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by

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THE ABSTRACT CAUCHY PROBLEM FOR THE FRACTIONAL EVOLUTION EQUATION

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Abstract

We give necessary and sufficient conditions for an unbounded closed operator $A$ in a Banach space $X$ such that the abstract Cauchy problem for the fractional evolution equation

$$D^\alpha u = Au, \quad 0 < \alpha < 1,$$

can be solved. We also present conditions on $A$ such that the fractional time evolution is holomorphic. A relation between the solutions of the problem for different $\alpha$, $0 < \alpha \leq 1$, is obtained. In particular this relation shows that the problem has a holomorphic solution whenever $A$ generates just a $C_0$-semigroup. Some examples are given.

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1. Introduction

Consider a linear closed operator $A$ in a Banach space $X$ with dense domain $D(A) \subset X$. Let $0 < \alpha < 1$. Given $x \in X$ we investigate the following Cauchy problem in $X$:

$$D^\alpha u(t) = Au(t),$$

$$u(0) = x.\quad (1.1)$$

Here $D^\alpha_t$ is the regularized fractional derivative of order $\alpha$, $0 < \alpha < 1$,

$$D^\alpha_t u(t) = J_t^{1-\alpha} \frac{d}{dt} u(t),\quad (1.3)$$

where $J_t^{\alpha}$ is the fractional integration operator defined by

$$J_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) \, d\tau, \quad \alpha > 0.\quad (1.4)$$

The connection between $D^\alpha_t$ and the Riemann-Liouville fractional derivative

$$D^\alpha_t u(t) = \frac{d}{dt} J_t^{1-\alpha} u(t)\quad (1.5)$$

is given by
\[ D^\alpha u(t) = D^\alpha (u(t) - u(0)). \] (1.6)

The definition (1.5) is the most commonly adopted in mathematical oriented papers and text-books (see e.g. [16]). The definition (1.3)\(^1\) is more suitable for problems of physical interest (see [9], [13]) where the conventional initial conditions are expressed in terms of integer derivatives.

For \( \alpha = 1 \) the problem (1.1-2) is the classical abstract Cauchy problem and there is a vast amount of literature devoted to it and its equivalent formulation – the semigroup theory (see [6], [17], [22] and the references there). The case \( \alpha = 2 \) has been investigated in detail in [7], higher order equations have been also discussed. The abstract Cauchy problem for non-integer values of \( \alpha \) is investigated in [10] and [11] (for \( 0 < \alpha < 1 \)), [12] (for \( \alpha > 0 \)), [19] and [20] (for \( 0 < \alpha \leq 2 \)). More general evolutionary equations are studied in the recent monograph [18].

For integer \( \alpha \) the following results are classical:

1. \( \alpha = 1 \): \( A \) is the generator of a strongly continuous (\( C_0 \)) semigroup (the theorem of Hille-Yosida, see [6], [17], [22]);
2. \( \alpha = 2 \): \( A \) is the generator of a strongly continuous cosine function (this implies: \( A \)-generator of a holomorphic semigroup, see [6], p.100);
3. \( \alpha = 3, 4, \ldots \): \( A \) is a bounded operator (see [6], p.99).

Roughly speaking, the set of "permitted" operators \( A \) "shrinks" as \( \alpha \) increases. We will show that this holds in some sense also for \( 0 < \alpha \leq 1 \).

Using ideas related to the theory of first and second order abstract differential equations ([6], [7]) we give in Theorem 3.1. necessary and sufficient conditions for solvability of the Cauchy problem (1.1-2), extending the conditions of the Hille-Yosida theorem from \( \alpha = 1 \) to \( \alpha \in (0, 1) \). (Let us note that Theorem 3.1. also follows as a particular case of a more general theorem given in [18], p. 43.) Theorem 4.1. gives a sufficient condition when the solution is analytic in some sector of the complex plane containing the semiaxis \( t > 0 \) and describes the properties of the solution in this case. For \( \alpha = 1 \) this result coincides with a well-known result concerning analytic semigroups (see [6], [17], [22]). Using the representations of the Mittag-Leffler functions ([4], p.5-6), we give in Theorem 4.2. a relation between the solutions of (1.1-2) for different \( \alpha, \) \( 0 < \alpha \leq 1, \) in terms of an auxiliary function. In particular this relation proves that if \( A \) is the generator of a \( C_0 \)-semigroup then the problem (1.1-2) with \( 0 < \alpha < 1 \) has an analytic solution. Some examples show that the analytic properties of the solution can be achieved for operators \( A \) that do not generate \( C_0 \)-semigroup.

\(^1\) This (alternative to Riemann-Liouville) definition of fractional derivative was originally introduced by Caputo [1,2] in the late sixties and adopted by Caputo and Mainardi [3] in the framework of the theory of Linear Viscoelasticity.
2. Preliminaries

We summarize some properties of the fractional integrals and derivatives (for more details see [9], [13] and [16]). Similarly to the ordinary differentiation and integration we have:

\[ D_0^\alpha J_0^\alpha u(t) = u(t); \quad J_0^\alpha D_0^\alpha u(t) = u(t) - u(0), \quad 0 < \alpha < 1. \] (2.1)

Simple results valid for \( t > 0 \) are:

\[ J_0^\alpha t^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\alpha + \beta}, \quad D_0^\alpha t^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}, \quad \alpha > 0, \beta > -1. \] (2.2)

If \( \beta - \alpha + 1 \in \{0, -1, -2, \ldots\} \) then \( \Gamma(\beta + 1)/\Gamma(\beta - \alpha + 1) = 0 \) in (2.2). In particular \( D_0^\alpha t^{\alpha - 1} = 0, \quad D_0^\alpha 1 = t^{-\alpha}/\Gamma(1 - \alpha) \), while \( D_0^\alpha 1 = 0, \quad \alpha > 0. \)

Using the properties of the Laplace transform it is not difficult to prove that

\[ \mathcal{L}\{D_0^\alpha u\}(\lambda) = \lambda^\alpha \mathcal{L}\{u\}(\lambda) - \lambda^{\alpha - 1} u(0). \] (2.3)

