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A behavioral approach to the $H_\infty$ optimal control problem

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Abstract

This paper considers the general $H_\infty$ optimal control problem from a behavioral perspective. A formalization of this problem is given that departs from the usual $H_\infty$ control paradigm in the sense that system variables of the plant are treated in a symmetric way, without distinguishing control inputs, measurements, exogenous inputs and to-be-controlled variables. Interconnection variables are introduced and controllers are allowed to constrain the interconnection variables of the plant. Necessary and sufficient conditions are given for the existence of controllers that achieve an $H_\infty$ control objective for a given linear time-invariant plant. The set of all such controllers is parametrized by means of $J$-spectral factorizations. © 1997 Elsevier Science B.V.

Keywords: $H_\infty$ optimal control; Linear systems; Behavioral theory; $J$-spectral factorizations

1. Introduction

The usual formulation of the $H_\infty$ optimal control problem involves the design of a control system for a generalized plant in which four sets of signals are distinguished. Exogenous disturbances and controls are the two sets of input signals. To-be-controlled variables and measurements act as the two sets of output signals of the plant. The control inputs are typically generated by actuators and the measurements are produced by sensors. The $H_\infty$ optimal control problem then amounts to synthesizing an operator called the controller which maps the measurements to the controls so as to achieve an internally stable closed-loop system which has minimal $H_\infty$ or $L_2$-induced norm when viewed as a mapping from exogenous disturbances to to-be-controlled outputs. The corresponding signal-flow configuration of the closed-loop system is shown in Fig. 1 and is generally accepted as a convenient and basic starting point for many paradigms in control theory.

In the behavioral theory, it has been advocated that the input–output structure is not a natural starting point for many problems in modeling, optimal control, and system identification. From a general modeling point of view, the distinction between causes (inputs) and effects (outputs) may not always be clear and in fact many models derived from first principles are actually

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not in input–output form. Also for many applications in control, the causality structure in the signal-flow diagram of Fig. 1 is a suggested one that does not need to correspond to the physical reality.

In this paper we consider a different, and in principle more general, approach to the $H_\infty$ optimal control problem. The main feature of this approach is to depart from the input–output and transfer function setting of this problem and instead adhere to an entirely symmetrical treatment of system variables. In particular, we do not distinguish between controls and measurements, but instead introduce interconnection variables to define the link between plant and controller. Furthermore, exogenous disturbances and to-be-controlled variables are lumped into external variables through which the controlled system interacts with its environment. In this setting, a controller is no longer a mapping from measurements to controls, but instead a set of laws which constrain the interconnection variables so as to achieve a desired control objective. In the framework presented here, “control” therefore amounts to restricting the interconnection variables of a plant. We formalize the general $H_\infty$ optimal control problem in such a setting and show that a complete solution can be obtained by using well known factorization results. One of the most important consequences of this technique is that the partitioning of interconnection variables in measurement and control components is not required prior to the synthesis of $H_\infty$ controllers, but may take place a posteriori, after the controller has been designed.

This paper is motivated by earlier work on control in a behavioral context [10, 20, 24, 26], and by papers on $H_2$, $H_\infty$ and $L_1$ optimal control in this formalism [2–4, 11–13, 16–19, 25, 29]. The papers [2, 3] concentrate on a state space formulation of this problem, whereas [17–19] consider polynomial differential equations to formalize the $H_\infty$ control problem in a behavioral framework. In [2, 3, 18, 19] different methods are employed to solve the full-information and full-control cases and the external variables of the controlled system are still assumed to be partitioned in input and output components. Here, we propose a different behavioral perspective to the $H_\infty$ control problem. We consider the general partial information case and we refrain from making assumptions on the partitioning of exogenous and endogenous variables. $J$-spectral factorization techniques are employed to solve the $H_\infty$ control problem in this setting. In [12, 13], similar techniques are used to solve the general $H_\infty$ optimal control problem in a behavioral setting, but in these works the input–output partitioning of signals is still assumed. Here, we will generalize these results to allow for a symmetrical treatment of system variables. A preliminary version of this paper appeared in [30].

2. A motivating example

As a motivating example, consider the controller design for an active suspension of a transport vehicle as described, with slight variations, in [9]. The system is depicted in Fig. 2 and modeled by the equations

\[
\begin{align*}
 m_2q''_2 + b_2(q''_2 - q'_1) + k_2(q_2 - q_1) - F &= 0, \\
 m_1q''_1 + b_2(q'_1 - q''_2) + k_2(q_1 - q_2) + k_1(q_1 - q_0) + F &= 0,
\end{align*}
\]

where $F$ (resp. $-F$) is a force acting on the chassis mass $m_2$ (the axle mass $m_1$). The controller $\Sigma_c$ is a device that constrains the three interconnection variables

\[
\begin{pmatrix}
 q'_2 \\
 q_2 - q_1 \\
 F
\end{pmatrix}
\]

with $q_2 - q_1$ the distance between chassis and axle, and $q'_2$ the acceleration of the chassis mass $m_2$. Let

\[
\begin{pmatrix}
 q_2 - q_1 \\
 F
\end{pmatrix}
\]

with $q_2 - q_1$ the distance between chassis and axle, and $q'_2$ the acceleration of the chassis mass $m_2$. Let

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\begin{pmatrix}
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 F
\end{pmatrix}
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\[
\begin{pmatrix}
 q_2 - q_1 \\
 F
\end{pmatrix}
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\begin{pmatrix}
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\begin{pmatrix}
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 F
\end{pmatrix}
\]

with $q_2 - q_1$ the distance between chassis and axle, and $q'_2$ the acceleration of the chassis mass $m_2$. Let

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\begin{pmatrix}
 q_2 - q_1 \\
 F
\end{pmatrix}
\]

with $q_2 - q_1$ the distance between chassis and axle, and $q'_2$ the acceleration of the chassis mass $m_2$. Let

\[
\begin{pmatrix}
 q_2 - q_1 \\
 F
\end{pmatrix}
\]
achieve a stable interconnected system in which the partitioned external variables satisfy the inequality

$$||z||_2 < \gamma ||d||_2$$  \tag{2.1}

where $\gamma \geq 0$ is to be chosen as small as possible and the norms are the standard norms in the Hilbert space of square-integrable functions. If we interpret this control problem in the context of the configuration of Fig. 1, then the controller $\Sigma_c$ is required to be an operator mapping selected components of the interconnection variables $w_{in}$ (called measurements) to its complementary components (called the controls). However, for this example, such a partitioning of $w_{in}$ is neither unique, nor obvious. Indeed, the force $F$ may be generated by an electrical actuator, which makes $F$ a control input to the plant. Alternatively, for a hydraulic actuator, $q_2 - q_1$ is a more appropriate choice of control input for the plant. Likewise, the acceleration $q_2$ also qualifies as feasible control input. The interconnection variables $\dot{q}_2$ and $q_2 - q_1$ are quantities which are relatively easy to measure and thus qualify as measurements. In an operator theoretic setting, three standard $H_\infty$ control problems can be formulated in which the controller $\Sigma_c$ is either one of the mappings:

$$\Sigma_c : (\dot{q}_2, q_2 - q_1)^T \mapsto F, \quad (2.2a)$$

$$\Sigma_c : (F, q_2 - q_1)^T \mapsto \dot{q}_2, \quad (2.2b)$$

$$\Sigma_c : (\dot{q}_2, F)^T \mapsto q_2 - q_1. \quad (2.2c)$$

Two obvious questions arise. First, does a particular partitioning of the interconnection variables in controller inputs and outputs influence the achievable performance $\gamma$? Second, is such a partition necessary to synthesize $H_\infty$ optimal controllers?

