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Two Approximations for the Steady-State Probabilities and the Sojourn-Time Distribution of the $M/D/c$ Queue with State-Dependent Feedback

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Eindhoven, December 1996
The Netherlands
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Abstract

In the $M/D/c$ queue with state-dependent feedback, a customer is only allowed to depart from the system if his service has been successful. Otherwise, the customer must be re-serviced immediately. The probability that a customer's service is successful depends on the number of customers in service at the moment the service is finished. The application behind this type of feedback queue is a real-time database where transactions must be rerun if their data was changed by other transactions during the execution. In this paper, two different approximations for the steady-state probabilities and the sojourn-time distribution of the $M/D/c$ queue with state-dependent feedback are studied. The first approximation is based on an embedded Markov chain and uses the well-known residual-life approximation for the remaining service times of the customers in service. The second approximation is similar to the exact analysis of the ordinary $M/D/c$ queue. Comparison with simulation shows, that both approximations are very accurate for a wide range of system parameters, even for heavily loaded systems.

1 Introduction

Consider the $M/D/c$ queue with $c \geq 1$ servers where customers arrive according to a Poisson process with rate $\lambda$. The service times of the customers are all equal to $D$. In the ordinary $M/D/c$ queue, a customer whose service is completed departs from the system immediately (so with probability 1). In the queueing model considered in this paper, a customer whose service is completed departs from the system immediately with probability $p(n)$, but is fed back to the server for a new service with probability $1 - p(n)$ (with $0 < p(n) \leq 1$). Here $n$ is the number of customers in service just prior to the service completion epoch. We call this queueing system an $M/D/c$ queue with state-dependent feedback.

The queueing model is depicted in Figure 1. At most $c$ customers can be served at the same time, the others have to wait in a queue. The waiting room is unbounded. If a customer's service is unsuccessful, the customer is immediately fed back to his server for a rerun. Customers do not depart from the system until they have received a successful service.

We are interested in the steady-state probabilities and the sojourn-time distribution of this queueing system. For stability, the customer arrival rate should not exceed the average number of customers that leaves the system per time unit when all $c$ servers are busy. So we assume that $\lambda D < cp(c)$. 

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The feedback mechanism studied in this paper is unconventional in two respects. Firstly, a customer immediately restarts service when he is fed back, so he does not have to rejoin the queue to await a new service. Although this immediate-restart mechanism has no consequences for the distribution of the queue length, it does change the sojourn-time distribution. Secondly, the feedback probability depends on the number of customers in service just before the service completion epoch. Known (and analyzed) feedback mechanisms are either Bernoulli (i.e., the success probability is fixed at $p$) or depend on the number of service runs already received by the customer.

The application behind the $M/D/c$ queue with state-dependent feedback studied in this paper is a real-time database (RTDB) with optimistic concurrency control (OCC) where transactions are processed in parallel (concurrently) as much as possible.

A transaction on a database is a sequence of operations, such as reading, calculating, and writing, on a set of data. If during the processing of a transaction other transactions overwrite (some of) the data in use by the transaction, it becomes unsuccessful and will have to be rerun. So success of a transaction depends on the number of transactions that were present during its execution.

We found that the very complicated behavior of this RTDB with OCC can be well approximated by modeling it as an $M/D/c$ queue with state-dependent feedback. For details, see Sassen and van der Wal [1996a].

To our knowledge, the $M/D/c$ queue with state-dependent feedback has not received any attention in literature. On the one hand, this is caused by the uncommon feedback mechanism. On the other hand, even for multi-server queues with conventional feedback mechanisms almost no results seem to be available. A possible reason for this is that multi-server queues without feedback are already so difficult to analyze, that they deserve full attention. An exception is of course the $M/M/c$ queue, which (both with and without Bernoulli feedback) has a steady-state distribution of product form.

Research on queueing models with feedback was initiated by the pioneering paper of Takács [1963] for the $M/G/1$ queue with Bernoulli feedback. From then on, many feedback variants for this
single-server queue have been analyzed. References can be found in the paper of Hunter [1989]. Hunter obtained an expression for the Laplace-Stieltjes transform of the sojourn-time distribution in Markov renewal and birth-death queues with feedback. Van den Berg and Boxma [1991] obtained results for the sojourn-time distribution in an $M/G/1$ processor-sharing queue. The only multi-server queue with feedback for which we know the sojourn-time distribution was analyzed is the $M/M/2$ queue with Bernoulli feedback, see Montazer-Haghighi [1977].

In this paper, we study two approximations for the system. The first approximation, discussed in section 2, is an embedded Markov chain approach that uses a residual-life approximation for the remaining service times of the customers in service. The state of the system is only reviewed just after service completion epochs. The second approximation, considered in section 3, resembles the exact analysis of the ordinary $M/D/c$ queue by observing the system state at the start and at the end of a slot of length $D$. From the two approximations for the steady-state probabilities, we compute approximations for the sojourn-time distribution in section 4. Section 5 compares both approximations with values resulting from a simulation of the model. Section 6 contains some concluding remarks.

