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The abstract Cauchy problem for the fractional evolution equation

by

E. Bazhlekov
THE ABSTRACT CAUCHY PROBLEM FOR THE FRACTIONAL EVOLUTION EQUATION

E. Bazhlekova*

Abstract

We give necessary and sufficient conditions for an unbounded closed operator \( A \) in a Banach space \( X \) such that the abstract Cauchy problem for the fractional evolution equation

\[
D^\alpha u = Au, \quad 0 < \alpha < 1,
\]

can be solved. We also present conditions on \( A \) such that the fractional time evolution is holomorphic. A relation between the solutions of the problem for different \( \alpha \), \( 0 < \alpha \leq 1 \), is obtained. In particular this relation shows that the problem has a holomorphic solution whenever \( A \) generates just a \( C_0 \)-semigroup. Some examples are given.

Mathematics Subject Classification: 12H22, 26A33, 35F10, 47D06.

Key Words and Phrases: fractional calculus, semigroup theory, Mittag-Leffler function, Wright function.

1. Introduction

Consider a linear closed operator \( A \) in a Banach space \( X \) with dense domain \( D(A) \subset X \). Let \( 0 < \alpha < 1 \). Given \( x \in X \) we investigate the following Cauchy problem in \( X \):

\[
D_t^\alpha u(t) = Au(t),
\]

\[
u(0) = x.
\]

Here \( D_t^\alpha \) is the regularized fractional derivative of order \( \alpha \), \( 0 < \alpha < 1 \),

\[
D_t^\alpha u(t) = J_t^{1-\alpha} \frac{d}{dt} u(t),
\]

where \( J_t^\alpha \) is the fractional integration operator defined by

\[
J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) \, d\tau, \quad \alpha > 0.
\]

The connection between \( D_t^\alpha \) and the Riemann-Liouville fractional derivative

\[
D_t^\alpha u(t) = \frac{d}{dt} J_t^{1-\alpha} u(t)
\]
is given by
\[ D_\alpha^\alpha u(t) = D_\alpha^\alpha (u(t) - u(0)). \] (1.6)

The definition (1.5) is the most commonly adopted in mathematical oriented papers and text-books (see e.g. [16]). The definition (1.3)\(^1\) is more suitable for problems of physical interest (see [9], [13]) where the conventional initial conditions are expressed in terms of integer derivatives.

For \(\alpha = 1\) the problem (1.1-2) is the classical abstract Cauchy problem and there is a vast amount of literature devoted to it and its equivalent formulation – the *semigroup theory* (see [6], [17], [22] and the references there). The case \(\alpha = 2\) has been investigated in detail in [7], higher order equations have been also discussed. The abstract Cauchy problem for non-integer values of \(\alpha\) is investigated in [10] and [11] (for \(0 < \alpha < 1\)), [12] (for \(\alpha > 0\)), [19] and [20] (for \(0 < \alpha \leq 2\)). More general *evolutionary equations* are studied in the recent monograph [18].

For integer \(\alpha\) the following results are classical:

The Cauchy problem (1.1-2) can be solved if and only if

- \(\alpha = 1\): \(A\) is the generator of a strongly continuous (*C\(_0\)*) semigroup (the *theorem of Hille-Yosida*, see [6], [17], [22]);
- \(\alpha = 2\): \(A\) is the generator of a strongly continuous cosine function (this implies: \(A\)-generator of a holomorphic semigroup, see [6], p.100);
- \(\alpha = 3, 4, \ldots\): \(A\) is a bounded operator (see [6], p.99).

Roughly speaking, the set of “permitted” operators \(A\) “shrinks” as \(\alpha\) increases. We will show that this holds in some sense also for \(0 < \alpha \leq 1\).

Using ideas related to the theory of first and second order abstract differential equations ([6], [7]) we give in Theorem 3.1. necessary and sufficient conditions for solvability of the Cauchy problem (1.1-2), extending the conditions of the Hille-Yosida theorem from \(\alpha = 1\) to \(\alpha \in (0, 1)\). (Let us note that Theorem 3.1. also follows as a particular case of a more general theorem given in [18], p. 43.) Theorem 4.1. gives a sufficient condition when the solution is analytic in some sector of the complex plane containing the semiaxis \(t > 0\) and describes the properties of the solution in this case. For \(\alpha = 1\) this result coincides with a well-known result concerning analytic semigroups (see [6], [17], [22]). Using the representations of the Mittag-Leffler functions ([4], p.5-6), we give in Theorem 4.2. a relation between the solutions of (1.1-2) for different \(\alpha\), \(0 < \alpha \leq 1\), in terms of an auxiliary function. In particular this relation proves that if \(A\) is the generator of a \(C\(_0\)*-semigroup then the problem (1.1-2) with \(0 < \alpha < 1\) has an analytic solution. Some examples show that the analytic properties of the solution can be achieved for operators \(A\) that do not generate \(C\(_0\)*-semigroup.

\(^1\) This (alternative to Riemann-Liouville) definition of fractional derivative was originally introduced by Caputo [1,2] in the late sixties and adopted by Caputo and Mainardi [3] in the framework of the theory of Linear Viscoelasticity.
2. Preliminaries

We summarize some properties of the fractional integrals and derivatives (for more details see [9], [13] and [16]). Similarly to the ordinary differentiation and integration we have:

\[ D_0^\alpha J_0^\alpha u(t) = u(t); \quad J_0^\alpha D_0^\alpha u(t) = u(t) - u(0), \quad 0 < \alpha < 1. \]  

(2.1)

Simple results valid for \( t > 0 \) are:

\[ J_0^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\alpha + \beta}, \quad D_0^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}, \quad \alpha > 0, \beta > -1. \]  

(2.2)

If \( \beta - \alpha + 1 \in \{0, -1, -2, \ldots \} \) then \( \Gamma(\beta + 1)/\Gamma(\beta - \alpha + 1) = 0 \) in (2.2). In particular \( D_0^\alpha t^{\alpha - 1} = 0, \quad D_0^\alpha 1 = t^{-\alpha}/\Gamma(1 - \alpha), \quad \) while \( D_0^\alpha 1 = 0, \alpha > 0. \)

