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Bates, J.W.; Montgomery, D.C.

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Toroidal visco-resistive magnetohydrodynamic steady states contain vortices

Jason W. Bates and David C. Montgomery
Department of Physics & Astronomy, Dartmouth College, Hanover, New Hampshire 03755-3528

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Poloidal velocity fields seem to be a fundamental feature of resistive toroidal magnetohydrodynamic (MHD) steady states. They are a consequence of force balance in toroidal geometry, do not require any kind of instability, and disappear in the “straight cylinder” (infinite aspect ratio) limit. If a current density \( j \) results from an axisymmetric toroidal electric field that is irrotational inside a torus, it leads to a magnetic field \( B \) such that \( \nabla \times (j \times B) \) is nonvanishing, so that the Lorentz force cannot be balanced by the gradient of any scalar pressure in the equation of motion. In a steady state, finite poloidal velocity fields and toroidal vorticity must exist. Their calculation is difficult, but explicit solutions can be found in the limit of low Reynolds number. Here, existing calculations are generalized to the more realistic case of no-slip boundary conditions on the velocity field and a circular toroidal cross section. The results of this paper strongly suggest that discussions of confined steady states in toroidal MHD must include flows from the outset. © 1998 American Institute of Physics. [S1070-664X(98)02407-0]

I. INTRODUCTION

Theoretical or computational discussions of allowable magnetohydrodynamic (MHD) steady states are often carried out in the ideal limit. Viscosity and resistivity are set equal to zero, and the only MHD requirement remaining for zero-flow equilibrium is one of mechanical force balance, such that the Lorentz force \( j \times B \) (where \( j \) is the current density and \( B \) is the magnetic field) is balanced by the gradient of a scalar fluid pressure, \( \nabla p \). In the ideal treatment, the electric field also vanishes, and neither Ohm’s law nor Faraday’s law imposes any additional requirements. For axisymmetric toroidal geometry, an elliptic partial differential equation (the Grad-Shafranov equation) for the magnetic flux function governs the (many) possible ideal equilibria.

In this paper, we treat nonideal (dissipative) toroidal MHD steady states which are required to obey Ohm’s law and Faraday’s law as well as mechanical force balance. These states differ considerably from their ideal MHD counterparts, and from treatments which omit one or more of these three crucial ingredients. Except in the case of what we have previously considered\(^2\) to be unphysical resistivity profiles, \( \nabla \times (j \times B) \) is nonvanishing and so \( j \times B \) cannot be balanced by the gradient of any scalar pressure.\(^3\) Finite velocity fields and an attendant viscous drag are sufficient for force balance, and in an earlier calculation restricted to toroids with rectangular cross sections and ideally smooth walls,\(^3\) have been shown to give rise to a characteristic “double smoke ring” pair of toroidal vortex rings.

These vortex rings have no analogue in the straight-cylinder approximation (infinite toroidal aspect ratio), and could not be expected to appear in a perturbation theory in which the small parameter is the inverse toroidal aspect ratio. Here, we replace the previously oversimplified boundary conditions by considerably more realistic ones: toroidal no-slip walls with a circular cross section. It should be noted that other kinds of velocity fields, including toroidal ones, may result in real confinement experiments from the combined effects of charge separation and electric fields. Though these may be even more important in actual devices, they are not to be sought in MHD descriptions, and our intent here is only one of clarifying the MHD framework.