## 2 Approximation I

The inter-arrival times are exponentially distributed so have the memoryless property. However, the time between two service completions (successful or unsuccessful) is not memoryless. For an exact analysis of the steady-state probabilities of the $M/D/c$ queue with state-dependent feedback, the system should be described by the state-vector $(w(t), r_1(t), r_2(t), \ldots, r_c(t))$ with $w(t)$ the number of waiting customers and $r_i(t)$ the remaining service time of the customer at server $i$ at time $t$ ($r_i(t) = 0$ if server $i$ is free), $i = 1, \ldots, c$. We are not very optimistic about the chances of an exact analysis of this system.

Therefore, we introduce the following approximation assumption regarding the time until the next service completion epoch. The assumption is similar to the approximation assumption Tijms et al. [1981] used for the $M/G/c$ queue.

**Approximation Assumption**

1 a) If just after a successful service completion epoch $k$ customers are in the system with $1 \leq k < c$, then the time until the next service completion epoch is distributed as the minimum of $k$ independent random variables, each uniformly distributed over $(0, D)$.

b) If just after an unsuccessful service completion epoch $k$ customers are in the system with $1 \leq k < c$, then the time until the next service completion epoch is distributed as the minimum of the deterministic variable $D$ and $k - 1$ independent random variables, each uniformly distributed over $(0, D)$.

2 If just after a successful or unsuccessful service completion epoch $k \geq c$ customers are in the system, then the time until the next service completion epoch equals $D/c$ with probability 1.
In other words, when $k < c$, the approximation assumption states that the remaining service time of each service in progress is distributed as the equilibrium excess distribution of the original service time. The equilibrium excess distribution of a deterministic variable $D$ is a uniform distribution over $(0, D)$. When $k \geq c$, the approximation assumption states that the system behaves like an $M/D/1$ system with feedback, in which the single server works $c$ times as fast as each of the $c$ servers in the original system.

This type of approximation assumption, based on the equilibrium excess distribution of the service times, is well known and was first applied successfully for approximating the steady-state probabilities of the $M/G/c$ queue by Tijms et al. [1981]. We will show that the approximation assumption also yields very accurate results for the $M/D/c$ queue with state-dependent feedback. If the success probability $p(n)$ equals 1 for all $n$, then the approximation assumption and our analysis reduces to Case A of the approximative analysis of Tijms et al. [1981] for the ordinary $M/D/c$ queue.

The above assumption enables us to model the $M/D/c$ system with state-dependent feedback by an embedded Markov chain that only considers the system just after service completion epochs. The possible states of this embedded Markov chain are:

$(k, 0)$: just after an unsuccessful service completion epoch $k$ customers are in the system, $k \geq 1$. One of the services has just started.

$(k, 1)$: just after a successful service completion epoch $k$ customers are in the system, $k \geq 0$. If $k \geq c$, a new service has just started. Otherwise, all services were already in progress.

We want to compute the steady-state probabilities of the embedded Markov chain. Once the steady state at service completion epochs has been found, the steady-state probabilities at arbitrary epochs in time are calculated very easily.

Let $R_j$ be distributed as the minimum of $j$ independent, uniform$(0, D)$-distributed random variables ($j = 1, \ldots, c - 1$). Let $R_j(t)$ be the distribution function of $R_j$. Then

$$R_j(t) = \begin{cases} 
1 - \left(1 - \frac{t}{D}\right)^j & , \quad t < D \\
1, & , \quad t \geq D
\end{cases}$$

for $1 \leq j \leq c - 1$.

Define for $j = 1, \ldots, c - 1$ and $\ell \geq 0$ the probability $a[j, \ell]$ as the probability that $\ell$ customers arrive during $R_j$. Also, define $a[0, \ell]$ with $\ell \geq 0$ as the probability that $\ell$ customers arrive during $D$, and $a[c, \ell]$ with $\ell \geq 0$ as the probability that $\ell$ customers arrive during $D/c$. Since the arrival process is Poisson with intensity $\lambda$,

$$a[0, \ell] = e^{-\lambda D} \frac{\lambda^\ell}{\ell!} \quad \text{and} \quad a[c, \ell] = e^{-\lambda D/c} \frac{(\lambda D/c)^\ell}{\ell!}.$$
Computing $a[j, \ell]$ for $j = 1, \ldots, c - 1$ is more cumbersome, but can be done by conditioning on $R_j$.

$$a[j, \ell] = \int_0^D e^{-\lambda t} \frac{(\lambda t)^\ell}{\ell!} dR_j(t)$$

$$= \sum_{i=0}^{j-1} \frac{(\ell + i)!}{\ell!} \frac{j(-1)^i}{(j - 1)!} \left[ 1 - \sum_{m=0}^{\ell+i} \frac{e^{-\lambda D (\lambda D)^m}}{m!} \right], \quad 1 \leq j \leq c - 1, \ell \geq 0.$$ 

Here we applied the binomial of Newton and the useful identity

$$\int_0^D \lambda e^{-\lambda t} (\lambda t)^k \frac{dt}{k!} = 1 - \sum_{m=0}^k \frac{(-1)^m e^{-\lambda D (\lambda D)^m}}{m!}.$$ 

