case can be found in van der Laan (1992). In the present paper we use a loss-function approach similar to the one used by van der Laan and van Eeden (1993, 1996) and obtain the expected loss for the case where \( k = 3 \).

2 The loss function

Given positive numbers \( p_1 \) and \( p_2 \) and non-negative numbers \( \varepsilon_1 \) and \( \varepsilon_2 \), the loss incurred when the subset \( d \subset \{ \pi_1, \ldots, \pi_k \} \), is selected is, for \( D = \{ i \mid \pi_i \in d \} \), defined as follows.
For the proofs of our formulas for these probabilities, we need the following results

a) for positive numbers \( A \) and \( B \) with \( A \neq 1, B \neq 1 \) and \( A \neq B \)

\[
I = \int_0^\infty \frac{dz}{(z+1)^2(z+A)(z+B)}
\]

\[
= \frac{1}{B-A} \left[ \frac{1}{A-1} - \frac{1}{B-1} - \frac{\log A}{(A-1)^2} + \frac{\log B}{(B-1)^2} \right].
\]

(3.1)

This result follows immediately from Theorem 1 of van der Laan (1992) which says that, for \( r > 0, r \neq 1 \) and integers \( k > 0 \) and \( m \geq 0, \)

\[
r^{k-1} \int_0^\infty (s+1)^{-2}(s+r)^{-k+1} ds = 1 - \frac{k-1}{r} C_{k-1}(r),
\]

(3.2)

b) for positive numbers \( A \) and \( B \) with \( A \neq 1, B \neq 1 \) and \( A \neq B \)

\[
L_1(\theta, d) = (\theta[k] - \epsilon_1 - \max_{i \in D} \theta_i)^{p_1} I(\theta[k] - \max_{i \in D} \theta_i > \epsilon_1),
\]

\[
L_2(\theta, d) = (\max_{i \in D} \theta_i - \epsilon_2 - \min_{i \in D} \theta_i)^{p_2} I(\max_{i \in D} \theta_i - \min_{i \in D} \theta_i > \epsilon_2),
\]

where \( \theta[1] \leq \ldots \leq \theta[k] \) are the ordered \( \theta_i \)'s and \( \theta = (\theta_1, \ldots, \theta_k) \). Then the loss is given by

\[
L(\theta, d) = L_1(\theta, d) + L_2(\theta, d).
\]

(2.2)

In order to determine the expected loss we need, for each possible \( d \), its probability of being the chosen subset of populations. Or, equivalently, for each possible \( D \) the probability that it is the chosen subset of indices. In the next section we determine these probabilities for \( k = 3 \) and for logistic populations differing only in their location parameters.

3 The basic probabilities

Without loss of generality it can be assumed that the cumulative distribution functions \( F_i \) and the densities \( f_i \) are, for \( i = 1, \ldots, k, -\infty < x < \infty \) and \( -\infty < \theta < \infty \), given by

\[
F_i(x) = \left( 1 + e^{-(x-\theta_i)} \right)^{-1}
\]

\[
f_i(x) = e^{-(x-\theta_i)} \left( 1 + e^{-(x-\theta_i)} \right)^2.
\]

The expected loss can be obtained from (2.1), (2.2) and the probabilities of selecting the subsets \( D \subset \{1, \ldots, k\} \).

For the proofs of our formulas for these probabilities, we need the following results
where

$$C_m(r) = \left( \frac{r}{r-1} \right)^{m+1} \left\{ \ln r - \sum_{i=1}^{m} \frac{1}{i} \left( 1 - \frac{1}{r} \right)^i \right\}$$

with $$\sum_{i=1}^{m}(1-r^{-1})^i/i = 0$$ for $$m \leq 0$$. In van der Laan (1992) Theorem 1 is formulated for $$r > 1$$, but it is easy to see that the proof of this theorem is also correct for $$0 < r < 1$$;

b) for $$A = 1, A \neq B (\Rightarrow B \neq 1)$$

$$I = \frac{B - 3}{2(B - 1)^2} + \frac{\ln B}{(B - 1)^3};$$  \hspace{1cm} (3.3)

c) for $$A = B, A \neq 1 (\Rightarrow B \neq 1)$$

$$I = \frac{A + 1}{A(A - 1)^2} - 2 \frac{\ln A}{(A - 1)^3};$$  \hspace{1cm} (3.4)

d) for $$A = B = 1$

$$I = \frac{1}{3};$$  \hspace{1cm} (3.5)