Perhaps the earliest and most influential treatment of MHD flows in toroidal geometry appears in an unpublished report by Pfirsch and Schlüter from over thirty years ago.\(^5\) These authors incorporated the effect of finite electrical resistivity (but not viscosity) on MHD phenomena by calculating perturbatively from Ohm’s law a velocity in terms of the current density and magnetic field. Pfirsch and Schlüter did not, however, reintroduce their velocity field into the MHD equation of motion, which simply remained as \( j \times B = \nabla p \), as in ideal MHD. Inattention to the requirement of Faraday’s law that the electric field be irrotational in steady state made it unnecessary to discover the peculiar nature of the resistivity profile that would be implied.\(^2\) We have found this development less than wholly convincing. In particular, a net outward mass flux was found to be inevitable and had to be compensated by unspecified “sources” within the magneto-fluid. It appears that the results may be to some extent tied up with the inverse aspect ratio expansion that was performed. In the calculation presented in this paper, we shall see that no inherent outward mass flux is implied. A later treatment by Grad and Hogan\(^6\) sought to analyze confinement in a tokamak as the slow evolution from one quasi-steady state to another, but suffered some of the same limitations. A wide variety of other discussions of steady-state flows, sometimes involving the plasma as having crossed some instability threshold, also appears in the literature (see, for example, Refs. 7–12). In some instances, a balance between the electric field and the velocity-dependent term in Ohm’s law is
assumed, leaving no connection between zeroth order toroidal currents and the imposed toroidal voltages which, in the laboratory, typically drive them. We make no attempt to survey or to critique this literature except to note that in no case does the flow pattern presented here (and in Ref. 4) appear.

The velocity fields presented in this paper derive from the equation of motion, using a low (kinetic and magnetic) Reynolds number approximation, and their calculation retains the effects of both Ohm’s law and Faraday’s law. The low Reynolds number approximation is equivalent to one of high viscosity. The viscous stress tensor is a very uncertain quantity for confined fusion plasmas of current interest, and there are no adequate measurements of it; the principal theoretical calculation in the high collisionality limit is due to Braginskii and Balescu.\textsuperscript{13,14} The largest term in the viscous stress tensor, the so-called “ion parallel” viscosity, can be estimated as an ion mean free path times an ion thermal speed. For the present generation of tokamak experiments, this is an extremely large number; if it were to be taken seriously, it would more than justify a low Reynolds number treatment, but doubts remain as to just what viscous term in the equation of motion really is appropriate, if any. The advantage of the low Reynolds number approximation is that it makes it possible to include the effects of finite resistivity and viscosity at every step of the calculation, even with the complications of nonrectangular boundaries and no-slip boundary conditions on the velocity field. These are thought to be superior to the previous (convenient) choice of rectangular toroidal cross sections and stress-free boundaries.\textsuperscript{4}

The remainder of this paper is organized as follows. In Sec. II, we derive the electric field, current density, magnetic field, and velocity for our axisymmetric toroidal MHD model. In the interest of obtaining explicit steady-state expressions for all of these vector quantities, we imagine that the torus is a perfect conductor coated on the inside with a thin layer of dielectric material (so that a finite tangential electric field is allowed there). We ignore, as is unfortunately customary and necessary, any departures from axisymmetry that result from the slits and slots that must be cut in the conductor to allow the applied electric field to penetrate into the magnetofluid. In this case, the appropriate boundary conditions to be enforced are “magnetic, no-slip” ones, which means that the velocity vector and the normal components of the magnetic field and current density vanish on the toroidal surface. The low Reynolds number approximation used in carrying out these calculations can be thought of as giving the lowest order contribution in a perturbative treatment in which the squares of the viscous Lundquist number $M$ and the Hartmann number $H$ appear as small parameters.\textsuperscript{4} The precise meaning of this statement will be made clear presently. Finally, in Sec. III, we remark on the implications of this work for future studies of MHD equilibria.

II. A “SLOW-FLOW” CALCULATION

Using toroidal coordinates $(\xi, \eta, \varphi)$ defined\textsuperscript{15} by $x = R_0 \sqrt{1 - \xi^2 \cos \varphi / (1 - \xi \cos \eta)}$, $y = R_0 \sqrt{1 - \xi^2 \sin \varphi / (1 - \xi \cos \eta)}$, and $z = -R_0 \xi \sin \eta / (1 - \xi \cos \eta)$, where $(x, y, z)$ are standard Cartesian coordinates, we solve the one-fluid, incompressible, uniform-density MHD equations in a toroidal region which is axisymmetric ($\partial / \partial \varphi = 0$). The toroidal variable $\xi$ is similar to a minor radius, $\eta$ is a poloidal angle, and $\varphi$ is the usual toroidal angle, equivalent to the azimuthal angle of cylindrical coordinates, $(r, \varphi, z)$; see Fig. 1. The major and minor radii of the torus have dimensions $r_0$ and $a$, respectively, and the scale length featured in the transformation to toroidal coordinates is defined by $R_0 = \sqrt{r_0^2 - a^2}$. We assume the magnetofluid occupies the region $(r - r_0)^2 + z^2 < a^2$, where $r^2 = x^2 + y^2$. The bounding toroidal wall is denoted by $\xi = \xi_w = a/r_0$.