Now the steady-state vector $\pi$ of the embedded Markov chain is the unique non-negative solution to the balance equations

$$\pi(k, 1) = \sum_{\ell=0}^k p(k+1)a[k-\ell, \ell]\pi(k-\ell+1, 0) + p(k+1)a[0, k]\pi(0, 1)$$

$$+ \sum_{\ell=0}^k p(k+1)a[k-\ell+1, \ell]\pi(k-\ell+1, 1), \quad 0 \leq k \leq c - 2$$

$$\pi(c-1, 1) = \sum_{\ell=1}^c p(c)a[c-1-\ell, \ell]\pi(c-\ell, 0) + p(c)a[c, 0]\pi(c, 0)$$

$$+ \sum_{\ell=0}^{c-1} p(c)a[c-\ell, \ell]\pi(c-\ell, 1) + p(c)a[0, c-1]\pi(0, 1)$$

$$\pi(k, 0) = \sum_{\ell=0}^{k-1} (1-p(k))a[k-\ell-1, \ell]\pi(k-\ell, 0) + (1-p(k))a[0, k-1]\pi(0, 1)$$

$$+ \sum_{\ell=0}^{k-1} (1-p(k))a[k-\ell, \ell]\pi(k-\ell, 1), \quad 1 \leq k \leq c - 1$$

$$\pi(k, 1) = \sum_{\ell=0}^{k-c+1} p(c)a[c, \ell]\pi(k-\ell+1, 0) + \sum_{\ell=k-c+2}^k p(c)a[k-\ell, \ell]\pi(k-\ell+1, 0)$$

$$+ \sum_{\ell=0}^{k-c} p(c)a[\min\{k-\ell+1, c\}, \ell]\pi(k-\ell+1, 1) + p(c)a[0, k]\pi(0, 1), \quad k \geq c$$

$$\pi(k, 0) = \sum_{\ell=0}^{k-1} (1-p(c))a[c, \ell]\pi(k-\ell, 0) + \sum_{\ell=k-c+1}^{k-1} (1-p(c))a[k-\ell-1, \ell]\pi(k-\ell, 0)$$

$$+ \sum_{\ell=0}^{k-1} (1-p(c))a[\min\{k-\ell, c\}, \ell]\pi(k-\ell, 1) + (1-p(c))a[0, k-1]\pi(0, 1), \quad k \geq c$$

together with the normalization equation

$$\sum_{k=1}^{\infty} [\pi(k, 0) + \pi(k, 1)] + \pi(0, 1) = 1.$$ 

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The balance equations can be solved by truncating the state space at a large level $M$ (say), so at the states $(M, 0)$ and $(M - 1, 1)$, and rejecting customers that find $M$ customers in the system.

Another way to solve the balance equations is by exploiting the geometric-tail behavior of the embedded Markov chain, as in Tijms and van de Coevering [1991]. In Appendix A we show that the Markov chain has a single geometric tail. Thus there exist a large $M$ and a $\tau \in (0, 1)$ such that for $k \geq M$, $\pi(k, 0) \approx \pi(M, 0)\tau^{k-M}$ and $\pi(k, 1) \approx \pi(M, 1)\tau^{k-M}$. From the balance equations for $\pi(k, 1)$ and $\pi(k, 0)$ for $k \geq c$, we find (see Appendix A) that $\tau$ is the unique root of the equation

$$1 - p(c)(1 - y) = \exp(\lambda D(1 - 1/y)/c)$$

on the interval $(0, 1)$. When $p(c) = 1$, this equation simplifies to

$$1/y = \exp(-\lambda D(1 - 1/y)/c),$$

the equation for the geometric tail of the ordinary $M/D/c$ queue.

Computing $\tau$ from (1) and substituting $\pi(k, 0) = \pi(M, 0)\tau^{k-M}$ and $\pi(k, 1) = \pi(M, 1)\tau^{k-M}$ for $k \geq M$ in the balance equations and in the normalization equation leads to a system of $2M + 1$ linear equations. This system can easily be solved since $M$ does not have to be very large to obtain reasonable accuracy of the solution. Typically, the value of $M$ required by the geometric-tail approach to obtain some desired accuracy is much smaller than the value of $M$ required when solving for the steady-state probabilities by truncating the state space, especially when the traffic intensity $\rho$ is large, see Tijms and van de Coevering [1991].

Next, we show how the steady-state probabilities of the $M/D/c$ queue with state-dependent feedback can be computed from the steady-state probabilities of the embedded Markov chain. Denote by $\varphi_l(k)$ the fraction of departing customers that leaves $k$ customers behind in the system. Once $\pi(i, 1)$ is known for $i \geq 0$, $\varphi_l(k)$ can be computed as

$$\varphi_l(k) = \frac{\pi(k, 1)}{\sum_{i \geq 0} \pi(i, 1)}, \quad k \geq 0.$$ 

Since customers arrive one at a time and are served one at a time, the fraction of real departures that leaves $k$ customers behind equals the fraction of new customers that finds $k$ customers in the system upon arrival. Further, because of the Poisson arrival process, we have by the PASTA property (Wolff [1982]) that the long-term fraction of time that $k$ customers are in the system equals the fraction of arrivals that finds $k$ customers in the system.

