The abstract Cauchy problem for the fractional evolution equation

Citation for published version (APA):

Document status and date:
Published: 01/01/1998

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
RANA 98-14
July 1998

The abstract Cauchy problem for the fractional evolution equation

by

E. Bazhlekova
THE ABSTRACT CAUCHY PROBLEM FOR THE FRACTIONAL EVOLUTION EQUATION

E. Bazhlekova

Abstract

We give necessary and sufficient conditions for an unbounded closed operator \( A \) in a Banach space \( X \) such that the abstract Cauchy problem for the fractional evolution equation

\[
D^\alpha u = Au, \quad 0 < \alpha < 1,
\]

can be solved. We also present conditions on \( A \) such that the fractional time evolution is holomorphic. A relation between the solutions of the problem for different \( \alpha, 0 < \alpha \leq 1 \), is obtained. In particular this relation shows that the problem has a holomorphic solution whenever \( A \) generates just a \( C_0 \)-semigroup. Some examples are given.

Mathematics Subject Classification: 12H22, 26A33, 35F10, 47D06.

Key Words and Phrases: fractional calculus, semigroup theory, Mittag-Leffler function, Wright function.

1. Introduction

Consider a linear closed operator \( A \) in a Banach space \( X \) with dense domain \( D(A) \subset X \). Let \( 0 < \alpha < 1 \). Given \( x \in X \) we investigate the following Cauchy problem in \( X \):

\[
D_t^\alpha u(t) = Au(t),
\]

\[
u(0) = x.
\]

Here \( D_t^\alpha \) is the regularized fractional derivative of order \( \alpha, 0 < \alpha < 1 \),

\[
D_t^\alpha u(t) = J_t^{1-\alpha} \frac{d}{dt} u(t),
\]

where \( J_t^{\alpha} \) is the fractional integration operator defined by

\[
J_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) \, d\tau, \quad \alpha > 0.
\]

The connection between \( D_t^\alpha \) and the Riemann-Liouville fractional derivative

\[
D_t^\alpha u(t) = \frac{d}{dt} J_t^{1-\alpha} u(t)
\]

(1.5)
is given by

$$D^\alpha_t u(t) = D^\alpha_t (u(t) - u(0)).$$ (1.6)

The definition (1.5) is the most commonly adopted in mathematical oriented papers and text-books (see e.g. [16]). The definition (1.3)\(^1\) is more suitable for problems of physical interest (see [9], [13]) where the conventional initial conditions are expressed in terms of integer derivatives.

For \(\alpha = 1\) the problem (1.1-2) is the classical abstract Cauchy problem and there is a vast amount of literature devoted to it and its equivalent formulation – the semigroup theory (see [6], [17], [22] and the references there). The case \(\alpha = 2\) has been investigated in detail in [7], higher order equations have been also discussed. The abstract Cauchy problem for non-integer values of \(\alpha\) is investigated in [10] and [11] (for \(0 < \alpha < 1\)), [12] (for \(\alpha > 0\)), [19] and [20] (for \(0 < \alpha \leq 2\)). More general evolutionary equations are studied in the recent monograph [18].

For integer \(\alpha\) the following results are classical:

The Cauchy problem (1.1-2) can be solved if and only if

- \(\alpha = 1\): \(A\) is the generator of a strongly continuous \((C_0)\) semigroup (the theorem of Hille-Yosida, see [6], [17], [22]);
- \(\alpha = 2\): \(A\) is the generator of a strongly continuous cosine function (this implies: \(A\)-generator of a holomorphic semigroup, see [6], p.100);
- \(\alpha = 3, 4, \ldots\): \(A\) is a bounded operator (see [6], p.99).

Roughly speaking, the set of "permitted" operators \(A\) "shrinks" as \(\alpha\) increases. We will show that this holds in some sense also for \(0 < \alpha \leq 1\).

Using ideas related to the theory of first and second order abstract differential equations ([6], [7]) we give in Theorem 3.1. necessary and sufficient conditions for solvability of the Cauchy problem (1.1-2), extending the conditions of the Hille-Yosida theorem from \(\alpha = 1\) to \(\alpha \in (0, 1)\). (Let us note that Theorem 3.1. also follows as a particular case of a more general theorem given in [18], p. 43.) Theorem 4.1. gives a sufficient condition when the solution is analytic in some sector of the complex plane containing the semiaxis \(t > 0\) and describes the properties of the solution in this case. For \(\alpha = 1\) this result coincides with a well-known result concerning analytic semigroups (see [6], [17], [22]). Using the representations of the Mittag-Leffler functions ([4], p.5-6), we give in Theorem 4.2. a relation between the solutions of (1.1-2) for different \(\alpha\), \(0 < \alpha \leq 1\), in terms of an auxiliary function. In particular this relation proves that if \(A\) is the generator of a \(C_0\)-semigroup then the problem (1.1-2) with \(0 < \alpha < 1\) has an analytic solution. Some examples show that the analytic properties of the solution can be achieved for operators \(A\) that do not generate \(C_0\)-semigroup.

\(^1\) This (alternative to Riemann-Liouville) definition of fractional derivative was originally introduced by Caputo [1,2] in the late sixties and adopted by Caputo and Mainardi [3] in the framework of the theory of Linear Viscoelasticity.
2. Preliminaries

We summarize some properties of the fractional integrals and derivatives (for more details see [9], [13] and [16]). Similarly to the ordinary differentiation and integration we have:

\[
D_t^\alpha J_t^\alpha u(t) = u(t); \quad J_t^\alpha D_t^\alpha u(t) = u(t) - u(0), \quad 0 < \alpha < 1. \quad (2.1)
\]

Simple results valid for \( t > 0 \) are:

\[
J_t^\alpha t^\alpha = \Gamma(\alpha + 1) t^{\alpha + 1}, \quad D_t^\alpha t^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - 1)} t^{\alpha - 1}, \quad \alpha > 0, \quad \beta > -1. \quad (2.2)
\]

If \( \beta - \alpha + 1 \in \{0, -1, -2, \ldots\} \) then \( \Gamma(\beta + 1)/\Gamma(\beta - \alpha + 1) = 0 \) in (2.2). In particular \( D_t^\alpha t^{\alpha - 1} = 0, \quad D_t^\alpha 1 = t^{-\alpha}/\Gamma(1 - \alpha), \) while \( D_t^\alpha 1 = 0, \quad \alpha > 0. \)

