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A GENERAL INVERSION CELL, OBTAINED FROM TWO RANDOM CONTRA-PARALLELOGRAM LINKAGES, INTERCONNECTED AT THE VERTICES OF A PARALLELOGRAM

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Abstract—It is possible to interconnect two dissimilar contra-parallelgram four-bar linkages by four turning-joints in a manner that permits (overconstrained) motion. These joints are located at the four vertices of a parallelogram. During the motion, the angles enclosed between the sides of this parallelogram do not vary, but the lengths, of which their product remains a constant, do. The overconstrained linkage mechanism, leads to the derivation of an entirely new *inversion mechanism*. Its inversion-cell is a 6-bar linkage of Watt's type, simply obtained from the combined linkage by erasing two interconnected bars with their turning-joints. One may show that *Peaucellier's* inversion-cell just appears as a very special case.

INTRODUCTION

There are many examples of *overconstrained* linkage mechanisms. One type is the one obtained by excessive interconnection of two four-bar linkages. Kempe [1] as well as Burmester [2] succeeded in doing this in different ways. Both obtained overconstrained 8-bar linkages with 2 excessive turning-joints. They took *reflected-* and *directly similar four-bars* and interconnected corresponding sides with a *common* turning-joint. One particular type, however, was not noticed, namely the one derived by the author at the occasion of the *7th World Congress on the Theory of Machines and Mechanisms*, Sevilla in 1987 [3]. Then, the author took two *random* and thus *dissimilar contra-parallelgram linkages* and interconnected corresponding sides with common turning-joints. To get this done, he first replaced the contra-parallelgrams in a rigid way namely such that they got a *common* axis of symmetry. Then, one of them was shifted along the axis until $\angle (BD, BD)$ became π radians. (See the obtained, relative position of the contra-parallelgrams $AS\bar{A}S$ and $CRC\bar{R}$ in Fig. 1.) In this position, the contra-parallelgrams were said to be *perspective*, since apparently corresponding joints were aligned with a common *perspectivity center* T . (Thus, $\square AS\bar{A}S \overset{T}{\square} \square CRC\bar{R}$.) To prove this it is sufficient to show that the intersection point T , defined as $(AC \times SR)$, actually joins the axis of symmetry. (Use Desargues' Theorem about perspective triangles to complete the proof.) Anyway, in this position, corresponding sides of the two contra-parallelgrams intersected at the points $D, \bar{B}, B$ and $\bar{D}$ of a straight-line running parallel to the diagonals of the contra-parallelgrams. (See Ref. [3].) It appeared possible to turn the intersections into turning-joints, interconnecting these two particular four-bar linkages in an excessive way.

Reobserving the overconstrained 8-bar linkage from a different angle, one notes that actually, two *inversion-cells* of Hart are interconnected at these aligned turning-joints meeting the property that $|OP| \cdot |OQ| = \text{constant}$ with $O \equiv D, P \equiv \bar{B}, Q \equiv B$ (and $U \equiv \bar{D}$). Thus, as has been shown already by the author [4], the common turning-joints coincide with the points O, P, Q (and U) of Hart's *inversion-cell*. In fact, by interconnecting two of these cells at these points, one regains the combined linkage mentioned earlier. Clearly, the *four* times overconstrained linkage mechanism may still be used as an *inversion* mechanism. (Four times, if we count turning-joints for two coordinates in design.)

At this point, it becomes possible to obtain even *new* inversion-cells by erasing the two ternary links interconnected at the fourth point $\bar{D} \equiv U$. (See again Fig. 1 and also Ref. [4].) The result

Fig. 1. Two perspective anti-parallelogram linkages excessively interlocked through 4 turning-joints, forming an aligned parallelogram $DB\bar{D}B$.

appeared to be a generalization of *Peaucellier's* inversion linkage. On the other hand, the two contra-parallelogram linkages excessively interconnected as demonstrated in Fig. 1 (as well as at the World Congress in Sevilla), will overcome their stretched or aligned positions through the very existence of their twin-brother. One may even erase *one* of the 8 bars, and still maintain this practical property. Then, a 7-bar, now singular overconstrained linkage chain, is obtained, still doing its job of maintaining contra-parallelogram motion. The three adjoined bars then prevent transition into the parallelogram-mode.

