Behavioral controllability for convolution systems --
with an application to time-delay systems

by

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Abstract

In this paper we study controllability of autoregressive convolution systems in a behavioral framework. We show that right-invertibility of the system representation matrix over the convolution algebra of Schwartz distributions with compact support is a sufficient condition for controllability. Furthermore, a pointwise full row rank condition on the Fourier transform of the representation matrix is necessary for controllability. In general, these conditions are not equivalent. However, for a large class of time-delay systems, satisfying a monicity condition on one of the minors of the representation matrix, a modification of the Corona Theorem is used to prove that in this situation both conditions are equivalent. This result is applied to obtain a generalization of the Hautus-test for time-delay systems in pseudo-state form, containing both distributed- and (in)commensurable point-delays.

Keywords: Convolution systems, time-delay systems, behavioral systems, behavioral controllability, matrices of distributions, Liouville numbers.
1 Introduction

In the behavioral approach to dynamical systems, introduced by J.C. Willems (see e.g. [19], [20], [14]), a system is described by a triple \((T, W, B)\). Here \(T\) is the time-axis, \(W\) the space in which the signals take their values, and \(B\)—the behavior—is a subset of the signal space \(W^T\): it consists of all time-trajectories satisfying the laws governing the system.

In the behavioral setup, no a priori distinction is made between signals serving as input or output. Instead, all signals are treated in the same way. From a classical point of view, this abandoning of the notions of input and output seems a problem for the definition of controllability. For time-invariant systems, Willems solved this issue in [19], [20] by presenting an alternative definition of behavioral controllability.

Definition 1.1 ([14, p. 153]) Let \(B\) be the behavior of a time-invariant dynamical system. This system is called behaviorally controllable if for any two trajectories \(w_1, w_2 \in B\) there exist a \(t_1 \geq 0\) and a trajectory \(w \in B\) with the property

\[
    w(t) = \begin{cases} 
        w_1(t) & t \leq 0, \\
        w_2(t - t_1) & t \geq t_1.
    \end{cases}
\]

(1)

Definition 1.1 shows that behavioral controllability is not dependent on the specific choice of inputs and outputs, but is an intrinsic property of the behavior of a system.

For autoregressive differential systems, i.e. for systems that can be described as the kernel of a polynomial matrix \(P\) in the differentiation operator \(\frac{d}{dt}\), there exist easily verifiable conditions to check behavioral controllability, that are very similar to the celebrated Hautus-test ([10]) for state-space systems.

Theorem 1.2 Let \(P \in \mathbb{R}[s]^{n \times m}\) and assume that \(P\) has full row rank. Define the behavior \(B\) of the autoregressive differential system corresponding to \(P\) by

\[
    B(P) = \{ w \in C^\infty(\mathbb{R})^m \mid P(\frac{d}{dt})w = 0 \},
\]

i.e. \(B(P)\) consists of all smooth solutions of the \(n\)-tuple of differential equations in \(m\) variables determined by \(P\). Then the following statements are equivalent:

(i) \(B(P)\) is behaviorally controllable.

(ii) \(B(P)\) has an image representation, i.e. there exist \(q \in \mathbb{N}\) and \(Q \in \mathbb{R}[s]^{m \times q}\) such that

\[
    B(P) = \text{Range} \left( Q(\frac{d}{dt}) \right) = \{ Q(\frac{d}{dt})v \mid v \in C^\infty(\mathbb{R})^q \}.
\]

(iii) \(P\) is right-invertible over \(\mathbb{R}[s]\).

(iv) \(\forall \lambda \in \mathbb{C} : \text{rank}(P(\lambda)) = n\).

Theorem 1.2 is also valid in more general settings. For example, the assumption that \(P\) has full row rank may be omitted (provided that conditions (iii) and (iv) are changed accordingly), and instead of considering smooth solutions only, one may also include weak solutions (i.e. solutions in \(L^1_{\text{loc}}\), see [14, p. 34]) in the behavior. For that purpose, the equation \(P(\frac{d}{dt})w = 0\) should be given a distributional interpretation.

In the literature (see e.g. [19], [14, pp. 154-155 and pp. 229-230]), the proof of Theorem 1.2 is based on the Smith form of the matrix \(P\) over \(\mathbb{R}[s]\). Therefore it seems that the fact
that the polynomial ring $\mathbb{R}[s]$ is an elementary divisor ring is of crucial importance for the validity of Theorem 1.2. This observation was used in [7] to generalize the rank condition for controllability to delay-differential systems with commensurable delays. In a different way, the same result was also obtained in [16]; there the fact that behavioral controllability is equivalent with the existence of an image representation plays an important role.

In this paper we propose an alternative scheme to prove Theorem 1.2 that does not depend on the construction of a Smith form. In this way it is possible to extend the result to more general types of dynamics, described by matrices over a ring $\mathcal{R}$ more general than $\mathbb{R}[s]$. It is not necessary that $\mathcal{R}$ is an elementary divisor ring; instead other properties are required: the ring $\mathcal{R}$ should be an integral domain, equivalence of system representations over $\mathcal{R}$ should be related to division properties over $\mathcal{R}$, and, most importantly, the ring $\mathcal{R}$ should allow for a type of Nullstellensatz or Corona Theorem: there should be a relationship between the ideal generated by a finite number of elements from the ring $\mathcal{R}$ and the zeros of these elements. The last requirement is the most restrictive one; in many interesting situations this condition is not fulfilled, and additional requirements are needed to derive a useful result.