Using the properties of the Laplace transform it is not difficult to prove that

\[
\mathcal{L}\{D_t^\alpha u\}(\lambda) = \lambda^\alpha \mathcal{L}\{u\}(\lambda) - \lambda^{\alpha - 1} u(0). \quad (2.3)
\]

The Mittag-Leffler functions, defined by the following series

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C} \quad (2.4)
\]

are entire functions which provide a simple generalization of the exponential function \( e^z = E_1(z) \) and play an important role in the theory of fractional differential equations. Similarly to the differential equation \( d/dt(e^{\omega t}) = \omega e^{\omega t} \) the Mittag-Leffler function \( E_\alpha(z) \) satisfies the more general differential relation

\[
D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha). \quad (2.5)
\]

The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

\[
\mathcal{L}\{E_\alpha(\omega t^\alpha)\}(\lambda) = \frac{\lambda^{\alpha - 1}}{\lambda^\alpha - \omega} \quad (2.6)
\]

and with their asymptotic expansions as \( z \to \infty \) (see [4],[5]). If \( 0 < \alpha < 2, \beta > 0 \)

\[
E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| \leq \frac{1}{2} \alpha \pi, \quad (2.7)
\]

\[
E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < (1 - \frac{1}{2} \alpha) \pi, \quad (2.8)
\]

where

\[
\varepsilon_{\alpha,\beta}(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad z \to \infty.
\]

**Definition 3.1.** A strong solution of the abstract differential equation (1.1) in \( t \geq 0 \) \((t > 0)\) is an \( X \)-valued function \( u(t) \) such that \( u(t) \) is continuous for \( t \geq 0 \), \( u(t) \in D(A) \) and \( D_t^2 u(t) \) is continuous for \( t \geq 0 \) \((t > 0)\) and (1.1) is satisfied for \( t \geq 0 \) \((t > 0)\).

**Definition 3.2.** ([12]) A Cauchy problem is called solvable in a sharply uniformly well-posed manner (SUWP-solvable) if:

(a) There exists a dense subspace \( D \) of \( X \) such that for any \( x \in D \) there exists a solution \( u(t) \) of (1.1) in \( t \geq 0 \) satisfying (1.2);

(b) There exist constants \( M \geq 1, \omega \geq 0 \), not depending on \( u(t) \) and \( x \), such that

\[
\|u(t)\| \leq Me^{\omega t}\|u(0)\|, \quad t \geq 0.
\]

for any solution \( u(t) \) of (1.1) in \( t \geq 0 \).

**Definition 3.3.** Let \( 0 < \alpha \leq 1, M \geq 1, \omega \geq 0 \). An operator \( A \) is said to belong to \( C^{\alpha}(M, \omega) \) if the Cauchy problem (1.1-2) satisfies (a) and (b).

Denote \( C^{\alpha}(\omega) = \bigcup \{ C^{\alpha}(M, \omega); M \geq 1 \} \), \( C^{\alpha} = \bigcup \{ C^{\alpha}(\omega); \omega \geq 0 \} \). Let us note that \( A \in C^{1} \) iff \( A \) generates a \( C_0 \)-semigroup.

In analogy with the case \( \alpha = 1 \) we define a propagator \( P_t^\alpha \) of a SUWP-solvable Cauchy problem, i.e. a bounded operator \( P_t^\alpha \) such that \( u(t) = P_t^\alpha x \) for any solution \( u(t) \) of (1.1-2).

Let \( \rho(A) \) denote the resolvent set and \( R(\lambda) = (\lambda I - A)^{-1} \) the resolvent operator of \( A \), as usual.

**Theorem 3.1.** Let \( 0 < \alpha \leq 1, M \geq 1, \omega \geq 0 \). Then \( A \in C^{\alpha}(M, \omega) \) iff \( R(\lambda^\alpha) \) exists in the half plane \( \text{Re} \lambda > \omega \) and

\[
\| (\lambda^{\alpha-1} R(\lambda^\alpha))^{(n)} \| \leq M \frac{n!}{(\text{Re} \lambda - \omega)^{n+1}}, \quad \text{Re} \lambda > \omega, \quad n = 0, 1, \ldots
\]

**Proof.** Assume that the problem (1.1-2) is SUWP-solvable and let \( P_t^\alpha \) be the corresponding propagator, so that

\[
\| P_t^\alpha \| \leq Me^{\omega t}, \quad t \geq 0.
\]

Fix \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > \omega \) and define an operator on \( X \) by

\[
S(\lambda)x = \int_0^\infty e^{-\lambda t} P_t^\alpha x \, dt, \quad x \in X.
\]