The Mittag-Leffler functions, defined by the following series

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C} \] (2.4)

are entire functions which provide a simple generalization of the exponential function \( e^z = E_1(z) \) and play an important role in the theory of fractional differential equations. Similarly to the differential equation \( \frac{d}{dt}(e^{\omega t}) = \omega e^{\omega t} \) the Mittag-Leffler function \( E_\alpha(z) \) satisfies the more general differential relation

\[ D_0^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha). \] (2.5)

The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

\[ \mathcal{L}\{E_\alpha(\omega t^\alpha)\}(\lambda) = \frac{\lambda^{\alpha - 1}}{\lambda^\alpha - \omega} \] (2.6)

and with their asymptotic expansions as \( z \to \infty \) (see [4],[5]). If \( 0 < \alpha < 2, \beta > 0 \)

\[ E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| < \frac{1}{2} \alpha \pi, \] (2.7)

\[ E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < (1 - \frac{1}{2} \alpha) \pi, \] (2.8)

where

\[ \varepsilon_{\alpha,\beta}(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad z \to \infty. \]

DEFINITION 3.1. A strong solution of the abstract differential equation (1.1) in \( t \geq 0 \) \((t > 0)\) is an \( X \)-valued function \( u(t) \) such that \( u(t) \) is continuous for \( t \geq 0 \), \( u(t) \in D(A) \) and \( D_A \) \( u(t) \) is continuous for \( t \geq 0 \) \((t > 0)\) and (1.1) is satisfied for \( t \geq 0 \) \((t > 0)\).

DEFINITION 3.2. ([12]) A Cauchy problem is called solvable in a sharply uniformly well-posed manner (SUWP-solvable) if:

(a) There exists a dense subspace \( D \) of \( X \) such that for any \( x \in D \) there exists a solution \( u(t) \) of (1.1) in \( t \geq 0 \) satisfying (1.2);

(b) There exist constants \( M \geq 1 \), \( \omega \geq 0 \), not depending on \( u(t) \) and \( x \), such that

\[
\|u(t)\| \leq Me^{\omega t}\|u(0)\|, \quad t \geq 0.
\]

for any solution \( u(t) \) of (1.1) in \( t \geq 0 \).

DEFINITION 3.3. Let \( 0 < \alpha \leq 1 \), \( M \geq 1 \), \( \omega \geq 0 \). An operator \( A \) is said to belong to \( C^\alpha(M,\omega) \) if the Cauchy problem (1.1-2) satisfies (a) and (b).

Denote \( C^\alpha(\omega) = \bigcup\{C^\alpha(M,\omega); \ M \geq 1\} \), \( C^\alpha = \bigcup\{C^\alpha(\omega); \ \omega \geq 0\} \). Let us note that \( A \in C^1 \) iff \( A \) generates a \( C_0 \)-semigroup.

In analogy with the case \( \alpha = 1 \) we define a propagator \( P_t^\alpha \) of a SUWP-solvable Cauchy problem, i.e. a bounded operator \( P_t^\alpha \) such that \( u(t) = P_t^\alpha x \) for any solution \( u(t) \) of (1.1-2).

Let \( \rho(A) \) denote the resolvent set and \( R(\lambda) = (\lambda I - A)^{-1} \) the resolvent operator of \( A \), as usual.

THEOREM 3.1. Let \( 0 < \alpha \leq 1 \), \( M \geq 1 \), \( \omega \geq 0 \). Then \( A \in C^\alpha(M,\omega) \) iff \( R(\lambda) \) exists in the half plane \( \text{Re}\lambda > \omega \) and

\[
\|((\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)})\| \leq M\frac{n!}{\text{Re}\lambda - \omega)^{n+1}}, \ \text{Re}\lambda > \omega, \ n = 0, 1, \ldots
\]

Proof. Assume that the problem (1.1-2) is SUWP-solvable and let \( P_t^\alpha \) be the corresponding propagator, so that

\[
\|P_t^\alpha\| \leq Me^{\omega t}, \quad t \geq 0.
\]

Fix \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > \omega \) and define an operator on \( X \) by

\[
S(\lambda)x = \int_0^\infty e^{-\lambda t}P_t^\alpha x \ dt, \quad x \in X.
\]

Since the norm of the integrand is bounded by \( M\|x\|e^{(\omega - \text{Re}\lambda)t} \), \( S(\lambda) \) is a bounded operator in \( X \). Assume now that \( x \in D \). Then \( P_t^\alpha x \) is a solution of (1.1-2).
and this together with the closedness of $A$ and (2.3) implies that $S(\lambda)x \in D(A)$ and $AS(\lambda)x = \lambda^\alpha S(\lambda)x - \lambda^{\alpha-1}x$. Using again the closedness of $A$ it follows $S(\lambda)x \subseteq D(A)$ and

$$\lambda^{1-\alpha}(\lambda^\alpha I - A)S(\lambda) = I. \tag{3.4}$$

In particular (3.4) shows that $(\lambda^\alpha I - A) : D(A) \to X$ is onto. It is injective as well. To see this assume there exists $x \in D(A)$ with $Ax = \lambda^\alpha x$. Then from (2.5) $u(t) = E_\beta(\lambda^\alpha t^\alpha)x$ is a solution of (1.1-2). Since $|\arg(\lambda^\alpha t^\alpha)| \leq \frac{1}{2}\alpha \pi$ when $\text{Re}\lambda > \omega \geq 0$, $t > 0$, from (2.7) we have

$$|E_\beta(\lambda^\alpha t^\alpha)| \geq \frac{1}{\alpha} e^{\text{Re}\lambda t} - \varepsilon > M e^{\omega t}, \ t > T,$$

that contradicts (3.1) unless $x = 0$. Thus we have proved that $R(\lambda^\alpha)$ exists when $\text{Re}\lambda > \omega$ and $S(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha)$, that is

$$\lambda^{\alpha-1}R(\lambda^\alpha)x = \int_0^\infty e^{-\lambda t}P_t^\alpha x \ dt, \ x \in X. \tag{3.5}$$

After easily justified differentiation under the integral sign we obtain

$$(\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)}x = (-1)^n \int_0^\infty t^n e^{-\lambda t}P_t^\alpha x \ dt, \ n = 0, 1, \ldots, \ x \in X, \ \text{Re}\lambda > \omega,$$

that together with (3.3) gives (3.2).

