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Extensions to the Context Tree Weighting Method

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Abstract — We modify the basic context tree weighting method so that past symbols are not needed, and that the context tree depth is infinite. For stationary ergodic sources we now achieve entropy.

I. Introduction

The context tree weighting method [2] was first presented at the previous IEEE ISIT in San Antonio, Texas. It appears to be an efficient implementation for weighting all the coding distributions (universal over the parameters) corresponding to FSMX models, whose maximum depth does not exceed d . An FSMX model is determined by a proper and complete set S of postfixes. Together these postfixes form a tree that grows towards $-\infty$. Each sequence $\dots x_{t-2}x_{t-1}$ has a unique postfix in S , i.e. it passes through a unique leaf in the corresponding tree. This postfix (leaf) determines the probability that x_t , i.e. the next symbol generated by the (binary) source, is a 1.

The analysis of this weighting method turns out to be very straightforward (see [3]). It shows that the performance is as good as we can possibly hope, not only asymptotically but also for finite sequence lengths. Here we will propose two extensions to the basic weighting method.

II. Coding without knowledge of past symbols

In its basic form the context tree weighting method needs, for every processed symbol x_t for $t = 1, 2, \dots, T$, its context $x_{t-d}x_{t-d+1}\dots x_{t-1}$, where d is the depth of the context tree. This implies that for processing x_1 we must have access to $x_{1-d}, x_{2-d}, \dots, x_0$. If this is not the case, the straightforward way is to start processing only after having seen a full context, i.e. to start processing with x_{d+1} . The first context $x_1x_2\dots x_d$ is sent to the decoder in an uncoded way, requiring d bits.

In the ideal situation where we do know the model S , we could send a symbol x_t in an uncoded way, if its context $x_1x_2\dots x_{t-1}$ does not have a postfix in S . The number of uncoded symbols $\Delta_S(x_1^T)$ now depends on the postfix set S and on the actual sequence x_1^T , thus not knowing the past costs $\Delta_S(x_1^T)$ bits, if we do know S . Note that $\Delta_S(x_1^T) \leq d$ for models S that fit into a context tree of depth d .

Our first result is now that, also in the case where we do not know the model S , we lose not more than $\Delta_S(x_1^T)$ bits, if we do not have access to the past. In other words the *starting redundancy* is not more than $\Delta_S(x_1^T)$. This is demonstrated by modifying the context tree weighting method. The key idea is to assume that the symbols $x_{1-d}, x_{2-d}, \dots, x_0$ all have value ϵ , and to consider a ternary context tree in which weighted probabilities are updated as usual, the difference being that now every node in principle has three children.

III. Infinite depth context tree methods

A second unpleasant property of the basic context tree weighting method is that the depth d of the context tree is finite. Only for models that fit into this finite context tree, the weighting method achieves its desirable redundancy bounds.

Our second result is that we can generalize the basic context tree weighting method to the situation where the context tree

depth is *infinite*, and still achieve a storage complexity which is not larger than linear in the sequence length T . The first observation that leads to this result is that we have T contexts each starting (ending) with a (semi-infinite) sequence of ϵ 's, which therefore all differ from each other. The second observation is that the context tree need only contain non-unique nodes that split, or nodes that are unique with a non-unique parent (a node is said to be unique if the subsequence to which it corresponds does only occur once in $\dots \epsilon \epsilon x_1 x_2 \dots x_T$, a node splits if it has more than one child).

It can now be shown that our modified context tree does not contain more than T (internal) nodes. For processing x_t it is however necessary that the sequence $x_1x_2\dots x_{t-1}$ is available in memory. This results in a total storage complexity not larger than linear in T . Note that this relates to the concept DAWG proposed by Blumer et al. [1].

The performance is as expected, the model redundancy being upperbounded by $2|S| - 1$ bits, now for *all* S .

IV. Achieving entropy for arbitrary stationary ergodic sources

Now that we can use the context tree weighting algorithm without knowing past symbols and for infinite depth context trees, we can show that this method achieves entropy for arbitrary stationary and ergodic sources. Let $c(x_1^T, S) = \frac{|S|}{2} \log_2 \frac{T}{|S|} + |S| + 2|S| - 1 + \Delta_S(x_1^T) + 2$, i.e. the upper bound for the total parameter, model, starting and coding redundancy for large T . Take $S = \{0, 1\}^d$ for arbitrary d , then

$$\begin{aligned} L(x_1^T) &\leq \sum_{s \in S} (a_s \log_2 \frac{a_s + b_s}{a_s} + b_s \log_2 \frac{a_s + b_s}{b_s}) + c(x_1^T, S) \\ &\leq \sum_{s \in S} (a_s \log_2 \frac{1}{1 - \pi_s} + b_s \log_2 \frac{1}{\pi_s}) + c(x_1^T, S), \end{aligned}$$

where a_s and b_s are the number of occurrences of $s0$ resp. $s1$ in x_1^T , and $\pi_s = P_a(X_{d+1} = 1 | X_1^d = s)$, i.e. the actual probability for $s \in S$. Using the ergodic theorem we obtain (entropy in bits) that

$$L(x_1^T)/T \leq H(X_{d+1} | X_1 \dots X_d) \text{ with probability 1}$$

for $T \rightarrow \infty$ for all d . Since $\lim_{d \rightarrow \infty} H(X_{d+1} | X_1 \dots X_d) = H_\infty(X)$ it follows, that for stationary ergodic sources the codeword-length $L(x_1^T)$ divided by the sequence length is, with probability one, not larger than the entropy of the source.

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