Since the norm of the integrand is bounded by \( M \| x \| e^{(\omega - \text{Re} \lambda)t} \), \( S(\lambda) \) is a bounded operator in \( X \). Assume now that \( x \in D \). Then \( P_t^\alpha x \) is a solution of (1.1-2)
and this together with the closedness of $A$ and (2.3) implies that $S(\lambda)x \in D(A)$ and $AS(\lambda)x = \lambda^\alpha S(\lambda)x - \lambda^{\alpha-1}x$. Using again the closedness of $A$ it follows $S(\lambda)X \subseteq D(A)$ and
\[ \lambda^{1-\alpha}(\lambda^\alpha I - A)S(\lambda) = I. \quad (3.4) \]
In particular (3.4) shows that $(\lambda^\alpha I - A) : D(A) \to X$ is onto. It is injective as well. To see this assume there exists $x \in D(A)$ with $Ax = \lambda^\alpha x$. Then from (2.5) $u(t) = E_\alpha(\lambda^\alpha t^\alpha)x$ is a solution of (1.1-2). Since $|\arg(\lambda^\alpha t^\alpha)| \leq \frac{1}{2}\pi$ when $\Re \lambda > \omega \geq 0$, $t > 0$, from (2.7) we have
\[ |E_\alpha(\lambda^\alpha t^\alpha)| \geq \frac{1}{\alpha}e^{\Re \lambda t} - \varepsilon > M e^{\omega t}, \quad t > T, \]
that contradicts (3.1) unless $x = 0$. Thus we have proved that $R(\lambda^\alpha)$ exists when $\Re \lambda > \omega$ and $S(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha)$, that is
\[ \lambda^{\alpha-1}R(\lambda^\alpha)x = \int_0^\infty e^{-\lambda t}P_t^\alpha x \, dt, \quad x \in X. \quad (3.5) \]
After easily justified differentiation under the integral sign we obtain
\[ (\lambda^{\alpha-1}R(\lambda^\alpha))^{(n)}x = (-1)^n \int_0^\infty t^n e^{-\lambda t}P_t^\alpha x \, dt, \quad n = 0, 1, \ldots, \quad x \in X, \quad \Re \lambda > \omega, \]
that together with (3.3) gives (3.2).
Suppose now that $R(\lambda^\alpha)$ exists in the half plane $\Re \lambda > \omega$ and the conditions (3.2) hold. We begin by constructing certain smooth solutions of (1.1-2). Let $x \in D(A^m)$ for some $m > 2/\alpha$ and define $u(t)$ as the inverse Laplace transform of $\lambda^{\alpha-1}R(\lambda^\alpha)x$. Using the well-known formula
\[ R(\lambda)x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}Ax + \ldots + \frac{1}{\lambda^m}A^{m-1}x + \frac{1}{\lambda^m}R(\lambda)A^mx, \]
we have
\[ u(t) = x + \frac{t^\alpha}{\Gamma(\alpha + 1)}Ax + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}A^2x + \ldots + \frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha + 1)}A^mx + I(t), \quad (3.6) \]
where
\[ I(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^m} \lambda^{\alpha-1}R(\lambda^\alpha)A^mx \, d\lambda, \quad \omega' > \omega. \quad (3.7) \]
Since $\|e^{\lambda t}\lambda^{\alpha-\beta-1}R(\lambda^\alpha)A^mx\| \leq Ce^{\omega'|\lambda|^{-\beta}}$ for $\Re \lambda = \omega'$ the integral
\[ \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^\beta} \lambda^{\alpha-1}R(\lambda^\alpha)A^mx \, d\lambda \]
converges absolutely and uniformly for $\beta > 1$ with respect to $t$ on compact subsets of $t \geq 0$. Therefore, taking $\beta = ma$ and $\beta = ma - 1$, we see that $I(t)$ and $I'(t)$
are continuous for \( t \geq 0 \), from (1.3) the same is true for \( D_t^\alpha I(t) \), hence for \( u(t) \) and \( D_t^\alpha u(t) \).

Let now \( t = 0 \) and \( \Lambda_R \) be the part of the circle \(|\lambda| = R\) to the right of the line \( \text{Re}\lambda = \omega'\). Then

\[
||I(0)|| \leq \frac{C}{2\pi} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{d\lambda}{|\lambda|^{m\alpha}} = \frac{C}{2\pi} \lim_{R \to \infty} \int_{\Lambda_R} \frac{d\lambda}{|\lambda|^{m\alpha}} \leq C \lim_{R \to \infty} \frac{1}{R^{m\alpha - 1}} = 0,
\]

so that \( u(0) = x \). It remains to check (1.1). Denote

\[
I_r(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t} - \alpha E_\alpha(\lambda^{\alpha}t^\alpha)}{\lambda^\beta} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda, \quad t > 0.
\]

Since the integrand is a holomorphic function with respect to \( \lambda \) for \( \text{Re}\lambda > \omega \) we can shift the path of integration to \( \Lambda_R, R = \sqrt{r^2 + \omega'^2}\). But from (2.7) it follows that \( \alpha E_\alpha(\lambda^{\alpha}t^\alpha) = e^{\lambda t} + O((|\lambda|t)^{-\alpha}) \) for \( \text{Re}\lambda > \omega, \ t > 0, \lambda \to \infty \). Hence

\[
||I_r(t)|| \leq \frac{C'}{2\pi} \int_{\Lambda_R} \frac{d\lambda}{|\lambda|^{\beta + \alpha}} \leq \frac{C'}{R^{\beta + \alpha - 1}} \to 0, \quad r \to \infty, \ \beta > 1 - \alpha, \ t > 0,
\]

that is if \( \beta > 1 - \alpha, \ t > 0 \)

\[
\frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{e^{\lambda t}}{\lambda^\beta} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{E_\alpha(\lambda^{\alpha}t^\alpha)}{\lambda^\beta} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda.
\]

Taking \( \beta = m\alpha \) we have

\[
I(t) = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{E_\alpha(\lambda^{\alpha}t^\alpha)}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda, \quad t > 0,
\]

and after differintegration under the integral sign justified on the basis of the asymptotic expansion (2.7) for \( d/dt E_\alpha(\lambda^{\alpha}t^\alpha) = \lambda^{\alpha - 1} E_\alpha(\lambda^{\alpha}t^\alpha) \), the uniform convergence of (3.8) for \( \beta = m\alpha - 1 \) on compacts of \( t \geq 0 \) and Fubini’s theorem we obtain

\[
D_t^\alpha I(t) = \frac{\alpha}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{\lambda^{\alpha} E_\alpha(\lambda^{\alpha}t^\alpha)}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda, \quad t > 0.
\]

Again from (3.9) with \( \beta = (m - 1)\alpha \) we have

\[
D_t^\alpha I(t) = \frac{1}{2\pi i} \int_{\omega' - i\infty}^{\omega' + i\infty} \frac{\lambda^{\alpha} e^{\lambda t}}{\lambda^{m\alpha}} \lambda^{\alpha - 1} R(\lambda^\alpha)A^m x d\lambda, \quad t > 0.
\]

On the other hand \( A \) can be applied under the integral sign in (3.7) with the convergence of the resulting integral, so that \( u(t) \in D(A) \) and using (1.6), (2.2) and (3.11) we have

\[
D_t^\alpha u(t) - Au(t) =
\]
\[-\frac{t^{(m-1)\alpha}}{\Gamma((m-1)\alpha + 1)} A^m x + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda^{\alpha}} \lambda^{\alpha - 1} R(\lambda^{\alpha})(\lambda^{\alpha} I - A) A^m x \, d\lambda = 0,\]

hence \( u(t) \) is a strong solution of (1.1-2) in \( t \geq 0 \) for \( x \in D(A^m) \).