For this application, the design of an active suspension $H_\infty$ controller for a tractor-semitrailer, the physical parameters are given in Table 1. A feasible set of weighting parameters reflecting road, passenger comfort and physical system characteristics, is given in Table 2. With these specifications, the transfer function of the plant in Fig. 1 is proper\(^1\) if either $F$ or $\dot{q}_2$ is selected as control input; it is non-proper if $q_2 - q_1$ is taken as control variable. This implies that an $H_\infty$ optimal controller, Eq. (2.2c), cannot be calculated with the methods advocated in, e.g., [5]. Within the class of stabilizing, proper controllers, Eqs. (2.2a) and (2.2b), the minimal values of $\gamma$ for which Eq. (2.1) holds turn out to coincide and are given by $\gamma_{\text{prop}}^{\text{opt}} = 0.1284$. The algorithm developed in this paper shows that, within the class of stabilizing and possibly non-proper controllers, Eqs. (2.2a)–(2.2c), the minimal values of $\gamma$ achieving Eq. (2.1) in the controlled system all coincide and are given by $\gamma_{\text{non-prop}}^{\text{opt}} = 0.1284$. In particular, $\gamma_{\text{prop}}^{\text{opt}} = \gamma_{\text{non-prop}}^{\text{opt}}$ for this example.

The point of this example is that the optimal achievable $H_\infty$ norm is independent of a partitioning of the interconnection variables in controls and measurements. The example suggests that such a partitioning (which corresponds to a choice of actuators and sensors) is not required prior to the synthesis of $H_\infty$ controllers but may take place a posteriori, after the control system has been synthesized, without compromising performance. This observation does not only have conceptual advantages but is also of great practical relevance when costs of sensors and actuators play a role in control system design. In comparison, optimal $H_2$ performance levels do depend on the partitioning of interconnection variables in measurements and controls. The minimal achievable $H_2$ norm of the closed-loop system obtained by interconnecting the system with a proper stabilizing controller, Eq. (2.2a) is given by 0.7718. However, this norm is $\infty$ for any proper stabilizing controller of the form, Eqs. (2.2b) or (2.2c). We refer to [9, 21, 22] for more details on this control problem and for further considerations on appropriate selections of actuators and sensors.

3. System interconnections

For reasons of exposition, we focus on discrete-time dynamical systems in the sequel. The generalization to continuous-time systems is straightforward and discussed in Remark 5.13 below.

We follow the basic definition of a dynamical system in the behavioral theory and consider

---

\(^{1}\) A transfer function $G$ is proper if $\lim_{|s| \to \infty} G(s) < \infty$. 

---

<table>
<thead>
<tr>
<th>Table 1 Physical parameters</th>
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<tbody>
<tr>
<td>$m_1$ (kg)</td>
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<tr>
<td>1.5 x $10^3$</td>
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</table>

<table>
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<tr>
<th>Table 2 Scaling parameters</th>
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<tr>
<td>$S_1$</td>
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<td>250/s + 125</td>
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dynamical systems \( \Sigma = (T, W, \mathcal{B}) \) with discrete-time set \( T = \mathbb{Z} \), finite-dimensional real-valued signal spaces \( W = \mathbb{R}^q \), \( q > 0 \), and behaviors \( \mathcal{B} \) which are linear, shift-invariant and closed \(^2\) subsets of \( W^T \). This class of systems has been a main topic of investigation in the behavioral theory of linear systems (see, e.g., [23]). We will denote this class by \( \mathbb{L}^q \) or \( \mathbb{L} \) if the dimension of the signal space is clear from the context.

It has been shown in [23] that \( \mathbb{L}^q \) admits a parametrization by means of real polynomial matrices with \( q \) columns. Precisely, \( \Sigma = (T, W, \mathcal{B}) E \mathcal{A}_q \) if and only if there exists \( L > 0 \), \( p > 0 \), and a polynomial \( \Theta(z) = \sum_{i=0}^L \theta_i z^i \) with \( \Theta_i \in \mathbb{R}^{p \times q} \), such that

\[
\mathcal{B} = \{ w : T \rightarrow \mathbb{R}^q | \Theta(z)w = 0 \} = \ker \Theta(z). \quad (3.1)
\]

Here, \( \Theta(z) \) is to be interpreted as a polynomial operator in the left-shift \( \sigma \) (i.e., \( [w\sigma](t) = w(t+1) \)) acting on the signal space \( (\mathbb{R}^q)^T \). \( \Theta \) is called an autoregressive (AR) representation of \( \Sigma \) and the rows of \( \Theta \) are called the laws of the system.

Let \( \Sigma_1 = (T, W_1 \times W_{\text{int}}, \mathcal{B}_1) \) and \( \Sigma_2 = (T, W_2 \times W_{\text{int}}, \mathcal{B}_2) \) be two dynamical systems with identical time sets. Suppose that the signal spaces of \( \Sigma_1 \) and \( \Sigma_2 \) are Cartesian products which have a common non-empty factor \( W_{\text{int}} \), called the interconnection space.

**Definition 3.1.** The interconnection of \( \Sigma_j = (T, W_j \times W_{\text{int}}, \mathcal{B}_j) \), \( j = 1, 2 \), is the system \( \Sigma_1 \cap \Sigma_2 := (T, W_1 \times W_2 \times W_{\text{int}}, \mathcal{B}_1 \cap \mathcal{B}_2) \) where

\[
\mathcal{B}_1 \cap \mathcal{B}_2 := \{ (w_1, w_2), w_{\text{int}} \} | (w_j, w_{\text{int}}) \in \mathcal{B}_j, \quad j = 1, 2 \}. \quad (3.2)
\]

Signals \( w_{\text{int}} \) are called the interconnection variables.

It is important to emphasize that neither the external variables \( w_1 \) and \( w_2 \) nor the interconnection variables \( w_{\text{int}} \) are assumed to be partitioned in inputs and outputs. Similarly, the causality structure among these variables is irrelevant. Note that \( \mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_1 \cap \mathcal{B}_2 \) whenever both \( W_1 \) and \( W_2 \) are void. If we view \( w_1 \) and \( w_2 \) in Eq. (3.2) as exogenous and \( w_{\text{int}} \) as endogenous variables then \( \Sigma_1 \cap \Sigma_2 \) induces the system \( \Sigma_1 \wedge \Sigma_2 = (T, W_1 \times W_2, \mathcal{B}_1 \wedge \mathcal{B}_2) \) where

\[
\mathcal{B}_1 \wedge \mathcal{B}_2 := \{ (w_1, w_2) | \exists w_{\text{int}} \text{ such that } (w_1, w_2, w_{\text{int}}) \in \mathcal{B}_1 \cap \mathcal{B}_2 \}
\]

The following result is standard [23, 24].

**Theorem 3.2** (AR representations of interconnections). Let \( \Sigma_1 \in \mathbb{L}^{q_1+q_{\text{int}}} \) and \( \Sigma_2 \in \mathbb{L}^{q_2+q_{\text{int}}} \). Then

1. \( \Sigma_1 \cap \Sigma_2 \in \mathbb{L}^{q_1+q_{\text{int}}} \).
2. \( \Sigma_1 \wedge \Sigma_2 \in \mathbb{L}^{q_1+q_{\text{int}}} \).
3. If \( (\Theta_j, \Theta_{\text{int},j}) \) defines an autoregressive representation of \( \Sigma_j \), \( j = 1, 2 \), then

\[
\begin{pmatrix}
\Theta_1(\sigma) & 0 \\
0 & \Theta_2(\sigma) \end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \end{pmatrix}
= 0.
\]

is an autoregressive representation of \( \Sigma_1 \cap \Sigma_2 \).

**Remark 3.3.** There exists a natural extension of autoregressive polynomial representations to rational kernel representations of dynamical systems. Since every polynomial \( \Theta \) with coefficients in \( \mathbb{R}^{p \times q} \) is a rational function which is analytic for \( z \in \mathbb{C}, |z| < 1 \), the difference equation \( \Theta(z)w = 0 \) can equivalently be interpreted as a convolution equation \( \Theta \ast w = 0 \) where the convolution kernel \( \Theta : \mathbb{Z} \rightarrow \mathbb{R}^{p \times q} \) is the inverse \( z \)-transform of \( \Theta \). The latter interpretation in terms of convolution kernels allows for general rational functions \( \Theta \).