Although we believe that our approach to behavioral controllability may be successful in other situations, the goal of this paper is to find a generalization of Theorem 1.2 for delay-differential systems with incommensurable delays. For this we first consider a far more general type of linear time-invariant continuous-time systems: the class of autoregressive convolution systems as introduced in [6]. The behaviors of these systems are described by the kernel of a set of convolution equations. Alternatively, they may be considered as systems over the convolution algebra $\mathcal{E}'(\mathbb{R})$ of Schwartz distributions with compact support. Linear systems with point- and distributed time-delays may be considered as AR-convolution systems. In Section 2 we start with a short introduction to this class of systems.

In Section 3 we study behavioral controllability for AR-convolution systems. Unfortunately it is not possible to extend the results of Theorem 1.2 completely to this far more general class of systems. Instead we will find a necessary and a sufficient condition for behavioral controllability, but these conditions are not equivalent in general. The bottleneck is the fact that for the ring $\mathcal{E}'(\mathbb{R})$ and the corresponding ring of Fourier transforms the Corona Theorem (which may be considered as a Nullstellensatz for rings of analytic functions) is not valid. In Section 4 we will provide a counterexample that proves this claim. Fortunately, in the literature a modification of the Corona Theorem for the ring $\mathcal{E}'(\mathbb{R})$ is available, due to Hörmander (see [11]). An explicit description of this result is given in Section 4, indicating that in several interesting situations, the necessary and the sufficient condition for controllability, derived in Section 3, are equivalent.

In Section 5 we return to the question of controllability for delay-differential systems with (in)commensurable point delays. If these systems satisfy an additional condition on the structure of the system defining equations, it is possible to combine the results of Section 3 and Section 4 to obtain a complete generalization of Theorem 1.2. The corresponding method to verify controllability is very similar to the Hautus-test. Some of the computational issues that are involved in the actual application of this test are discussed. Furthermore, the result is extended to a class of delay-differential systems containing both (in)commensurable point- and distributed time-delays. This may be considered as a generalization of the commensurable delays case, treated in [7] and [16]. Note however that the results in [7] and [16] hold for any delay-differential system with commensurable delays, i.e. without an additional requirement on the structure of the system defining equations.

Our result may be applied to the class of time-delay systems with (in)commensurable
point delays, written in pseudo-state form,

\[ \dot{x}(t) = \sum_{j=1}^{N} A_j x(t-h_j) + B_j u(t-h_j), \]

with \( A_j \in \mathbb{R}^{n \times n}, \ B_j \in \mathbb{R}^{n \times m}, \ (j = 1, \ldots, N), \) and arbitrary time-delays \( h_1, \ldots, h_N \geq 0 \) (i.e. \( h_1, \ldots, h_N \) are not necessarily commensurable). For equations of the form (2) it is possible to generalize Theorem 1.2 completely, thus showing that in this situation behavioral controllability is equivalent with \textit{spectral controllability}, see e.g. [2], [13], and [12]. This observation for delay systems in pseudo-state form provides new insight in the solution of a classical question of controllability: when is it possible to steer system (2) from an arbitrary pseudo-state \( x_1 \) to any other pseudo-state \( x_2 \)? Provided that the pseudo-state \( x_1 \) has a history compatible with (2) and pseudo-state \( x_2 \) admits a future, behavioral controllability (and thus spectral controllability) is a sufficient condition for the solution of this problem. This stresses once more the fundamental importance of the notion of spectral controllability, that may be regarded as a generalization of the Hautus-test to time-delay systems in pseudo-state form.

2 AR-convolution systems

In this section we give a short introduction to autoregressive convolution systems. This class of systems was originally defined in [6]. For an exhaustive treatment we therefore refer to this more detailed paper.

Let \( \mathcal{E}(\mathbb{R}) \) denote the Fréchet space of all infinitely differentiable complex-valued functions on \( \mathbb{R} \), under the topology of compact convergence in all derivatives. So a sequence \( (f_n)_{n \in \mathbb{N}} \) in \( \mathcal{E}(\mathbb{R}) \) converges to \( f \in \mathcal{E}(\mathbb{R}) \) if \( (f_n) \) converges uniformly to \( f \) on every compact subset \( \Omega \subseteq \mathbb{R} \), and the same holds for all derivatives of \( f_n \) and \( f \), respectively. Throughout this paper we assume the signal space to be \( \mathcal{E}(\mathbb{R})^m \), where \( m \) denotes the number of variables involved. For \( t \in \mathbb{R} \) we denote by \( \sigma_t \) the \textit{t-shift operator} on \( \mathcal{E}(\mathbb{R})^m \):

\[ \sigma_t : \mathcal{E}(\mathbb{R})^m \to \mathcal{E}(\mathbb{R})^m : \quad (\sigma_t f)(s) = f(s + t). \]

Without causing any confusion, the same notation \( \sigma_t \) is used simultaneously for different values of \( m \). A continuous linear operator \( L : \mathcal{E}(\mathbb{R})^m \to \mathcal{E}(\mathbb{R})^n \) is called \textit{shift-invariant} if \( L \circ \sigma_t = \sigma_t \circ L \) for all \( t \in \mathbb{R} \).

\textbf{Definition 2.1} An \textit{autoregressive convolution system} in \( \mathcal{E}(\mathbb{R})^m \) is a system whose behavior is described by the kernel of a continuous linear shift-invariant operator \( L : \mathcal{E}(\mathbb{R})^m \to \mathcal{E}(\mathbb{R})^n \); the behavior consists of all \( w \in \mathcal{E}(\mathbb{R})^m \) for which \( Lw = 0 \).

Note that Definition 2.1 automatically implies that an AR-convolution system is time-invariant. If \( w \in \mathcal{E}(\mathbb{R})^m \) satisfies \( Lw = 0 \), and \( t \in \mathbb{R} \), then also \( L(\sigma_t w) = (L \circ \sigma_t)w = (\sigma_t \circ L)w = 0 \). In fact, the class of AR-convolution systems is one of the most general classes of linear time-invariant systems with signal space \( \mathcal{E}(\mathbb{R})^m \).