To prove the inequality (3.1) we use the Post-Widder inversion formula:

**Lemma 3.1.** Let \( u(t) \) be a \( X \) valued continuous function defined in \( t \geq 0 \) such that \( u(t) = O(\exp(\gamma t)) \) as \( t \to \infty \) for some \( \gamma \) and let \( (Lu)(\lambda) \) be the Laplace transform of \( u(t) \). Then

\[
    u(t) = \lim_{n \to \infty} \left( \frac{-1}{n!} \right) \left( \frac{n}{t} \right)^{n+1} (Lu)(\lambda) \left( \frac{n}{t} \right).
\]

uniformly on compacts of \( t > 0 \).

Routine calculations show that \( u(t) \) satisfies the conditions of Lemma 3.1. with \( \gamma = \omega' \) and \( (Lu)(\lambda) = \lambda^{\alpha - 1} R(\lambda^{\alpha}) x \). Then, from (3.12) and the inequalities (3.2) it follows

\[
    \|u(t)\| \leq M\|x\| \lim_{n \to \infty} \frac{(n/t)^{n+1}}{(n/t - \omega)^{n+1}} = M\|x\| \lim_{n \to \infty} \left(1 - \frac{\omega t}{n}\right)^{-(n+1)} = M\|x\| \epsilon^{\omega t}.
\]

Next we define \( \tilde{P}^\alpha_t x = u(t) \) for \( x \in D(A^m), \ t \geq 0 \), where \( u(t) \) is the solution obtained. Since \( D(A) \) is dense in \( X \) and \( g(A) \neq \emptyset \) then \( D(A^m) \) is dense in \( X \) and \( \tilde{P}^\alpha_t \) can be extended to a bounded operator in \( X \) without increase of norm, that is (3.3) holds for \( \tilde{P}^\alpha_t \). The proof will be complete if we show that any solution \( u(t) \) of (1.1-2) admits the representation \( u(t) = \tilde{P}^\alpha_t u(0) \). Since (1.1) must hold for \( t = 0 \) there is no solution for \( x \notin D(A) \) and it is enough to establish that \( \tilde{P}^\alpha_t x \) is the unique solution of (1.1-2) for any \( x \in D(A) \).

Actually for the uniqueness of the solution a weaker condition suffices:

**Theorem 3.2.** ([5]) If \( \text{Re} \lambda \) exists on the half line \( \lambda > \delta \ (\delta > 0) \) and

\[
    \limsup_{\lambda \to \infty} (1/\lambda) \ln \|R(\lambda^{\alpha})\| = 0
\]

then the solution of the problem (1.1-2) is unique.

It is not difficult to check that the inequalities (3.2) imply (3.13) and so we have uniqueness of the solution of (1.1-2).

Next we show that \( \tilde{P}^\alpha_t x \) satisfies (1.1-2) for any \( x \in D(A) \). If \( \lambda \in g(A) \) it follows immediately from the definition of \( \tilde{P}^\alpha_t \) that

\[
    R(\lambda) \tilde{P}^\alpha_t x = \tilde{P}^\alpha_t R(\lambda)x, \ x \in D(A^m), \ t \geq 0.
\]

By the usual continuity argument (3.14) must hold for all \( x \in X \). Noting that \( D(A) = R(\lambda) X \), (3.14) implies \( \tilde{P}^\alpha_t D(A) \subseteq D(A), \ t \geq 0 \) and

\[
    A\tilde{P}^\alpha_t x = \tilde{P}^\alpha_t Ax, \ x \in D(A), \ t \geq 0.
\]
If we apply \( J_\alpha \) to both sides of (1.1) and use (1.2), (2.1) and (3.15) we have

\[ \hat{P}_t^\alpha x - x = J_\alpha \hat{P}_t^\alpha A x, \quad x \in D(A^m). \]

Applying now \( R(\lambda) \) to both sides of this equality and using (3.14) it follows

\[ R(\lambda)(\hat{P}_t^\alpha x - x) = J_\alpha \hat{P}_t^\alpha A R(\lambda)x, \quad x \in D(A^m). \]

Since on both sides bounded operators are applied to \( x \), the last equality must hold for all \( x \in X \). Making use of the fact that \( R(\lambda) \) is one-to-one, we conclude that

\[ \hat{P}_t^\alpha x - x = J_\alpha A \hat{P}_t^\alpha x, \quad x \in D(A). \]

Therefore \( \hat{P}_t^\alpha x \) is the (unique) solution of (1.1-2).

**Remark 3.1.** Inequalities (3.2) follow from their real counterparts

\[ \|(\lambda^{\alpha-1} R(\lambda^\alpha))^{(n)}\| \leq M \frac{n!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0, 1, \ldots \]

The proof uses Taylor series expansion for the function \( \lambda^{\alpha-1} R(\lambda^\alpha) \).

**Corollary 3.1.** If \( A \in C^1(\omega) \) then \( A \in C^\alpha(\omega^{1/\alpha}) \) for any \( \alpha \in (0,1) \).

**Proof.** By induction in \( n \) we obtain the following representation

\[ (\lambda^{\alpha-1} R(\lambda^\alpha))^{(n)} = (-1)^n \sum_{k=1}^{n+1} c_{k,n}^\alpha \lambda^{k\alpha-n-1} R(\lambda^\alpha)^k, \quad n = 0, 1, \ldots, \]  \hspace{1cm} (3.16)

where \( c_{k,n}^\alpha \geq 0 \) are constants, satisfying \( c_{1,0}^\alpha = 1, \ c_{1,n}^\alpha = (n-\alpha)c_{1,n-1}^\alpha, \ c_{k,n}^\alpha = (n-k\alpha)c_{k,n}^{\alpha-1} + \alpha(k-1)c_{k-1,n-1}^\alpha, \ k = 2, \ldots, n, \ c_{n+1,n}^\alpha = n c_{n,n}^\alpha, \ n = 1, 2, \ldots. \)

Then inequalities (3.2) follow using (3.16). Since Corollary 3.1. is a particular case of the more general Theorem 4.2., we omit the details of the proof.