4. Control objectives

Let \( \Sigma_1 \in \mathbb{L}^{q_1+q_{\text{int}}} \) be a given system and suppose that \( \Sigma_2 \in \mathbb{L}^{q_2+q_{\text{int}}} \) is a dynamical system that needs to be synthesized so as to achieve a certain objective for the interconnected systems \( \Sigma_1 \cap \Sigma_2 \) and \( \Sigma_1 \wedge \Sigma_2 \). Following standard terminology, we refer to \( \Sigma_1 \) as the **plant** and to \( \Sigma_2 \) as the **controller**. The interconnections \( \Sigma_1 \cap \Sigma_2 \) and \( \Sigma_1 \wedge \Sigma_2 \) are called **controlled systems**.

A control objective is specified in terms of a **minimal** and a **maximal requirement** on the behavior of the controlled system. These two requirements are meant to specify desirable properties of the controlled system. Roughly speaking, the minimal requirement formalizes that the controlled system should be “rich enough” in that it contains at least a specified set of trajectories. For example, it may be a minimal requirement of the controlled system that exogenous signals, such as references and disturbances, are not restricted by the controller. The maximal requirement specifies performance of the controlled system. Since desired performance is often quantified in terms of functionals defined on specific signals (like impulse responses, step responses, frequency responses or norm-bounded disturbances) it seems reasonable to consider subsets of the controlled system behavior to specify perfor-
mance. The maximal requirement formalizes the idea that a suitable subset of the controlled behavior should exhibit desired behavior. We formalize this as follows.

**Definition 4.1 (Control objectives).** A control objective is a quadruple \( C = (\mathcal{S}_{\text{min}}, \mathcal{S}_{\text{min}}, \mathcal{S}_{\max}, \mathcal{S}_{\max}) \) of subsets of \((\mathbb{R}^q)^T\). A controller \( \Sigma_c \) is said to achieve the control objective \( C \) for the plant \( \Sigma_p \) if the behavior \( \mathcal{S} \) of \( \Sigma_p \) satisfies the two inclusions

\[ \mathcal{S}_{\text{min}} \subseteq \mathcal{B} + \mathcal{S}_{\text{min}}, \quad (4.1a) \]
\[ \mathcal{B} \cap \mathcal{S}_{\max} \subseteq \mathcal{S}_{\max}. \quad (4.1b) \]

In words, Eqs. (4.1a) and (4.1b) state that a suitable extension of the controlled system behavior contains at least the trajectories specified by \( \mathcal{S}_{\text{min}} \) and a suitable restriction of the controlled system behavior satisfies the constraints defined by \( \mathcal{S}_{\max} \). Note that the inclusions, Eqs. (4.1a) and (4.1b), are dual.

### 4.1. The \( \mathcal{H}_\infty \) control problem

The \( \mathcal{H}_\infty \) control problem is formalized in this setting as follows. Let \( \ell^+_2 \) denote the space of square summable trajectories \( w : \mathbb{Z} \to \mathbb{R}^q \) which vanish for \( t < 0 \). Let \( Q = Q^T \) be a real symmetric full rank indefinite matrix of dimension \( q \times q \) and suppose that

\[ Q = Q_+ - Q_- \]

where \( Q_+ \geq 0 \) and \( Q_- \geq 0 \) are such that \( q_+ := \text{rank } Q_+ \) and \( q_- := \text{rank } Q_- \) satisfy \( q_+ + q_- = q \). Given such a \( Q \), the \( \mathcal{H}_\infty \) control objective \( C_\infty \) is defined by the quadruple

\[ \mathcal{R}_{\text{min}} := Q_- \ell^+_2, \quad (4.2a) \]
\[ \mathcal{S}_{\text{min}} := \ell^+_2, \quad (4.2b) \]
\[ \mathcal{R}_{\max} := \ell^+_2, \quad (4.2c) \]
\[ \mathcal{S}_{\max} := \{ w \in \ell^+_2 \mid \exists \varepsilon > 0 \text{ such that } \langle w, Qw \rangle \geq \varepsilon \langle w, w \rangle \}, \quad (4.2d) \]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in the space of square summable sequences.

### 4.2. Comparison to the standard \( \mathcal{H}_\infty \) problem

The standard input–output formulation of the \( \mathcal{H}_\infty \) control problem refers to Fig. 1 in which the external variables are partitioned in exogenous inputs \( d \) and to-be-controlled outputs \( z \). Likewise, interconnection variables are partitioned in measurements \( y \) and controls \( u \). The (suboptimal) \( \mathcal{H}_\infty \) control problem then amounts to finding a proper transfer function mapping \( y \) to \( u \) such that the closed-loop system is stable and for some \( \varepsilon > 0 \)

\[ \gamma^2 \| d \|^2 - \| z \|^2 \leq \left\langle \begin{pmatrix} d \\ z \end{pmatrix}, \begin{pmatrix} y^T & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix} \right\rangle \leq \varepsilon \| d \|^2 \quad (4.3) \]

for all \( d \in \ell^+_2 \). The constant \( \gamma > 0 \) is then an upper bound of the \( \mathcal{H}_\infty \) norm of the closed-loop transfer function. See [5] for a detailed account on this problem. Our formulation in Definition 4.2 differs in various aspects from this standard input–output formulation:

1. In our setting all signals are treated in a symmetric way. This implies that causality and properness constraints are not imposed either on the plant or the controller. Both the controller and the plant are modeled as a set of (behavioral) equations and the
controlled system is naturally defined as the solution set of both sets of equations. In particular, this enables the use of non-proper “weighting filters” which are often desired in $\mathcal{H}_\infty$ control designs where frequency response shaping is necessary.

2. The inclusion, Eq. (4.1b), with Eqs. (4.2c) and (4.2d), corresponds to Eq. (4.3) with $Q = \text{diag}(\gamma^2, -I)$. However, in the absence of an input–output partitioning, we allow for general full-rank-indefinite matrices $Q$ to specify $\mathcal{C}_\infty$ performance of the controlled system. In contrast to the standard $\mathcal{H}_\infty$ control problem, the question of existence of controllers that achieve $\mathcal{C}_\infty$ is therefore not necessarily a one-parameter search problem in $\gamma$.

3. By taking a suitable basis of $N_q$, Eq. (4.1b) is equivalent to $w_+ + w_- = \varepsilon \forall \|w\|_2$ where $w = (w_+^T, w_-^T)^T$ is partitioned in a basis of eigenvectors corresponding to positive and negative eigenvalues of $Q$. In such a partition, Eq. (4.1a) implies that the component $w_+$ qualifies as input in the sense that it is a free variable in $\mathcal{C}_+$. In contrast to the standard $\mathcal{H}_\infty$ control problem, the question of existence of controllers that achieve $\mathcal{C}_\infty$ is therefore not necessarily a one-parameter search problem in $\gamma$.

4. Internal stability of the controlled system is not formalized as an input–output concept in our setting. It is shown in Theorem 5.2 below that the constraints, Eqs. (4.1a) and (4.1b), on the $\ell_2$ trajectories of the controlled system actually imply $\ell_2$ stability in an input–output sense.

5. In the standard input–output setting an $\mathcal{H}_\infty$ controller is a mapping from $\dim(y)$ inputs to $\dim(u)$ outputs. Such a controller imposes, in some well-defined sense, $\dim(u)$ independent constraints on the interconnection variables $(u, y)$. This number is fixed a priori in the standard formulation but is unspecified in our formulation of the $\mathcal{H}_\infty$ control problem. We will see that all controllers that achieve $\mathcal{C}_\infty$, whenever they exist, have the same number of independent control laws. This number is defined by $\Sigma_p$ and $Q$ only. See Remark 5.11.