GENERALIZATION OF THE COMBINED CONTRA-PARALLELOGRAMS

In 1971, the author showed the *mechanical cognation* between the inversion cells of Hart and Peaucellier [5]. He did this by application of his *cognate-theory* at 6-bar linkages of Watt's type [9]. Note that two linkages are said to be cognated if there is a pure mechanical way to interconnect the two of them in an overconstrained way by a certain number of common turning-joints and common links. Observed separately, the two linkages must be able to carry out identical objectives in every corresponding position. (Consider, for instance, Roberts' Law resulting into three four-bar *curve-cognates*, producing identical curves.) In the course of his proof, the author similarly showed mechanical cognation between Hart's inversion-cell and its more *generalized* form, the so-called *quadruplanar inversion-cell* of Sylvester and Kempe. This was simply done by *stretch-rotation*, being basically a *similarity transformation of sub-chains* and therefore a powerful tool to prove cognate theory as, for instance in the case of Roberts' Law.

The quadruplanar inversion-cell is to be obtained as follows (see Fig. 2, or alternatively, Fig. 5 of Ref. [5]):

- Start from Hart's inversion-cell $AS\bar{A}\bar{S}$ containing the aligned points O, P, Q and U.
- Stretch-rotate the contra-parallelogram $AS\bar{A}\bar{S}$ about A with the *randomly* chosen complex factor AS'/AS into the contra-parallelogram $AS'\bar{A}'\bar{S}'$.
- Attach S' to link ADS and S' to link AQS .
- Stretch-rotate the linkage-dyad PSD about D with the complex factor DS'/DS into the dyad $P'S'D$.
- Attach P' to $S'\bar{A}'$ and note the similarities of the triangles $P'S'\bar{A}'$ and $DS'A$.
- Stretch-rotate the linkage-dyad $Q\bar{S}U$ into $Q\bar{S}'Q'$ about Q.

Fig. 2. Quadruplanar inversion-cell of Sylvester and Kempe obtained from Hart's cell of inversion through cognation.

(g) Finally, attach Q' to $\bar{S}'\bar{A}'$.

(h) Note that the quadrilateral $DP'Q'Q$ represents a parallelogram with fixed angles but varying sides for which $|OP'| \cdot |OQ| = \text{constant}$ (i.e. independent of a motion variable).

As the combined 8-bar linkage of Fig. 1 contains two contra-parallelgrams, and therefore also two of cells of Hart, generalization of each of these cells must be possible. Say, *random* stretch-rotation and generalization turns the first contra-parallelgram linkage into some quadruplanar inversion-cell of Sylvester and Kempe. The latter then contains four new fundamental points, to wit: the old points $O \equiv D$ and $B \equiv Q$ in addition to the new points $D' \equiv P'$ and $B' \equiv Q'$ obtained with the transformation. The new points, P' , O , Q and Q' , replacing the former points P , O , Q and U of Hart's cell, do form a *parallelogram* with varying sides enclosing constant angles, namely $\pi - \beta$ and β , the top-angle of the attached triangles. Further, the product of two different lengths of these sides, is independent of any motion variable. (See again Fig. 2.) The newly obtained fundamental points are now to be used as *common turning-joints* between the first and the second quadruplanar inversion-cell of Sylvester and Kempe. Permissible only, if the second one, obtained by generalization of the second contra-parallelgram, obtains the same fundamental points P' , O , Q and Q' after transformation. Whence the (second) generalization of the second contra-parallelgram, is not a randomly chosen one like the first generalization, but fully depends on the first generalization.

The details of the generalization now directly applied on the kinematic chain of Fig. 1, will be given through the assignments (see Fig. 3):