The proofs of (3.3) - (3.5) are straightforward and therefore omitted.

In the following theorems $$\mu_1 = \theta_1 - \theta_1, \mu_2 = \theta_3 - \theta_2, a = e^{-\mu_1}, b = e^{-\mu_2}$$ and $$q = e^{-c}$$.  

**Theorem 3.1** The probability $$p_1$$ of choosing, with the rule given in (1.1), the subset $$D_1 = \{1\}$$ is, for $$b \neq 1$$ and $$aq \neq 1 (\Rightarrow abq \neq 1)$$, given by

$$p_1 = P_\theta(D_1) = \frac{a^2bq^2}{(aq - 1)(abq - 1)} - \frac{abq}{b - 1} \left\{ \frac{\ln aq}{(aq - 1)^2} - \frac{\ln abq}{(abq - 1)^2} \right\}.$$

**Proof.** The probability of choosing the subset $$\{1\}$$ equals

$$p_1 = P_\theta(X_1 \geq X_{[3]} - c, X_2 < X_{[3]} - c, X_3 < X_{[3]} - c)$$

$$= P_\theta(X_2 < X_1 - c, X_3 < X_1 - c)$$

$$= \int_{-\infty}^{\infty} F_2(x - c)F_3(x - c)f_1(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x+\theta_1}}{(1 + e^{-x+\theta_2+c})(1 + e^{-x+\theta_2+c})(1 + e^{-x+\theta_1})^2} dx$$

3
Theorem 3.2

For the case where \( q \neq a \) and \( ab \neq 1 \) \((\Rightarrow bq \neq 1, (q/a) \neq bq)\), the probability \( p_2 \) of choosing the subset \( D_2 = \{2\} \) is given by

\[
p_2 = P_s(D_2) = \frac{bq^2}{(q-a)(bq-1)} - \frac{bq}{ab-1} \left\{ \frac{\ln (q/a)}{(aq-1)^2} - \frac{\ln bq}{(abq-1)^2} \right\},
\]

where the sixth equality is obtained from (3.1).

When the conditions of Theorem 3.1 are not satisfied, the probability \( p_1 \) can be obtained by applying (3.3) - (3.5) to (3.6) as follows:

a) when \( aq = 1, b \neq 1 \) \((\Rightarrow abq \neq 1)\), (3.3) gives

\[
p_1 = \frac{b(b-3)}{2(b-1)^2} + b \frac{\ln b}{(b-1)^3};
\]

b) when \( abq = 1, b \neq 1 \) \((\Rightarrow aq \neq 1)\), (3.3) gives

\[
p_1 = aq \left[ \frac{aq-3}{2(aq-1)^2} + \frac{\ln aq}{(aq-1)^3} \right] = \frac{1-3b}{2(1-b)^2} - b^2 \frac{\ln b}{(1-b)^3};
\]

c) when \( b = 1, aq \neq 1 \) \((\Rightarrow abq \neq 1, aq = abq)\), (3.4) gives

\[
p_1 = \frac{aq(aq+1)}{(aq-1)^2} - 2a^2 q^2 \frac{\ln aq}{(aq-1)^3};
\]

d) when \( aq = 1, abq = 1 \), (3.5) gives

\[
p_1 = \frac{1}{3}.
\]
Proof. The probability of choosing the subset \( \{2\} \) equals