Suppose now that $R(\lambda^\alpha)$ exists in the half plane $\text{Re}\lambda > \omega$ and the conditions (3.2) hold. We begin by constructing certain smooth solutions of (1.1-2). Let $x \in D(A^m)$ for some $m > 2/\alpha$ and define $u(t)$ as the inverse Laplace transform of $\lambda^{\alpha-1}R(\lambda^\alpha)x$. Using the well-known formula

$$R(\lambda)x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}Ax + \ldots + \frac{1}{\lambda^m}A^{m-1}x + \frac{1}{\lambda^m}R(\lambda)A^mx,$$

we have

$$u(t) = x + \frac{t^\alpha}{\Gamma(\alpha + 1)}Ax + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}A^2x + \ldots + \frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha + 1)}A^mx + I(t), \tag{3.6}$$

where

$$I(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^\alpha} \lambda^{\alpha-1}R(\lambda^\alpha)A^mx \ d\lambda, \ \omega' > \omega. \tag{3.7}$$

Since $\|e^{\lambda t}\lambda^{\alpha-\beta-1}R(\lambda^\alpha)A^mx\| \leq C e^{\omega t}|\lambda|^{-\beta}$ for $\text{Re}\lambda = \omega'$ the integral

$$\frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^\beta} \lambda^{\alpha-1}R(\lambda^\alpha)A^mx \ d\lambda \tag{3.8}$$

converges absolutely and uniformly for $\beta > 1$ with respect to $t$ on compact subsets of $t \geq 0$. Therefore, taking $\beta = m\alpha$ and $\beta = m\alpha - 1$, we see that $I(t)$ and $I'(t)$
are continuous for \( t \geq 0 \), from (1.3) the same is true for \( \mathbf{D}_t^\alpha I(t) \), hence for \( u(t) \) and \( \mathbf{D}_t^\alpha u(t) \).

Let now \( t = 0 \) and \( \Lambda_R \) be the part of the circle \(|\lambda| = R\) to the right of the line \( \text{Re}\lambda = \omega' \). Then

\[
\|I(0)\| \leq \frac{C}{2\pi} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{|d\lambda|}{|\lambda|^{m\alpha}} \leq \frac{C}{2\pi} \lim_{R \to \infty} \int_{\Lambda_R} \frac{|d\lambda|}{|\lambda|^{m\alpha}} \leq \frac{C}{2\pi} \lim_{R \to \infty} \frac{1}{R^{m\alpha - 1}} = 0,
\]

so that \( u(0) = x \). It remains to check (1.1). Denote

\[
I_r(t) = \frac{1}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{e^{\lambda t} - \alpha E_\alpha(\lambda^\omega t^\omega)}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda, \quad t > 0.
\]

Since the integrand is a holomorphic function with respect to \( \lambda \) for \( \text{Re}\lambda > \omega \) we can shift the path of integration to \( \Lambda_R \), \( R = \sqrt{\omega^2 + \omega'^2} \). But from (2.7) it follows that \( \alpha E_\alpha(\lambda^\omega t^\omega) = e^{\lambda t} + O((|\lambda|t)^{-\alpha}) \) for \( \text{Re}\lambda > \omega, \ t > 0, \ \lambda \to \infty \). Hence

\[
\|I_r(t)\| \leq \frac{C'}{2\pi} \int_{\Lambda_R} \frac{|d\lambda|}{|\lambda|^{\beta + \alpha}} \leq \frac{C'}{R^{\beta + \alpha - 1}} \to 0, \quad r \to \infty, \ \beta > 1 - \alpha, \ t > 0,
\]

that is if \( \beta > 1 - \alpha, \ t > 0 \)

\[
\frac{1}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{e^{\lambda t} - \alpha E_\alpha(\lambda^\omega t^\omega)}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda = \frac{\alpha}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{E_\alpha(\lambda^\omega t^\omega)}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda.
\]

(3.9)

Taking \( \beta = m\alpha \) we have

\[
I(t) = \frac{\alpha}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{E_\alpha(\lambda^\omega t^\omega)}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda, \quad t > 0,
\]

(3.10)

and after differintegration under the integral sign justified on the basis of the asymptotic expansion (2.7) for \( d/dt E_\alpha(\lambda^\omega t^\omega) = \lambda^\omega t^\omega - 1 E_\alpha,\alpha(\lambda^\omega t^\omega) \), the uniform convergence of (3.8) for \( \beta = m\alpha - 1 \) on compacts of \( t \geq 0 \) and Fubini's theorem we obtain

\[
\mathbf{D}_t^\alpha I(t) = \frac{\alpha}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{\lambda^\alpha E_\alpha(\lambda^\omega t^\omega)}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda, \quad t > 0.
\]

Again from (3.9) with \( \beta = (m - 1)\alpha \) we have

\[
\mathbf{D}_t^\alpha I(t) = \frac{1}{2\pi i} \int_{\omega'' - i\infty}^{\omega'' + i\infty} \frac{\lambda^\alpha e^{\lambda t}}{\lambda^\beta} \lambda^\omega - 1 R(\lambda^\omega) A^m x d\lambda, \quad t > 0.
\]

(3.11)

On the other hand \( A \) can be applied under the integral sign in (3.7) with the convergence of the resulting integral, so that \( u(t) \in \mathcal{D}(A) \) and using (1.6), (2.2) and (3.11) we have

\[
\mathbf{D}_t^\alpha u(t) - Au(t) =
\]
\[- \frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)} A^m x + \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^{\alpha-1} R(\lambda^\alpha)(\lambda^\alpha I - A) A^m x} d\lambda = 0,\]

hence \( u(t) \) is a strong solution of (1.1-2) in \( t \geq 0 \) for \( x \in D(A^m) \).