- Start the design with the two (four times) interconnected contra-parallelgram chains $AS\bar{A}S$ and $CR\bar{C}R$ as demonstrated in Fig. 1.
- Stretch-rotate the contra-parallelgram linkage $AS\bar{A}S$ about A with the *random* and complex factor $f_s = S'A/SA$, resulting into the contra-parallelgram linkage $AS'\bar{A}'S'$ similar to $AS\bar{A}S$.
- Form the rigid and similar triangles ASS' and $A\bar{S}\bar{S}'$.
- Form the rigid and mutually similar triangles $\bar{A}'D'S'$, ADS' , ABS' and $\bar{A}'B'\bar{S}'$.
- Note that $|OP'| \cdot |OQ| = (|DS'|/|DS|)|OP| \cdot |OQ| = |S'D'| \cdot |DA| \{ (|S'\bar{A}'|^2/|S'A|^2) - 1 \} = \text{constant}$, whereas also $\angle QOP' = \pi - \beta = \text{constant}$.
- Remark that the linkage-dyad DSP has been stretch-rotated about D into the linkage-dyad $DS'P'$, making $\triangle DS'P' \sim \triangle DSP$.
- Now, stretch-rotate the linkage-dyad DRP into $DR'P'$, similarly making $\triangle DR'P' \sim \triangle DRP$. Thus, $\square S'P'R'D \sim \square SPRD$, whence the four-bar $SPRD$ has been stretch-rotated about its turning-joint D into the four-bar $S'P'R'D$.
- Stretch-rotate the rigid $\triangle CRR'$ into the similar $\triangle C\bar{R}\bar{R}'$ (also rigid), determining turning-joint $\bar{R}'$.
- Stretch-rotate the contra-parallelgram chain $CR\bar{C}R$ about C into the similar contra-parallelgram-linkage $CR'\bar{C}'R'$.
- Form the rigid and mutual similar triangles $\bar{C}'R'D'$, $CR'D$, $CR'Q$ and $\bar{C}'\bar{R}'Q'$.

Fig. 3. Two dissimilar quadruplanar inversion-cells interlinked at $DD'B'B$ forming a parallelogram for which $\overline{OP'}$, $\overline{OQ} = \text{constant}$ while $\angle QOP' = \pi - \beta$.

The two contra-parallelogram-linkages $AS'\bar{A}'S'$ and $CR'\bar{C}'R'$ now possess the *common turning-joints* D , B , B' and D' . Clearly, the common joints form a parallelogram with varying sides enclosing angles being *independent* of the motion variable. Though the two contra-parallelogram-linkages, contained in the 8-bar, are not similar, the two quadruplanar inversion-cells (of Sylvester and Kempe) may be regarded as *cognates*, having four common turning-joints forming a common parallelogram $DBB'D'$, whereas both sustain the identical property of inverting planar curves in the same way. The 8-bar linkage-combination, just designed, represents the most generalized possibility to interconnect two entirely different contra-parallelogram linkages in this excessive way. Studying the number of degrees of freedom in design of this 8-bar, one notes that the 2nd quadruplanar inversion-cell is completely determined by the first and the linkage-dyad $DR'D'$, the latter being linked to the first cell at D and D' . Thus, *two* additional coordinates for joint R' in addition to those determining the first quadruplanar inversion-cell. Both cells possess the same top-angles β (eventually differing π radians) of the rigidly attached triangles, but the corresponding base-angles are generally different. Further note that *any* pair of triangles with a common vertex possess either the same top-angle at that vertex, or have otherwise top-angles differing π radians.

An easier running but more direct, design of the combined (8-bar) linkage-configuration, would read something like this (see Fig. 3):

- Start with the choice of a *random* four-bar $ABCD$.
- Choose a random turning-joint S' in the moving plane of DA .
- Draw the similar triangles $AS'D$ and $AS'B$, and determine point S' .
- Next, draw the contra-parallelogram $AS'\bar{A}'S'$.
- Reflect the four-bar $ABCD$ about a perpendicular of DB and shift her reflection to the four-bar $\bar{A}'D'\bar{C}'B'$. (Verify the similarity of the triangles $AS'B$ and $\bar{A}'S'B'$, further note that $\square DBB'D'$ represents a parallelogram.)
- Determine R' such that $\angle CDR' = \pi - \angle S'DA = \angle \bar{C}'D'R'$ (i.e. $\triangle CDR' \sim \triangle \bar{C}'D'R'$).
- Draw the contra-parallelogram $CR'\bar{C}'R'$ and draw its similar and rigid triangles CDR' , $\bar{C}'D'R'$, $\bar{C}'B'R'$ and CBR' .