\[
p_2 = \int_{-\infty}^{\infty} F_1(x - c) F_3(x - c) f_2(x) dx
\]

\[
= \int_{-\infty}^{\infty} \frac{e^{-x+\theta_2}}{(1 + e^{-x+\theta_1+c})(1 + e^{-x+\theta_3+c})(1 + e^{-x+\theta_2})^2} dx
\]

\[
= \frac{bq^2}{a} \int_{0}^{\infty} \frac{dz}{(z + 1)^2(z + (q/a))(z + bq)}
\]

\[
= \frac{bq}{ab - 1} \left[ \frac{a}{q - a} - \frac{1}{bq - 1} \cdot \frac{\ln (q/a)}{((q/a) - 1)^2} + \frac{\ln bq}{(bq - 1)^2} \right]
\]

\[
= \frac{bq^2}{(q - a)(bq - 1)} - \frac{bq}{ab - 1} \left\{ \frac{\ln (q/a)}{((q/a) - 1)^2} - \frac{\ln bq}{(bq - 1)^2} \right\},
\]

where the fourth equality is obtained from (3.1). \( \square \)

When the conditions of Theorem 3.2 are not satisfied, the probability \( p_2 \) can be obtained applying (3.3) - (3.5) to (3.7) as follows:

a) when \( q = a, bq \neq 1 \) (\( \Rightarrow (q/a) \neq bq \)), (3.3) gives

\[
p_2 = bq \left[ \frac{bq - 3}{2(bq - 1)^2} + \frac{\ln bq}{(bq - 1)^3} \right];
\]

b) when \( bq = 1, q \neq a \) (\( \Rightarrow bq \neq (q/a), ab \neq 1 \)), (3.3) gives

\[
p_2 = \frac{1 - 3ab}{2(1 - ab)^2} - a^2b^2 \frac{\ln ab}{(1 - ab)^3};
\]

c) when \( q/a = bq \neq 1 \), (3.4) gives

\[
p_2 = (bq)^2 \left[ \frac{bq + 1}{bq(bq - 1)^2} - 2 \frac{\ln bq}{(bq - 1)^3} \right];
\]

d) when \( q/a = bq = 1 \) (or \( ab = bq = 1 \)), (3.5) gives

\[
p_2 = \frac{1}{3}.
\]
Theorem 3.3 Assume that \( a \neq 1, q \neq b, q \neq ab \Rightarrow (q/ab) \neq (q/b) \) then the probability of choosing the subset \( D_3 = \{3\} \) is given by

\[
p_3 = P_3(D_3) = \frac{q^2}{(q-ab)(q-b)} - \frac{q}{b-ab} \left\{ \frac{\ln(q/b)}{((q/b) - 1)^2} - \frac{\ln(q/ab)}{((q/ab) - 1)^2} \right\}.
\]

Proof. The probability of choosing the subset \( \{3\} \) equals

\[
p_3 = \int_{-\infty}^{\infty} F_1(x-c) F_2(x-c) f_3(x) \, dx
= \int_{-\infty}^{\infty} \frac{e^{-x+\theta_3}}{(1 + e^{-x+\theta_1+c})(1 + e^{-x+\theta_2+c})(1 + e^{-x+\theta_3})^2} \, dx
= \frac{q^2}{ab^2} \int_0^\infty \frac{dz}{(z+1)^2(z+(q/ab))(z+(q/b))}
= \frac{q}{b-ab} \left[ \frac{1}{(q/b)-1} - \frac{1}{(q/ab)-1} - \frac{\ln(q/b)}{((q/b) - 1)^2} + \frac{\ln(q/ab)}{((q/ab) - 1)^2} \right],
\]

where the third equality is obtained from (3.1). \( \square \)