Hence, the probabilities $\varphi_l(k)$ are our first approximation for the steady-state probabilities of the $M/D/c$ queue with state-dependent feedback.

### 3 Approximation II

In Approximation I, the time between successive service completions is approximated. Since the state of the system is observed at every service completion epoch, the success probability is known exactly
(namely, \( p(k) \) if \( k \) customers are in service at a service completion epoch). Approximation II, which will be discussed next, can be considered as the opposite to Approximation I, because it is exact with respect to time but inexact with respect to the success probability.

Let us explain Approximation II. Just as in the exact analysis of the ordinary \( M/D/c \) queue by Crommelin [1932], we observe the state of the system every \( D \) time units. Since the service times are constant and equal to \( D \), any customer in service at some time \( t \) will have completed his service — either successfully or unsuccessfully — at time \( t + D \). The customers present at time \( t + D \) are exactly those customers who completed an unsuccessful service during \( (t, t + D] \), plus the customers who were either waiting in queue at time \( t \) or who arrived in \( (t, t + D] \). Hence, we can relate the number of customers in the system at time \( t + D \) to the number in the system at time \( t \).

To do this, let \( q_k(u) \) be the probability that \( k \) customers are in the system at time \( u \). Also, let \( a[\ell] \) be the probability that \( \ell \) customers arrive in \( (t, t + D] \), so \( a[\ell] = e^{-\lambda D} (\lambda D)^\ell / \ell! \) for \( \ell \geq 0 \). Finally, let \( B_i^j \) denote the probability that \( i \) services are completed successfully during a time-interval \( (0, D] \), given that \( j \) customers are in the system at the start of the interval. How to find \( B_i^j \) is discussed in detail below, but first we state the relation between the number of customers present at time \( t \) and at time \( t + D \).

By conditioning on the state at time \( t \) we find

\[
q_k(t + D) = \sum_{j=0}^{c+k} q_j(t) \sum_{i=\max\{0, j-k\}}^{\min\{j, c\}} B_i^j a[k - j + i] \quad \text{for} \quad k \geq 0.
\]

Next, by letting \( t \to \infty \) in these equations and noting that \( q_k(u) \to q_k \) as \( u \to \infty \), it follows that the time-average probabilities \( q_k \) satisfy the linear equations

\[
q_k = \sum_{j=0}^{c+k} q_j \sum_{i=\max\{0, j-k\}}^{\min\{j, c\}} B_i^j a[k - j + i], \quad k \geq 0, \quad \sum_{k=0}^{\infty} q_k = 1.
\]

In the same way as done in Appendix A for the balance equations of Approximation I, it can be proved that the probabilities \( q_k \) have a geometric tail, i.e., \( q_k \approx q_{k-1} \tau \) as \( k \to \infty \). The geometric-tail factor \( \tau \) is exactly equal to the tail of Approximation I, so \( \tau \) is the root of equation (1) on the interval \((0, 1)\). Hence, the probabilities \( q_k \) can be computed by choosing a large \( M \) and substituting \( q_k = q_M \tau^{k-M} \) in (2) for \( k \geq M \).

It remains to specify the probability \( B_i^j \), that is, the probability that \( i \) services are completed successfully during a time-interval \((0, D]\) if \( j \) customers are present at time 0. The relations (2) are exact if we have an exact expression for \( B_i^j \). Of course, in the special case that \( p(n) = 1 \) for all \( n \), \( B_i^j = 1 \) for \( i = \min\{c, j\} \) and \( B_i^j = 0 \) otherwise. Then the model reduces to the ordinary \( M/D/c \) queue and the analysis is exact. However, for the general \( M/D/c \) queue with state-dependent feedback, it is not possible to compute the exact value of \( B_i^j \) if the system state is observed only after every \( D \) time units.
The probability that a service is successful depends on the number of customers present at the moment
the service is completed. This number is not known exactly, because the system state is not observed
at service completion epochs.

Therefore, we studied the following approximation for $B_i$. We approximated $B_i$ by the probabil-
ity that a binomial$(\min\{c,j\}, p(\min\{c,j\}))$ distributed random variable equals $i$. This approximation
ignores that the number of customers present changes during $(0,D]$. Comparison of the resulting ap-
proximation for the steady-state probabilities with simulation indicated, that the approximation is quite
accurate. However, as we found from numerical experiments, a slight improvement of this approxima-
tion can be achieved by basing the approximation of $B_i$ on the expected number of customers present
halfway the interval (so at time $D/2$) instead of at the start of the interval (so at time $0$). In Appendix
B we show how this can be done.

The steady-state probabilities $q_k$ obtained by solving (2) are our second approximation for the
steady-state probabilities of the $M/D/c$ queue with state-dependent feedback. We denote these prob-
abilities by $\varphi(k), k \geq 0$.

4 The Sojourn-Time Distribution

Define $S$ as the sojourn time of an arbitrary customer. Using the approximation assumption of section 2
and Approximation I or II for the steady-state probabilities of the $M/D/c$ queue with state-dependent
feedback, we approximate the distribution of $S$.