Using the properties of the Laplace transform it is not difficult to prove that

\[ \mathcal{L}\{D_0^\alpha u\}(\lambda) = \lambda^\alpha \mathcal{L}\{u\}(\lambda) - \lambda^{\alpha - 1} u(0). \]  

(2.3)

The Mittag-Leffler functions, defined by the following series

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C} \]  

(2.4)

are entire functions which provide a simple generalization of the exponential function \( e^z = E_1(z) \) and play an important role in the theory of fractional differential equations. Similarly to the differential equation \( d/dt(e^{\omega t}) = \omega e^{\omega t} \) the Mittag-Leffler function \( E_\alpha(z) \) satisfies the more general differential relation

\[ D_0^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha). \]  

(2.5)

The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

\[ \mathcal{L}\{E_\alpha(\omega t^\alpha)\}(\lambda) = \frac{\lambda^{\alpha - 1}}{\lambda^\alpha - \omega} \]  

(2.6)

and with their asymptotic expansions as \( z \to \infty \) (see [4],[5]). If \( 0 < \alpha < 2, \beta > 0 \)

\[ E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| \leq \frac{1}{2} \alpha \pi, \]  

(2.7)

\[ E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < (1 - \frac{1}{2} \alpha) \pi, \]  

(2.8)

where

\[ \varepsilon_{\alpha,\beta}(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad z \to \infty. \]

**Definition 3.1.** A strong solution of the abstract differential equation (1.1) in $t \geq 0$ ($t > 0$) is an $X$-valued function $u(t)$ such that $u(t)$ is continuous for $t \geq 0$, $u(t) \in D(A)$ and $Df_t u(t)$ is continuous for $t \geq 0$ ($t > 0$) and (1.1) is satisfied for $t \geq 0$ ($t > 0$).

**Definition 3.2.** ([12]) A Cauchy problem is called solvable in a sharply uniformly well-posed manner (SUWP-solvable) if:

(a) There exists a dense subspace $D$ of $X$ such that for any $x \in D$ there exists a solution $u(t)$ of (1.1) in $t \geq 0$ satisfying (1.2);

(b) There exist constants $M \geq 1$, $\omega \geq 0$, not depending on $u(t)$ and $x$, such that

$$\|u(t)\| \leq Me^{\omega t}\|u(0)\|, \quad t \geq 0. \quad (3.1)$$

for any solution $u(t)$ of (1.1) in $t \geq 0$.

**Definition 3.3.** Let $0 < \alpha \leq 1$, $M \geq 1$, $\omega \geq 0$. An operator $A$ is said to belong to $C^\alpha (M, \omega)$ if the Cauchy problem (1.1-2) satisfies (a) and (b).

Denote $C^\alpha (\omega) = \bigcup \{C^\alpha (M, \omega); \quad M \geq 1\}$, $C^\alpha = \bigcup \{C^\alpha (\omega); \quad \omega \geq 0\}$. Let us note that $A \in C^1$ iff $A$ generates a $C_0$-semigroup.

In analogy with the case $\alpha = 1$ we define a propagator $P^\alpha_t$ of a SUWP-solvable Cauchy problem, i.e. a bounded operator $P^\alpha_t$ such that $u(t) = P^\alpha_t x$ for any solution $u(t)$ of (1.1-2).

Let $\rho(A)$ denote the resolvent set and $R(\lambda) = (\lambda I - A)^{-1}$ the resolvent operator of $A$, as usual.

**Theorem 3.1.** Let $0 < \alpha \leq 1$, $M \geq 1$, $\omega \geq 0$. Then $A \in C^\alpha (M, \omega)$ iff $R(\lambda)$ exists in the half plane $Re \lambda > \omega$ and

$$\|\lambda^{\alpha-1} R(\lambda) \|^{(n)} \leq \frac{M n!}{(Re \lambda - \omega)^{n+1}}, \quad Re \lambda > \omega, \quad n = 0, 1, \ldots \quad (3.2)$$

**Proof.** Assume that the problem (1.1-2) is SUWP-solvable and let $P^\alpha_t$ be the corresponding propagator, so that

$$\|P^\alpha_t\| \leq Me^{\omega t}, \quad t \geq 0. \quad (3.3)$$

Fix $\lambda \in \mathbb{C}$ with $Re \lambda > \omega$ and define an operator on $X$ by

$$S(\lambda) x = \int_0^\infty e^{-\lambda t} P^\alpha_t x \, dt, \quad x \in X. \quad$$

Since the norm of the integrand is bounded by $M\|x\|e^{(\omega - Re \lambda) t}$, $S(\lambda)$ is a bounded operator in $X$. Assume now that $x \in D$. Then $P^\alpha_t x$ is a solution of (1.1-2)
and this together with the closedness of \( A \) and (2.3) implies that \( S(\lambda)x \in D(A) \) and \( AS(\lambda)x = \lambda^\alpha S(\lambda)x - \lambda^{\alpha - 1}x \). Using again the closedness of \( A \) it follows \( S(\lambda)X \subseteq D(A) \) and
\[
\lambda^{1 - \alpha}(\lambda^\alpha I - A)S(\lambda) = I.
\]

In particular (3.4) shows that \((\lambda^\alpha I - A) : D(A) \to X\) is onto. It is injective as well. To see this assume there exists \( x \in D(A) \) with \( Ax = \lambda^\alpha x \). Then from (2.5) \( u(t) = E_\alpha(\lambda^\alpha t^\alpha)x \) is a solution of (1.1-2). Since \(|\arg(\lambda^\alpha t^\alpha)| \leq \frac{1}{2}\alpha\pi\) when \( \text{Re}\lambda > \omega \geq 0, t > 0 \), from (2.7) we have
\[
|E_\alpha(\lambda^\alpha t^\alpha)| \geq \frac{1}{\alpha}e^{\text{Re}\lambda t} - \varepsilon > M e^\omega t, \quad t > T,
\]
that contradicts (3.1) unless \( x = 0 \). Thus we have proved that \( R(\lambda^\alpha) \) exists when \( \text{Re}\lambda > \omega \) and \( S(\lambda) = \lambda^{\alpha - 1}R(\lambda^\alpha) \), that is
\[
\lambda^{\alpha - 1}R(\lambda^\alpha)x = \int_0^\infty e^{-\lambda t}P_t^\alpha x \, dt, \quad x \in X.
\]

After easily justified differentiation under the integral sign we obtain
\[
(\lambda^{\alpha - 1}R(\lambda^\alpha))^{(n)}x = (-1)^n \int_0^\infty t^n e^{-\lambda t}P_t^\alpha x \, dt, \quad n = 0, 1, \ldots, \quad x \in X, \text{ Re}\lambda > \omega,
\]
that together with (3.3) gives (3.2).