\textbf{Example 2.2} Consider a system with \( m \) variables \( w_1, \ldots, w_m \), described by a set of \( n \) equations of the form

\[ \sum_{i=1}^{m} \sum_{j=1}^{N} \int_{a}^{b} g_{ij}(\tau) w^{(j)}_i(t-\tau) \, d\tau + \sum_{i=1}^{m} \sum_{j=1}^{N} \sum_{k=1}^{k} h_{ij} \tau_k w^{(j)}_i(t-\tau_k) = 0, \]
where \( a, b \in \mathbb{R}, \tau_1, \ldots, \tau_k \in \mathbb{R} \), and for all \( i = 1, \ldots, m, j = 1, \ldots, N \) and \( \ell = 1, \ldots, k \): \( g_{ij} \in L^1[a, b] \) and \( h_{ij\ell} \in \mathbb{R} \). This set of equations is an AR-representation of a differential-difference system with point- and distributed time-delays. If only solutions \( w \) in the signal space \( \mathcal{E}(\mathbb{R})^m \) are considered, formula (4) describes the behavior of an AR-convolution system. The continuous linear shift-invariant operator \( L \) characterizing the behavior is determined by the left-hand side of equation (4).

In the algebraic terminology used in [9], an AR-convolution system is regarded as a system over the convolution algebra \( \mathcal{E}'(\mathbb{R}) \). Here \( \mathcal{E}'(\mathbb{R}) \) denotes the dual space of \( \mathcal{E}(\mathbb{R}) \) (i.e. the space of all continuous linear functionals from \( \mathcal{E}(\mathbb{R}) \) to \( \mathbb{C} \)), consisting of all Schwartz distributions on \( \mathbb{R} \) with compact support. To clarify the relationship between \( \mathcal{E}'(\mathbb{R}) \) and AR-convolution systems, we need the following definition.

**Definition 2.3** For every \( F \in \mathcal{E}'(\mathbb{R}) \) the convolution operator \( \sigma[F] \) corresponding to \( F \) is defined as

\[
\sigma[F] : \mathcal{E}(\mathbb{R}) \longrightarrow \mathcal{E}(\mathbb{R}) : (\sigma[F]x)(t) = F(x(t)).
\]

(5)

Stated differently, \( \sigma[F] \) is the operator that maps \( w \in \mathcal{E}(\mathbb{R}) \) to the convolution of \( \hat{F} \) and \( w \), where \( \hat{F} \) denotes the reflection of \( F \), (so \( \hat{F}(s) = F(-s) \) if \( F \) is a function). Usually, this convolution is denoted by \( F \ast w \), but with the present notation, the introduction of \( \hat{F} \) is avoided. Note that for \( w \in \mathcal{E}(\mathbb{R}) \) the function \( \sigma[F]w \) is again an element of \( \mathcal{E}(\mathbb{R}) \) (see e.g. [15, p. 63]).

Next, we extend Definition 2.3 to the multivariable case. For \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \), the convolution operator \( \sigma[P] : \mathcal{E}(\mathbb{R})^m \longrightarrow \mathcal{E}(\mathbb{R})^n \) maps \( w \in \mathcal{E}(\mathbb{R})^m \) to \( y = \sigma[P]w \in \mathcal{E}(\mathbb{R})^n \) with

\[
y_i = \sum_{j=1}^m \sigma[P_{ij}]w_j, \quad (i = 1, \ldots, n).
\]

Stated differently, \( P \) maps \( w \in \mathcal{E}(\mathbb{R})^m \) to \( v \in \mathbb{C}^n \) with \( v_i = \sum_{j=1}^m P_{ij}w_j \), so \( y = \sigma[P]w \) satisfies \( y(t) = (\sigma[P]w)(t) = P(\sigma_t w) \).

It is obvious that every convolution operator is continuous, linear and shift-invariant. The next theorem states that also the opposite is true.

**Theorem 2.4** ([6], [15]) \( L : \mathcal{E}(\mathbb{R})^m \longrightarrow \mathcal{E}(\mathbb{R})^n \) is a continuous linear shift-invariant operator if and only if there exists a matrix \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \) such that \( L = \sigma[P] \).

According to Theorem 2.4, every AR-convolution system may be represented by a matrix \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \). The corresponding behavior is given by

\[
B(P) := \ker(\sigma[P]) = \{ w \in \mathcal{E}(\mathbb{R})^m \mid \sigma[P]w = 0 \}.
\]

(6)

It is clear that formula (6) uniquely determines the behavior represented by the matrix \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \). Note however that the opposite is not true: the same behavior may be characterized by different representation matrices.

Let \( (\mathcal{A}, +, \circ) \) denote the algebra of all continuous linear shift-invariant operators from \( \mathcal{E}(\mathbb{R}) \) to \( \mathcal{E}(\mathbb{R}) \), with respect to addition and composition of maps. According to Theorem 2.4, the mapping \( \sigma : \mathcal{E}'(\mathbb{R}) \longrightarrow \mathcal{A} : \sigma(F) = \sigma[F] \) is bijective. In fact, this mapping becomes an algebra isomorphism between \( (\mathcal{A}, +, \circ) \) and the convolution algebra \( (\mathcal{E}'(\mathbb{R}), +, \ast) \) if we define the convolution product of two elements of \( \mathcal{E}'(\mathbb{R}) \) in the following way.
Definition 2.5 Let \( F_1, F_2 \in \mathcal{E}'(\mathbb{R}) \), and let \( H \in \mathcal{E}'(\mathbb{R}) \) be the distribution such that
\[
\sigma[F_1] \circ \sigma[F_2] = \sigma[H].
\]
Then \( H \) is called the **convolution product** of \( F_1 \) and \( F_2 \), and is denoted by \( H = F_1 \ast F_2 \).