When the conditions of Theorem 3.3 are not satisfied the probability \( p_3 \) can be obtained by applying (3.3) - (3.5) to (3.8) as follows:

a) when \( q = ab, a \neq 1 \Rightarrow q \neq b \), (3.3) gives

\[
p_3 = \frac{a(a-3)}{2(a-1)^2} + a \frac{\ln a}{(a-1)^3};
\]

b) when \( q = b, a \neq 1 \Rightarrow q \neq ab \), (3.3) gives

\[
p_3 = \frac{1-3a}{2(1-a)^2} - a^2 \frac{\ln a}{(1-a)^3};
\]

c) when \( a = 1, q \neq b \Rightarrow q \neq ab \), (3.4) gives

\[
p_3 = \frac{q(q+b)}{(q-b)^2} - 2bq^2 \frac{\ln(q/b)}{(q-b)^3};
\]

d) when \( q = b, a = 1 \), (3.5) gives

\[
p_3 = \frac{1}{3};
\]
Theorem 3.4 Under the conditions of the Theorems 3.1 and 3.2, the probability of choosing the subset \( D_{12} = \{1, 2\} \) is given by

\[
p_{12} = P_0(D_{12}) = \frac{a^2bq}{(a-1)(abq-1)} - \frac{abq}{bq-1} \left\{ \frac{\ln a}{(a-1)^2} - \frac{\ln abq}{(abq-1)^2} \right\} \\
+ \frac{bq}{(1-a)(bq-1)} - \frac{bq}{abq-1} \left\{ \frac{-\ln a}{((1/a)-1)^2} - \frac{\ln bq}{(bq-1)^2} \right\} \\
- \frac{a^2bq^2}{(aq-1)(abq-1)} + \frac{abq}{b-1} \left\{ \frac{\ln aq}{(aq-1)^2} - \frac{\ln abq}{(abq-1)^2} \right\} \\
- \frac{bq^2}{(q-a)(bq-1)} + \frac{bq}{ab-1} \left\{ \frac{\ln (q/a)}{((q/a)-1)^2} - \frac{\ln bq}{(bq-1)^2} \right\}.
\]

Proof. The probability \( p_{12} \) equals

\[
p_{12} = P_0(X_1, X_2 \geq X_{[3]} - c, X_3 < X_{[3]} - c) \\
= P_0(X_1, X_2 \geq X_1 - c, X_1 = \max_{1 \leq j \leq 3} X_j, X_3 < X_1 - c) \\
+ P_0(X_1, X_2 \geq X_2 - c, X_2 = \max_{1 \leq j \leq 3} X_j, X_3 < X_2 - c) \\
= P_0(X_2 \geq X_1 - c, X_3 < X_1 - c, X_2 < X_1, X_3 < X_1) \\
+ P_0(X_1 \geq X_2 - c, X_3 < X_2 - c, X_1 < X_2, X_3 < X_2) \\
= P_0(X_1 - c \leq X_2 < X_1, X_3 < X_1 - c) + P(X_2 - c \leq X_1 < X_2, X_3 < X_2 - c) \\
= \int_{-\infty}^{\infty} \{F_2(x) - F_2(x - c)\}F_3(x - c)f_1(x)dx \\
+ \int_{-\infty}^{\infty} \{F_1(x) - F_1(x - c)\}F_3(x - c)f_2(x)dx \\
= \int_{-\infty}^{\infty} F_2(x)F_3(x - c)f_1(x)dx \\
+ \int_{-\infty}^{\infty} F_1(x)F_3(x - c)f_2(x)dx - p_1 - p_2 \\
= I_1 + I_2 - p_1 - p_2,
\]

with

\[
I_1 = \int_{-\infty}^{\infty} \frac{e^{-x+\theta_1}}{(1 + e^{-x+\theta_2})(1 + e^{-x+\theta_2+c})(1 + e^{-x+\theta_1})^2}dx.
\]
where, for each of \( 1_1 \) and \( 1_2 \), the last equality follows from (3.1). 