Suppose now that \( R(\lambda^\alpha) \) exists in the half plane \( \text{Re}\lambda > \omega \) and the conditions (3.2) hold. We begin by constructing certain smooth solutions of (1.1-2). Let \( x \in D(A^m) \) for some \( m > 2/\alpha \) and define \( u(t) \) as the inverse Laplace transform of \( \lambda^{\alpha - 1}R(\lambda^\alpha)x \). Using the well-known formula
\[
R(\lambda)x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}Ax + \ldots + \frac{1}{\lambda^m}A^{m-1}x + \frac{1}{\lambda^m}R(\lambda)A^m x,
\]
we have
\[
u(t) = x + \frac{t^\alpha}{\Gamma(\alpha + 1)}Ax + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}A^2 x + \ldots + \frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha + 1)}A^m x + I(t),
\]
where
\[
I(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} e^{\lambda t} \lambda^{\alpha - 1}R(\lambda^\alpha)A^m x \, d\lambda, \quad \omega' > \omega.
\]
Since \(|e^{\lambda t}\lambda^{\alpha - \beta}R(\lambda^\alpha)A^m x| \leq C e^{\omega t} |\lambda|^{-\beta} \) for \( \text{Re}\lambda = \omega' \) the integral
\[
\frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} e^{\lambda t} \lambda^{\alpha - 1}R(\lambda^\alpha)A^m x \, d\lambda
\]
converges absolutely and uniformly for \( \beta > 1 \) with respect to \( t \) on compact subsets of \( t \geq 0 \). Therefore, taking \( \beta = m\alpha \) and \( \beta = m\alpha - 1 \), we see that \( I(t) \) and \( I'(t) \)
are continuous for $t \geq 0$, from (1.3) the same is true for $D_t^\alpha I(t)$, hence for $u(t)$ and $D_t^\alpha u(t)$.

Let now $t = 0$ and $\Lambda_R$ be the part of the circle $|\lambda| = R$ to the right of the line $\text{Re}\lambda = \omega'$. Then

$$\|I(0)\| \leq \frac{C}{2\pi} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{|d\lambda|}{|\lambda|^{m\alpha}} \leq \frac{C}{2\pi} \lim_{R \to \infty} \int_{\Lambda_R} \frac{|d\lambda|}{|\lambda|^{m\alpha}} \leq C \lim_{R \to \infty} \frac{1}{R^{m\alpha - 1}} = 0,$$

so that $u(0) = x$. It remains to check (1.1). Denote

$$I_r(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t} - \alpha E_\alpha(\lambda^\alpha t^\alpha)}{\lambda^\beta} \lambda^{\alpha - 1} R(\lambda^\alpha) A^m x d\lambda, \quad t > 0.$$

Since the integrand is a holomorphic function with respect to $\lambda$ for $\text{Re}\lambda > \omega$ we can shift the path of integration to $\Lambda_R$, $R = \sqrt{r^2 + \omega'^2}$. But from (2.7) it follows that $\alpha E_\alpha(\lambda^\alpha t^\alpha) = e^{\lambda t} + O((|\lambda| t)^{-\alpha})$ for $\text{Re}\lambda > \omega$, $t > 0$, $\lambda \to \infty$. Hence

$$\|I_r(t)\| \leq \frac{C'}{2\pi} \int_{\Lambda_R} \frac{|d\lambda|}{|\lambda|^\beta + \alpha} \leq \frac{C'}{R^\beta + \alpha - 1} \to 0, \quad r \to \infty, \beta > 1 - \alpha, \quad t > 0,$$

that is if $\beta > 1 - \alpha$, $t > 0$

$$\frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t} - \alpha E_\alpha(\lambda^\alpha t^\alpha)}{\lambda^\beta} \lambda^{\alpha - 1} R(\lambda^\alpha) A^m x d\lambda = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{E_\alpha(\lambda^\alpha t^\alpha)}{\lambda^\beta} \lambda^\alpha - 1 R(\lambda^\alpha) A^m x d\lambda.$$

(3.9)

Taking $\beta = m\alpha$ we have

$$I(t) = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{E_\alpha(\lambda^\alpha t^\alpha)}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha) A^m x d\lambda, \quad t > 0,$$

(3.10)

and after differintegration under the integral sign justified on the basis of the asymptotic expansion (2.7) for $d/dt E_\alpha(\lambda^\alpha t^\alpha) = \lambda^\alpha t^\alpha - 1 E_\alpha(\lambda^\alpha t^\alpha)$, the uniform convergence of (3.8) for $\beta = m\alpha - 1$ on compacts of $t \geq 0$ and Fubini's theorem we obtain

$$D_t^\alpha I(t) = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{\lambda^\alpha E_\alpha(\lambda^\alpha t^\alpha)}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha) A^m x d\lambda, \quad t > 0.$$

Again from (3.9) with $\beta = (m - 1)\alpha$ we have

$$D_t^\alpha I(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{\lambda^\alpha e^{\lambda t}}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha) A^m x d\lambda, \quad t > 0.$$

(3.11)

On the other hand $A$ can be applied under the integral sign in (3.7) with the convergence of the resulting integral, so that $u(t) \in D(A)$ and using (1.6), (2.2) and (3.11) we have

$$D_t^\alpha u(t) - Au(t) =$$
\[
-\frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha + 1)} A^m x + \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^{\alpha-1}} R(\lambda^\omega)(\lambda^\omega I - A) A^m x \, d\lambda = 0,
\]

hence \( u(t) \) is a strong solution of (1.1-2) in \( t \geq 0 \) for \( x \in D(A^m) \).