The governing MHD equations\textsuperscript{16} in a familiar set of dimensionless (“Alfvénic”) units are

\begin{align}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= \mathbf{j} \times \mathbf{B} - \nabla p + \nu \nabla^2 \mathbf{v}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\mathbf{j} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \frac{\mathbf{j}}{\sigma},
\end{align}

where $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Here, $\mathbf{v}$ is the fluid velocity and $\mathbf{E}$ is the electric field. In Eqs. (1) and (4), $\nu$ and $\sigma^{-1}$ are the reciprocals of a viscous and resistive Lundquist number, $M$ and $S$, respectively. In terms of laboratory (cgs) quantities, they are defined by $\nu^{-1} = M = C_a L / \nu$ and $\sigma = \sigma = 4 \pi \sigma C_a L / c$, where $C_a$ is an Alfvén speed (based on the rms magnetic field), $L$ is a scale length (e.g., the minor radius of the torus), $\nu$ is the kinematic viscosity, $\sigma$ is the electrical conductivity, and $c$ is the speed of light. Note that for reasons of tractability we have assumed a simple scalar form for the Newtonian viscous term in Eq. (1).

In general, Eqs. (1)–(4) comprise a nonlinear set of partial differential equations that are difficult to solve. Our approach has been to seek steady-state solutions ($\partial / \partial t = 0$), allowing for finite values of resistivity and viscosity (both taken to be spatially uniform scalars), and assuming $M^2 \ll 1$ and $H^2 \ll 1$, where $H = \sqrt{MS}$ is the Hartmann number. In this
limit, one can justify the neglect of the inertia term \( \mathbf{v} \cdot \nabla \mathbf{v} \) in the equation of motion, Eq. (1), as well as the \( \mathbf{v} \times \mathbf{B} \) term in Ohm’s law, Eq. (4).

To understand this better, we recall that a convenient measure of the magnitude of MHD dissipation coefficients is given by the following ratios of averaged quantities:

\[
\frac{\langle |\mathbf{v} \cdot \nabla \mathbf{v}| \rangle}{\langle |\nabla \mathbf{v}|^2 \rangle} \sim \frac{vL}{v} = R_e, \quad (5)
\]

\[
\frac{\langle |\mathbf{v} \times \mathbf{B}| \rangle}{\langle |\nabla \sigma| \rangle} \sim \frac{vL}{\eta^*} = R_m. \quad (6)
\]

where \( R_e \) and \( R_m \) are the kinetic and magnetic Reynolds numbers, respectively, \( v \) is a typical flow velocity, and \( \eta^* \equiv c^2/4\pi \sigma \) is the ‘‘magnetic diffusivity.’’ The lowest order neglect of the \( \mathbf{v} \cdot \nabla \mathbf{v} \) and \( \mathbf{v} \times \mathbf{B} \) terms in the equation of motion and Ohm’s law, respectively, is justified provided that both \( R_e \) and \( R_m \) are small compared to unity. The inequalities \( R_e \ll 1 \) and \( R_m \ll 1 \) can be shown to be implied by \( M^2 \ll 1 \) and \( H^2 = MS \ll 1 \), respectively, by using the equation of motion to estimate a typical flow speed as \( v \sim C_0 M \). Note that these inequalities can be satisfied even for large values of the Lundquist number \( S \) provided that the viscous Lundquist number \( M \) is sufficiently small.