To prove the inequality (3.1) we use the Post-Widder inversion formula:

**Lemma 3.1.** Let \( u(t) \) be a \( X \) valued continuous function defined in \( t \geq 0 \) such that \( u(t) = O(\exp(\gamma t)) \) as \( t \to \infty \) for some \( \gamma \) and let \( (\mathcal{L}u)(\lambda) \) be the Laplace transform of \( u(t) \). Then

\[
u(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left( \frac{\lambda}{n} \right)^{n+1} (\mathcal{L}u)(n) \left( \frac{n}{t} \right)
\]

(3.12)

uniformly on compacts of \( t > 0 \).

Routine calculations show that \( u(t) \) satisfies the conditions of Lemma 3.1. with \( \gamma = \omega' \) and \( (\mathcal{L}u)(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha) x \). Then, from (3.12) and the inequalities (3.2) it follows

\[
\|u(t)\| \leq M\|x\| \lim_{n \to \infty} \frac{(n/t)^{n+1}}{(n/t - \omega)^{n+1}} = M\|x\| \lim_{n \to \infty} \left( 1 - \frac{\omega t}{n} \right)^{-(n+1)} = M\|x\| e^{\omega t}.
\]

Next we define \( \hat{P}_t^\alpha x = \hat{P}_t^\alpha u \) for \( x \in D(A^m), \ t \geq 0 \), where \( u(t) \) is the solution obtained. Since \( D(A) \) is dense in \( X \) and \( g(A) \neq \emptyset \) then \( D(A^m) \) is dense in \( X \) and \( \hat{P}_t^\alpha \) can be extended to a bounded operator in \( X \) without increase of norm, that is (3.3) holds for \( \hat{P}_t^\alpha \). The proof will be complete if we show that any solution \( u(t) \) of (1.1-2) admits the representation \( u(t) = \hat{P}_t^\alpha u(0) \). Since (1.1) must hold for \( t = 0 \) there is no solution for \( x \notin D(A) \) and it is enough to establish that \( \hat{P}_t^\alpha x \) is the unique solution of (1.1-2) for any \( x \in D(A) \).

Actually for the uniqueness of the solution a weaker condition suffices:

**Theorem 3.2.** ([5]) If \( \text{Re}\lambda \) exists on the half line \( \lambda > \delta \ (\delta > 0) \) and

\[
\limsup_{\lambda \to \infty} (1/\lambda) \ln \| R(\lambda^\alpha) \| = 0
\]

(3.13)

then the solution of the problem (1.1-2) is unique.

It is not difficult to check that the inequalities (3.2) imply (3.13) and so we have uniqueness of the solution of (1.1-2).

Next we show that \( \hat{P}_t^\alpha x \) satisfies (1.1-2) for any \( x \in D(A) \). If \( \lambda \in g(A) \) it follows immediately from the definition of \( \hat{P}_t^\alpha \) that

\[
R(\lambda) \hat{P}_t^\alpha x = \hat{P}_t^\alpha R(\lambda) x, \ x \in D(A^m), \ t \geq 0.
\]

(3.14)

By the usual continuity argument (3.14) must hold for all \( x \in X \). Noting that \( D(A) = R(\lambda) X \), (3.14) implies \( \hat{P}_t^\alpha D(A) \subseteq D(A), \ t \geq 0 \) and

\[
A \hat{P}_t^\alpha x = \hat{P}_t^\alpha Ax, \ x \in D(A), \ t \geq 0.
\]

(3.15)
If we apply $J_t^\alpha$ to both sides of (1.1) and use (1.2), (2.1) and (3.15) we have
\[ \hat{P}_t^\alpha x - x = J_t^\alpha \hat{P}_t^\alpha Ax, \quad x \in D(A^m). \]

Applying now $R(\lambda)$ to both sides of this equality and using (3.14) it follows
\[ R(\lambda)(\hat{P}_t^\alpha x - x) = J_t^\alpha \hat{P}_t^\alpha AR(\lambda)x, \quad x \in D(A^m). \]

Since on both sides bounded operators are applied to $x$, the last equality must hold for all $x \in X$. Making use of the fact that $R(\lambda)$ is one-to-one, we conclude that
\[ \hat{P}_t^\alpha x - x = J_t^\alpha A \hat{P}_t^\alpha x, \quad x \in D(A). \]

Therefore $\hat{P}_t^\alpha x$ is the (unique) solution of (1.1-2).

**Remark 3.1.** Inequalities (3.2) follow from their real counterparts
\[ ||(\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)}|| \leq M \frac{n!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0, 1, \ldots \]

The proof uses Taylor series expansion for the function $\lambda^{\alpha-1}R(\lambda^\alpha)$.

**Corollary 3.1.** If $A \in C^1(\omega)$ then $A \in C^\alpha(\omega^{1/\alpha})$ for any $\alpha \in (0, 1)$.

**Proof.** By induction in $n$ we obtain the following representation
\[ (\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)} = (-1)^n \sum_{k=1}^{n+1} c_{k,n}^{\alpha} \lambda^{k\alpha - n - 1}R(\lambda^\alpha)^k, \quad n = 0, 1, \ldots, \quad (3.16) \]

where $c_{k,n}^{\alpha} \geq 0$ are constants, satisfying $c_{1,0}^{\alpha} = 1$, $c_{n,n}^{\alpha} = (n-\alpha)c_{n-1,n-1}^{\alpha}$, $c_{k,n}^{\alpha} = (n-\alpha)c_{k,n-1}^{\alpha} + \alpha(k-1)c_{k-1,n-1}^{\alpha}$, $k = 2, \ldots, n$, $c_{n+1,n}^{\alpha} = an^{\alpha}c_{n,n-1}^{\alpha}$, $n = 1, 2, \ldots$.

Then inequalities (3.2) follow using (3.16). Since Corollary 3.1. is a particular case of the more general Theorem 4.2., we omit the details of the proof.

From (3.5), (3.12) and (3.16) we obtain the following representation:

**Corollary 3.2.**
\[ P_t^\alpha x = \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{n+1} c_{k,n}^{\alpha} (I - (t/n)^\alpha A)^{-k} x, \quad (3.17) \]

where $0 < \alpha \leq 1$, $c_{k,n}^{\alpha}$ are the constants in (3.16).