To prove the inequality (3.1) we use the Post-Widder inversion formula:

**Lemma 3.1.** Let \( u(t) \) be a \( X \) valued continuous function defined in \( t \geq 0 \) such that \( u(t) = O(\exp(\gamma t)) \) as \( t \to \infty \) for some \( \gamma \) and let \( (Lu)(\lambda) \) be the Laplace transform of \( u(t) \). Then

\[
u(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left( \frac{n}{t} \right)^{n+1} (Lu)^{(n)} \left( \frac{n}{t} \right)
\]

uniformly on compacts of \( t > 0 \).

Routine calculations show that \( u(t) \) satisfies the conditions of Lemma 3.1. with \( \gamma = \omega' \) and \( (Lu)(\lambda) = \lambda^{\alpha-1} R(\lambda^\omega)x \). Then, from (3.12) and the inequalities (3.2) it follows

\[
\|u(t)\| \leq M\|x\| \lim_{n \to \infty} \frac{(n/t)^{n+1}}{(n/t - \omega)^{n+1}} = M\|x\| \lim_{n \to \infty} \left( \frac{1 - \omega t}{n} \right)^{-(n+1)} = M\|x\|e^{\omega t}.
\]

Next we define \( \hat{P_t}^\omega x = u(t) \) for \( x \in D(A^m) \), \( t \geq 0 \), where \( u(t) \) is the solution obtained. Since \( D(A) \) is dense in \( X \) and \( g(A) \neq \emptyset \) then \( D(A^m) \) is dense in \( X \) and \( \hat{P_t}^\omega \) can be extended to a bounded operator in \( X \) without increase of norm, that is (3.3) holds for \( \hat{P_t}^\omega \). The proof will be complete if we show that any solution \( u(t) \) of (1.1-2) admits the representation \( u(t) = \hat{P_t}^\omega u(0) \). Since (1.1) must hold for \( t = 0 \) there is no solution for \( x \notin D(A) \) and it is enough to establish that \( \hat{P_t}^\omega x \) is the unique solution of (1.1-2) for any \( x \in D(A) \).

Actually for the uniqueness of the solution a weaker condition suffices:

**Theorem 3.2.** ([5]) If \( \text{Re} \lambda \) exists on the half line \( \lambda > \delta \) (\( \delta > 0 \)) and

\[
\limsup_{\lambda \to \infty} \frac{(1/\lambda) \ln \| R(\lambda^\omega) \|}{\lambda} = 0
\]

then the solution of the problem (1.1-2) is unique.

It is not difficult to check that the inequalities (3.2) imply (3.13) and so we have uniqueness of the solution of (1.1-2).

Next we show that \( \hat{P_t}^\omega x \) satisfies (1.1-2) for any \( x \in D(A) \). If \( \lambda \in g(A) \) it follows immediately from the definition of \( \hat{P_t}^\omega \) that

\[
R(\lambda) \hat{P_t}^\omega x = \hat{P_t}^\omega R(\lambda)x, \quad x \in D(A^m), \quad t \geq 0.
\]

By the usual continuity argument (3.14) must hold for all \( x \in X \). Noting that \( D(A) = R(\lambda)X \), (3.14) implies \( \hat{P_t}^\omega D(A) \subseteq D(A), \quad t \geq 0 \) and

\[
A \hat{P_t}^\omega x = \hat{P_t}^\omega Ax, \quad x \in D(A), \quad t \geq 0.
\]
If we apply $J^\alpha_t$ to both sides of (1.1) and use (1.2), (2.1) and (3.15) we have

$$\hat{P}^\alpha_t x - x = J^\alpha_t \hat{P}^\alpha_t Ax, \quad x \in D(A^m).$$

Applying now $R(\lambda)$ to both sides of this equality and using (3.14) it follows

$$R(\lambda)(\hat{P}^\alpha_t x - x) = J^\alpha_t \hat{P}^\alpha_t A R(\lambda)x, \quad x \in D(A^m).$$

Since on both sides bounded operators are applied to $x$, the last equality must hold for all $x \in X$. Making use of the fact that $R(\lambda)$ is one-to-one, we conclude that

$$\hat{P}^\alpha_t x - x = J^\alpha_t A \hat{P}^\alpha_t x, \quad x \in D(A).$$

Therefore $\hat{P}^\alpha_t x$ is the (unique) solution of (1.1-2).\hfill\Box

**Remark 3.1.** Inequalities (3.2) follow from their real counterparts

$$\|(\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)}\| \leq M \frac{n!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0, 1, \ldots$$

The proof uses Taylor series expansion for the function $\lambda^{\alpha-1}R(\lambda^\alpha)$.

**Corollary 3.1.** If $A \in C^1(\omega)$ then $A \in C^\alpha(\omega^{1/\alpha})$ for any $\alpha \in (0, 1)$.

**Proof.** By induction in $n$ we obtain the following representation

$$(\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)} = (-1)^n \sum_{k=1}^{n+1} c^{k,n}_{k,n} \lambda^{k\alpha-n-1}R(\lambda^\alpha)^k, \quad n = 0, 1, \ldots, \quad (3.16)$$

where $c^{k,n}_{k,n} \geq 0$ are constants, satisfying $c^{0,0}_{1,0} = 1$, $c^{n}_{1,n} = (n - \alpha)c^{n}_{1,n-1}$, $c^{n}_{k,n} = (n - k\alpha)c^{n}_{k,n-1} + \alpha(k - 1)c^{n}_{k-1,n-1}$, $k = 2, \ldots, n$, $c^{n}_{n+1,n} = \alpha c^{n}_{n,n-1}$, $n = 1, 2, \ldots$. Then inequalities (3.2) follow using (3.16). Since Corollary 3.1. is a particular case of the more general Theorem 4.2., we omit the details of the proof.\hfill\Box

From (3.5), (3.12) and (3.16) we obtain the following representation:

**Corollary 3.2.**

$$P^\alpha_t x = \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{n+1} c^{k,n}_{k,n} (I - (t/n)^\alpha A)^{-k} x, \quad (3.17)$$

where $0 < \alpha \leq 1$, $c^{k,n}_{k,n}$ are the constants in (3.16).