The condition \( M^2 \ll 1 \) is tantamount to the statement that the ‘‘viscous momentum diffusion’’ time \( L^2/v \) is short compared to the Alfvén transit time \( L/C_0 \) for the system. Usually, this limit is not considered relevant to tokamak equilibria, but such a conclusion is suspect in that it relies on by-passing the largest viscosity coefficients in the Braginskii-Balescu\(^{13,14} \) viscous stress tensor. Nevertheless, it clearly would be desirable in this model to relax the low viscous Lundquist and Hartmann number approximations, as well as the assumptions of uniform density and constant transport coefficients (resistivity and viscosity), but doing so takes the problem outside the realm of present analytical tractability. The situation becomes even more formidable if one attempts to include a local thermodynamic equation of state for determining the pressure of the magnetofluid. In that case, an energy equation must also be satisfied, which greatly complicates the system of governing equations to be solved.\(^{17} \)

At present, no one seems close to treating the full MHD problem in toroidal geometry in the face of realistic boundary conditions. In this paper, we have sought to include as much of the existing complexity in our model as possible while still being able to demonstrate explicit solutions at every stage of the calculation.

The first step in this problem is to specify the electric field \( \mathbf{E} \) in the magnetofluid. We assume that the electric field is induced via transformer action by a central solenoid that passes through the center of the torus (perpendicular to its midplane) and produces a time-proportional magnetic flux. The resulting electric field within the torus is irrotational as it must be for a steady state. (We point out that the electric field is not irrotational within the central solenoid, but this does not present a contradiction since the interior of the torus is a non-simply-connected region.) Within the magnetofluid, \( \mathbf{E} \) will have the form

\[
\mathbf{E} = \frac{E_0 r_0}{r} \hat{\phi} + \nabla \Phi(r, z),
\]

where \( E_0 \) is a reference electric field at \( r = r_0 \), \( \hat{\phi} \) is a unit vector in the \( \phi \)-direction, \( r \) is the distance from the toroidal axis (\( z \)-axis), and \( \Phi \) is an additional scalar potential function yet to be determined. Substituting Eq. (7) into Eq. (4) and neglecting the \( \mathbf{v} \times \mathbf{B} \) term shows that \( \Phi \) obeys Laplace’s equation since \( r = \text{const} \) and \( \nabla \cdot \mathbf{j} = 0 \). From the boundary condition \( \mathbf{j} \cdot \hat{n} = 0 \), where \( \hat{n} \) is an outward-pointing unit vector on the wall of the torus, we see that \( \Phi \) must obey a homogeneous Neumann boundary condition, and thus is at most a constant. Hence, the electric field is simply given by \( \mathbf{E} = (E_0 r_0 / r) \hat{\phi} \) to lowest order in the square of the Hartmann number. Since the conductivity is taken to be constant, the current density is easily determined from the simplified Ohm’s law: \( \mathbf{j} = \sigma \mathbf{E} = (\sigma E_0 r_0 / r) \hat{\phi} \).

\[ \text{A. The magnetic field} \]

With the current density known, Ampère’s law now can be used to find the magnetic field \( \mathbf{B} \). Since \( \mathbf{B} \) is axisymmetric and divergence-free, we can represent it in terms of a magnetic flux function \( \chi \):

\[
\mathbf{B} = \frac{B_0 r_0}{r} \hat{\phi} + \nabla \chi \times \nabla \varphi, \quad (8)
\]

where \( B_0 \) is a reference magnetic field at \( r = r_0 \). The first term on the right hand side of Eq. (8) is a vacuum toroidal magnetic field that has no effect on establishing the properties of the equilibrium in this problem. It will, however, greatly influence the stability of that equilibrium. In addition, this toroidal magnetic field would be important in our description if tensor conductivities, anisotropic pressures and/or poloidal current densities were allowed, but we do not consider such possibilities here; see, however, Kamp et al.\(^{18} \)

Substitution of Eq. (8) into Eq. (2) yields

\[
\Delta \chi = r^2 \nabla \cdot (r^{-2} \nabla \chi) = -\sigma E_0 r_0. \quad (9)
\]