From (3.5),(3.12) and (3.16) we obtain the following representation:

**Corollary 3.2.**

\[ P_t^\alpha x = \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{n+1} c_{k,n}^\alpha (I - (t/n)^\alpha A)^{-k} x, \]  \hspace{1cm} (3.17)

where \( 0 < \alpha \leq 1, \ c_{k,n}^\alpha \) are the constants in (3.16).

Next we express the operator \( A \) in terms of the propagator of the Cauchy problem (1.1-2).

**Proposition 3.1.** If \( P_t^\alpha \) is the propagator of (1.1-2) and \( x \in D(A) \) then

\[ Ax = \Gamma(\alpha + 1) \lim_{t \to 0} \frac{P_t^\alpha x - x}{t^\alpha}. \]  \hspace{1cm} (3.18)
Proof. For a function \( v(t) \) continuous in \( t \geq 0 \) we have: \( \lim_{t \to 0} v(t) = \Gamma(\alpha + 1) \lim_{t \to 0} t^{-\alpha} J_\alpha^t v(t) \). Taking \( v(t) = D_t^\alpha P_t^\alpha x \) and using (1.1) and (2.1) we obtain (3.18). □

Incidentally (3.18) shows that \( P_t^\alpha x \) is not differentiable at \( t = 0^+ \).

The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.3.** Let \( A \) be an operator such that \( R(\lambda^\alpha) \) exists in the half-plane \( \Re \lambda > \omega, \), \( \tilde{P}_t^\alpha \) an operator-valued function strongly continuous in \( t \geq 0 \) and such that \( \|\tilde{P}_t^\alpha\| \leq M e^{\omega t} \), \( t \geq 0 \). Assume that for each \( x \in X \)

\[
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha x \, dt = \lambda^{\alpha-1} R(\lambda^\alpha) x, \quad \Re \lambda > \omega.
\]

(3.19)

Then, \( A \in C^\omega(\omega) \) and \( \tilde{P}_t^\alpha \) is the propagator of (1.1-2).

**4. Holomorphic solutions**

**Theorem 4.1.** Let \( 0 < \alpha \leq 1 \). Assume \( \varrho(A) \supset \Sigma_{\delta,\alpha} = \{ \lambda : |\arg \lambda| < \alpha(\pi/2 + \delta); \lambda \neq 0 \} \) for some \( 0 < \delta \leq \pi/2 \) and

\[
\|R(\lambda)\| \leq C/|\lambda|, \quad \lambda \in \Sigma_{\delta,\alpha}.
\]

Then, \( A \in C^\omega(0) \) and the corresponding propagator \( P_t^\alpha \) has the following additional properties:

(a) \( P_t^\alpha \) can be extended analytically into the sector \( \Delta_\delta = \{ |\arg t| < \delta; t \neq 0 \} \);

(b) \( \|P_t^\alpha\| \) is uniformly bounded in every closed subsector \( \Delta_{\delta-\zeta} \) of \( \Delta_\delta \);

(c) \( P_t^\alpha x \to x \) as \( t \downarrow 0, t \in \Delta_{\delta-\zeta}, x \in X \);

(d) For any \( x \in X, P_t^\alpha x \in D(A) \) and \( \|AP_t^\alpha x\| \leq M(1 + t^{-\alpha}), t > 0 \);

(4.1)

(e) For any \( x \in X, P_t^\alpha x \) is a strong solution of (1.1-2) in \( t > 0 \).

**Proof.** Let \( t \in \Delta_{\delta-\zeta} \) for some \( \varepsilon \in (0, \delta) \) and \( \theta \in (\delta - \varepsilon/2, \delta) \), \( \varrho > 0 \). Set

\[
\tilde{P}_t^\alpha = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha) \, d\lambda,
\]

(4.2)

where \( \Gamma = \{ re^{-i(\pi/2 + \theta)}; \varrho \leq r < \infty \} \cup \{ ge^{i\varphi}; -(\pi/2 + \theta) \leq \varphi \leq \pi/2 + \theta \} \cup \{ re^{i(\pi/2 + \theta)}; \varrho \leq r < \infty \} \) is oriented counterclockwise. If \( \lambda \in \Gamma \) then \( \lambda^\alpha \in \Sigma_{\delta,\alpha} \), and so \( R(\lambda^\alpha) \) exists and from (4.1)

\[
\|\lambda^{\alpha-1} R(\lambda^\alpha)\| \leq C/|\lambda|, \quad \lambda \in \Gamma.
\]

(4.3)
Let \( q = 1/|t| \) and \( a = \sin \epsilon/2 \). Then from (4.2) and (4.3) it follows
\[
\|\hat{P}_t^\alpha\| \leq \frac{C}{2\pi} \int_{\Gamma} e^{\Re(\lambda) \frac{|d\lambda|}{|\lambda|}} \leq \frac{C}{\pi} \int_0^\infty e^{-a_r \frac{dr}{r}} + \frac{C}{\pi} \int_0^\pi e^{\cos \varphi} d\varphi \leq M. \tag{4.4}
\]
This estimate shows that the integral in (4.2) is absolutely convergent for \( t \in \tilde{\Delta}_{\delta-\epsilon} \), hence (a) and (b) are satisfied for \( \hat{P}_t^\alpha \).

Now fix \( \lambda, \Re \lambda > 0 \), and take \( q \) in (4.3) such that \( q < \Re \lambda \). Then, using (4.2), we have
\[
\int_0^\infty e^{-\lambda t} \hat{P}_t^\alpha dt = \frac{1}{2\pi i} \int_{\Gamma} \mu^{\alpha-1} R(\mu^\alpha) \int_0^\infty e^{-(\lambda-\mu)t} dt \frac{-1}{\mu - \lambda} d\mu = \frac{\mu^{\alpha-1} R(\mu^\alpha)}{\mu - \lambda} \Rightarrow \frac{\lambda^{\alpha-1} R(\lambda^\alpha)}{\lambda - \lambda},
\]
where Fubini’s theorem and Cauchy’s integral formula in the region of the right of \( \Gamma \) are used. So we have proved that the conditions of Theorem 3.3. are fulfilled with \( \omega = 0 \). Therefore \( A \in C^\alpha(0) \) and the corresponding propagator \( P_t^\alpha = \hat{P}_t^\alpha \).