\[
I_2 = \int_{-\infty}^{\infty} \frac{e^{-z+\theta_2}}{(1+e^{-z+\theta_1})(1+e^{-z+\theta_3})(1+e^{-z+\theta_2})^2} \, dx
\]

\[
= \int_{0}^{\infty} \frac{dz}{(z + 1)^2(1 + az)(1 + (z/bq))}
\]

\[
= \frac{bq}{a} \int_{0}^{\infty} \frac{dz}{(z + 1)^2(z + (1/a))(z + bq)}
\]

\[
= \frac{bq}{(1-a)(bq-1)} - \frac{bq}{abq-1} \left\{ \frac{-\ln a}{((1/a)-1)^2} - \frac{\ln bq}{(bq-1)^2} \right\}.
\]

For the cases where the conditions of Theorem 3.4 are not satisfied, \( I_1 \) and \( I_2 \) can be obtained by applying (3.3) - (3.5) to (3.9) and (3.10) as follows:

a) when \( a = 1, bq \neq 1 \), (3.3) gives

\[
I_1 = bq \left[ \frac{bq - 3}{2(bq - 1)^2} + \frac{\ln bq}{(bq - 1)^3} \right] \quad \text{and} \quad I_2 = I_1;
\]

b) when \( abq = 1, bq \neq 1 \), (3.3) for \( I_1 \) and (3.4) for \( I_2 \) give

\[
I_1 = a \left[ \frac{a - 3}{2(a - 1)^2} + \frac{\ln a}{(a - 1)^3} \right],
\]

\[
I_2 = \frac{a + 1}{(a - 1)^2} - 2a \frac{\ln a}{(a - 1)^3}.
\]

c) when \( bq = 1, a \neq 1 \), (3.4) for \( I_1 \) and (3.3) for \( I_2 \) give

\[
I_1 = a^2 \left[ \frac{a + 1}{a(a - 1)^2} - 2 \frac{\ln a}{(a - 1)^3} \right],
\]

\[
I_2 = 0.
\]
\[ I_2 = \frac{1 - 3a}{2(1 - a)^2} - a^2 \frac{\ln a}{(a - 1)^3}; \]

d) when \( a = 1, bq = 1 \), (3.5) gives

\[ I_1 = I_2 = \frac{1}{3}. \]

**Theorem 3.5** Assume the conditions of the Theorems 3.1 and 3.3 are fulfilled, then the probability \( p_{13} \) of choosing the subset \( D_{13} = \{1, 3\} \) is given by

\[
p_{13} = P_2(D_{13}) = \frac{a^2 bq}{(aq - 1)(ab - 1)} - \frac{abq}{b - q} \left\{ \frac{\ln aq}{(aq - 1)^2} - \frac{\ln ab}{(ab - 1)^2} \right\}
\]
\[
+ \frac{q}{(1 - ab)(q - b)} - \frac{q}{abq - b} \left\{ \frac{-\ln ab}{((1/ab) - 1)^2} - \frac{\ln(q/b)}{((q/b) - 1)^2} \right\}
\]
\[
- \frac{a^2 bq^2}{(aq - 1)(abq - 1)} + \frac{abq}{b - 1} \left\{ \frac{\ln aq}{(aq - 1)^2} - \frac{\ln ab}{(abq - 1)^2} \right\}
\]
\[
- \frac{q^2}{(q - ab)(q - b)} + \frac{q}{b - ab} \left\{ \frac{\ln(q/b)}{((q/b) - 1)^2} - \frac{\ln(q/ab)}{((q/ab) - 1)^2} \right\}.
\]