Definition 2.5 describes an alternative way to introduce the convolution product on \( \mathcal{E}'(\mathbb{R}) \), originally defined in [18, Chapter VI]. Note that the present definition is easily generalized to the multivariable case: for \( P_1 \in \mathcal{E}'(\mathbb{R})^{n \times m} \) and \( P_2 \in \mathcal{E}'(\mathbb{R})^{m \times p} \) we have \( \sigma[P_1] \circ \sigma[P_2] = \sigma[P_1 \ast P_2] \).

An alternative description of the action of a convolution operator on \( \mathcal{E}(\mathbb{R}) \) may be given in the frequency domain. For this purpose we define the Fourier transform of an element \( F \in \mathcal{E}'(\mathbb{R}) \) as follows.

**Definition 2.6** For \( \omega \in \mathbb{C} \), let \( e^{\omega t} \) denote the exponential function \( e^{\omega t} = e^{-i\omega t} \). Let \( F \in \mathcal{E}'(\mathbb{R}) \), and define the complex function \( FF \) by
\[
(FF)(\omega) := F(e^{\omega}).
\]
Then \( FF \) is called the **Fourier transform** of \( F \).

Again, generalization of Definition 2.6 to the multivariable case is straightforward. For \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \) we define \( FP \) by \( (FP)_{ij} = F(P_{ij}) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

The Fourier transform of \( F \in \mathcal{E}'(\mathbb{R}) \) is a holomorphic function of \( \omega \) (see e.g. [15, p. 65]), and can be regarded as the spectral description of the operator \( \sigma[F] \) corresponding to \( F \). Let \( \omega \in \mathbb{C} \), then for every \( t \in \mathbb{R} \):
\[
(\sigma[F]e^{\omega})(t) = F(\sigma_t e^{\omega}) = e^{-i\omega t} F(e^{\omega}) = (FF)(\omega) \cdot e^{\omega}(t),
\]
and thus \( \sigma[F]e^{\omega} = (FF)(\omega)e^{\omega} \). We conclude that for every \( \omega \in \mathbb{C} \), \( e^{\omega} \) is an eigenfunction of \( \sigma[F] \) with spectral value \((FF)(\omega)\). In particular, if \((FF)(\omega) = 0\), then \( e^{\omega} \in B(F) \).

For \( F, G \in \mathcal{E}'(\mathbb{R}) \) it is obvious that \( F(F + G) = FF + FG \). A similar result is valid for the convolution \( F \ast G \) and the product of \( FF \) and \( FG \): \( FF \ast FG \) (see e.g. [6, Lemma 2.9]). Therefore the Fourier transformation is a homomorphism from the convolution algebra \( (\mathcal{E}'(\mathbb{R}), +, \ast) \) to the function algebra \( H(\mathbb{C}) \) of holomorphic functions on the complex plane. However, not every holomorphic function is the Fourier transform of an element of \( \mathcal{E}'(\mathbb{R}) \). The functions \( FF \) with \( F \in \mathcal{E}'(\mathbb{R}) \) satisfy a growth condition, specified in the next definition.

**Definition 2.7** The **Paley-Wiener algebra** \( PW(\mathbb{C}) \) consists of all functions \( f \in H(\mathbb{C}) \), for which there exist \( C, a > 0 \) and \( N \in \mathbb{N} \cup \{0\} \), all depending on \( f \), such that for all \( \omega \in \mathbb{C} \):
\[
|f(\omega)| \leq C \cdot (1 + |\omega|)^N e^{a|\text{Im}\omega|}.
\]
Stated differently: \( PW(\mathbb{C}) \) consists of all holomorphic functions of exponential type, that are polynomially bounded on the real axis.

The Paley-Wiener algebra describes exactly those holomorphic functions that are Fourier transforms of elements of \( \mathcal{E}'(\mathbb{R}) \).

**Theorem 2.8** (Paley-Wiener-Schwartz) For all \( f \in H(\mathbb{C}) \):
\[
f \in PW(\mathbb{C}) \iff (\exists F \in \mathcal{E}'(\mathbb{R}) : FF = f).
\]
For a proof of this fundamental result we refer to the classical sources [18, p. 271] or [4, p. 156]. The theorem implies that the algebra homomorphism \( F : \mathcal{E}'(\mathbb{R}) \to PW(\mathbb{C}) \) is in fact an isomorphism. Since the Paley-Wiener algebra \((PW(\mathbb{C}), +, \cdot)\) is a commutative function algebra, this implies that also the operator algebra \((\mathcal{A}, +, \circ)\) and the convolution algebra \((\mathcal{E}'(\mathbb{R}), +, \ast)\) are commutative. In this paper on convolution systems, we may therefore switch arbitrarily between the algebras \(\mathcal{A}, \mathcal{E}'(\mathbb{R}),\) and \(PW(\mathbb{C}).\) So we are free to use the characterization that is most appropriate for solving the problem at hand.

We end this section with an example to illustrate the transformations between the different algebras.