5. Solution of the $\mathcal{H}_\infty$ control problem

In this section, the first two results characterize $\ell_2$-behaviors of dynamical systems and the $\mathcal{H}_\infty$ control objective, Eqs. (4.1a) and (4.1b). We then present a complete solution to the $\mathcal{H}_\infty$ control problem in terms of $J$-spectral factors. Throughout this section, we identify an element $w \in \ell_2^+$ with its restriction $w|_{[0, \infty)} \in \ell_2(Z_+, \mathbb{R}^q)$ and naturally embed $\ell_2(Z_+, \mathbb{R}^q)$ in the Hilbert space $L_2(Z_+, \mathbb{R}^q)$. Denote by $\hat{w}$ the $z$-transform $\hat{w}(z) = \sum_{t=-\infty}^{\infty} w(t)z^{-t}$ of $w \in \ell_2$ and let $L_\infty$ and $L_2$ be the (normed) spaces of complex vector-valued functions that are, resp., bounded and square integrable on the unit circle $z \in \mathbb{C}$, $|z| = 1$. The Hardy space $L_2$ consists of those elements in $L_2$ which have bounded analytic continuation outside the unit circle (including $\infty$). The Hardy spaces $L_\infty$ and $L_\infty$ consist of those elements of $L_\infty$ which have bounded analytic continuation in $|z| > 1$ and $|z| < 1$, respectively. The prefixes $\mathcal{B}$ and $\mathcal{U}$ will be used to denote rational elements and units, respectively. $\mathcal{C}_+ := \{z \in \mathbb{C} | |z| \geq 1 \} \cup \{\infty\}$.

The $\ell_2$-behavior of a dynamical system $\Sigma = (T, \mathbb{R}^q, \mathcal{B}) \in L_\infty$. The restriction $\mathcal{B} \cap \ell_2^+$ is in general not injective. That is, there are systems $\Sigma', \Sigma'' \in L_\infty$ whose behaviors are unequal and satisfy $\mathcal{B}' \cap \ell_2^+ = \mathcal{B}'' \cap \ell_2^+$. It has been shown in Ref. [28] that in that case the difference between $\mathcal{B}'$ and $\mathcal{B}''$ amounts to unstable and uncontrollable dynamics. The $\ell_2$-behavior of $\Sigma \in L_\infty$ can be characterized as an $\ell_2$-system in the sense that $\mathcal{B} \cap \ell_2^+$ can be represented as the kernel or as the image of an analytic operator.

Theorem 5.1 (Representations of $\ell_2$ behaviors). Let $\Sigma = (T, \mathbb{R}^q, \mathcal{B}) \in L_\infty$. Then there exists $\Theta, \Psi \in \mathcal{B} \mathcal{H}_\infty^+$ such that

\[
\mathcal{B} \cap \ell_2^+ = \mathcal{B} \ker(\Theta) := \{w \in \ell_2^+ | \Theta \hat{w} = 0\}
\]

\[
= \mathcal{B} \im(\Psi) := \{w \in \ell_2^+ | \hat{w} \in \mathcal{H}_\infty\},
\]

where $\Theta$ and $\Psi$ are viewed as multiplicative operators $\mathcal{H}_2 \to \mathcal{H}_2$. Moreover, $\Theta$ and $\Psi$ can be chosen to have full row and full column rank, respectively.

Proof. Infer from [28, Theorem 5.7] that there exists $\Theta_0 \in \mathcal{H}_\infty$ such that $\mathcal{B} \cap \ell_2^+ = \{w \in \ell_2^+ | \Theta_0 \hat{w} = 0\}$. Let $\Theta_0 = \Psi^{-1} \Theta$ be a left-coprime factorization of $\Theta_0$ over $\mathcal{H}_\infty^+$ and $\Psi_0 \hat{w} = 0$ implies that $\Psi \Theta_0 \hat{w} = \Theta_0 \hat{w} = 0$. Conversely, if $\Theta_0 \hat{w} = 0$ then also $\Theta_0 \hat{w} = \Psi^{-1} \Theta_0 \hat{w} = 0$. Hence, ker $\Theta = \ker \Theta_0$ from which we conclude that Eq. (5.1) holds. The existence of image representations, Eq. (5.2), is proven in [28, Theorem 6.7].

Theorem 5.2 (Characterization of $\mathcal{C}_\infty$). Let $\Sigma = (T, \mathbb{R}^q, \mathcal{B}) \in L_\infty$ and let $\mathcal{C}_\infty$ be as defined in Eqs. (4.2a)-(4.2d). Suppose that a full row rank
Theorem 5.3. Let \( \mathcal{B} \in \mathcal{H}_\infty \) and \( \mathcal{B}_\text{full} \) be the \( \mathcal{K}_\infty \)-behavior of the plant \( \mathcal{P}_p \in \mathcal{L}_{d\to c} \) which achieves \( \mathcal{C}_\infty \) for \( \mathcal{P}_p \) only if \( \mathcal{B}_\text{full} \subseteq \mathcal{B} \). Moreover, \( \mathcal{B}_\text{full} \) must be square and invertible as a rational function. Consequently, \( \mathcal{B}_\text{full} \) is uniquely determined by \( \mathcal{P}_p \) so that there exists a mapping \( H : \mathcal{H}_2 \to \mathcal{H}_2 \) with the property that \( \mathcal{B}_\text{full} = H \mathcal{B}_\text{full} \) and \( H^{-1} \mathcal{B}_\text{full} = \mathcal{B}_\text{full} \). This theorem implies the existence of \( c > 0 \) such that Eq. (5.3) holds which in turn is equivalent to Eq. (4.1b).

The equivalence (1 \( \iff \) 3) is the dual of (1 \( \iff \) 2) and is proven in a similar way. We omit the details. \( \square \)

We return to the \( \mathcal{H}_\infty \) control problem. Infer from Theorem 5.1 that the \( \mathcal{K}_\infty \)-behavior of the plant \( \mathcal{P}_p \) admits a projection \( \mathcal{B}_\text{proj} \) where \( \mathcal{P}_p \in \mathcal{L}_{d\to c} \) has full row rank, say \( p \), and is partitioned as

\[
\begin{bmatrix} \mathcal{P}_p & \mathcal{P}_q \end{bmatrix} = \begin{bmatrix} \mathcal{P}_p \mathcal{P}_q \end{bmatrix}
\]

with \( \mathcal{P}_p \) and \( \mathcal{P}_q \) of dimension \( p \times q \) and \( q \times q \), respectively. Likewise, the \( \mathcal{K}_\infty \)-behavior of a controller \( \mathcal{C}_c \in \mathcal{L}_{c\to d} \) can be represented as \( \mathcal{B}_\text{ent} \) where \( \mathcal{C}_c \in \mathcal{L}_{c\to d} \). Since \( \mathcal{P}_p \mathcal{C}_c \) is the \( \mathcal{K}_\infty \)-behavior of the controlled system \( \mathcal{P}_p \mathcal{C}_c \in \mathcal{L}_{d\to c} \), the \( \mathcal{K}_\infty \)-behavior of the controlled system \( \mathcal{P}_p \mathcal{C}_c \) is represented by

\[
\mathcal{B}_\text{full} = \mathcal{B}_\text{proj} \begin{bmatrix} \mathcal{P}_c \mathcal{P}_q \end{bmatrix}
\]

In this setting of \( \mathcal{K}_\infty \)-behaviors, the \( \mathcal{H}_\infty \) control problem thus amounts to synthesizing \( \mathcal{C}_c \), given \( \mathcal{P}_p \), so as to achieve the objective \( \mathcal{C}_\infty \). A simple necessary solvability condition for this problem is given as follows.

Theorem 5.3. Let \( \mathcal{B}_\text{proj} \) be the \( \mathcal{K}_\infty \)-behavior of the plant \( \mathcal{P}_p \) which achieves \( \mathcal{C}_\infty \) for \( \mathcal{P}_p \) only if \( \mathcal{B}_\text{proj} \subseteq \mathcal{B}_\text{max} \). Then there exists a controller \( \mathcal{C}_c \in \mathcal{L}_{c\to d} \) which achieves \( \mathcal{C}_\infty \) for \( \mathcal{P}_p \) only if \( \mathcal{B}_\text{proj} \subseteq \mathcal{B}_\text{max} \).

Proof. Let the controlled system behavior \( \mathcal{B}_\text{full} = \mathcal{B}_\text{proj} \) satisfy Eqs. (4.1a) and (4.1b). Define \( \mathcal{B}' = \{ w_{\text{ext}, 0} \} \). Then \( \mathcal{B}' \subseteq \mathcal{B}_\text{proj} \)

\[
\Theta_+ \Theta_+ - \Theta_- \Theta_- = \Theta_- (H \Theta_+ - I) \Theta_- < 0 \quad (5.6)
\]

on the unit circle.