Next we express the operator $A$ in terms of the propagator of the Cauchy problem (1.1-2).

**Proposition 3.1.** If $P^\alpha_t$ is the propagator of (1.1-2) and $x \in D(A)$ then

$$Ax = \Gamma(\alpha + 1) \lim_{t \downarrow 0} \frac{P^\alpha_t x - x}{t^\alpha}. \quad (3.18)$$
Proof. For a function \( v(t) \) continuous in \( t \geq 0 \) we have: \( \lim_{t \to 0^+} v(t) = \Gamma(\alpha + 1) \lim_{t \to 0^+} t^{-\alpha} J_\alpha v(t) \). Taking \( v(t) = D^\alpha \rho_t \) and using (1.1) and (2.1) we obtain (3.18).

Incidentally (3.18) shows that \( \rho_t^\alpha x \) is not differentiable at \( t = 0^+ \).

The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.3.** Let \( A \) be an operator such that \( R(\lambda^\alpha) \) exists in the half-plane \( \Re \lambda > \omega \), \( \hat{\rho}_t^\alpha \) an operator-valued function strongly continuous in \( t \geq 0 \) and such that \( \| \hat{\rho}_t^\alpha \| \leq M e^{\omega t}, \quad t \geq 0 \). Assume that for each \( x \in X \)

\[
\int_0^\infty e^{-\lambda t} \hat{\rho}_t^\alpha x \, dt = \lambda^{\alpha-1} R(\lambda^\alpha) x, \quad \Re \lambda > \omega.
\]

Then, \( A \in C^0(\omega) \) and \( \hat{\rho}_t^\alpha \) is the propagator of (1.1-2).

Proof. We obtain the inequalities (3.2) in the same way as from (3.5). Applying Theorem 3.1, it results that \( A \in C^\alpha \). Let \( \hat{\rho}_t^\alpha \) be the propagator of (1.1-2). Then (3.19) holds for both \( \hat{\rho}_t^\alpha \) and \( \hat{\rho}_t^\alpha = \hat{\rho}_t^\alpha \) follows from the uniqueness of the Laplace transform. □

### 4. Holomorphic solutions

**Theorem 4.1.** Let \( 0 < \alpha \leq 1 \). Assume \( \varphi(A) \supset \Sigma_{\delta, \alpha} = \{ \lambda : |\arg \lambda| < \alpha(\pi/2 + \delta); \lambda \neq 0 \} \) for some \( 0 < \delta \leq \pi/2 \) and

\[
\| R(\lambda) \| \leq C/|\lambda|, \quad \lambda \in \Sigma_{\delta, \alpha}.
\]

Then, \( A \in C^\alpha(0) \) and the corresponding propagator \( \rho_t^\alpha \) has the following additional properties:

(a) \( \rho_t^\alpha \) can be extended analytically into the sector \( \Delta_\delta = \{ |\arg t| < \delta; t \neq 0 \} \);

(b) \( \| \rho_t^\alpha \| \) is uniformly bounded in every closed subsector \( \Delta_{\delta - \epsilon} \) of \( \Delta_\delta \);

(c) \( \rho_t^\alpha x \to x \) as \( t \downarrow 0 \), \( t \in \Delta_{\delta - \epsilon}, \quad x \in X \);

(d) For any \( x \in X \), \( \rho_t^\alpha x \in D(A) \) and \( \| A \rho_t^\alpha \| \leq M(1 + t^{-\alpha}) \), \( t > 0 \);

(e) For any \( x \in X \), \( \rho_t^\alpha x \) is a strong solution of (1.1-2) in \( t > 0 \).

**Proof.** Let \( t \in \Delta_{\delta - \epsilon} \) for some \( \epsilon \in (0, \delta) \) and \( \theta \in (\delta - \epsilon/2, \delta), \quad \varphi > 0 \). Set

\[
\hat{\rho}_t^\alpha = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha) \, d\lambda,
\]

where \( \Gamma = \{ re^{-i(\pi/2 + \theta)}; \quad \varphi \leq r < \infty \} \cup \{ re^{i\varphi}; - (\pi/2 + \theta) \leq \varphi \leq \pi/2 + \theta \} \cup \{ re^{i(\pi/2 + \theta)}; \varphi \leq r < \infty \} \) is oriented counterclockwise. If \( \lambda \in \Gamma \) then \( \lambda^\alpha \in \Sigma_{\delta, \alpha} \), and so \( R(\lambda^\alpha) \) exists and from (4.1)

\[
\| \lambda^{\alpha-1} R(\lambda^\alpha) \| \leq C/|\lambda|, \quad \lambda \in \Gamma.
\]
Let $\varrho = 1/|t|$ and $a = \sin \varepsilon/2$. Then from (4.2) and (4.3) it follows

$$
\|\tilde{P}_t^\alpha\| \leq \frac{C}{2\pi} \int_{\Gamma} e^{\Re(\lambda)} \frac{|d\lambda|}{|\lambda|} \leq \frac{C}{\pi} \int_{r_1}^{r_2} e^{-ar} \frac{dr}{r} + \frac{C}{\pi} \int_0^\pi e^{\cos \varphi} d\varphi \leq M. \quad (4.4)
$$

This estimate shows that the integral in (4.2) is absolutely convergent for $t \in \Delta_{\delta-\varepsilon}$, hence (a) and (b) are satisfied for $\tilde{P}_t^\alpha$.