Next we express the operator $A$ in terms of the propagator of the Cauchy problem (1.1-2).

**Proposition 3.1.** If $P_t^\alpha$ is the propagator of (1.1-2) and $x \in D(A)$ then
\[ Ax = \Gamma(\alpha + 1) \lim_{t \to 0} \frac{P_t^\alpha x - x}{t^\alpha}. \quad (3.18) \]
Proof. For a function $v(t)$ continuous in $t \geq 0$ we have: $\lim_{t \to 0} v(t) = \Gamma(\alpha + 1) \lim_{t \to 0} t^{-\sigma} J_\sigma v(t)$. Taking $v(t) = D^\omega P_t^\alpha x$ and using (1.1) and (2.1) we obtain (3.18).

Incidentally (3.18) shows that $P_t^\alpha x$ is not differentiable at $t = 0^+$.

The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.3.** Let $A$ be an operator such that $R(\lambda^\alpha)$ exists in the half-plane $\Re\lambda > \omega$, $\hat{P}_t^\alpha$ an operator-valued function strongly continuous in $t \geq 0$ and such that $\|\hat{P}_t^\alpha\| \leq M e^{\omega t}$, $t \geq 0$. Assume that for each $x \in X$

$$
\int_0^\infty e^{-\lambda t} \hat{P}_t^\alpha x \, dt = \lambda^{\alpha - 1} R(\lambda^\alpha) x, \quad \Re\lambda > \omega.
$$

Then, $A \in C^\alpha(\omega)$ and $\hat{P}_t^\alpha$ is the propagator of (1.1-2).

Proof. We obtain the inequalities (3.2) in the same way as from (3.5). Applying Theorem 3.1, it results that $A \in C^\alpha$. Let $P_t^\alpha$ be the propagator of (1.1-2). Then (3.19) holds for both $P_t^\alpha$ and $\hat{P}_t^\alpha$ and $P_t^\alpha = \hat{P}_t^\alpha$ follows from the uniqueness of the Laplace transform.

4. Holomorphic solutions

**Theorem 4.1.** Let $0 < \alpha \leq 1$. Assume $\phi(A) \supset \Sigma_{\delta,\alpha} = \{\lambda : |\arg\lambda| < \alpha(\pi/2 + \delta); \lambda \neq 0\}$ for some $0 < \delta \leq \pi/2$ and

$$
\|R(\lambda)\| \leq C/|\lambda|, \quad \lambda \in \Sigma_{\delta,\alpha}.
$$

Then, $A \in C^\alpha(0)$ and the corresponding propagator $P_t^\alpha$ has the following additional properties:

(a) $P_t^\alpha$ can be extended analytically into the sector $\Delta_\delta = \{ |\arg t| < \delta; t \neq 0\}$;
(b) $\|P_t^\alpha\|$ is uniformly bounded in every closed subsector $\Delta_{\delta - \epsilon}$ of $\Delta_\delta$;
(c) $P_t^\alpha x \to x$ as $t \downarrow 0$, $t \in \Delta_{\delta - \epsilon}$, $x \in X$;
(d) For any $x \in X$, $P_t^\alpha x \in D(A)$ and $\|AP_t^\alpha\| \leq M(1 + t^{-\alpha})$, $t > 0$;
(e) For any $x \in X$, $P_t^\alpha x$ is a strong solution of (1.1-2) in $t > 0$.

Proof. Let $t \in \Delta_{\delta - \epsilon}$ for some $\epsilon \in (0, \delta)$ and $\theta \in (\delta - \epsilon/2, \delta)$, $\varphi > 0$. Set

$$
\hat{P}_t^\alpha = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha - 1} R(\lambda^\alpha) \, d\lambda,
$$

where $\Gamma = \{r e^{-i(\pi/2 + \theta)}; \varphi \leq r < \infty\} \cup \{r e^{i\varphi}; -(\pi/2 + \theta) \leq \varphi \leq \pi/2 + \theta\} \cup \{r e^{i(\pi/2 + \theta)}; \varphi \leq r < \infty\}$ is oriented counterclockwise. If $\lambda \in \Gamma$ then $\lambda^\alpha \in \Sigma_{\delta,\alpha}$, and so $R(\lambda^\alpha)$ exists and from (4.1)

$$
\|\lambda^{\alpha - 1} R(\lambda^\alpha)\| \leq C/|\lambda|, \quad \lambda \in \Gamma.
$$
Let $\varrho = 1/|t|$ and $a = \sin \varepsilon/2$. Then from (4.2) and (4.3) it follows
\[ \|\tilde{P}_t^\alpha\| \leq \frac{C}{2\pi} \int_\Gamma \frac{e^{Re(\lambda t)}}{|\lambda|} d\lambda \leq \frac{C}{\pi} \int_1^\infty e^{-ar} \frac{dr}{r} + \frac{C}{\pi} \int_0^\infty e^{\cos \varphi} d\varphi \leq M. \tag{4.4} \]

This estimate shows that the integral in (4.2) is absolutely convergent for $t \in \bar{\Delta}_{\delta - \varepsilon}$, hence (a) and (b) are satisfied for $\tilde{P}_t^\alpha$.

Now fix $\lambda, \Re \lambda > 0$, and take $\varrho$ in (4.3) such that $\varrho < \Re \lambda$. Then, using (4.2), we have
\[ \int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha dt = \frac{1}{2\pi i} \int_\Gamma \lambda^{\alpha-1} R(\mu^\alpha) \int_0^\infty e^{-(\lambda-\mu)t} dt d\mu = \frac{-1}{2\pi i} \int_\Gamma \lambda^{\alpha-1} R(\mu^\alpha) x \mu d\mu = \lambda^{\alpha-1} R(\lambda^\alpha), \]
where Fubini’s theorem and Cauchy’s integral formula in the region of the right of $\Gamma$ are used. So we have proved that the conditions of Theorem 3.3. are fulfilled with $\omega = 0$. Therefore $A \in C^\prime(0)$ and the corresponding propagator $\tilde{P}_t^\alpha = \tilde{P}_t^\alpha$.