Next for \( x \in D(A) \) by Cauchy’s theorem and Cauchy’s integral formula if \( t \in \tilde{\Delta}_{\delta-\epsilon}, t \downarrow 0 \)
\[
P_t^\alpha x - x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda^{\alpha-1} R(\lambda^\alpha) x - \lambda^{-1} x) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha) A x d\lambda \rightarrow 0,
\]
that together with (b) implies (c).

Now, for \( t > 0 \) and \( q = 1/t \), applying \( A \) to both sides of (4.2) and using the identity \( AR(\lambda) = \lambda R(\lambda) - I \) we have
\[
AP_t^\alpha = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} A R(\lambda^\alpha) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha) d\lambda - \frac{e^{-\alpha}}{\Gamma(1-\alpha)}.
\]
Taking \( q = 1/t \) and estimating \( AP_t^\alpha \) in the same way as in (4.4) we obtain (d).

At the end, applying \( J_t^\alpha \) to both sides of (1.1) we have
\[
P_t^\alpha x - x = J_t^\alpha P_t^\alpha Ax, \quad x \in D(A). \tag{4.5}
\]
But from (d) \( J_t^\alpha P_t^\alpha A \) is a bounded operator for \( t > 0 \). Then (4.5) holds for any \( x \in X, t > 0 \). This implies (e). \( \Box \)

**Proposition 4.1.** If \( \varrho(A) \supset \{ \lambda: \Re \lambda > 0 \} \) and for some constant \( M \)
\[
\|R(\lambda)\| \leq M/\Re \lambda, \quad \Re \lambda > 0. \tag{4.6}
\]
then for any \( \alpha \in (0, 1) \) \( A \) satisfies the hypotheses of Theorem 4.1. with \( 0 < \delta < \min\{ (1/\alpha - 1)\pi/2, \pi/2 \} \).
Proof. Fix $\alpha$ and $\delta$ satisfying the conditions mentioned above. Then $\alpha(\pi/2 + \delta) < \pi/2$ and so $\Sigma_{\alpha, \delta} \subset \rho(A)$. Taking $\beta$, $\alpha(\pi/2 + \delta) < \beta < \pi/2$ we obtain

$$||R(\lambda)|| \leq \frac{M}{\text{Re}\lambda} = \frac{M}{|\lambda| \cos \theta} < \frac{M}{|\lambda| \cos \beta}, \quad \lambda \in \Sigma_{\alpha, \delta},$$

where $\theta = \arg \lambda$ and (4.1) holds with $C = M/\cos \beta$. □

The next theorem describes the relation between the solutions of the Cauchy problem (1.1-2) for different values of $\alpha$, $0 < \alpha \leq 1$.

**Theorem 4.2.** Let $0 < \alpha < \beta \leq 1$, $\gamma = \alpha/\beta$, $\omega \geq 0$. If $A \in C^{\beta}(\omega)$ then $A \in C^{\alpha}(\omega^{1/\gamma})$ and the following representation holds

$$P_t^{\alpha}x = \frac{1}{t^\gamma} \int_0^\infty \Phi_{\gamma} \left( \frac{s}{t^\gamma} \right) P_s^{\beta}x \, ds, \quad t > 0, \quad (4.7)$$

where

$$\Phi_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \quad z \in \mathbb{C}, \quad (4.8)$$

is an entire function. Moreover $P_t^{\alpha}$ can be analytically extended to the sector $\Delta_\theta$, $\theta = \min\{(1/\gamma - 1)\pi/2, \pi/2\}$ and for any $x \in X$ $P_t^{\alpha}x$ is a strong solution of (1.1-2) in $t > 0$.

Proof. We summarize some properties of $\Phi_{\gamma}(z)$ (for more details see [4], p. 5, [13], [15] and [21]). It is a particular case of a special function known as Wright function ([5]), entire in the complex plane:

$$W(z; \gamma, \delta) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\gamma n + \delta)} = \frac{1}{2\pi i} \int_{\Gamma} \mu^{-\delta} \exp(\mu + z \mu^{-\gamma}) \, d\mu, \quad \gamma > -1, \quad \delta \in \mathbb{C},$$

where $\Gamma$ is a contour which starts at $-\infty$, encircles the origin once counterclockwise and turns to its starting point. Obviously $\Phi_{\gamma}(z) = W(-z; -\gamma, 1 - \gamma)$ and the integral representation

$$\Phi_{\gamma}(z) = \frac{1}{2\pi i} \int_{\Gamma} \mu^{\gamma - 1} \exp(\mu - z \mu^{-\gamma}) \, d\mu, \quad z \in \mathbb{C}, \quad (4.9)$$

holds. For real values of $z$ we have ([4], p. 5):

$$\Phi_{\gamma}(t) \geq 0, \quad t \geq 0. \quad (4.10)$$

Let us denote $\varphi_{t, \gamma}(s) = (1/t^\gamma)\Phi_{\gamma}(s/t^\gamma)$. Then the following formula is true ([4], p. 5):

$$E_{\gamma}(\omega t^\gamma) = \int_0^{\infty} \varphi_{t, \gamma}(s)e^{\omega s} \, ds, \quad t > 0, \quad 0 < \gamma < 1. \quad (4.11)$$
From the asymptotic expansion of the Wright function given in [21] (this function is called there \textit{generalized Bessel function}) we obtain the inequality:

\[ |W(-z; -\gamma, \delta)| \leq B \exp(-bz^{1/(1-\gamma)}) \quad (4.12) \]

for \(|\arg z| < \min\{(1-\gamma)3\pi/2, \pi\}, |z| \to \infty, \ 0 < \gamma < 1, \ \delta \in \mathbb{C}\), where \(B = B(\gamma, \delta), \ b = b(\gamma, \delta)\) are positive constants.