1 \( \iff \) 2: Suppose that statement 2 holds. Then for all \( \mathcal{B}_\text{proj} \in \mathcal{L}_{d\to c} \) there exist \( \mathcal{B}_\text{proj} \in \mathcal{L}_{d\to c} \) such that \( \mathcal{B}_\text{proj} \mathcal{C}_c + \Theta_+ \mathcal{B}_\text{proj} = 0 \). The inclusion, Eq. (4.1a), then immediately follows. Since \( \mathcal{B}_\text{proj} \mathcal{C}_c + \Theta_+ \mathcal{B}_\text{proj} = \mathcal{B}_\text{proj} \), we must have that \( \mathcal{B}_\text{proj} \) is injective. But then \( \mathcal{B}_\text{proj} \in \mathcal{L}_{d\to c} \) is uniquely determined by \( \mathcal{B}_\text{proj} \in \mathcal{L}_{d\to c} \) so that there exist a mapping \( H : \mathcal{H}_2 \to \mathcal{H}_2 \) with the property that \( \mathcal{B}_\text{proj} = H \mathcal{B}_\text{proj} \) if and only if \( (\mathcal{B}_\text{proj}, w_\mathcal{B}_\text{proj}) \) satisfies Eq. (5.4). Since \( \mathcal{B}_\text{proj} \) is full rank, \( \mathcal{B}_\text{proj} \) is necessarily square and invertible as a rational function. Hence, \( H = -\Theta_+ \mathcal{B}_\text{proj} \) and Eq. (5.6) implies that \( \mathcal{B}_\text{proj} \in \mathcal{L}_{d\to c} \) for all \( \mathcal{B}_\text{proj} \). The latter implies the existence of \( \varepsilon > 0 \) such that Eq. (5.3) holds which in turn is equivalent to Eq. (4.1b).

The equivalence (1 \( \iff \) 3) is the dual of (1 \( \iff \) 2) and is proven in a similar way. We omit the details. \( \square \)
and thus satisfies $\mathcal{B}' \subseteq \mathcal{L}_{\text{max}}$. Now infer from Eq. (5.7) that $\mathcal{B}' = \mathcal{B}_{\text{ker}}(\Theta_{\text{ext}})$ which yields the result. □

Note that $\mathcal{B}_{\text{ker}}(\Theta_{\text{ext}})$ consists of those external $\ell_2$ trajectories $w_{\text{ext}}$ for which $(w_{\text{ext}}, 0)$ belongs to the $\ell_2$ behavior of the plant. Theorem 5.3 therefore states that strict positivity of this set is a necessary solvability condition for the $\mathcal{L}_\infty$ control problem. We make the following assumption.

**Assumption 5.4.** $\Theta_{\text{ext}}(z)$ and $\Theta_{\text{int}}(z)$ have, respectively, full row and full column rank for all $z \in C_+$. The interpretation of this assumption is that the two sub-systems $\mathcal{B}_{\text{ker}}(\Theta_{\text{ext}})$ and $\mathcal{B}_{\text{int}}(\Theta_{\text{int}})$ admit input-output representations which are minimum phase. The next step is to relate the necessary solvability condition of Theorem 5.3 to $J$-spectral factorization theory. The existence of $J$-spectral factors will be a key step in transforming the $\mathcal{L}_\infty$ control problem into an equivalent, but simpler, problem in which $\Theta_{\text{ext}}$ is replaced by the identity. All solutions to the $\mathcal{L}_\infty$ control problem are then generated by solving the latter problem. Let $J_{r,s}$ denote the signature matrix $J_{r,s} := \text{diag}(I_r, -I_s)$.

**Theorem 5.5.** Let $\Theta \in \mathcal{M}_{\infty}^{(p \times q)}$ have rank $p$ for all $z \in C_+$, and let $p_+ = p - q_-$. Then $\mathcal{B}_{\text{ker}}(\Theta) \subseteq \mathcal{L}_{\text{max}}$ if and only if there exists $W \in \mathcal{M}_{\infty}^{p \times p}$ such that

$$
\Theta^*Q_-^\dagger \Theta^\dagger = W_{p+q_-}W^\dagger,
$$

$$
\Theta^*Q_+\mathcal{H}_2 \subseteq \Theta^*Q_-\mathcal{H}_2,
$$

$$
\text{rank } \Theta^*Q_- = \text{rank } Q_-,
$$

where $\Theta^* := (0 \ I_{q_-}) W_1^\dagger \Theta$. \(\text{Proof.} \ (\Rightarrow) \) Suppose such a $W$ exists. Define $\Theta' := (0 \ I_{q_-}) \Theta''$ and $\Theta'' = W^\dagger \Theta$. Then $\Theta', \Theta'' \in \mathcal{M}_{\infty}$ and we obtain from [28, Theorem 5.5] that $\mathcal{B}_{\text{ker}}(\Theta) = \mathcal{B}_{\text{ker}}(\Theta'') \subseteq \mathcal{B}_{\text{ker}}(\Theta')$. By Eq. (5.8a), $\Theta'^{-1}Q_-^\dagger \Theta'^\dagger = -I_{q_-} < 0$ on the unit circle. By Theorem 5.2, it thus follows that $\mathcal{B}_{\text{ker}}(\Theta) \subseteq \mathcal{B}_{\text{ker}}(\Theta') \subseteq \mathcal{L}_{\text{max}}$, as desired.

$$
\Theta^*Q_+ \mathcal{H}_2 \subseteq \Theta^*Q_- \mathcal{H}_2.
$$

$$
\text{rank } \Theta^*Q_- = \text{rank } Q_-,
$$

$$
\text{where } \Theta^* = (0 \ I_{q_-}) W_1^\dagger \Theta.
$$

Now, infer from (a transposed version of) [7, Theorem 2.4] (or [6, Theorem 3.1]) that there exists $U \in \mathcal{M}_{\infty}$ such that

$$
\Theta^* U_{q_-} \Theta^* = U J_{p_+ q_-} U^\dagger,
$$

where $U_{11}$, the upper-left $p_+ \times p_+$ block of $U$, belongs to $\mathcal{M}_{\infty}$. But then $W := U_{11}^{-1} U_2^{-1} U$ belongs to $\mathcal{M}_{\infty}$, and has the required property, Eq. (5.8a). Next, consider $\Theta'$. Substitution of $W$ yields that rank $\Theta'^*Q_- = \text{rank } (0 \ I_{q_-}) U_{11}^{-1} (0 \ I_{q_-})^\dagger = q_-$, which shows Eq. (5.8c). Finally, to prove Eq. (5.8b), it suffices to prove that for all $\hat{w}_+ \in \mathcal{H}_2$ there exists $\hat{w}_- \in \mathcal{H}_2$ such that

$$
\hat{w}_+ + \Theta^* Q_- \hat{w}_- = 0,
$$

$$
\langle w_+, w_+ \rangle + \langle w_-, w_- \rangle \geq \varepsilon (\|w_+\|^2 + \|w_-\|^2)
$$

for some $\varepsilon > 0$. Observe that $\Theta_-$ must be injective as an operator on $\mathcal{H}_2$. Indeed, $\Theta^* \hat{w}_- = 0$ implies that $(0, w_-) \in U \mathcal{B}_{\text{ker}}(\Theta)$ which, using Eq. (5.8b), yields that $\hat{w}_- = 0$. Hence, using Hermite forms and the fact that $\Theta$ has full rank as a rational matrix, there exists $U_1 \in \mathcal{M}_{\infty}$ such that

$$
U_1 \Theta_- = \left( \begin{array}{c}
0 \\
\Theta_-
\end{array} \right),
$$

where $\text{rank } \Theta_- = q_-$. Since $\Theta(z)$ has constant rank for $z \in C_+$, also $\Theta_-(z)$ has constant rank for $z \in C_+$ which yields that $\Theta'_- \in \mathcal{M}_{\infty}$. Consequently, $U_2 := \text{diag}(I, (\Theta'_-)^{-1}) \in \mathcal{M}_{\infty}$ and

$$
\Theta_0 := U_2 U_1 (\Theta_+ \Theta_-) = \left( \begin{array}{cc}
\Theta_1 & 0 \\
\Theta_2 & I
\end{array} \right),
$$

where $\Theta_1(z)$ will have rank $p_z$ for all $z \in C_+$. Using Eq. (5.9b), it follows that $\langle \hat{w}_+, (I - \Theta_{22}^\dagger \Theta_2) \hat{w}_+ \rangle > 0$, for all non-zero $\hat{w}_+ \in \ker \Theta_$. Stated otherwise,

$$
\|(\Theta_2 + \Theta_1) \|_{\ker \Theta} \|_{\infty} < 1
$$

for any $P \in \mathcal{M}_\infty$ of appropriate dimension. Since rank $\Theta_1(z) = p_z$ for all $z \in C_+$, $\Theta_1$ is surjective as a mapping from $\mathcal{H}_2 \rightarrow \mathcal{H}_2$. It is well known from the solution of the so-called two-block or unilateral model matching problem (see, e.g., [5, 15]), that then there exists $P \in \mathcal{M}_{\infty}$ such that

$$
\|(\Theta_2 + P \Theta_1) \|_{\infty} = \|(\Theta_2 + P \Theta_1) \|_{\ker \Theta} \|_{\infty} < 1.
$$