Let the random variable $L$ denote the steady-state number of customers in the system. Denote our
approximation for the distribution of $L$ by $\{\varphi(k), k \geq 0\}$. (This can be either $\varphi_1(k)$ or $\varphi_2(k)$.)
According to Little's theorem, $E[L] = \lambda E[S]$. Hence, we compute our approximation for $E[S]$ as

$$E[S] = \frac{1}{\lambda} \sum_k k\varphi(k).$$

To approximate the distribution of $S$, we need to approximate the distribution of the waiting time and
the total service time of a customer.

Let us first discuss the service-time distribution. Every service run of a customer takes $D$ time. The
probability that another run is needed depends on the number of customers in service at the moment
the present run is finished. This number is not known beforehand. Therefore, we approximate the
total service time of a customer $A$ by pretending the number of busy servers remains constant from the
moment $A$'s service starts. Then the service time is geometrically distributed.

Next, we discuss the waiting-time distribution. If a customer $A$ finds $i \geq c$ customers in the sys-
tem upon arrival, he has to wait until $i - c + 1$ service completions have been successful. Using part
2 of the approximation assumption of section 2, the time between the arrival of $A$ and the next service
completion is approximately uniform$(0, D)$-distributed. With probability $p(c)$, that service is success-
ful. Then $A$ still has to wait for $i - c$ successful service completions. With probability $1 - p(c)$,
that service is unsuccessful. Then $A$ still has to wait for $i - c + 1$ successful service completions.
As long as all servers are busy, the number of service completions needed for \( j \) successful services is negative-binomially distributed with parameters \( j \) and \( p(e) \). Hence, using part 2 of the approximation assumption, the time needed for \( j \) successful service completions (starting just after a service completion epoch) is \( D/e \) times a negative-binomial(\( j, p(e) \)) distributed variable.

Denote by \( G_i \), a geometrically distributed random variable with success probability \( p(i) \), denote by \( N_j \), a negative-binomially distributed variable with parameters \( j \) and \( p(e) \), and let \( U(0, a) \) be a uniform(\( 0, a \))-distributed random variable. Summarizing the above discussion, the approximation we suggest is as follows.

**Total Service-Time Distribution**  
If a customer \( A \) sees \( i \) other servers busy at the start of his first service run, the distribution of the total service time of \( A \) is approximated by \( D G_{i+1} \).

**Waiting-Time Distribution**  
If a customer \( A \) finds \( i \geq c \) customers in the system upon arrival, the waiting-time distribution of \( A \) is approximated by

\[
\begin{cases} 
U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c} & \text{w.p. } p(e) \\
U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c+1} & \text{w.p. } 1 - p(e).
\end{cases}
\]

Our approximation for the sojourn-time distribution thus is

\[
P(S \leq t) = \sum_{i=0}^{c-1} \varphi(i)P(D G_{i+1} \leq t) + p(c) \sum_{i=c}^{\infty} \varphi(i)P(U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c} + D G_c \leq t) \\
+(1 - p(c)) \sum_{i=c}^{\infty} \varphi(i)P(U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c+1} + D G_c \leq t).
\]

Similarly, for a direct approximation of \( E[S^2] \) we propose

\[
E[S^2] = \sum_{i=0}^{c-1} \varphi(i)E[D^2 G_{i+1}^2] + p(c) \sum_{i=c}^{\infty} \varphi(i)E[(U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c} + D G_c)^2] \\
+(1 - p(c)) \sum_{i=c}^{\infty} \varphi(i)E[(U(0, \frac{D}{c}) + \frac{D}{c} N_{i-c+1} + D G_c)^2].
\]

Note, that we could have approximated \( E[S] \) in this way as well. However, we already have an approximation for \( E[S] \) from Little’s theorem.

5 **Numerical Results**

We tested Approximation I and II by comparing them with a simulation of the system. Without loss of generality, the service time \( D \) of the customers was taken equal to 1. The success probabilities \( p(n) \) were the fixed points of the equations

\[(1 - bp(n))^{n-1} = p(n), \quad n = 1, \ldots, c,
\]

in which \( b \) is a constant in \([0, 1]\). This fixed-point equation originates from a model for a RTDB, see
Sassen and Van der Wal [1996a]. Table 1 contains the success probabilities for 3 different choices of \( b \), namely \( b = 0.01, 0.1, \) and 0.2. The ordinary \( M/D/c \) with \( p(n) = 1 \) for all \( n \) corresponds with \( b = 0 \).

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<td>0.488</td>
<td>0.461</td>
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</table>

Table 1: Success probabilities \( p(n) \) for various \( b \)

By applying Approximation I and II, we obtained approximations \( \varphi_I \) and \( \varphi_{II} \) for the steady-state probabilities of the \( M/D/c \) queue with state-dependent feedback. Using the approximation \( \varphi \), we computed the expected waiting time \( E[W_q] \) and the delay probability \( P(\text{wait}) \) respectively as

\[
E[W_q] = \sum_{k=1}^{\infty} (k-c) \varphi(k) / \lambda \quad \text{(via Little's theorem)}
\]

and

\[
P(\text{wait}) = \sum_{k=1}^{\infty} \varphi(k).
\]

Further, we computed \( E[S] \) and \( \text{sdev}(S) \) (the standard deviation of \( S \)) as described in section 4.