This establishes the design of the generalized 8-bar containing two quadruplanar inversion-cells of Sylvester and Kempe being interconnected through the 4 turning-joints D, D', B' and B , forming a parallelogram for which $|DD'| \cdot |DB| = \text{constant}$ as well as $\angle BDD' = \pi - \beta = \text{constant}$ (see

Fig. 3). The (four times) overconstrained, now generalized inversion-cell corresponds even to an *infinite* number of cells as demonstrated in Fig. 1. One of them may be obtained as follows:

- (h) Take a *random* point S at AD or at its prolongation.
- (i) Make the four-bar $S\bar{B}RD$ similar to the four-bar $S'D'R'D$, yielding the points $\bar{B} \equiv P$ and R .
- (j) Determine T at the intersection of AC and SR .
- (k) Draw the symmetry-axis, joining T , running normal to BD .
- (l) Determine the image-points $\bar{A}$, $\bar{D}$, $\bar{S}$, $\bar{R}$ and $\bar{C}$.

The design of the generalized 8-bar clearly needs 10 co-ordinates, i.e. *eight* for the random choice of the four-bar $ABCD$ and *two* for the random choice of point S' in the plane attached to DA . (The choice of point S is not necessary for the determination of the generalized 8-bar.)

NEW INVERSION-CELL

The general 8-bar linkage, obtained from the four times overconstrained linkage chain of Fig. 1, naturally, is quadruple overconstrained too. As a consequence, *two* ternary links having a *common* turning-joint, say the ones with their top-angle at their common turning-joint B' , may be omitted. The 8-bar linkage chain then turns into a *six*-bar linkage of Watt's type (see Fig. 4). The remaining linkage then consists of a *random* four-bar, $ABCD$, and the linkage-dyad $S'D'R'$. The turning-joints S' and R' of this dyad, are the vertices of the rigid triangles $S'DA$ and $R'DC$, rigidly attached to the respective sides DA and DC of the initial four-bar $ABCD$. Both triangles attached to the four-bar, possess either the same angle β at the common vertex D , or have angles at D being each other's supplement. The other corresponding angles are generally different. Only in case the initial four-bar $ABCD$ has perpendicular diagonals AC and BD , the attached pair of triangles are similar. (Then, the four-bar $R'D'S'D$ possesses perpendicular diagonals too; thus, $R'S' \perp D'D$ in that case.)

The *general* design of the *inversion-cell* may now be obtained through the following assignments (see Figs 3 and 4):

- (1) Start the design with a *random* four-bar $ABCD$.
- (2) Choose a random turning-joint S' in the moving plane of DA .
- (3) Determine the image AB_i of AB with respect to the normal of BD passing through A .
- (4) Make $\triangle DS'D'$ similar to $\triangle DAB_i$ and establish the location of turning-joint $D' \equiv P'$.

Fig. 4. New inversion cell with a random 4-bar $ABCD$ and a random joint S' . $\overline{OP'}$, $\overline{OQ}$ = constant.

Fig. 5. Special case with $T \rightarrow \infty$ (reflected 4-bars with perpendicular diagonals).

- (5) Determine the image CB_j of CB with respect to the normal of BD joining C .
- (6) Make $\triangle DR'D'$ similar to $\triangle DCB_j$, and determine turning-joint R' .

It is now possible to prove that the so-obtained cell indeed meets the two conditions for *geometric inversion* for *any* value of the motion-variable, namely:

$$|OP'| \cdot |OQ| = |DA| \cdot |DS'| \{1 - (|S'\bar{A}'|^2 / |S'A|^2)\} = \text{constant, and } \angle QOP' = \pi - \beta = \text{constant.}$$

(Note, that by agreement $O \equiv D$, $P' \equiv D'$, $Q \equiv B$, $Q' \equiv B'$, $U \equiv \bar{D}$ and $P \equiv \bar{B}$.)

Taking point $O \equiv D$ as a fixed turning-joint, *any* circle traced by P' will then be transformed into the inverse circle, traced by point Q , and conversely.

Apart from the *four* sides of the random four-bar, the inversion-cell is determined by 2 additional design-degrees of freedom, namely *two* for the random choice of S' .

In the particular case for which the four-bar $ABCD$ has perpendicular diagonals, the rigid triangles $R'DC$ and $S'DA$ become similar, whereas the 2nd four-bar $S'D'R'D$, comprised in the cell, will also get perpendicular diagonals.