Next for $x \in D(A)$ by Cauchy’s theorem and Cauchy’s integral formula if $t \in \bar{\Delta}_{\delta - \varepsilon}, t \downarrow 0$
\[ \tilde{P}_t^\alpha x - x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^{\alpha-1} R(\lambda^\alpha) x - \lambda^{-1} x) d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{-1} R(\lambda^\alpha) Ax d\lambda \to \frac{1}{2\pi i} \int_\Gamma \lambda^{-1} R(\lambda^\alpha) Ax d\lambda = 0, \]
that together with (b) implies (c).

Now, for $t > 0$ and $\varrho = 1/t$, applying $A$ to both sides of (4.2) and using the identity $AR(\lambda) = \lambda R(\lambda) - I$ we have
\[ AP_t^\alpha = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} AR(\lambda^\alpha) d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{2\alpha-1} R(\lambda^\alpha) d\lambda - \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}. \]
Taking $\varrho = 1/t$ and estimating $AP_t^\alpha$ in the same way as in (4.4) we obtain (d).

At the end, applying $J_t^\alpha$ to both sides of (1.1) we have
\[ P_t^\alpha x - x = J_t^\alpha P_t^\alpha Ax, \quad x \in D(A). \tag{4.5} \]
But from (d) $J_t^\alpha P_t^\alpha A$ is a bounded operator for $t > 0$. Then (4.5) holds for any $x \in X, t > 0$. This implies (e).\( \Box \)

**Proposition 4.1.** If $\varrho(A) \supset \{\lambda : \Re \lambda > 0\}$ and for some constant $M$
\[ \|R(\lambda)\| \leq M/\Re \lambda, \quad \Re \lambda > 0. \tag{4.6} \]
then for any $\alpha \in (0, 1)$ $A$ satisfies the hypotheses of Theorem 4.1. with $0 < \delta < \min\{(1/\alpha - 1)\pi/2, \pi/2\}$. 

Proof. Fix $\alpha$ and $\delta$ satisfying the conditions mentioned above. Then $\alpha(\pi/2+\delta) < \pi/2$ and so $\Sigma_{\alpha,\delta} \subset \rho(A)$. Taking $\beta$, $\alpha(\pi/2+\delta) < \beta < \pi/2$ we obtain

$$||R(\lambda)|| \leq \frac{M}{\text{Re} \lambda} = \frac{M}{|\lambda| \cos \theta} < \frac{M}{|\lambda| \cos \beta}, \quad \lambda \in \Sigma_{\alpha,\delta},$$

where $\theta = \text{arg} \lambda$ and (4.1) holds with $C = M/\cos \beta$.\hfill $\Box$

The next theorem describes the relation between the solutions of the Cauchy problem (1.1-2) for different values of $\alpha$, $0 < \alpha \leq 1$.

**Theorem 4.2.** Let $0 < \alpha < \beta \leq 1$, $\gamma = \alpha/\beta$, $\omega \geq 0$. If $A \in C^\beta(\omega)$ then $A \in C^\alpha(\omega^{1/\gamma})$ and the following representation holds

$$P_t^{\alpha}x = \frac{1}{t^{\gamma}} \int_0^\infty \Phi_\gamma \left( \frac{s}{t^{\gamma}} \right) \partial_s^{\beta} x ds, \quad t > 0,$$

where

$$\Phi_\gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n z^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \quad z \in \mathbb{C},$$

is an entire function. Moreover $P_t^{\alpha}$ can be analytically extended to the sector $\Delta_\theta$, $\theta = \text{min}\{(1/\gamma - 1)\pi/2, \pi/2\}$ and for any $x \in X$ $P_t^{\alpha}x$ is a strong solution of (1.1-2) in $t > 0$.

Proof. We summarize some properties of $\Phi_\gamma(z)$ (for more details see [4], p. 5, [13], [15] and [21]). It is a particular case of a special function known as Wright function ([5]), entire in the complex plane:

$$W(z; \gamma, \delta) = \sum_{n=0}^\infty \frac{z^n}{n! \Gamma(\gamma n + \delta)} = \frac{1}{2\pi i} \int_\Gamma \mu^{-\delta} \exp(\mu + z \mu^{-\gamma}) d\mu, \quad \gamma > -1, \delta \in \mathbb{C},$$

where $\Gamma$ is a contour which starts at $-\infty$, encircles the origin once counterclockwise and turns to its starting point. Obviously $\Phi_\gamma(z) = W(-z; -\gamma, 1 - \gamma)$ and the integral representation

$$\Phi_\gamma(z) = \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} \exp(\mu - z \mu^{-\gamma}) d\mu, \quad z \in \mathbb{C},$$

holds. For real values of $z$ we have ([4], p. 5):

$$\Phi_\gamma(t) \geq 0, \quad t \geq 0.$$ \hspace{1cm} (4.10)

Let us denote $\varphi_{t,\gamma}(s) = (1/t^{\gamma}) \Phi_\gamma(s/t^{\gamma})$. Then the following formula is true ([4], p. 5):

$$E_\gamma(\omega t^{\gamma}) = \int_0^\infty \varphi_{t,\gamma}(s) e^{\omega s} ds, \quad t > 0, \quad 0 < \gamma < 1.$$ \hspace{1cm} (4.11)
From the asymptotic expansion of the Wright function given in [21] (this function is called there *generalized Bessel function*) we obtain the inequality:

$$|W(-z; -\gamma, \delta)| \leq B \exp(-bz^{1/(1-\gamma)}) \quad (4.12)$$

for $|\arg z| < \min\{(1 - \gamma)3\pi/2, \pi\}$, $|z| \to \infty$, $0 < \gamma < 1$, $\delta \in \mathbb{C}$, where $B = B(\gamma, \delta)$, $b = b(\gamma, \delta)$ are positive constants.