We looked at systems with \( c = 2, 4, 8, \) and 10. Table 2 and 3 show the results for \( E[W_q] \), \( P(\text{wait}) \), \( E[S] \), and \( \text{sdev}(S) \)

<table>
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<tr>
<th>( c )</th>
<th>( \lambda_1 )</th>
<th>( b )</th>
<th>( \rho_I )</th>
<th>( E[W_q] )</th>
<th>( P(\text{wait}) )</th>
<th>( E[S] )</th>
<th>( \text{sdev}(S) )</th>
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<td></td>
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<tr>
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<td>1.53</td>
<td>1.54</td>
<td>0.84</td>
<td>0.83</td>
</tr>
</tbody>
</table>

Table 2: Analysis versus Simulation for \( c = 2 \) and \( c = 4 \)
The input parameters were the number of servers \( c \), the arrival intensity per server \( \lambda_1 (\text{so } \lambda_1 = \lambda/c) \), and \( b \) (representing the choice of success probabilities). Define

\[
\rho_L := \frac{\lambda D}{c p(1)} = \frac{\lambda_1 D}{p(1)} \quad \text{and} \quad \rho_U := \frac{\lambda D}{c p(c)} = \frac{\lambda_1 D}{p(c)}.
\]

Given that \( p(n) \) is decreasing in \( n \), we have for the actual server utilization \( \rho \), that \( \rho_L \leq \rho \leq \rho_U \). The values of \( \rho_U \) are also tabulated in Table 2 and 3.

The parameter \( b \) was chosen at 0, 0.01, 0.1, and 0.2. The arrival intensity per server, \( \lambda_1 \), was varied such, that for every choice of \( b \) systems with utilizations \( \rho_U \) from 0.50 to (about) 0.95 were investigated. As \( b \) increases (keeping \( c \) and \( \lambda_1 \) fixed), \( p(c) \) decreases, so \( \rho_U \) becomes larger. For \( b = 0 \), the system is an \( M/D/c \) queue without feedback. The results of Approximation I are then identical to the results for the \( M/D/c \) queue as obtained by Case A of TUMS et al. [1981]. If \( b = 0 \), the steady-state probabilities produced by Approximation II are exact. Since \( \mathbb{E}[W_q] \), \( P(\text{wait}) \), and \( E[S] \) are derived directly from the steady-state probabilities, they are also exact for Approximation II if \( b = 0 \). In the tables, their values are equal to the simulated values. The standard deviation and distribution of \( S \) however are not exact, as explained in section 4.

<table>
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<tr>
<th>( c )</th>
<th>( \lambda_1 )</th>
<th>( b )</th>
<th>( \rho_U )</th>
<th>( \mathbb{E}[W_q] )</th>
<th>( P(\text{wait}) )</th>
<th>( E[S] )</th>
<th>sdev(( S ))</th>
</tr>
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<td>2.68 2.67 2.66</td>
<td>3.58 3.56 3.54</td>
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</tr>
</tbody>
</table>

Table 3: Analysis versus Simulation for \( c = 8 \) and \( c = 10 \)
The simulation results in Table 2 and 3 are accurate up to the last digit shown. The number of customers simulated was such, that the width of the 95% confidence interval is smaller than the last shown decimal place. For instance, a simulated value of 2.84 for $E[S]$ means, that the 95% confidence interval lies inside [2.83, 2.85].

Table 2 and 3 clearly show that both approximative analyses of the $M/D/c$ queue with state-dependent feedback are very accurate, even for high utilizations.

For $c = 2$ and $c = 4$, the relative differences between Approximation I and simulation of $E[S]$ are all below 2.7%. The differences between Approximation II and simulation of $E[S]$ are also typically below 2.7%, but exceptional cases are $b = 0.1$ or 0.2 with $\rho_U \leq 0.60$, where differences up to 6% occur. For all $\rho_U$, Approximation II is very accurate if $b \leq 0.01$. The differences in sdev($S$) between Approximation I [II] and simulation are all below 10%. In both approximations, the high differences of 6 to 10% occur in cases where sdev($S$) $< 0.40$. Again for Approximation II, the cases with $b = 0.1$ or 0.2 with $\rho_U \leq 0.60$ give the worst results with inaccuracies of 6 to 10%.

For $c = 8$ and $c = 10$, both approximations are very accurate for $E[S]$: all differences with simulation are smaller than 2%. Approximation II outperforms Approximation I for $b = 0$ and 0.01, whereas Approximation I is slightly better than II for $b = 0.1$ and 0.2. The inaccuracies of both I and II in estimating sdev($S$) typically are below 5%, where the best estimates (with relative differences $< 2\%$) occur if $\rho_U$ is high. Bad exceptions for both I and II are the cases where sdev($S$) $< 0.15$, but these cases are not very interesting. Also, Approximation II shows a difference up to 10% in sdev($S$) compared to simulation for $b = 0.1$ or 0.2 with $\rho_U \leq 0.60$.