**Example 2.9** Let \( \tau \in \mathbb{R}, \) and consider the distribution \( F = \delta_{\tau} - \delta'_0 \in \mathcal{E}'(\mathbb{R}). \) \( F \) represents the continuous linear functional on \( \mathcal{E}(\mathbb{R}) \) given by

\[
F : \mathcal{E}(\mathbb{R}) \to \mathbb{C} : \quad F(x) = x(\tau) + \dot{x}(0).
\]

The corresponding convolution operator \( \sigma[F] \) is a delay-differential operator on \( \mathcal{E}(\mathbb{R}). \) For \( x \in \mathcal{E}(\mathbb{R}) \) and \( t \in \mathbb{R} \) we have

\[(\sigma[F]x)(t) = F(\sigma_t x) = x(t + \tau) + \dot{x}(t),\]

so

\[\sigma[F] : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}) : \quad \sigma[F]x = \sigma_t x + \dot{x},\]

and thus \( \sigma[F] = \sigma_t + \frac{d}{dt}. \) The Fourier transform of \( F \) is given by

\[\mathcal{F}F : \mathbb{C} \to \mathbb{C} : \quad (\mathcal{F}F)(\omega) = F(e^{i\omega}) = e^{-i\omega\tau} - i\omega.\]

### 3 A necessary and a sufficient condition for controllability

In this section we study behavioral controllability for AR-convolution systems. Our goal is to extend Theorem 1.2 to this large class of linear time-invariant systems. Unfortunately, a complete generalization of this result can not be obtained.

**Theorem 3.1** Let \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \) and assume that the convolution operator \( \sigma[P] : \mathcal{E}(\mathbb{R})^{m} \to \mathcal{E}(\mathbb{R})^{n} \) is surjective. Consider the following four statements:

(i) \( B(P) \) is behaviorally controllable,

(ii) \( B(P) \) has an image representation, i.e. there exist \( q \in \mathbb{N} \) and \( Q \in \mathcal{E}'(\mathbb{R})^{m \times q} \) such that

\[B(P) = \text{Range} (\sigma[Q]) = \{\sigma[Q]v \mid v \in \mathcal{E}(\mathbb{R})^{q}\},\]

(iii) \( P \) is right-invertible over \( \mathcal{E}'(\mathbb{R}), \)

(iv) \( \forall \lambda \in \mathbb{C} : \text{rank}\left(\mathcal{F}P(\lambda)\right) = n. \)

Then (iii) \( \implies \) (ii) \( \implies \) (i) \( \implies \) (iv).

Note that the implication (iv) \( \implies \) (iii) does not hold in general. So Theorem 3.1 only provides a necessary and a sufficient condition for behavioral controllability. In this section we focus on the proof of Theorem 3.1. The rest of the paper is mainly devoted to the
implication (iv) \(\implies\) (iii). For the proof of this implication a Corona Theorem for the Paley-Wiener algebra \(PW(C)\) would be needed, but in the literature it is known (see [11]) that such a result does not exist. To overcome this difficulty we have to impose additional requirements on the structure of the system representation matrix \(P\).

The condition under which Theorem 3.1 is valid —surjectivity of the convolution operator \(\sigma[P]—\) is not very restrictive. Moreover, this condition is only required in the proof of (i) \(\implies\) (iv). In most practical situations, e.g. for delay-differential systems with (in)commensurable point delays, a convolution operator \(\sigma[P]\) is surjective if the matrix \(P\) over \(\mathcal{E}'(\mathbb{R})\) has full row rank. This rank condition is a standing assumption throughout the paper; otherwise the statements (iii) and (iv) are violated anyway. For a detailed discussion of the surjectivity of convolution operators we refer to [6].

Next we will prove the three implication statements of Theorem 3.1. Moreover, the exact conditions that are needed in these proofs are specified explicitly.

**Lemma 3.2** Let \(P \in \mathcal{E}'(\mathbb{R})^{n \times m}\) and assume that \(P\) is right-invertible over \(\mathcal{E}'(\mathbb{R})\). Then there exists a matrix \(Q \in \mathcal{E}'(\mathbb{R})^{m \times m}\) such that \((\sigma[P]) = \text{Range}(\sigma[Q])\).

**Proof:** Let \(R \in \mathcal{E}'(\mathbb{R})^{m \times n}\) be a right-inverse of \(P\), and define \(Q \in \mathcal{E}'(\mathbb{R})^{m \times m}\) by \(Q = I_m - RP\). We show that \(\text{Range}(\sigma[Q]) = B(P)\).

Let \(w \in B(P)\). Then \(\sigma[P]w = \sigma[I - R]w = \sigma - \sigma[R]\sigma[P]w = w\). Hence \(w = \sigma[Q]w \in \text{Range}(\sigma[Q])\).

Conversely, if \(w \in \text{Range}(\sigma[Q])\), then there exists \(v \in \mathcal{E}(\mathbb{R})^m\) such that \(w = \sigma[Q]v\), and we have

\[\sigma[P]w = \sigma[P]\sigma[Q]v = \sigma[P]((I - \sigma[R])\sigma[P])v = \sigma[P]v - \sigma[P]\sigma[R]\sigma[P]v = 0.\]

Note that if \(R \in \mathcal{E}'(\mathbb{R})^{m \times n}\) is a right-inverse of \(P \in \mathcal{E}'(\mathbb{R})^{n \times m}\), then the convolution operator \(\sigma[Q]\), with \(Q = I_m - RP\), is a projection of \(\mathcal{E}(\mathbb{R})^m\) onto \(B(P)\). Indeed, \(\sigma[Q]^2 = \sigma[Q]\) and we have just shown above that \(\text{Range}(\sigma[Q]) = B(P)\).

Next we prove that any behavior that allows an image representation is behaviorally controllable.

**Lemma 3.3** For any \(Q \in \mathcal{E}'(\mathbb{R})^{m \times q}\), the behavior \(\text{Range}(\sigma[Q]) = \{\sigma[Q]v \mid v \in \mathcal{E}(\mathbb{R})^q\}\) is behaviorally controllable.