Now, infer from (a transposed version of) [7, Theorem 2.4] (or [6, Theorem 3.1]) that there exists $U \in \mathcal{M}_{\infty}$ such that

$$
\Theta_U \in \mathcal{M}_{\infty},
$$

where $U_{11}$, the upper-left $p_+ \times p_+$ block of $U$, belongs to $\mathcal{M}_{\infty}$. But then $W := U_{11}^{-1} U_2^{-1} U$ belongs to $\mathcal{M}_{\infty}$, and has the required property, Eq. (5.8a). Next, consider $\Theta'$. Substitution of $W$ yields that rank $\Theta'^*Q_- = \text{rank } (0 \ I_{q_-}) U_{11}^{-1} (0 \ I_{q_-})^\dagger = q_-$, which shows Eq. (5.8c). Finally, to prove Eq. (5.8b), it suffices to prove that for all $\hat{w}_+ \in \mathcal{H}_2$ there exists $\hat{w}_- \in \mathcal{H}_2$ such that

$$
(0 \ I_{q_-}) U_{11}^{-1} \left( \begin{array}{c}
\Theta_1 & 0 \\
\Theta_2 & I
\end{array} \right) \left( \begin{array}{c}
\hat{w}_+ \\
\hat{w}_-
\end{array} \right) = \left( \begin{array}{c}
0 \\
0
\end{array} \right).
$$

To see this, let $\hat{w}_+ \in \mathcal{H}_2$, $V = U_{11}^{-1}$ and partition $U$ and $V$ conformably with $\Theta_0$. Since $U_{11} \in \mathcal{M}_{\infty}$ and $V_{22}^{-1} = U_{22} - U_{21} U_{11}^{-1} U_{12}$, it follows that $V_{22} \in \mathcal{M}_{\infty}$. Thus, $\hat{w}_- := -[V_{22} \ V_2] \Theta_2 \Theta_2 \hat{w}_+$ belongs to $\mathcal{H}_2$.
and one easily checks that Eq. (5.10) holds. This completes the proof. □

The expression, Eq. (5.8a) is called a \( J \)-spectral factorization of \( \Theta Q^{-1} \Theta \). It has been shown in [6] that spectral factors \( W \in \mathcal{U} \mathcal{H}^+ \) satisfying Eq. (5.8a) are unique up to multiplication by constant matrices \( M \) which satisfy \( M^* J_{p,q} M = J_{p,q} \). Moreover, in the same paper it is shown that whenever Eqs. (5.8b) and (5.8c) hold for one solution \( W \in \mathcal{U} \mathcal{H}^+ \) of Eq. (5.8a), then these conditions hold for all such solutions. Clearly, this property facilitates the verifiability of the factorizability conditions, Eqs. (5.8a)-(5.8c). It should be emphasized that it cannot be inferred from Theorems 5.2 and 5.5 that \( \Theta Q^{-1} \Theta \) < 0 if \( \ker((\Theta)) \subset \sigma_{\text{max}} \). As a matter of fact, \( \Theta Q^{-1} \Theta \) has \( p_+ \) positive eigenvalues on every point of the unit circle. The proof of the ‘if’ part of Theorem 5.5 shows that \( \Theta \) defined in Eqs. (5.8a)-(5.8c) has an interesting system theoretic interpretation: if \( \ker((\Theta)) \subset \sigma_{\text{max}} \) then \( \Theta \) has the property that \( \Theta^2 > 0 \) and is a proper superset of \( \ker((\Theta)) \). That is, \( \ker((\Theta)) \) is an extension of a strictly positive subspace that satisfies both the inclusions, Eqs. (4.1a) and (4.1b).

**Corollary 5.6.** Let \( \ker((\Theta)) \subset \mathcal{U} \mathcal{H}^+ \) represent the \( \ell_2 \) behavior of the plant \( \Sigma_p \in \mathcal{U} \mathcal{H}_{q+}^+ \). Then the \( \mathcal{H}_\infty \) control problem admits a solution \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) only if there exists \( \Theta \in \mathcal{U} \mathcal{H}_\infty^+ \) with \( \Theta = \Theta \). The necessary condition for solvability of the \( \mathcal{H}_\infty \) control problem is therefore solved if there exists a controller \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) which achieves the objective \( \mathcal{C}_\infty \) for the plant \( \Sigma_p \).

**Proof.** By Theorem 5.3, \( \ker((\Theta)) \subset \mathcal{U} \mathcal{H}^+ \) is a necessary condition for solvability of the \( \mathcal{H}_\infty \) control problem. Theorem 5.5 with \( \Theta = \Theta \), then yields the result. □

Suppose that the necessary condition of Corollary 5.6 holds. That is, there exists \( \Theta \in \mathcal{U} \mathcal{H}^+ \) such that Eqs. (5.8a)-(5.8c) hold with \( \Theta = \Theta \). The following result then reduces the solution of the general \( \mathcal{H}_\infty \) control problem to the solution of an \( \mathcal{H}_\infty \) problem in which \( \Theta = \Theta \). Define \( Q' = J_{p,q} \) and \( Q' = J_{q,p} \) where \( Q' = (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) has rank \( p_+ \) and \( Q' = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) has rank \( q_- \). Define the control objective \( \mathcal{C}_\infty \) by the quadruple

\[
\begin{align*}
\mathcal{H}'_{\text{min}} & := Q'_2, \\
\mathcal{H}'_{\text{min}} & := Q'_2, \\
\mathcal{H}'_{\text{max}} & := Q'_2.
\end{align*}
\]

\( \mathcal{H}'_{\text{max}} := \{ w \in \ell_2^+ \mid \exists \varepsilon > 0 \text{ such that } \langle w, Q' w \rangle \geq \varepsilon \langle w, w \rangle \}. \) (5.11d)

**Theorem 5.7** (Reduction theorem). Let \( \ker((\Theta)) \subset \mathcal{U} \mathcal{H}^+ \) represent the \( \ell_2 \) behavior of the plant \( \Sigma_p \in \mathcal{U} \mathcal{H}_{q+}^+ \) and suppose there exists \( W \in \mathcal{U} \mathcal{H}_\infty^+ \) such that Eqs. (5.8a)-(5.8c) with \( \Theta = \Theta \). Define \( \Sigma'_p := (T, \mathcal{R}^{p+}, \mathcal{R}'_p) \) with

\[
\mathcal{R}'_p := \ker((I \Psi_{\text{int}}), \Psi_{\text{int}} := W^{-1} \Theta_{\text{int}})
\]

and let \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) be an arbitrary controller. Then the following statements are equivalent:

1. \( \Sigma_c \) achieves the \( \mathcal{H}_\infty \) control objective \( \mathcal{C}_\infty \) for the plant \( \Sigma_p \).
2. \( \Sigma_c \) achieves the \( \mathcal{H}_\infty \) control objective \( \mathcal{C}_\infty \) for the plant \( \Sigma'_p \).