(Note that, according to Ref. [8], the perpendicularity between the diagonals of the four-bars remains true for the entire motion of the cell.) In that case 1 degree-of-freedom-in-design has been lost. See also Fig. 8, or Fig. 10 of Ref. [4] in this particular case.

In the *very* particular case for which the four-bar $ABCD$ represents a *kite* and further $R' = C$, one regains *Peaucellier's* inversion-cell, also containing 6 links (see again Ref. [4]).

PARTICULAR CASE

An interesting case occurs when the perspectively-center T of Fig. 1 goes to the infinite. Thus, when $T = T^\infty$. Then, all four-bars, with the exception of the two contra-parallelograms, have *perpendicular diagonals* (see Fig. 5 and also Fig. 8 of Ref. [3]).

Note that once the diagonals of a four-bar linkage are perpendicular, they *remain perpendicular* throughout the complete motion; a proposition easily proved through the knowledge that it is possible to move *any* linkage-polygon, though its turning-joints remain at perpendicular axes causing each side to produce the *elliptic* motion (see Figs 6 and 14 of Ref. [6], and also Ref. [8]).

Random choice of S at AD

Fig. 6. Derivation of a generalized configuration containing 4 four-bar linkages with perpendicular diagonals.

Generalization of Fig. 5, like we did with Fig. 1, interconnecting two quadruplanar inversion-cells of Sylvester and Kempe, produces equally similar four-bars as in the case of Fig. 1 (see Fig. 6):

$$\square S'D'R'D \sim \square S\bar{B}RD, (\square S'B'R'B \sim \square S\bar{D}R\bar{B}) \text{ and } \square \bar{A}'B'C'D' \sim \square \bar{A}DC\bar{B}.$$

(Corresponding sides of $\bar{A}'B'C'D'$ run even parallel to those of $\bar{A}DC\bar{B}$!) We even find that two pairs of reflected equal four-bars, namely ($S'D'R'D$ with $S'B'R'B$) as well as the pair ($ABCD$ with $\bar{A}'D'C'B'$) all possess perpendicular diagonals. As Fig. 5 further shows that $\triangle ADC \sim \triangle SDR$, the similarity between the four-bars $\square S'D'R'D$ and $\square S\bar{B}RD$ just results into the equations:

$$R'D/S'D = RD/SD = CD/AD.$$

As finally, $\angle R'DC = \pi - \angle RDR' = \pi - \angle SDS' = \angle S'DA = \beta$, we find that in this case, $\triangle R'DC \sim \triangle S'DA$.

Since *each* of these triangles are similar to three others, each of them being attached to a different side of a contra-parallelogram, all eight triangles, i.e. $\triangle ADS'$, $\triangle ABS'$, $\triangle CBR'$, $\triangle CDR'$, $\triangle C'D'R'$, $\triangle \bar{A}'D'S'$, $\triangle \bar{A}'B'S'$ and $\triangle \bar{C}'B'R'$, are all mutually similar (see Fig. 7). This shows that when two pairs of reflected equal four-bars, contained in the generalized 8-bar, all have *perpendicular diagonals*, not only the top-angles (β), but also the corresponding base-angles (α) have to be the same in that particular case.

(Thus, $\angle DAS' = \angle BASS' = \angle \dots = \alpha$. See Fig. 7 and also Ref. [4].)

Fig. 7. Special case with 8 similar triangles.

DOS CONEXIONES DE PARALELOGRAMOS NO SIMILARES CONECTADAS EN LOS VERTICES DE UN PARALELOGRAMO

Resumen—A pesar de una supuesta limitación del movimiento, se pueden conectar dos conexiones de contrapalelogramos completamente diferentes de una manera excesiva con hasta cuatro puntos de gira (normales), colocados en los vértices de un paralelogramo. Durante el movimiento los ángulos encerrados entre los lados de este paralelogramo no varían, pero la longitud sí varía aunque el producto continúa constante. El superlimitado mecanismo de conexión en general lleva a la derivación de un mecanismo de inversión completamente nuevo. Su célula de inversión es una conexión de seis barras de tipo Watt, obtenida simplemente del punto de gira conectado, eliminando dos barras conectadas con sus puntos de gira. Se puede probar que la célula invertida de *Peaucellier* sólo aparece como un caso muy especial.