Now fix $\lambda$, $\Re \lambda > 0$, and take $\varrho$ in (4.3) such that $\varrho < \Re \lambda$. Then, using (4.2), we have

$$
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha dt = \frac{1}{2\pi i} \int_{\Gamma} \frac{\mu^{\alpha-1} R(\mu^\alpha)}{\mu - \lambda} e^{-(\lambda - \mu)t} dt d\mu =
$$

$$
= \frac{-1}{2\pi i} \int_{\Gamma} \frac{\mu^{\alpha-1} R(\mu^\alpha)x}{\mu - \lambda} e^{\lambda t} d\mu = \lambda^{\alpha-1} R(\lambda^\alpha),
$$

where Fubini's theorem and Cauchy's integral formula in the region of the right of $\Gamma$ are used. So we have proved that the conditions of Theorem 3.3. are fulfilled with $\omega = 0$. Therefore $A \in C^1(0)$ and the corresponding propagator $P_t^\alpha = \tilde{P}_t^\alpha$.

Next for $x \in D(A)$ by Cauchy's theorem and Cauchy's integral formula if $t \in \Delta_{\delta-\varepsilon}$, $t \downarrow 0$

$$
P_t^\alpha x - x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha)x - \lambda^{-1}x =
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{-1} R(\lambda^\alpha)Ax d\lambda \rightarrow \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} R(\lambda^\alpha)Ax d\lambda = 0,
$$

that together with (b) implies (c).

Now, for $t > 0$ and $\varrho = 1/t$, applying $A$ to both sides of (4.2) and using the identity $AR(\lambda) = \lambda R(\lambda) - I$ we have

$$
AP_t^\alpha = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} A R(\lambda^\alpha) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{2\alpha-1} R(\lambda^\alpha) d\lambda - \frac{t^{\alpha}}{\Gamma(1 - \alpha)}.
$$

Taking $\varrho = 1/t$ and estimating $AP_t^\alpha$ in the same way as in (4.4) we obtain (d).

At the end, applying $J_t^\alpha$ to both sides of (1.1) we have

$$
P_t^\alpha x - x = J_t^\alpha P_t^\alpha Ax, \quad x \in D(A). \quad (4.5)
$$

But from (d) $J_t^\alpha P_t^\alpha A$ is a bounded operator for $t > 0$. Then (4.5) holds for any $x \in X$, $t > 0$. This implies (e).$\Box$

**Proposition 4.1.** If $\varrho(A) \supset \{\lambda : \Re \lambda > 0\}$ and for some constant $M$

$$
\|R(\lambda)\| \leq M/\Re \lambda, \quad \Re \lambda > 0. \quad (4.6)
$$

then for any $\alpha \in (0, 1)$ $A$ satisfies the hypotheses of Theorem 4.1. with $0 < \delta < \min\{(1/\alpha - 1)\pi/2, \pi/2\}$. 
Proof. Fix \( \alpha \) and \( \delta \) satisfying the conditions mentioned above. Then \( \alpha(\pi/2+\delta) < \pi/2 \) and so \( \Sigma_{\alpha,\delta} \subset \rho(A) \). Taking \( \beta, \alpha(\pi/2+\delta) < \beta < \pi/2 \) we obtain

\[
||R(\lambda)|| \leq \frac{M}{\Re \lambda} = \frac{M}{|\lambda| \cos \theta} < \frac{M}{|\lambda| \cos \beta}, \ \lambda \in \Sigma_{\alpha,\delta},
\]

where \( \theta = \arg \lambda \) and (4.1) holds with \( C = M/\cos \beta, \square \)

The next theorem describes the relation between the solutions of the Cauchy problem (1.1-2) for different values of \( \alpha, 0 < \alpha \leq 1 \).

**Theorem 4.2.** Let \( 0 < \alpha < \beta \leq 1, \gamma = \alpha/\beta, \omega \geq 0 \). If \( A \in \mathcal{C}^\beta(\omega) \) then \( A \in \mathcal{C}^\alpha(\omega^{1/\gamma}) \) and the following representation holds

\[
P_t^\alpha x = \frac{1}{t^\gamma} \int_0^\infty \Phi_{\gamma} \left( \frac{s}{t^\gamma} \right) P_s^\beta x \, ds, \quad t > 0,
\]

where

\[
\Phi_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \quad z \in \mathbb{C},
\]

is an entire function. Moreover \( P_t^\alpha \) can be analytically extended to the sector \( \Delta_\theta, \theta = \min\{(1/\gamma - 1)\pi/2, \pi/2\} \) and for any \( x \in X \) \( P_t^\alpha x \) is a strong solution of (1.1-2) in \( \mathbb{T} > 0 \).

Proof. We summarize some properties of \( \Phi_{\gamma}(z) \) (for more details see [4], p. 5, [13], [15] and [21]). It is a particular case of a special function known as Wright function ([5]), entire in the complex plane:

\[
W(z; \gamma, \delta) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\gamma n + \delta)} = \frac{1}{2\pi i} \int_{\Gamma} \mu^{-\delta} \exp(\mu + z\mu^{-\gamma}) \, d\mu, \quad \gamma > -1, \ \delta \in \mathbb{C},
\]

where \( \Gamma \) is a contour which starts at \( -\infty \), encircles the origin once counterclockwise and turns to its starting point. Obviously \( \Phi_{\gamma}(z) = W(-z; -\gamma, 1 - \gamma) \) and the integral representation

\[
\Phi_{\gamma}(z) = \frac{1}{2\pi i} \int_{\Gamma} \mu^{-\gamma - 1} \exp(\mu - z\mu^{-\gamma}) \, d\mu, \quad z \in \mathbb{C},
\]

holds. For real values of \( z \) we have ([4], p. 5):

\[
\Phi_{\gamma}(t) \geq 0, \quad t \geq 0.
\]

Let us denote \( \varphi_{t,\gamma}(s) = (1/t^\gamma)\Phi_{\gamma}(s/t^\gamma) \). Then the following formula is true ([4], p. 5):

\[
E_{\gamma}(\omega t^\gamma) = \int_0^\infty \varphi_{t,\gamma}(s) e^{\omega s} \, ds, \quad t > 0, \quad 0 < \gamma < 1.
\]
From the asymptotic expansion of the Wright function given in [21] (this function is called there generalized Bessel function) we obtain the inequality:

$$|W(-z; -\gamma, \delta)| \leq B \exp(-bz^{1/(1-\gamma)})$$  \hspace{1cm} (4.12)

for $|\arg z| < \min\{(1-\gamma)3\pi/2, \pi\}$, $|z| \to \infty$, $0 < \gamma < 1$, $\delta \in \mathbb{C}$, where $B = B(\gamma, \delta)$, $b = b(\gamma, \delta)$ are positive constants.