We also compared our approximations for the sojourn-time distribution with simulation results. Table 4 displays $E[S]$, sdev($S$), $P(S > 5)$, and $P(S > 10)$. The systems considered in Table 4 are precisely those systems that have $E[S] > 2$ in Table 2 or 3. We conclude from the results in Table 4, that we obtained two excellent approximations for the sojourn-time distribution in the $M/D/c$ queue with state-dependent feedback.

Summarizing, we recommend to use Approximation II for $b \leq 0.01$, so when the system is still nearly an $M/D/c$ queue. For $b > 0.01$, both approximations are equally appropriate. The only exception to this is a combination of a high value of $b$ and a low value of $\rho_U$: then Approximation I is more accurate than Approximation II.

Finally, we point out that the approximations for the system behavior are not only very good, but also very quick. It took about 770 hours to run all simulations reported in Table 2 and 3 (on a Sun Sparc5), whereas all results for Approximation I and Approximation II together were generated in 12 minutes.
In this paper, we derived two approximations for the steady-state probabilities and the sojourn-time distribution of an $M/D/c$ queue with state-dependent feedback. Approximation I was based on an embedded Markov chain analysis and the well-known residual life approximation of TUMS et al. [1981] was used for the remaining service times of the customers in service. Approximation II resembled the exact analysis of the $M/D/c$ queue (CROMMELIN [1932]) by observing the state of the system after every $D$ time units.

The accuracy of the approximations was investigated for three different sequences of the feedback probabilities and for various system loads. The error made by the approximations for both the steady-state probabilities and the sojourn-time distribution typically is only a few percent. Hence, Approximation I is yet another example of the usefulness of the residual-life approximation for the remaining service times. (For an earlier example, see SASSEN et al. [1997].)

An important advantage of Approximation I is, that it is easily extendible to the $M/G/c$ queue with state-dependent feedback. That is, with the stipulation that the service time of a customer in a rerun is drawn freshly from the general service-time distribution. Then the approximation assumption to be used is identical to the ones TUMS et al. [1981] used for the ordinary $M/G/c$ queue (except for the required distinction between successful and unsuccessful services). However, if the service time of a customer in every rerun exactly equals the service time of that customer in his first run, as actually happens in real-time databases, then it is very difficult to give a good approximative analysis.

### Table 4: Distribution of the sojourn time $S$

<table>
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<th>c</th>
<th>$\lambda_1$</th>
<th>b</th>
<th>$p_D$</th>
<th>$E[S]$</th>
<th>sdev($S$)</th>
<th>$P(S &gt; 5)$</th>
<th>$P(S &gt; 10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Appl</td>
<td>Sim</td>
<td>Appl</td>
<td>Sim</td>
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of the system. SASSEN and VAN DER WAL [1996b] considered the $M/M/c$ queue with this type of feedback and derived a good approximation for not too heavily loaded systems. Notice, that in the $M/D/c$ queue with feedback the 'redraw' and 'no-redraw' cases coincide.

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References


Appendix A

We demonstrate under which conditions there exists a \( \tau \in (0, 1) \), such that \( \pi(k, 1) \approx \pi(k - 1, 1)\tau \) and \( \pi(k, 0) \approx \pi(k - 1, 0)\tau \) for \( k \to \infty \). Define the generating functions

\[
\prod_1(z) = \sum_{k=0}^{\infty} \pi(k, 1)z^k \quad \text{and} \quad \prod_0(z) = \sum_{k=1}^{\infty} \pi(k, 0)z^k \quad \text{for} \quad |z| \leq 1.
\]

From Tijms and Van de Goevering [1991], it follows that the steady-state probabilities \( \pi(k, 1) \) asymptotically exhibit a geometric-tail behavior if the following conditions are satisfied:

C0. The generating function \( \prod_1(z) \) is the ratio of two analytic functions \( A(z) \) and \( B(z) \) of which the domains of definition can be extended to a region \( |z| < R \) in the complex plane for some \( R > 1 \), and which have no common zeros.

C1. The equation \( B(x) = 0 \) has a real root \( x_0 \) on the interval \( (1, R) \).

C2. The function \( B(z) \) has no zeros in the domain \( 1 < |z| < x_0 \) of the complex plane.

C3. The zero \( z = x_0 \) of \( B(z) \) is of order one and is the only zero of \( B(z) \) on the circle \( |z| = x_0 \).

The geometric-tail factor \( \tau \) is then found as the reciprocal of \( x_0 \).

By writing

\[
\prod_1(z) = \sum_{k=0}^{c-1} \pi(k, 1)z^k + \prod_1^{\geq c}(z) \quad \text{where} \quad \prod_1^{\geq c}(z) := \sum_{k=c}^{\infty} \pi(k, 1)z^k,
\]

we see that if \( \prod_1^{\geq c}(z) \) satisfies conditions C0–C3 above, then \( \prod_1(z) \) satisfies these conditions. Analogously, the same applies for \( \prod_0^{\geq c}(z) := \sum_{k=c}^{\infty} \pi(k, 0)z^k \) and \( \prod_0(z) \).