In terms of toroidal coordinates, the solution of Eq. (9) is\(^{15,19} \)

\[
\chi = -\frac{\sigma E_0 r_0^2}{2} \left[ \frac{1}{r_0^2} - \frac{a^2}{r_0^4} \right] \frac{1 - \xi^2}{1 - \xi^2} \ln \sqrt{1 - \xi^2} \quad - \frac{1 - \xi^2}{\sqrt{1 - \xi^2}} \sum_{m=0}^{\infty} b_m T_m(\xi) \cos m \eta, \quad (10)
\]

where the special functions \( T_m(\xi) \) are related to associated Legendre functions of half-integer degree: \( T_m(\xi) = Q_{m-1}(\xi^{-1}) / \sqrt{\xi} \), and the \( b_m \)’s are constant real coefficients chosen to satisfy the magnetic boundary condition \( \mathbf{B} \cdot \hat{n} = 0 \). A listing of the first ten coefficients \( b_m \) for tori of three different aspect ratios appears in Table I, and a projection of the magnetic field lines onto a plane at a fixed toroi-
TABLE I. The first ten values of $b_m/(\sigma E_0 a_0^2)$ used in the calculation of the poloidal magnetic field for three different aspect ratios. Although the coefficients $b_m$ are seen to increase with $m$, the dependence of the special functions $T_{m1}$ on $m$ is such that the series in Eq. (10) is rapidly converging.

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</table>

The velocity field

Now that the magnetic field has been determined, the final step is to find the velocity field $v$ of the magnetofluid. Returning to the “slow-flow” steady-state equation of motion (with the $v \cdot \nabla v$ term neglected), we see that $v$ must obey

$$

\nu \nabla^2 v = \nabla p - j \times B.

$$

(11)

Taking the curl of this equation gives $-\nu \nabla^2 \omega = \nabla \times (j \times B)$, where $\omega = \nabla \times v$ is the vorticity. With axisymmetry, we may introduce the azimuthal stream function $\psi$ such that $v = \nabla \psi \times \nabla \varphi$, and $\omega = -\nabla^2 (\psi/\rho)$. Note that a toroidal component of velocity $v_\varphi$ could be included here, but is not required for force balance. Moreover, application of no-slip boundary conditions on the toroidal surface shows that $v_\varphi = 0$. This leads to a “biharmonic” equation for $\psi$,

$$

\nabla \frac{\psi}{\rho} \hat{\varphi} = \frac{1}{\nu} \nabla \times (j \times B) = - \frac{2 \sigma E_0 r_0}{\rho^2} B_r \hat{\varphi},

$$

(12)

where $B_r$ is the “$r$ component” of the magnetic field, and the operator $\nabla^4$ is defined by $(\nabla^2)^2$. Satisfaction of no-slip boundary conditions requires that $\psi$ and its normal derivative $\partial \psi/\partial n$ vanish on the wall of the torus, $\varphi = \varphi_w$.

An integral solution of Eq. (12) that is consistent with no-slip boundary conditions on the velocity field can be found by using a “torsional” Green’s function $G(x|x')$ for the biharmonic operator in axisymmetric toroidal geometry. This function is defined by

$$

\nabla^4 G(x|x') \hat{\varphi} = - \frac{4\pi}{J} \delta(\xi - \xi') \delta(\eta - \eta') \hat{\varphi},

$$

(13)

where $J$ is the Jacobian of the toroidal coordinate system. The Green’s function can be constructed such that its value and the value of its normal derivative vanish on the toroidal wall: $G|_{\varphi = 0} = \partial G/\partial n|_{\varphi = 0} = 0$. It is given by

$$

G = - \frac{R_0}{2\pi \sqrt{1 - \xi \cos \eta - 1 - \xi' \cos \eta}} \sum_{m = -\infty}^{\infty} \frac{\Gamma(m + 1/2)}{\Gamma(m + 3/2)} \Gamma(m + 1/2) \Gamma(m + 3/2) \Gamma(m + 1/2) \Gamma(m + 3/2)


\times g_m(\xi, \xi') \cos m(\eta - \eta'),

$$

(14)

where

$$

g_m(\xi, \xi') = m[T_{m+1,1}(\xi) T_{m-1,1}(\xi') - T_{m-1,1}(\xi) T_{m+1,1}(\xi')]


= m[T_{m+1,1}(\xi) T_{m-1,1}(\xi') - T_{m-1,1}(\xi') T_{m+1,1}(\xi)]


- T_{m+1,1}(\xi) [S_{m+1,1}(\xi) T_{m-1,1}(\xi') - S_{m-1,1}(\xi) T_{m+1,1}(\xi')]