Now define

$$\tilde{P}_t^\alpha x = \int_0^\infty \varphi_{t,\gamma}(s)P_s^\beta x \, ds, \quad t > 0.$$  \hspace{1cm} (4.13)

Our aim is to prove that $\tilde{P}_t^\alpha$ satisfies the conditions of Theorem 3.3. Since $A \in C^\beta$, there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$||P_s^\beta|| \leq Me^{\omega s}, \quad s \geq 0,$$  \hspace{1cm} (4.14)

and together with (4.10) and (4.11) we obtain

$$||\tilde{P}_t^\alpha|| \leq \int_0^\infty \varphi_{t,\gamma}(s)||P_s^\beta|| ds \leq M \int_0^\infty \varphi_{t,\gamma}(s)e^{\omega s} ds = ME_\gamma(\omega t^\gamma).$$  \hspace{1cm} (4.15)

We observe that a constant $C_\gamma$ exists, such that

$$E_\gamma(\omega t^\gamma) \leq C_\gamma \exp(\omega^{1/\gamma}t), \quad \omega \geq 0, \quad t \geq 0.$$  \hspace{1cm} (4.16)

Indeed, from (2.7) there exists a constant $C_1$ such that $E_\gamma(\omega t^\gamma) \leq C_1 \exp(\omega^{1/\gamma}t)$ for every $t \geq T$. From the continuity of the Mittag-Leffler function in $t \geq 0$ it follows that there exists a constant $C_2$ such that $E_\gamma(\omega t^\gamma) \leq C_2 \leq C_2 \exp(\omega^{1/\gamma}t)$ for $t \in [0, T]$. Hence, we can take $C_\gamma = \max\{C_1, C_2\}$. Now from (4.15) and (4.16) we have

$$||\tilde{P}_t^\alpha|| \leq M' \exp(\omega^{1/\gamma}t), \quad t \geq 0.$$

Taking $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \omega^{1/\gamma}$ we shall prove that $R(\lambda^\alpha)$ exists. If $\lambda = |\lambda|e^{i\theta}$ then $|\theta| < \pi/2$ and for such $\theta$ and $\gamma \in (0, 1)$ the inequality $\cos \gamma \theta \geq \cos^\gamma \theta$ holds. Then $\text{Re}(\lambda^\gamma) = |\lambda|^\gamma \cos \gamma \theta \geq |\lambda|^\gamma \cos^\gamma \theta = (\text{Re}\lambda)^\gamma > \omega$. Since $A \in C^\beta$, it follows from Theorem 3.1. that $R(\mu^\beta)$ exists in the half-plane $\text{Re}\mu > \omega$. Taking $\mu = \lambda^\gamma$ we obtain that $R(\lambda^\alpha) = R(\mu^\beta)$ exists.

It is plain that $\tilde{P}_t^\alpha$ is strongly continuous for $t > 0$. To prove the strong continuity at the origin we substitute $\sigma = s/t^\gamma$ in (4.13). Then, using the identity $\int_0^\infty \Phi_\gamma(\sigma) \, d\sigma = 1$ (that follows from (4.11) setting $\omega = 0$) and the strong continuity of $P_s^\beta$ at the origin, we have

$$\lim_{t \downarrow 0} \tilde{P}_t^\alpha x = \lim_{t \downarrow 0} \int_0^\infty \Phi_\gamma(\sigma)P_s^\beta x \, d\sigma = \int_0^\infty \Phi_\gamma(\sigma)x \, d\sigma = x.$$  \hspace{1cm} (4.17)
For $\Re \lambda > \omega^{1/\gamma}$, using (4.13) and interchanging the order of integration we have

$$
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha \, dt = \int_0^\infty \int_0^\infty e^{-\lambda t} \varphi_{t,\gamma}(s) \, dt \, ds. \tag{4.18}
$$

Substituting $\mu = \tau \tau$ in (4.9) and shifting the new contour $\Gamma^t = \Gamma / \tau$ to $\Gamma$ we get the integral representation

$$
\varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_{\Gamma} \tau^{\gamma - 1} \exp(\tau t - \tau^\gamma s) \, d\tau, \tag{4.19}
$$

Then, noting that $\Re(\lambda - \tau) > 0$ and using Cauchy’s formula it follows

$$
\int_0^\infty e^{-\lambda t} \varphi_{t,\gamma}(s) \, dt = \frac{1}{2\pi i} \int_{\Gamma} \tau^{\gamma - 1} \exp(-\tau^\gamma s) \int_0^\infty e^{-(\lambda - \tau)t} \, dt \, d\tau =
$$

$$
-\frac{1}{2\pi i} \int_{\Gamma} \tau^{\gamma - 1} \exp(-\tau^\gamma s) \, d\tau = \lambda^{\gamma - 1} \exp(-\lambda^\gamma s). \tag{4.20}
$$

Using now (3.5) for $P_\beta^\alpha$, we obtain from (4.18) and (4.20)

$$
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha \, dt = \lambda^{\gamma - 1} \int_0^\infty \exp(-\lambda^\gamma s) P_\beta^\alpha \, ds = \lambda^{\gamma - 1} \lambda^{(\beta - 1)} R(\lambda^\beta) = \lambda^{\alpha - 1} R(\lambda^\alpha).
$$

So we have proved the conditions of Theorem 3.3. Therefore $A \in C^\alpha(\omega^{1/\gamma})$ and the corresponding propagator $P_t^\alpha = \tilde{P}_t^\alpha$.