**Proof.** Variations of this result have been derived at many places. See, for instance, [12, Lemma 4.1.5; 7, Theorem 2.6] or [15, Lemma 5.4]. □

The \( \mathcal{H}_\infty \) control problem is therefore solved if there exists a controller \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) which achieves the objective \( \mathcal{C}_\infty \) for the plant \( \Sigma_p \). Consider

\[
\mathcal{B} := \ker((\Theta)) \subset \mathcal{U} \mathcal{H}^+
\]

By construction, \( \mathcal{B} \) admits an image representation of the form

\[
\mathcal{B} = \ker((\Theta)).
\]

By Theorem 5.5, the \( \ell_2 \) behavior of any controller \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) also admits an image representation of this form, i.e., for all \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) there exists \( \mathcal{B} \in \mathcal{U} \mathcal{H}_\infty^+ \) such that

\[
\mathcal{B} \cap \ell_2^+ = \ker((\mathcal{B})).
\]

It is then easily seen that the interconnection

\[
\mathcal{B} := \mathcal{B}' \wedge \mathcal{B} = \ker((\Psi_{\text{int}} \Psi_{\text{c}})).
\]

That is, the behavior of the interconnection admits an image representation that is simply the product of \( \Psi_{\text{int}} \) and \( \Psi_{\text{c}} \). The problem to find \( \Psi_{\text{c}} \), given \( \Psi_{\text{int}} \), such that \( \mathcal{B} \) satisfies the \( \mathcal{H}_\infty \) control objective \( \mathcal{C}_\infty \) is solved as follows.

**Theorem 5.8.** Let Assumption 5.4 hold and suppose there exists \( W \in \mathcal{U} \mathcal{H}_\infty^+ \) satisfying Eqs. (5.8a)-(5.8c) with \( \Theta = \Theta \). Let \( r_\infty = q_{\text{int}} - p_+ \). Then the following statements are equivalent:

1. There exists a controller \( \Sigma_c \in \mathcal{U} \mathcal{H}_\infty^+ \) which achieves the \( \mathcal{H}_\infty \) control objective \( \mathcal{C}_\infty \) for the plant \( \Sigma_p \).
2. There exists \( V \in \mathcal{U} \mathcal{H}^+_{\infty} \) such that
\[
\Psi^{-} Q' \Psi = V^{-} J_{p_{+}, r_{-}} V,
\]
\[
\ker Q' \subset \ker Q' \Psi',
\]
\[
\text{rank } Q' \Psi' = \text{rank } Q',
\]
where \( \Psi = \Psi_{\text{int}} \) and \( \Psi = \Psi_{V} = \Psi V^{-1}(I_{p_{+}} 0)^{T} \). Moreover, in that case all controllers achieving the \( \mathcal{H}^\infty \) control objective \( C_{\infty}^{'} \) for \( \Sigma_{p}^{'} \) are given by \( \mathcal{B}_{c} = \mathcal{B}_{\text{int}}(\Psi_{c}) \) where
\[
\Psi_{c} = V^{-1} \left( \begin{array}{c}
I_{p_{+}} \\
U \end{array} \right), \quad U \in \mathcal{U} \mathcal{H}^+_{\infty}, \quad \|U\|_{\infty} < 1.
\]

**Proof.** \( 2 \implies 1 \): Suppose that statement 2 holds. Let \( \Psi_{c} \) be given by Eq. (5.13) with \( U = 0 \). Then \( \Psi_{c} \in \mathcal{U} \mathcal{H}^+_{\infty} \) and \( \Psi' := \Psi_{\text{int}} \Psi_{c} \) has full column rank. Infer from Theorem 5.2 that \( \Sigma_{c} := (T, \mathcal{R}_{\text{out}}, \mathcal{B}_{\text{int}}(\Psi_{c})) \) achieves \( C_{\infty}^{'} \) for the plant \( \Sigma_{p}^{'} \). Let \( \mathcal{B}_{c} := \mathcal{B}_{\text{int}}(\Psi_{\text{int}} \Psi_{c}) \) and put \( \Psi = \Psi_{\text{int}} \). Then
\[
\mathcal{J}_{\text{min}}^{'} \subseteq \mathcal{B} + \mathcal{J}_{\text{min}}^{'} \subseteq \mathcal{B}_{\text{int}}(\Psi') + \mathcal{J}_{\text{min}}^{'}.
\]
Define \( \Psi_{+} := (I_{p_{+}} 0) \Psi \) and \( \Psi_{-} := (0 I_{q_{-}}) \Psi \). The inclusion, Eq. (5.14), implies that for all \( \tilde{w}_{+} \in \mathcal{H}_2 \), there exists \( v \in \mathcal{H}_2 \) such that \( \tilde{w}_{+} = \Psi_{+} v \). Consequently, \( \Psi_{+} \) is surjective as a multiplicative operator \( \mathcal{H}_2 \to \mathcal{H}_2 \). Since \( \Psi \) has full column rank as a rational matrix, there exists \( U_{1} \in \mathcal{U} \mathcal{H}_{\infty} \) such that
\[
\Psi_{+} U_{1} = (\Psi_{+}^{'} 0),
\]
where \( \Psi_{+}^{'} \) is square with rank \( p_{+} \). Moreover, since \( W^{-1} \in \mathcal{U} \mathcal{H}^+_{\infty} \), Assumption 5.4 implies that \( \Psi(z) = W^{-1}(z)\Theta_{\text{int}}(z) \) has full column rank for all \( z \in \mathbb{C}_{+} \). Hence, \( \Psi_{+}^{'} \in \mathcal{U} \mathcal{H}^+_{\infty} \) so that, with \( U_{2} := \text{diag}((\Psi_{+}^{'} 0)^{-1}, I) \Psi U_{1} U_{2} \) takes the form
\[
\Psi_{0} := \left( \begin{array}{c}
\Psi_{+}^{'} \\
\Psi_{-}^{'}
\end{array} \right) U_{1} U_{2} = \left( \begin{array}{cc}
I & 0 \\
0 & \Psi_{1} \Psi_{2}
\end{array} \right),
\]
where \( \Psi_{2}(z) \) will have rank \( r_{-} \) for all \( z \in \mathbb{C}_{+} \). Let
\[
U_{2}^{-1} U_{1}^{-1} \Psi_{c} = \left( \begin{array}{c}
\Psi_{c,1}^{'} \\
\Psi_{c,2}^{'}
\end{array} \right)
\]
be partitioned conformably with \( \Psi_{0} \). Using Eq. (5.14), it follows that \( \Psi_{c,1}^{'} \) is surjective. Since \( \Psi_{c} \) has full column rank, it follows that \( \Psi_{c,1}^{'} \in \mathcal{U} \mathcal{H}^+_{\infty} \). Consequently, \( \mathcal{B}_{\text{int}}(U_{2}^{-1} U_{1}^{-1} \Psi_{c}) = \mathcal{B}_{\text{int}}((U_{1}^{-1} \Psi_{c,2}^{'})) \) so that
\[
\mathcal{B} = \left\{ w \in \ell_{2} | \tilde{w} = \left( \begin{array}{c}
\tilde{w}_{+} \\
\tilde{w}_{-}
\end{array} \right) \in \left( \begin{array}{c}
\Psi_{1} + \Psi_{2} \Psi_{c,2}^{'}
\end{array} \right) \mathcal{H}_2 \right\}.
\]
As \( \mathcal{B} \subseteq \mathcal{J}_{\text{max}}^{'} \), it follows that \( \|\Psi_{1} + \Psi_{2} \Psi_{c,2}^{'}\|_{\infty} < 1 \). Infer from this and [6, Theorem 2.4] that there exists \( U \in \mathcal{U} \mathcal{H}_{\infty}^{+} \) such that \( \Psi_{0}^{'} Q \Psi_{0} = U^{-} J_{p_{+}, r_{-}} U \) and \( U_{22} \), the right-lower \( r_{-} \times r_{-} \) block of \( U \), belongs to \( \mathcal{U} \mathcal{H}^+_{\infty} \). But then \( \tilde{V} := UU_{2}^{-1} U_{1}^{-1} \) belongs to \( \mathcal{U} \mathcal{H}^+_{\infty} \) and satisfies Eq. (5.12a). Next, observe that rank \( Q'_{+} \Psi' = \text{rank} (I_{p_{+}} 0) U^{-1} (I_{p_{+}} 0)^{T} \) is \( p_{+} \), which yields Eq. (5.12c). Finally, let \( Z := U^{-1} \) be partitioned conformably with \( \Psi_{0} \). As in the proof of Theorem 5.8, Eq. (5.12b) is equivalent to \( Z_{11} \in \mathcal{U} \mathcal{H}_{\infty}^{+} \). Since \( Z_{11}^{-1} = U_{11}^{-1} - U_{12} U_{22} U_{21} \) and \( U_{22} \in \mathcal{U} \mathcal{H}_{\infty}^{+} \), it follows that \( Z_{11} \in \mathcal{U} \mathcal{H}_{\infty}^{+} \) and the proof is complete.