Now define

\[ \hat{P}^\alpha_t x = \int_0^\infty \varphi_{t,\gamma}(s) P^\beta_s x \, ds, \ t > 0. \quad (4.13) \]

Our aim is to prove that \(\hat{P}^\alpha_t\) satisfies the conditions of Theorem 3.3. Since \(A \in C^\beta\), there exist constants \(M \geq 1\) and \(\omega \geq 0\) such that

\[ \|P^\beta_s\| \leq Me^{\omega s}, \ s \geq 0, \quad (4.14) \]

and together with (4.10) and (4.11) we obtain

\[ \|\hat{P}^\alpha_t\| \leq \int_0^\infty \varphi_{t,\gamma}(s)\|P^\beta_s\| ds \leq M \int_0^\infty \varphi_{t,\gamma}(s)e^{\omega s} ds = ME_{\gamma}(\omega t^\gamma). \quad (4.15) \]

We observe that a constant \(C_{\gamma}\) exists, such that

\[ E_{\gamma}(\omega t^\gamma) \leq C_{\gamma} \exp(\omega^{1/\gamma}t), \ \omega \geq 0, \ t \geq 0. \quad (4.16) \]

Indeed, from (2.7) there exists a constant \(C_1\) such that \(E_{\gamma}(\omega t^\gamma) \leq C_1 \exp(\omega^{1/\gamma}t)\) for every \(t \geq T\). From the continuity of the Mittag-Leffler function in \(t \geq 0\) it follows that there exists a constant \(C_2\) such that \(E_{\gamma}(\omega t^\gamma) \leq C_2 \leq C_2 \exp(\omega^{1/\gamma}t)\) for \(t \in [0, T]\). Hence, we can take \(C_{\gamma} = \max\{C_1, C_2\}\). Now from (4.15) and (4.16) we have

\[ \|\hat{P}^\alpha_t\| \leq M \exp(\omega^{1/\gamma}t), \ t \geq 0. \]

Taking \(\lambda \in \mathbb{C}\) with \(\text{Re}\lambda > \omega^{1/\gamma}\) we shall prove that \(R(\lambda^\alpha)\) exists. If \(\lambda = |\lambda|e^{i\theta}\) then \(|\theta| < \pi/2\) and for such \(\theta\) and \(\gamma \in (0, 1)\) the inequality \(\cos \gamma \theta \geq \cos \gamma \theta\) holds. Then \(\text{Re}(\lambda^\gamma) = |\lambda|^\gamma \cos \gamma \theta \geq |\lambda|^\gamma \cos \gamma \theta = (\text{Re}\lambda)^\gamma > \omega\). Since \(A \in C^\beta\), it follows from Theorem 3.1. that \(R(\mu^\beta)\) exists in the half-plane \(\text{Re}\mu > \omega\). Taking \(\mu = \lambda^\gamma\) we obtain that \(R(\lambda^\alpha) = R(\mu^\beta)\) exists.

It is plain that \(\hat{P}^\alpha_t\) is strongly continuous for \(t > 0\). To prove the strong continuity at the origin we substitute \(\sigma = s/t^\gamma\) in (4.13). Then, using the identity \(\int_0^\infty \Phi_{\gamma}(\sigma) \, d\sigma = 1\) (that follows from (4.11) setting \(\omega = 0\)) and the strong continuity of \(P^\beta_t\) at the origin, we have

\[ \lim_{t \downarrow 0} \hat{P}^\alpha_t x = \lim_{t \downarrow 0} \int_0^\infty \Phi_{\gamma}(\sigma) P^\beta_{st^\gamma} x \, d\sigma = \int_0^\infty \Phi_{\gamma}(\sigma) x \, d\sigma = x. \quad (4.17) \]
For \( \text{Re}\lambda > \omega^{1/\gamma} \), using (4.13) and interchanging the order of integration we have

\[
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha dt = \int_0^\infty P_s^\beta \int_0^\infty e^{-\lambda t} \varphi_{t,\gamma}(s) dt ds. \tag{4.18}
\]

Substituting \( \mu = \tau t \) in (4.9) and shifting the new contour \( \Gamma' = \Gamma/t \) to \( \Gamma \) we get the integral representation

\[
\varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_\Gamma t^{-\gamma-1} \exp(\tau t - \tau^\gamma s) d\tau, \tag{4.19}
\]

Then, noting that \( \text{Re}(\lambda - \tau) > 0 \) and using Cauchy's formula it follows

\[
\int_0^\infty e^{-\lambda t} \varphi_{t,\gamma}(s) dt = \frac{1}{2\pi i} \int_\Gamma t^{-\gamma-1} \exp(-\tau^\gamma s) \int_0^\infty e^{-(\lambda - \tau)t} dt d\tau = \frac{-1}{2\pi i} \int_\Gamma \frac{t^{-\gamma-1} \exp(-\tau^\gamma s)}{\tau - \lambda} d\tau = \lambda^{-\gamma-1} \exp(-\lambda^\gamma s). \tag{4.20}
\]

Using now (3.5) for \( P_s^\beta \), we obtain from (4.18) and (4.20)

\[
\int_0^\infty e^{-\lambda t} \tilde{P}_t^\alpha dt = \lambda^{-\gamma-1} \int_0^\infty \exp(-\lambda^\gamma s) P_s^\beta ds = \lambda^{-\gamma-1} \lambda^\beta R(\lambda^\beta) = \lambda^{\alpha-1} R(\lambda^\alpha).
\]

So we have proved the conditions of Theorem 3.3. Therefore \( A \in C^\alpha(\omega^{1/\gamma}) \) and the corresponding propagator \( P_t^\alpha = \tilde{P}_t^\alpha \).

Using the integral representation (4.19) we have

\[
\frac{\partial}{\partial t} \varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_\Gamma t^{-\gamma} \exp(\tau t - \tau^\gamma s) d\tau = \frac{1}{t^{\gamma+1}} W\left(-\frac{s}{t^\gamma}; -\gamma, -\gamma\right).
\]

Now, applying the inequalities (4.12) and (4.14) and noting that \( 1/(\gamma - 1) > 1 \) for \( 0 < \gamma < 1 \) it easily follows that the integral in the right-hand side of (4.13) is absolutely convergent for \( t \in \Delta_\theta \) and the differentiation under the integral sign is possible, that is \( P_t^\alpha \) admits analytic extention into \( \Delta_\theta \).