Now define

$$\hat{P}_t^\alpha x = \int_0^\infty \varphi_{t,\gamma}(s) P_s^\delta x \, ds, \quad t > 0. \quad (4.13)$$

Our aim is to prove that $\hat{P}_t^\alpha$ satisfies the conditions of Theorem 3.3. Since $A \in \mathcal{C}^\beta$, there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|P_s^\delta\| \leq Me^{\omega s}, \quad s \geq 0, \quad (4.14)$$

and together with (4.10) and (4.11) we obtain

$$\|\hat{P}_t^\alpha\| \leq \int_0^{\infty} \varphi_{t,\gamma}(s)\|P_s^\delta\| ds \leq M \int_0^{\infty} \varphi_{t,\gamma}(s)e^{\omega s} ds = M E_{\gamma}(\omega t^\gamma). \quad (4.15)$$

We observe that a constant $C_{\gamma}$ exists, such that

$$E_{\gamma}(\omega t^\gamma) \leq C_{\gamma} \exp(\omega^{1/\gamma} t), \quad \omega \geq 0, \quad t \geq 0. \quad (4.16)$$

Indeed, from (2.7) there exists a constant $C_1$ such that $E_{\gamma}(\omega t^\gamma) \leq C_1 \exp(\omega^{1/\gamma} t)$ for every $t \geq T$. From the continuity of the Mittag-Leffler function in $t \geq 0$ it follows that there exists a constant $C_2$ such that $E_{\gamma}(\omega t^\gamma) \leq C_2 \leq C_2 \exp(\omega^{1/\gamma} t)$ for $t \in [0, T]$. Hence, we can take $C_{\gamma} = \max\{C_1, C_2\}$. Now from (4.15) and (4.16) we have

$$\|\hat{P}_t^\alpha\| \leq M' \exp(\omega^{1/\gamma} t), \quad t \geq 0.$$

Taking $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \omega^{1/\gamma}$ we shall prove that $R(\lambda^\alpha)$ exists. If $\lambda = |\lambda|e^{i\theta}$ then $|\theta| < \pi/2$ and for such $\theta$ and $\gamma \in (0, 1)$ the inequality $\cos \gamma \theta \geq \cos^\gamma \theta$ holds. Then $\text{Re}(\lambda^\gamma) = |\lambda|^\gamma \cos \gamma \theta \geq |\lambda|^\gamma \cos^\gamma \theta = (\text{Re}\lambda)^\gamma > \omega$. Since $A \in \mathcal{C}^\beta$, it follows from Theorem 3.1. that $R(\mu^\beta)$ exists in the half-plane $\text{Re}\mu > \omega$. Taking $\mu = \lambda^\gamma$ we obtain that $R(\lambda^\alpha) = R(\mu^\beta)$ exists.

It is plain that $\hat{P}_t^\alpha$ is strongly continuous for $t > 0$. To prove the strong continuity at the origin we substitute $\sigma = s/t^\gamma$ in (4.13). Then, using the identity $\int_0^{\infty} \Phi_{\gamma}(\sigma) \, d\sigma = 1$ (that follows from (4.11) setting $\omega = 0$) and the strong continuity of $P_s^\delta$ at the origin, we have

$$\lim_{t \downarrow 0} \hat{P}_t^\alpha x = \lim_{t \downarrow 0} \int_0^{\infty} \Phi_{\gamma}(\sigma) P_s^\delta x \, d\sigma = \int_0^{\infty} \Phi_{\gamma}(\sigma) x \, d\sigma = x. \quad (4.17)$$
For \( \text{Re}\lambda > \omega^{1/\gamma} \), using (4.13) and interchanging the order of integration we have
\[
\int_0^\infty e^{-\lambda t} \mathcal{P}_t^\alpha dt = \int_0^\infty \mathcal{P}_s^\beta \int_0^\infty e^{-\lambda t}\varphi_t(s) dt ds.
\] (4.18)
Substituting \( \mu = \lambda t \) in (4.9) and shifting the new contour \( \Gamma' = \Gamma/t \) to \( \Gamma \) we get the integral representation
\[
\varphi_t(s) = \frac{1}{2\pi i} \int_\Gamma \tau^{-1} \exp(\tau t - \tau^\gamma s) d\tau,
\] (4.19)
Then, noting that \( \text{Re}(\lambda - \tau) > 0 \) and using Cauchy’s formula it follows
\[
\int_0^\infty e^{-\lambda t} \varphi_t(s) dt = \frac{1}{2\pi i} \int_\Gamma \tau^{-1} \exp(-\tau^\gamma s) \int_0^\infty e^{-\lambda t} dt d\tau =
\frac{-1}{2\pi i} \int_\Gamma \frac{\tau^{-1} \exp(-\tau^\gamma s)}{\tau - \lambda} d\tau = \lambda^{-1} \exp(-\lambda^\gamma s).
\] (4.20)
Using now (3.5) for \( \mathcal{P}_s^\beta \), we obtain from (4.18) and (4.20)
\[
\int_0^\infty e^{-\lambda t} \mathcal{P}_t^\alpha dt = \lambda^{-1} \int_0^\infty \exp(-\lambda^\gamma s) \mathcal{P}_s^\beta ds = \lambda^{-1} \lambda^{\gamma(\beta - 1)} R(\lambda^\beta) = \lambda^{\alpha - 1} R(\lambda^\alpha).
\]
So we have proved the conditions of Theorem 3.3. Therefore \( A \in C^\alpha(\omega^{1/\gamma}) \) and the corresponding propagator \( \mathcal{P}_t^\alpha = \mathcal{P}_t^\alpha \).

Using the integral representation (4.19) we have
\[
\frac{\partial}{\partial t} \varphi_{t\gamma}(s) = \frac{1}{2\pi i} \int_\Gamma \tau^\gamma \exp(\tau t - \tau^\gamma s) d\tau = \frac{1}{t^{\gamma + 1}} W(-\frac{s}{t^\gamma}; -\gamma, -\gamma).
\]
Now, applying the inequalities (4.12) and (4.14) and noting that \( 1/(\gamma - 1) > 1 \) for \( 0 < \gamma < 1 \) it easily follows that the integral in the right-hand side of (4.13) is absolutely convergent for \( t \in \Delta_\theta \) and the differentiation under the integral sign is possible, that is \( \mathcal{P}_t^\alpha \) admits analytic extension into \( \Delta_\theta \).