The last statement can be found in [7, Theorem 2.4] or [6, Theorem 3.2]. □

The complete solution to the \( \mathcal{H}_{\infty} \) control problem can now be summarized from the previous results.

**Theorem 5.9.** If Assumption 5.4 holds, then there exists a controller \( \Sigma_{c} \in \mathcal{L}_{\text{int}}^{\infty} \) which achieves the control objective \( C_{\infty}^{'} \) for the plant \( \Sigma_{p}^{'} \) if and only if there exist \( J \)-spectral factors \( W \in \mathcal{U} \mathcal{H}_{\infty}^{+} \) and \( V \in \mathcal{U} \mathcal{H}_{\infty}^{+} \) such that Eqs. (5.8a)–(5.8c) with \( \Theta = \Theta_{\text{ext}} \), and Eqs. (5.12a)–(5.12c), with \( \Psi = \Psi_{\text{int}} \), hold. In that case, Eq. (5.13) parametrizes all such controllers.

**Proof.** The only if part and the last statement are immediate from Corollary 5.6, and Theorems 5.7 and 5.8. To see the if part, let Assumption 5.4 hold and let \( W, V \) satisfy the hypothesis. Then, by Theorem 5.8, there exists \( \Sigma_{c} \in \mathcal{L}_{\text{int}}^{\infty} \), which achieves \( C_{\infty}^{'} \) for \( \Sigma_{p}^{'} \). But then, by Theorem 5.7, \( \Sigma_{c} \) also achieves \( C_{\infty}^{'} \) for \( \Sigma_{p}^{'} \), which yields the result. □

**Remark 5.10.** The controllers, Eq. (5.13), can be synthesized by means of fast numerical calculations which exploit the relation between the \( J \)-spectral factors \( W \) and \( V \) in Eqs. (5.8a)–(5.8c) and (5.12a)–(5.12c) and solutions to algebraic Riccati equations. This relation is well known and documented in, e.g., [7, 8, 12, 14].

**Remark 5.11.** We emphasize that the numbers \( p_{+} \) and \( r_{-} \) which appear in the spectral factorizations are determined by \( Q \) and \( \Sigma_{p}^{'} \) only. All \( \mathcal{H}_{\infty} \) controllers, Eq. (5.13), achieving the control objective \( C_{\infty}^{'} \) for...
Zp define r_ independent constraints on the interconnection variables \( w_{\text{int}} \). It has been shown in [23] that any such controller admits a representation in input–output form where \( p_+ \) components of \( w_{\text{int}} \) are free variables (which qualify as inputs of the controller) and \( r_+ \) components of \( w_{\text{int}} \) are bound variables (corresponding to the outputs of the controller). These input–output representations need not be unique, but the dimensions \( p_+ \) and \( r_+ \) of their input and output spaces are. This means that there are many transfer functions describing the input–output behavior of the controller, some of them are proper, others not. See [23] for details. Two important conclusions can be derived from this. First, the number of sensors and actuators to solve the \( H_\infty \) control problem is uniquely determined by the control objective \( \Sigma_0 \) and the plant \( \Sigma_p \) and is given by \( p_+ \) and \( r_+ \), respectively. Stated otherwise, all controllers that achieve the objective \( \mathcal{C}_\infty \) define precisely \( r_+ \) independent restrictions on the interconnection variables. Secondly, all controllers, Eq. (5.13), define a regular interconnection in the sense that there exists \( \tau_0 > 0 \) such that whenever \((w_{\text{ext}}, w'_{\text{int}}) \) and \((w_{\text{ext}}, w''_{\text{int}}) \) belong to \( \mathcal{B}_p \cap \mathcal{B}_c \) with \( w_{\text{int}}(t) = w''_{\text{int}}(t) \) for \( t \leq \tau_0 \) then \( w'_{\text{int}} = w''_{\text{int}} \). This property means that none of the interconnection variables remains free in an interconnected system which meets the \( H_\infty \) control objective \( \mathcal{C}_\infty \). Thirdly, any of the controllers, Eq. (5.13), admit input–output representations by means of a proper or a non-proper transfer function.

**Remark 5.12.** In this paper the formalization of the \( H_\infty \) control objective involves the introduction of an indefinite inner product \( \langle \cdot, \cdot \rangle_Q \) where \( Q \in \mathbb{R}^{n \times n} \) is a symmetric-indefinite matrix. In Ref. [27] it is advocated that performance requirements are better expressed in terms of quadratic differential (or difference) forms \( Q : L_2(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathbb{R} \) defined by

\[
Q(w) := \sum_{i,j=1}^{n} \langle \sigma_i w, \sigma_j w \rangle Q_{ij},
\]

where \( Q_{ij} \in \mathbb{R}^{q \times q} \) is a symmetric matrix for \( i, j = 1, \ldots, n \). An alternative formulation for the \( H_\infty \) control problem is then obtained by putting

\[
\mathcal{J}_{\max} = \{ w \in \mathcal{H}_{\max} \mid \exists \varepsilon > 0 \text{ such that } Q(w) \geq \varepsilon \langle w, w \rangle \}
\]

in Eq. (4.2d). This avoids extending the signal space with differentials (or differences) of external variables although it is not essentially more general.

**Remark 5.13.** The theory developed here allows for a straightforward generalization to continuous time dynamical systems. Consider the class of systems whose laws take the form of ordinary differential equations. That is, let \( \Theta \in \mathbb{R}^{\times \mathbb{R}}[\mathbb{C}] \) be a polynomial matrix and consider the equation

\[
\Theta(d/dt)w = 0.
\]

The shorthand \( \Theta(d/dt) \) is used to refer to the polynomial differential operator \( \Theta(d/dt) = \sum_{i=0}^{\infty} \Theta_i(d/dt)^i \) and we call the solution set, i.e., the set of all \( w \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q) \) for which Eq. (5.15) holds in the sense of distributions, the behavior associated with \( \Theta \). Let \( \mathcal{L}^q \) denote the set of all systems \( \Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{H}) \) whose behavior can be represented in this way. The representation results of Theorems 3.2 and 5.1 then remain valid if the function spaces \( L_2, L_\infty \) and \( \mathcal{H} \) are understood as their continuous time analogues and if \( \tilde{w} \) is interpreted as the usual Laplace transform of \( w \). If Assumption 5.4 is replaced by the assumption that \( \Theta_{\text{ext}} \) and \( \Theta_{\text{int}} \) have, respectively, full row and full column rank in the closed right-half complex plane (including the point at infinity) then all results of Section (5) remain valid for the continuous-time \( H_\infty \) control problem.

6. Conclusion

In this paper the general \( H_\infty \) control problem has been formalized in a behavioral setting and it has been shown that \( H_\infty \) optimal controllers can be synthesized using J-spectral factorizations. The \( H_\infty \) control problem which has been studied here involves an entire symmetric treatment of system variables and departs from the prevailing input–output formalization of this problem. The idea to view “control” as imposing restrictions on a distinguished set of interconnection variables turned out to be useful to formalize general control objectives. We specialized such a setting to the \( H_\infty \) control problem and a parametrization is given of all controllers that achieve a \( H_\infty \) control objective. This parametrization involves two J-spectral factorizations.

Although the relation between \( H_\infty \) control, Nehari problems and J-spectral factorization theory is well known [6, 7, 12, 13], this paper provides a conceptual advantage in that no assumptions are made on input–output decompositions of the interconnection variables and on the causality structure of plant and controller.
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References