**Proof.** The probability \( p_{13} \) is given by

\[
p_{13} = \int_{-\infty}^{\infty} \{F_3(x) - F_3(x - c)\} F_2(x - c) f_1(x) dx
\]
\[
+ \int_{-\infty}^{\infty} \{F_1(x) - F_1(x - c)\} F_2(x - c) f_3(x) dx
\]
\[
= \int_{-\infty}^{\infty} F_3(x) F_2(x - c) f_1(x) dx
\]
\[
+ \int_{-\infty}^{\infty} F_1(x) F_2(x - c) f_3(x) dx - p_1 - p_3
\]
\[
= I_3 + I_4 - p_1 - p_3,
\]

with

\[
I_3 = \int_{-\infty}^{\infty} \frac{e^{-x+\theta_1}}{(1 + e^{-x+\theta_2})(1 + e^{-x+\theta_2+c})(1 + e^{-x+\theta_1})^2} dx
\]
\[
= \int_{0}^{\infty} \frac{dz}{(z + 1)^2((z/aq) + 1)((z/ab) + 1)}
\]
When the conditions of Theorem 3.5 are not satisfied, $I_3$ and $I_4$ can be obtained by applying (3.3) - (3.5) to (3.11) and (3.12) as follows:

a) when $aq = 1, q \neq b (\Rightarrow ab \neq 1)$, (3.3) for $I_3$ and (3.4) for $I_4$ give

$$I_3 = ab \left[ \frac{ab - 3}{2(ab - 1)^2} + \frac{\ln ab}{(ab - 1)^3} \right],$$

$$I_4 = \frac{q(q + b)}{(q - b)^2} - 2bq^2 \frac{\ln (q/b)}{(q - b)^3}.$$

b) when $ab = 1, q \neq b (\Rightarrow aq \neq 1)$, (3.3) gives

$$I_3 = aq \left[ \frac{aq - 3}{2(aq - 1)^2} + \frac{\ln aq}{(aq - 1)^3} \right],$$

$$I_4 = \frac{q(q - 3b)}{2(q - b)^2} + q^2 \frac{\ln (q/b)}{(q - b)^3}.$$

c) when $b = q, ab \neq 1 (\Rightarrow aq \neq 1)$, (3.4) for $I_3$ and (3.3) for $I_4$ give

$$I_3 = (ab)^2 \left[ \frac{ab + 1}{ab(ab - 1)^2} - 2 \frac{\ln ab}{(ab - 1)^3} \right],$$

$$I_4 = (aq - 1)(ab - 1) - abq \left\{ \frac{\ln aq}{(aq - 1)^2} - \frac{\ln ab}{(ab - 1)^2} \right\},$$

$$I_4 = \int_0^\infty \frac{e^{-x+\theta_3}}{(z + 1)^2(z + az)(1 + (bz/q))} dx$$

$$= \int_0^\infty \frac{dz}{(z + 1)^2(z + (1/ab))(1 + (b z/q))}$$

$$= \frac{q}{ab^2} \int_0^\infty \frac{dz}{(z + 1)^2(z + (1/ab))(z + (q/b))}$$

$$= \frac{q}{(1 - ab)(q - b)} - \frac{q}{abq - b} \left\{ - \frac{\ln ab}{((1/ab) - 1)^2} - \frac{\ln (q/b)}{((q/b) - 1)^2} \right\},$$

where, for each of $I_3$ and $I_4$, the last equality follows from (3.1). □
Proof. The probability $P_{23}$ is given by

$$I_4 = \frac{1 - 3ab}{2(1 - ab)^2} - (ab)^2 \frac{\ln ab}{(1 - ab)^3};$$

d) when $b = q, ab = 1 \Rightarrow aq = 1$, (3.5) give

$$I_3 = I_4 = \frac{1}{3}.$$

**Theorem 3.6** Under the conditions of the Theorems 3.2 and 3.3 we get, for the probability of choosing the subset $D_{23} = \{2, 3\}$

$$p_{23} = P_0(D_{23}) = \frac{bq}{(b-1)(q-a)} - \frac{bq}{q - ab} \left\{ \frac{\ln b}{(b-1)^2} - \frac{\ln (q/a)}{((q/a) - 1)^2} \right\}$$

$$+ \frac{q}{(1-b)(q-ab)} + \frac{q}{bq - ab} \left\{ \frac{\ln b}{((1/b) - 1)^2} + \frac{\ln (q/ab)}{((q/ab) - 1)^2} \right\}$$