Differintegrating under the integral sign in (4.19) we obtain
\[
D_t^\alpha \varphi_{t\gamma}(s) = \frac{1}{2\pi i} \int_\Gamma \tau^{\alpha + \gamma - 1} \exp(\tau t - \tau^\gamma s) d\tau = \frac{1}{t^{\gamma + \alpha}} W(-\frac{s}{t^{\gamma}}; -\gamma, 1 - \alpha - \gamma).
\]
Then, differintegrating (4.13) and applying again (4.12) and (4.14), it results that \( D_t^\alpha \mathcal{P}_t^\alpha \) is a bounded operator for \( t > 0 \). Since \( A \in C^\alpha \)
\[
D_t^\alpha \mathcal{P}_t^\alpha x = A\mathcal{P}_t^\alpha x, \quad x \in D(A), \ t > 0,
\]
that from (1.6), (2.2) and (4.17) is equivalent to
\[
D_t^\alpha \mathcal{P}_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} x = A\mathcal{P}_t^\alpha x, \quad x \in D(A), \ t > 0.
\] (4.21)
Taking \( x \in X \), a sequence \( \{x_n\} \subset D(A) \) exists such that \( x_n \to x \). Then from (4.21) and the boundedness of \( D_t^\alpha P_t^\alpha \) it follows

\[
AP_t^\alpha x_n = D_t^\alpha P_t^\alpha x_n - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}x_n \to D_t^\alpha P_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}x = y.
\]

Now from the closedness of \( A \) we have \( P_t^\alpha x \in D(A) \) and \( y = AP_t^\alpha x \). Therefore (4.21) holds for every \( x \in X \).

5. Examples

The next examples illustrate the theory in the preceding sections.

**Example 5.1.** Consider the fractional diffusion problem (see [8], [13]):

\[
D_t^\alpha u = \partial^2 u/\partial x^2, \quad -\infty < x < \infty, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(\pm \infty, t) = 0, \quad u(x, 0) = f(x).
\]

Let \( X = L^2(\mathbb{R}) \), \( Af = f'' \) with \( D(A) = W^{2,2}(\mathbb{R}) \). Since \( A \) generates a holomorphic semigroup, the conditions of Theorem 4.1. are satisfied and the solution has the properties (a)-(e). In this case the solution is given explicitly by (see [13]):

\[
u(x, t) = (1/2)t^{-\alpha/2} \int_{-\infty}^{\infty} \Phi_{\alpha/2}(|s|t^{-\alpha/2}) f(x-s) \, ds.
\]

Similar results hold for a more general fractional diffusion problem in \( \mathbb{R}^n \) with \( Af = \nabla^2 f \) (see [20]).

**Example 5.2.** Consider the problem

\[
D_t^\alpha u = -\partial u/\partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).
\]

Let \( X = L^2(0,1) \), \( Af = f' \) with \( D(A) = \{f \in W^{1,2}(0,1), \quad f(0) = 0\} \). It is well-known that \( A \) generates an uniformly bounded \( C_0 \)-semigroup and using Proposition 4.1. the properties (a)-(e) of the solution follow again. Moreover from (4.2) we obtain the following explicit representation of the solution:

\[
u(x, t) = t^{-\alpha} \int_0^x \Phi_\alpha(st^{-\alpha}) f(x-s) \, ds.
\]

**Example 5.3.** If \( X = L^2(0,1) \), \( A^i f = -e^{i\theta} f' \) with \( D(A^i) = \{f \in W^{1,2}(0,1), \quad f(0) = 0\} \) we have the problem

\[
D_t^\alpha u = -e^{i\theta} \partial u/\partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).
\]
with solution (obtained by Laplace transform method)
\[ u(x, t) = e^{-i\theta t - \alpha} \int_0^x \Phi_a (se^{-i\theta t - \alpha}) f(x-s) \, ds \]
for \(|\theta| \leq \theta'_a\), where \(\theta'_a = (1 - \alpha)\pi/2\). Therefore \(A'_\theta \in \mathcal{C}^\alpha\) iff \(|\theta| \leq \theta'_a\), while \(A'_\theta\) generates a \(C_0\)-semigroup only for \(\theta = 0\). Taking \(|\theta| < \theta'_a\), \(\theta \neq 0\), Theorem 4.2. implies that the solution is even holomorphic although \(A'_\theta\) does not generate a \(C_0\)-semigroup.

**Example 5.4.** If \(X = L^2(0, 1)\), \(A''_\theta f = e^{i\theta} f''\) with \(D(A''_\theta) = \{f \in W^{2,2}(0, 1), f(0) = f(1) = 0\}\) the corresponding problem is
\[ D^2 u = e^{i\theta} \partial^2 u / \partial x^2, 0 < x < 1, t \geq 0, 0 < \alpha < 1; u(0, t) = u(1, t) = 0, u(x, 0) = f(x). \]
Since \(A''_\theta\) has eigenvalues \(z_n = -e^{i\theta} n^2 \pi^2\) and eigenfunctions \(\sin n \pi x\), then if \(f(x) = \sum_{n=1}^\infty c_n \sin n \pi x\) the solution is
\[ u(x, t) = \sum_{n=1}^\infty c_n \sin n \pi x E_\alpha(-e^{i\theta} n^2 \pi^2 t\alpha). \]
From the asymptotic expansion of the Mittag-Leffler function (2.7-8) it is clear that \(A''_\theta \in \mathcal{C}^\alpha\) iff \(|\theta| \leq \theta''_a\), where \(\theta''_a = (1 - \alpha/2)\pi\). Moreover if \(|\theta| < \theta''_a\) the solution is holomorphic. So \(A''_\theta\) with \(|\theta| < \theta''_a\), \(\text{Re} \theta < 0\), is another example of an operator generating holomorphic solution of (1.1-2) but not generating \(C_0\)-semigroup (no right half-plane is free of spectra of \(A''_\theta\)).

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**References**