Using the integral representation (4.19) we have

$$
\frac{\partial}{\partial t} \varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_{\Gamma} \tau^{\gamma - 1} \exp(\tau t - \tau^\gamma s) \, d\tau = \frac{1}{t^{\gamma + 1}} W\left(-\frac{s}{t^\gamma}; -\gamma, -\gamma\right).
$$

Now, applying the inequalities (4.12) and (4.14) and noting that $1/(\gamma - 1) > 1$ for $0 < \gamma < 1$ it easily follows that the integral in the right-hand side of (4.13) is absolutely convergent for $t \in \Delta_\theta$ and the differentiation under the integral sign is possible, that is $P_t^\alpha$ admits analytic extension into $\Delta_\theta$.

Differintegrating under the integral sign in (4.19) we obtain

$$
D_t^\alpha \varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_{\Gamma} \tau^{\alpha + \gamma - 1} \exp(\tau t - \tau^\gamma s) \, d\tau = \frac{1}{t^{\gamma + \alpha}} W\left(-\frac{s}{t^\gamma}; -\gamma, 1 - \alpha - \gamma\right).
$$

Then, differintegrating (4.13) and applying again (4.12) and (4.14), it results that $D_t^\alpha P_t^\alpha$ is a bounded operator for $t > 0$. Since $A \in C^\alpha$

$$
D_t^\alpha P_t^\alpha x = A P_t^\alpha x, \quad x \in D(A), \quad t > 0,
$$

that from (1.6),(2.2) and (4.17) is equivalent to

$$
D_t^\alpha P_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} x = A P_t^\alpha x, \quad x \in D(A), \quad t > 0. \tag{4.21}
$$
Taking \( x \in X \), a sequence \( \{ x_n \} \subset D(A) \) exists such that \( x_n \to x \). Then from (4.21) and the boundedness of \( D_t^\alpha P_t^\alpha \) it follows

\[
AP_t^\alpha x_n = D_t^\alpha P_t^\alpha x_n - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x_n \to D_t^\alpha P_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x = y.
\]

Now from the closedness of \( A \) we have \( P_t^\alpha x \in D(A) \) and \( y = AP_t^\alpha x \). Therefore (4.21) holds for every \( x \in X \).

5. Examples

The next examples illustrate the theory in the preceding sections.

**Example 5.1.** Consider the fractional diffusion problem (see [8], [13]):

\[
D_t^\alpha u = \partial^2 u / \partial x^2, \quad -\infty < x < \infty, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(\mp \infty, t) = 0, \quad u(x, 0) = f(x).
\]

Let \( X = L^2(\mathbb{R}) \), \( Af = f'' \) with \( D(A) = W^{2,2}(\mathbb{R}) \). Since \( A \) generates a holomorphic semigroup, the conditions of Theorem 4.1. are satisfied and the solution has the properties (a)-(e). In this case the solution is given explicitly by (see [13]):

\[
u(x, t) = (1/2) t^{-\alpha/2} \int_{-\infty}^{\infty} \Phi_{\alpha/2}(|s| t^{-\alpha/2}) f(x - s) \, ds.
\]

Similar results hold for a more general fractional diffusion problem in \( \mathbb{R}^n \) with \( Af = \nabla^2 f \) (see [20]).

**Example 5.2.** Consider the problem

\[
D_t^\alpha u = -\partial u / \partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).
\]

Let \( X = L^2(0,1) \), \( Af = f' \) with \( D(A) = \{ f \in W^{1,2}(0,1), \quad f(0) = 0 \} \). It is well-known that \( A \) generates an uniformly bounded \( C_0 \)-semigroup and using Proposition 4.1. the properties (a)-(e) of the solution follow again. Moreover from (4.2) we obtain the following explicit representation of the solution:

\[
u(x, t) = t^{-\alpha} \int_0^x \Phi_{\alpha}(st^{-\alpha}) f(x - s) \, ds.
\]

**Example 5.3.** If \( X = L^2(0,1) \), \( A'_\phi f = -e^{i\theta} f' \) with \( D(A'_\phi) = \{ f \in W^{1,2}(0,1), \quad f(0) = 0 \} \) we have the problem

\[
D_t^\alpha u = -e^{i\theta} \partial u / \partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).
\]
with solution (obtained by Laplace transform method)

\[ u(x, t) = e^{-i\theta t - \alpha} \int_0^x \Phi_\alpha (se^{-i\theta t - \alpha}) f(x - s) \, ds \]

for \(|\theta| \leq \theta'\), where \(\theta' = (1 - \alpha)\pi/2\). Therefore \(A'_\theta \in C^\alpha\) iff \(|\theta| \leq \theta'\), while \(A'_\theta\) generates a \(C_0\)-semigroup only for \(\theta = 0\). Taking \(|\theta| < \theta'\), \(\theta \neq 0\), Theorem 4.2. implies that the solution is even holomorphic although \(A'_\theta\) does not generate a \(C_0\)-semigroup.

**Example 5.4.** If \(X = L^2(0, 1)\), \(A''_\theta f = e^{i\theta} f''\) with \(D(A''_\theta) = \{f \in W^{2,2}(0, 1), f(0) = f(1) = 1\}\) the corresponding problem is

\[ D_0^\alpha u = e^{i\theta} \partial^2 u / \partial x^2, 0 < x < 1, t \geq 0, 0 < \alpha < 1; u(0, t) = u(1, t) = 0, u(x, 0) = f(x). \]

Since \(A''_\theta\) has eigenvalues \(z_n = -e^{i\theta} n^2 \pi^2\) and eigenfunctions \(\sin n\pi x\), then if \(f(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x\) the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} c_n \sin n\pi x E_\alpha (-e^{i\theta} n^2 \pi^2 t^{\alpha}). \]

From the asymptotic expansion of the Mittag-Leffler function (2.7-8) it is clear that \(A''_\theta \in C^\alpha\) iff \(|\theta| \leq \theta''\), where \(\theta'' = (1 - \alpha/2)\pi\). Moreover if \(|\theta| < \theta''\) the solution is holomorphic. So \(A''_\theta\) with \(|\theta| < \theta''\), \(\Re \theta < 0\), is another example of an operator generating holomorphic solution of (1.1-2) but not generating \(C_0\)-semigroup (no right half-plane is free of spectra of \(A''_\theta\)).

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