$$- \frac{bq^2}{(q-a)(bq-1)} + \frac{bq}{ab - 1} \left\{ \frac{\ln (q/a)}{((q/a) - 1)^2} - \frac{\ln bq}{(bq - 1)^2} \right\}$$

$$- \frac{q^2}{(q-ab)(q-b)} + \frac{q}{b - ab} \left\{ \frac{\ln (q/b)}{((q/b) - 1)^2} - \frac{\ln (q/ab)}{((q/ab) - 1)^2} \right\}.$$
\[
I_6 = \int_0^\infty \frac{dz}{(z + 1)^2 (z + b)(z + (q/a))} = \frac{bq}{a} \int_0^\infty \frac{dz}{(z + 1)^2 (z + b)(z + (q/a))} - \frac{bq}{(b - 1)(q - a)} \left\{ \ln b - \frac{\ln (q/a)}{(b - 1)^2 - ((q/a) - 1)^2} \right\},
\]

\[
I_6 = \int_0^\infty \frac{dz}{(z + 1)^2 (z + b)(z + (q/a))} = \frac{q}{ab^2} \int_0^\infty \frac{dz}{(z + 1)^2 (z + (1/b))(z + (q/ab))} = \frac{q}{(1 - b)(q - ab)} + \frac{q}{bq - ab} \left\{ \ln b - \frac{\ln (q/a)}{(1/b - 1)^2 - ((q/ab) - 1)^2} \right\},
\]

where, for each of \(I_5\) and \(I_6\), the last equality follows from (3.1). \(\square\)

When the conditions of Theorem 3.6 are not satisfied, \(I_5\) and \(I_6\) can be obtained by applying (3.3) - (3.5) to (3.13) and (3.14) as follows:

a) when \(b = 1, a \neq q\), (3.3) gives
\[
I_5 = \frac{q(q - 3a)}{2(q - a)^2} + qa^2 \frac{\ln (q/a)}{(q - a)^3} \text{ and } I_6 = I_5;
\]

b) when \(q = a, b \neq 1\), (3.3) for \(I_5\) and (3.4) for \(I_6\) give
\[
I_5 = b \left[ \frac{b - 3}{2(b - 1)^3} + \frac{\ln b}{(b - 1)^3} \right], \quad I_6 = \frac{1 + b}{(1 - b)^3} + 2b \frac{\ln b}{(1 - b)^3};
\]

c) when \(q = ab, b \neq 1\), (3.4) for \(I_5\) and (3.3) for \(I_6\) give
\[
I_5 = b^2 \left[ \frac{b + 1}{b(b - 1)^2} - 2 \frac{\ln b}{(b - 1)^3} \right], \quad I_6 = \frac{1 - 3b}{2(1 - b)^2} - b^2 \frac{\ln b}{(1 - b)^3};
\]

d) when \(q = a, b = 1\), (3.5) gives
\[
I_5 = I_6 = \frac{1}{3}.
\]
Using these results, the expected loss $R(\theta, d) = \mathcal{E}_\theta L(\theta, d)$ can be computed. For the special case $\varepsilon_1 = \varepsilon_2 = \varepsilon, p_1 = p_2 = p$ we get, assuming without loss of generality that $\theta_1 < \theta_2 < \theta_3$,

$$R(\theta, d) = (\theta_3 - \theta_1 - \varepsilon)p\{p_{123} + p_{13} + p_1\}I(\theta_3 - \theta_1 > \varepsilon)$$

$$+ (\theta_3 - \theta_2 - \varepsilon)p\{p_{12} + p_{23} + p_2\}I(\theta_3 - \theta_2 > \varepsilon)$$

$$+ (\theta_2 - \theta_1 - \varepsilon)p_{12}I(\theta_2 - \theta_1 > \varepsilon).$$

References