Therefore, it is sufficient to show that C0–C3 hold for the functions \( \prod_1^{\geq c}(z) \) and \( \prod_0^{\geq c}(z) \). First we determine \( \prod_1^{\geq c}(z) \) and \( \prod_0^{\geq c}(z) \) from the balance equations for \( k \geq c \). Some tedious algebra yields

\[
\prod_1^{\geq c}(z) = \frac{p(c)A_1(z)}{B(z)} \quad \text{and} \quad \prod_0^{\geq c}(z) = \frac{(1 - p(c))A_0(z)}{B(z)},
\]

where

\[
A_1(z) = [(1 - p(c))A_c(z) - 1]z^cH + G(z),
\]

\[
A_0(z) = -p(c)A_c(z)z^cH + zG(z),
\]

\[
B(z) = z - zA_c(z)(1 - p(c)) - p(c)A_c(z)
\]

and

\[
A_c(z) = \sum_{\ell=0}^{\infty} a[c, \ell]z^\ell = \exp(-\lambda D(1 - z)/c)
\]

\[
H = \pi(0, 1)a[0, c - 1] + a[c, 0](\pi(e, 0) + \pi(c, 1)) +
\]

\[
+ \sum_{j=1}^{c-1}(a[j - 1, c - j]\pi(j, 0) + a[j, c - j]\pi(j, 1)) \quad (= \pi(c - 1, 1)/p(c))
\]
Indeed, the generating function \[ \prod_1^{z} \left[ \prod_0^{z} (z) \right] \] is the ratio of two analytic functions \( p(c)A_1(z) \) \( (1 - p(c))A_0(z) \) and \( B(z) \), the domains of which can be extended outside the unit circle, and which have no common zeros. The functions \( A_1(z) \) \( [A_0(z)] \) and \( B(z) \) are analytic in the whole complex plane, so condition C0 holds with \( R = \infty \). It can easily be verified that the equation \( B(x) = 0 \) has a unique real root on \((1, \infty)\), so condition C1 holds as well. Numerical experiments suggested that C2 and C3 are also satisfied, but we could not prove this analytically. Assuming C2 and C3 are true, the reciprocal of the geometric-tail factor \( \tau \) can be computed from the equation \( B(x) = 0 \) on \((1, \infty)\).

In particular, using the transformation \( y = 1/z \), it follows that \( \tau \) is the unique root of the equation \( 1 - p(c)(1 - y) = \exp(\lambda D(1 - 1/y)/c) \) on the interval \((0, 1)\).

Appendix B

We show how an approximation for \( B_i \) can be obtained, based on the expected number of customers present halfway the interval \((0, D)\) (so at time \( D/2 \)) instead of at the start of the interval (so at time 0). For notational convenience, define \( s_j \) as the number of customers in service if the total number of customers present is \( j \), so

\[
S_j = \min\{c, j\}.
\]

Suppose \( j \) customers are present at time 0. How many customers are present at time \( D/2 \)? On average, \( \lambda D/2 \) arrivals take place in \((0, D/2)\). Also, on average, a service completion occurs every \( D/(s_j + 1) \) time. Using this, we estimate the average number of successful service completions in \((0, D/2)\) by \( p(s_j)(s_j - 1)/2 \) if \( s_j + 1 \) is even, and by \( p(s_j)s_j/2 \) if \( s_j + 1 \) is odd. Denote by \( \bar{j} \) the average number of customers present at time \( D/2 \), given that \( j \) customers are present at time 0. Then, approximately,

\[
\bar{j} = \begin{cases} 
    j + (\lambda D/2) - p(s_j)(s_j - 1)/2 & \text{if } s_j \text{ is odd} \\
    j + (\lambda D/2) - p(s_j)s_j/2 & \text{if } s_j \text{ is even} 
\end{cases}
\]

The approximation we propose for \( B_i \) is the probability that a binomially distributed random variable equals \( i \), where the parameters of the binomial variable are \( s_j \) and \( p(\bar{j}) \). Since \( \bar{j} \) is not necessarily integer and smaller than \( c \), the function \( p(n) \) must be adapted. For any non-negative real-valued \( x \), let \( \lfloor x \rfloor \) denote the largest integer smaller than or equal to \( x \). Then, as an approximation, we redefine the success probability as \( \tilde{p}(x) \), with

\[
\tilde{p}(x) = \begin{cases} 
    (1 - (x - \lfloor x \rfloor))p(\lfloor x \rfloor) + (x - \lfloor x \rfloor))p(\lfloor x \rfloor + 1) & \text{if } 0 \leq x < c \\
    p(c) & \text{if } x \geq c.
\end{cases}
\]

Summarizing, we approximate \( B_i \) by the probability that a binomial(\( s_j, \tilde{p}(\bar{j}) \)) distributed variable equals \( i \), where \( s_j = \min\{c, j\} \), and \( \bar{j} \) and \( \tilde{p}(\bar{j}) \) are computed from (3) and (4), respectively.