- T_{m+1,1}(\xi) [S_{m+1,1}(\xi') T_{m-1,1}(\xi) - S_{m-1,1}(\xi') T_{m+1,1}(\xi)]


- [T_{m+1,1}(\xi) S_{m+1,1}(\xi) - T_{m+1,1}(\xi') S_{m+1,1}(\xi)]


+ T_{m+1,1}(\xi) S_{m+1,1}(\xi) - T_{m+1,1}(\xi') S_{m+1,1}(\xi),

$$

(15)

and

$$

\gamma_m(\xi, \xi') = m[T_{m+1,1}(\xi) T_{m-1,1}(\xi') - T_{m-1,1}(\xi) T_{m+1,1}(\xi')]


- T_{m+1,1}(\xi) [S_{m+1,1}(\xi) T_{m-1,1}(\xi') - S_{m-1,1}(\xi) T_{m+1,1}(\xi')]


+ T_{m+1,1}(\xi) T_{m-1,1}(\xi) S_{m+1,1}(\xi') - T_{m+1,1}(\xi') T_{m-1,1}(\xi) S_{m+1,1}(\xi)


+ T_{m+1,1}(\xi) S_{m+1,1}(\xi) - T_{m+1,1}(\xi') S_{m+1,1}(\xi)


+ T_{m+1,1}(\xi) S_{m+1,1}(\xi) - T_{m+1,1}(\xi') S_{m+1,1}(\xi).}

$$

(16)
Here, $S_m(\xi) = P_m(\xi^{-1})/\sqrt{\xi}$, where $P$ is a Legendre function, $e_m$ is Neumann’s number (equal to 1 if $m = 0$, and 2 otherwise), primes on the $T$ and $S$ functions denote differentiation with respect to $\xi$, and $\xi_+ (\xi_-)$ is the greater (lesser) of $\xi$ and $\xi^\prime$. To obtain the $m = 0$ term in the summation, we must compute $g_{m1}$ in the limit $m \to 0$. However, due to the parity of $B_\nu$, the $m = 0$ term is seen to make no contribution in this problem.

Given the above expression for the torsional Green’s function, it can be verified that the velocity stream function $\psi$ consistent with no-slip boundary conditions at $\xi = \xi_a$ is

$$\psi = \frac{r \sigma E \sigma f_0}{4 \pi^3 \nu} \int \left( \frac{B_\nu}{r} \right) G(\mathbf{x} | \mathbf{x}') d^3 \mathbf{x},$$

(17)

where the integration is performed over the volume of the torus. A plot of the stream function contours (and the associated flow pattern), obtained numerically from Eq. (17), appears in Fig. 3 for $\xi_a = 3/11$. Note the appearance of paired convection-like cells resembling a ‘‘double smoke ring’’ configuration. A theoretical calculation to extend this analysis beyond lowest order in $M^2$ and $H^2$ has been made for the case of rectangular stress-free boundaries.¹⁸

III. CONCLUSIONS

The principal implication of the results presented in this paper is to strongly suggest that analytical discussions of toroidal MHD confinement must necessarily feature finite velocity fields from the outset, as fundamental components of the force balance. The extent to which the presence of such flows modifies the panoply of instabilities in MHD theory will require time and experience to determine. It seems quite unlikely to us, though, to expect similarity between the linear stability analysis of resistive steady states with flow and those without.

Note added in proof. Since this article was submitted, we have become aware of two related theoretical calculations which may be mentioned in this context. Following the approach first adopted in Ref. 6, Grad, Hu, and Stevens,¹¹ and Grad, Hu, Stevens, and Turkel²⁵ have studied “slowly evolving” MHD states in which the determination of true stationary states was not attempted. Zero viscosity was assumed. Rosen and Greene²³ extended an investigation in the spirit of Pfirsch-Schlüter,³ with a somewhat intricate asymptotic analysis including compressible flow with sources, supersonic flow velocities, standing shocks, boundary layers, and poloidal and toroidal rotation, but apparently without the inclusion of viscosity in the equation of motion. We can take no position with regard to a possible relation of their solutions to ours or to those offered by Grad et al. All three approaches, it can be said, at least agree on the necessary role of velocity fields in any toroidal MHD current-carrying state.

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