**Proof:** Let \(w_1, w_2 \in \text{Range}(\sigma[Q])\). Then there exist \(y_1, y_2 \in \mathcal{E}(\mathbb{R})^q\) such that \(w_1 = \sigma[Q]y_1\) and \(w_2 = \sigma[Q]y_2\). Since \(Q\) is a matrix over \(\mathcal{E}'(\mathbb{R})\), the supports of all entries of \(Q\) are contained in a compact interval \([a, b]\). Choose \(t_1 > b - a\) and define \(z \in \mathcal{E}(\mathbb{R})^q\) by \(z = \sigma_{-t_1}y_2\). Next, choose \(y \in \mathcal{E}(\mathbb{R})^q\) such that

\[y(\tau) = \begin{cases} y_1(\tau) & \text{for } \tau \in (-\infty, b], \\ z(\tau) & \text{for } \tau \in [t_1 + a, \infty) \end{cases}\]

This is possible because \(t_1 + a > b\); take for example \(y = (1 - \varphi)y_1 + \varphi z\), where \(\varphi \in \mathcal{E}(\mathbb{R})\) is a sigmoidal function with the property that \(\varphi(\tau) = 0\) for \(\tau \leq b\) and \(\varphi(\tau) = 1\) for \(\tau \geq t_1 + a\). Now consider \(w = \sigma[Q]y \in \text{Range}(\sigma[Q])\). For \(t \leq 0\) we have

\[w(t) = Q(\sigma_{t}y) = Q(\sigma_{t}y_1) = (\sigma[Q]y_1)(t) = w_1(t),\]
and for $t \geq t_1$:
\[
w(t) = Q(\sigma_t y) = Q(\sigma_t z) = Q(\sigma_{t-t_1} y_2) = (\sigma[Q]y_2)(t-t_1) = w_2(t-t_1).
\]

Hence Range ($\sigma[Q]$) is behaviorally controllable. 

In the proof of the implication (i) $\implies$ (iv) we need the following result on the relationship between behavioral inclusion of convolution systems on the one hand, and division properties among the representation matrices on the other.

**Theorem 3.4** ([6]) Let $P \in E'(\mathbb{R})^{n \times m}$ and assume that the convolution operator $\sigma[P] : E(\mathbb{R})^m \rightarrow E(\mathbb{R})^n$ is surjective. Then for all $Q \in E'(\mathbb{R})^{q \times m}$ we have
\[
B(P) \subset B(Q)
\]
\[\iff \quad \text{There exists a unique } T \in E'(\mathbb{R})^{q \times n} \text{ such that } Q = T * P.\]

**Lemma 3.5** Let $P \in E'(\mathbb{R})^{n \times m}$ be such that the corresponding convolution operator $\sigma[P] : E(\mathbb{R})^m \rightarrow E(\mathbb{R})^n$ is surjective. If $B(P)$ is behaviorally controllable, then
\[
\forall \lambda \in \mathbb{C} : \text{rank}(FP)(\lambda) = n.
\]

**Proof:** (by contradiction) Let $B(P)$ be behaviorally controllable, and assume that there exists $\lambda \in \mathbb{C}$ such that $\text{rank}(FP)(\lambda) < n$. Then there is a vector $v \in \mathbb{C}^n \setminus \{0\}$ such that $v^T(FP)(\lambda) = 0$. So $\lambda$ is a zero of the vector $v^TFP \in PW(\mathbb{C})^{1 \times m}$, and there exists a $g \in PW(\mathbb{C})^{1 \times m}$ such that
\[
v^T(FP)(\omega) = (\omega - \lambda)g(\omega).
\]

Let $G \in E'(\mathbb{R})^{1 \times m}$ be the matrix of distributions corresponding to $g$, i.e. $FG = g$. We first show that $B(P) \subset B(G)$.

Let $w$ be an arbitrary trajectory in $B(P)$, and define $y = \sigma[G]w$. Since $(\omega - \lambda) = F(-i\delta_0 - \lambda \delta_0)(\omega)$ and $i(-i\delta_0 - \lambda \delta_0) = (i \frac{d}{dt} - \lambda)$, we have $v^T\sigma[P] = (i \frac{d}{dt} - \lambda)\sigma[G]$, and thus
\[
(i \frac{d}{dt} - \lambda)y = (i \frac{d}{dt} - \lambda)\sigma[G]w = v^T\sigma[P]w = 0.
\]

So, for any $w \in B(P)$, the corresponding trajectory $y = \sigma[G]w$ satisfies
\[
y(t) = e^{-i\lambda t}y(0). \quad (11)
\]

Next, let $w \in B(P)$. We will show that $y = \sigma[G]w = 0$. Fix $t_0 \in \mathbb{R}$. Since $G$ is a matrix over $E'(\mathbb{R})$, the supports of the entries of $G$ are contained in a fixed compact interval $[a, b]$. By assumption, $B(P)$ is behaviorally controllable, so there exist $t_1 > 0$ and $w_1 \in B(P)$ such that
\[
w_1(\tau) = \begin{cases} w(\tau) & \text{for } \tau \in (-\infty, t_0 + b], \\
0 & \text{for } \tau \in [t_0 + b + t_1, \infty). \end{cases}
\]

Define $y_1 := \sigma[G]w_1$. Then $y_1(t_0) = G(\sigma_{t_0}w_1) = G(\sigma_{t_0}w) = y(t_0)$, and for $t > t_0 + b + t_1 - a$:
\[
y_1(t) = G(\sigma_{t}w_1) = G(\sigma_{t}0) = 0.
\]

Now $w_1 \in B(P)$ implies $y_1(t) = e^{-i\lambda t}y_1(0)$ for all $t \in \mathbb{R}$, whence $y_1 \equiv 0$. We conclude that $y(t_0) = 0$. 