Differintegrating under the integral sign in (4.19) we obtain

\[
D_t^{\alpha} \varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_\Gamma t^{\alpha+\gamma-1} \exp(\tau t - \tau^\gamma s) d\tau = \frac{1}{t^{\gamma+\alpha}} W\left(-\frac{s}{t^\gamma}; -\gamma, 1 - \alpha - \gamma\right).
\]

Then, differintegrating (4.13) and applying again (4.12) and (4.14), it results that \( D_t^{\alpha} P_t^\alpha \) is a bounded operator for \( t > 0 \). Since \( A \in C^\alpha \)

\[
D_t^{\alpha} P_t^\alpha x = A P_t^\alpha x, \quad x \in D(A), \quad t > 0,
\]

that from (1.6), (2.2) and (4.17) is equivalent to

\[
D_t^{\alpha} P_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} x = A P_t^\alpha x, \quad x \in D(A), \quad t > 0. \tag{4.21}
\]
Taking $x \in X$, a sequence $\{x_n\} \subset D(A)$ exists such that $x_n \to x$. Then from (4.21) and the boundedness of $D_t^\alpha P_t^\alpha$ it follows

$$AP_t^\alpha x_n = D_t^\alpha P_t^\alpha x_n - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x_n \to D_t^\alpha P_t^\alpha x - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x = y.$$ 

Now from the closedness of $A$ we have $P_t^\alpha x \in D(A)$ and $y = AP_t^\alpha x$. Therefore (4.21) holds for every $x \in X$. □

5. Examples

The next examples illustrate the theory in the preceding sections.

EXAMPLE 5.1. Consider the fractional diffusion problem (see [8], [13]):

$$D_t^\alpha u = \partial^2 u / \partial x^2, \quad -\infty < x < \infty, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(x, 0) = f(x).$$

Let $X = L^2(R)$, $Af = f''$ with $D(A) = W^{2,2}(R)$. Since $A$ generates a holomorphic semigroup, the conditions of Theorem 4.1. are satisfied and the solution has the properties (a)-(e). In this case the solution is given explicitly by (see [13]):

$$u(x, t) = (1/2) t^{-\alpha/2} \int_{-\infty}^\infty \Phi_{\alpha/2}(|s|t^{-\alpha/2}) f(x - s) \, ds.$$ 

Similar results hold for a more general fractional diffusion problem in $R^n$ with $Af = \nabla^2 f$ (see [20]).

EXAMPLE 5.2. Consider the problem

$$D_t^\alpha u = -\partial u / \partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).$$

Let $X = L^2(0, 1)$, $Af = f'$ with $D(A) = \{f \in W^{1,2}(0, 1), \quad f(0) = 0\}$. It is well-known that $A$ generates an uniformly bounded $C_0$-semigroup and using Proposition 4.1. the properties (a)-(e) of the solution follow again. Moreover from (4.2) we obtain the following explicit representation of the solution:

$$u(x, t) = t^{-\alpha} \int_0^x \Phi_{\alpha}(st^{-\alpha}) f(x - s) \, ds.$$ 

EXAMPLE 5.3. If $X = L^2(0, 1)$, $A'_\phi f = -e^{i\theta} f'$ with $D(A'_\phi) = \{f \in W^{1,2}(0, 1), \quad f(0) = 0\}$ we have the problem

$$D_t^\alpha u = -e^{i\theta} \partial u / \partial x, \quad 0 < x < 1, \quad t \geq 0, \quad 0 < \alpha < 1; \quad u(0, t) = 0, \quad u(x, 0) = f(x).$$
with solution (obtained by Laplace transform method)

\[ u(x, t) = e^{-i\theta t - \alpha} \int_0^x \Phi_\alpha(se^{-i\theta t - \alpha})f(x-s) \, ds \]

for \(|\theta| \leq \theta'_\alpha\), where \(\theta'_\alpha = (1 - \alpha)\pi/2\). Therefore \(A'_\theta \in \mathcal{C}\) iff \(|\theta| \leq \theta'_\alpha\), while \(A'_\theta\) generates a \(C_0\)-semigroup only for \(\theta = 0\). Taking \(|\theta| < \theta'_\alpha\), \(\theta \neq 0\), Theorem 4.2. implies that the solution is even holomorphic although \(A'_\theta\) does not generate a \(C_0\)-semigroup.

**Example 5.4.** If \(X = L^2(0, 1)\), \(A''_\theta f = e^{i\theta f''}\) with \(D(A''_\theta) = \{f \in W^{2,2}(0, 1), f(0) = f(1) = 0\}\), the corresponding problem is

\[ D^2_\theta u = e^{i\theta} \partial^2 u/\partial x^2, \quad 0 < x < 1, \quad t > 0, \quad 0 < \alpha < 1; \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = f(x). \]

Since \(A''_\theta\) has eigenvalues \(z_n = -e^{i\theta}n^2\pi^2\) and eigenfunctions \(n\pi x\), then if \(f(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x\) the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} c_n \sin n\pi x E_\alpha(-e^{i\theta}n^2\pi^2 t^{\alpha}). \]

From the asymptotic expansion of the Mittag-Leffler function (2.7-8) it is clear that \(A''_\theta \in \mathcal{C}\) iff \(|\theta| \leq \theta''_\alpha\), where \(\theta''_\alpha = (1 - \alpha/2)\pi\). Moreover if \(|\theta| < \theta''_\alpha\) the solution is holomorphic. So \(A''_\theta\) with \(|\theta| < \theta''_\alpha\), \(\Re \theta < 0\), is another example of an operator generating holomorphic solution of (1.1-2) but not generating \(C_0\)-semigroup (no right half-plane is free of spectra of \(A''_\theta\)).

**Acknowledgments**

The author is grateful to Professor J. de Graaf for suggesting this problem and many helpful discussions. Thanks are also due to the referee for drawing our attention toward some relevant references.

**References**


16


*) Department of Mathematics and Computing Science,
Eindhoven University of Technology,
P.O. Box 513,
5600 MB Eindhoven – The NETHERLANDS

e-mail: emilia@win.tue.nl

Received: March, 1998