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Next we apply Theorem 3.4. Since $B(P) \subset B(G)$, and $\sigma[P]$ is surjective, there exists a unique $T \in \mathcal{E}'(\mathbb{R})^{1 \times n}$ such that $G = T \ast P$. In combination with (10) this yields the following equality in the frequency domain:

$$v^T \mathcal{F}P(\omega) = (\omega - \lambda)(\mathcal{F}G)(\omega) = (\omega - \lambda)(\mathcal{F}T)(\omega)(\mathcal{F}P)(\omega). \quad (12)$$

By assumption, $\sigma[P]$ is surjective, hence $P$ has full row rank. Therefore (12) implies $v^T = (\omega - \lambda)(\mathcal{F}T)(\omega)$, which contradicts that $v \in \mathbb{C}^n$ is a fixed nonzero vector.

Combination of Lemmas 3.2, 3.3, and 3.5 yields a proof of Theorem 3.1. We conclude that right-invertibility of the representation matrix $P$ over $\mathcal{E}'(\mathbb{R})$ is a sufficient condition for controllability, whereas a pointwise full rank condition on its Fourier transform is a necessary one. This last condition may be considered as a generalization of the Hautus-test to AR-convolution systems. Unfortunately, in this more general setting the Hautus-test is not proven to be a sufficient condition for controllability any longer.

4 A result on generating ideals for the ring $PW(\mathbb{C})$

The conditions (iii) and (iv) for behavioral controllability, as they are formulated in Theorem 3.1, are requirements on the representation matrix $P$ over $\mathcal{E}'(\mathbb{R})$ and its Fourier transform $\mathcal{F}P \in PW(\mathbb{C})^{n \times m}$. To study the relationship between both conditions, we first transform them into conditions on ideals in the rings $\mathcal{E}'(\mathbb{R})$ and $PW(\mathbb{C})$, respectively.

**Lemma 4.1** Let $P \in \mathcal{E}'(\mathbb{R})^{n \times m}$ and assume that $P$ is of full row rank. Denote the $n \times n$ minors of $P$ by $F_1, \ldots, F_N$. Then the following statements are equivalent:

(i) $P$ is right-invertible over $\mathcal{E}'(\mathbb{R})$.

(ii) The ideal in $\mathcal{E}'(\mathbb{R})$ generated by $F_1, \ldots, F_N$ is $\mathcal{E}'(\mathbb{R})$, i.e.

$$(F_1, \ldots, F_N)_{\mathcal{E}'(\mathbb{R})} = \mathcal{E}'(\mathbb{R}).$$

(iii) The ideal in $PW(\mathbb{C})$ generated by $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ is $PW(\mathbb{C})$, i.e.

$$(\mathcal{F}F_1, \ldots, \mathcal{F}F_N)_{PW(\mathbb{C})} = PW(\mathbb{C}).$$

The proof of Lemma 4.1 is straightforward (cp. [8, Lemma 3.5]); it is based on Cramer's rule and application of the Binet-Cauchy formula.

**Lemma 4.2** Let $P \in \mathcal{E}'(\mathbb{R})^{n \times m}$, and assume that $P$ has full row rank. Denote the $n \times n$ minors of $P$ by $F_1, \ldots, F_N$. Then the following statements are equivalent:

(i) $\forall \lambda \in \mathbb{C} : \text{rank}(\mathcal{F}P)(\lambda) = n$,

(ii) $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ do not have a common zero in $\mathbb{C}$.

Combination of Lemmas 4.1 and 4.2 gives an alternative argument to show that condition (iii) of Theorem 3.1 implies condition (iv). Indeed, if $(\mathcal{F}F_1, \ldots, \mathcal{F}F_N)_{PW(\mathbb{C})} = PW(\mathbb{C})$, then the functions $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have no common zeros. However, we are more interested in the implication in the opposite direction: if some functions $f_1, \ldots, f_n \in PW(\mathbb{C})$ do not have any common zeros, do they generate the whole ring $PW(\mathbb{C})$? Stated differently, is it possible to extend the Hilbert Nullstellensatz (for polynomials in several variables) or the Corona Theorem (for $H_\infty$-functions) to $PW(\mathbb{C})$? Unfortunately, the answer is no, as is illustrated in the next counterexample.
Example 4.3 ([5, pp. 319—320]) For \( \beta \in \mathbb{R} \) we consider the distributions \( H_\beta \in \mathcal{E}'(\mathbb{R}) \), given by \( H_\beta = 1_{[-2\beta,0]} \), i.e.

\[
H_\beta : \mathcal{E}(\mathbb{R}) \rightarrow \mathbb{C} : H_\beta(x) = \int_{-2\beta}^{0} x(\tau) \, d\tau.
\]

The Fourier transform of \( H_\beta \), \( (\mathcal{F}H_\beta)(\omega) = \int_{-2\beta}^{0} e^{-i\omega \tau} \, d\tau = 2e^{i\beta \omega} \cdot \frac{\sin(\beta \omega)}{\omega} ;
\]
the corresponding convolution operator is a delay operator with distributed time-delay.

Choose the distributions \( F_1 := \frac{1}{2}H_1 \) and \( F_2 := \frac{1}{2}H_\alpha \), with \( \alpha \in \mathbb{R}\setminus\{0\} \) fixed, and denote the corresponding Fourier transforms by \( \mathcal{F}H_1 \) and \( \mathcal{F}H_2 \), respectively. If \( \alpha \) is an irrational number, then \( f_1 \) and \( f_2 \) have no common zeros. Indeed, the zeros of \( f_1 \) are given by \( \{ k\pi \mid k \in \mathbb{Z}\setminus\{0\} \} \), and the zeros of \( f_2 \) by \( \{ \frac{\ell \pi}{\alpha} \mid \ell \in \mathbb{Z}\setminus\{0\} \} \). If \( \alpha \) is irrational, these sets are disjoint.

Next we assume that \( \alpha \) is a Liouville number. This is an irrational number with the property that for every \( n \in \mathbb{N} \) there exist \( p_n \in \mathbb{Z} \) and \( q_n \in \mathbb{N} \) with \( q_n > 1 \) such that

\[
|\alpha - \frac{p_n}{q_n}| < \frac{1}{(q_n)^n} \quad \text{for all } n \in \mathbb{N}.
\]

An example of a Liouville number is \( \sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k \). It follows from Liouville’s theorem (see e.g. [17, Sections 1 and 2] or Theorem A.1 in Appendix A), that any Liouville number is transcendental.

If \( \alpha \) is a Liouville number, the ideal in \( \mathcal{P}(\mathbb{C}) \) generated by \( \mathcal{F}H_1 \) and \( \mathcal{F}H_2 \) is not \( \mathcal{P}(\mathbb{C}) \), despite the absence of common zeros. To prove this assertion, we assume on the contrary that there exist \( g_1, g_2 \in \mathcal{P}(\mathbb{C}) \) such that

\[
f_1 \cdot g_1 + f_2 \cdot g_2 = 1.
\]

Let \( (p_n)_{n \in \mathbb{N}} \) and \( (q_n)_{n \in \mathbb{N}} \) be sequences of integers, satisfying formula (13). Then \( f_1(q_n \pi) = 0 \) for all \( n \in \mathbb{N} \), hence

\[
f_2(q_n \pi)g_2(q_n \pi) = 1 \quad \text{for all } n \in \mathbb{N}.
\]

Now

\[
|f_2(q_n \pi)| = \left| e^{\imath q_n \pi} \sin(\alpha q_n \pi) / q_n \pi \right| = \frac{1}{q_n \pi} |\sin(\alpha q_n \pi)| = \frac{1}{q_n \pi} \left| \sin \left( \left( \alpha - \frac{p_n}{q_n} \right) q_n \pi + p_n \pi \right) \right| \leq \frac{1}{q_n \pi} \left| \alpha - \frac{p_n}{q_n} \right| q_n \pi < \frac{1}{(q_n)^n}.
\]

So for all \( n \in \mathbb{N} \) we have \( |g_2(q_n \pi)| > (q_n)^n \). In particular, \( g_2 \) is not polynomially bounded on the real axis, i.e. there do not exist fixed \( C > 0 \) and \( N \in \mathbb{N} \) such that \( |g_2(q_n \pi)| \leq C \cdot (1 + |q_n \pi|)^N \) for all \( n \in \mathbb{N} \). We conclude that \( g_2 \notin \mathcal{P}(\mathbb{C}) \), and thus \( (f_1, f_2) \mathcal{P}(\mathbb{C}) \neq \mathcal{P}(\mathbb{C}) \).

Example 4.3 illustrates the main difficulty of the question whether a finite number of functions \( f_1, \ldots, f_n \in \mathcal{P}(\mathbb{C}) \) generates \( \mathcal{P}(\mathbb{C}) \). One has not only to verify whether there exist analytic functions \( g_1, \ldots, g_n \) such that

\[
\sum_{i=1}^{n} f_i g_i = 1,
\]

(14)
but additionally that \( g_1, \ldots, g_n \) belong to \( PW(C) \), i.e \( g_1, \ldots, g_n \) have to satisfy growth condition (9). So, if (14) holds with \( g_1, \ldots, g_n \in PW(C) \), then there exist \( C, a > 0 \) and \( N \in \mathbb{N} \) such that for arbitrary \( \omega \in \mathbb{C} \):

\[
1 = \left| \sum_{i=1}^{n} f_i(\omega)g_i(\omega) \right| \leq \sum_{i=1}^{n} |f_i(\omega)||g_i(\omega)| \leq C(1 + |\omega|)^N \exp(a |\omega|) \sum_{i=1}^{n} |f_i(\omega)|.
\]

Hence for all \( \omega \in \mathbb{C} \):

\[
\sum_{i=1}^{n} |f_i(\omega)| \geq \frac{1}{C (1 + |\omega|)^N} \exp(-a |\omega|).
\]

This indicates that the absence of common zeros is not sufficient for \( f_1, \ldots, f_n \) to be a generating set for \( PW(C) \). Additionally it is required that the functions \( |f_i(\omega)| \) do not decrease too fast around their zeros. In [11] it was shown that (15) is not only a necessary but also a sufficient condition for a finite number of functions \( f_1, \ldots, f_n \in PW(C) \) to be a generating set for \( PW(C) \). Therefore we may formulate the following modification of the Corona Theorem for the Paley-Wiener algebra.

**Theorem 4.4 ([11])** Let \( f_1, \ldots, f_n \in PW(C) \). The ideal in \( PW(C) \) generated by \( f_1, \ldots, f_n \) is equal to \( PW(C) \) if and only if there exist \( C, a > 0 \) and \( N \in \mathbb{N} \) such that for all \( \omega \in \mathbb{C} \):

\[
\sum_{i=1}^{n} |f_i(\omega)| \geq C (1 + |\omega|)^N \exp(-a |\omega|).
\]

Notice that in Example 4.3 condition (16) is violated by the functions \( f_1 \) and \( f_2 \) because

\[
|f_1(q_n \pi)| + |f_2(q_n \pi)| < \frac{1}{(q_n)^n}, \quad (n \in \mathbb{N}).
\]

Combination of Theorem 4.4 and Lemma 4.1 yields a sufficient condition for behavioral controllability. If a matrix \( P \in \mathcal{E}'(\mathbb{R})^{n \times m} \) has full row rank and the principal minors of \( FP \) satisfy condition (16), then \( B(P) \) is behaviorally controllable. Note however, that we did not prove the necessity of this condition. Furthermore, verification of (16) remains a difficult problem in general. From a computational point of view it is more attractive to check condition (ii) of Lemma 4.2: do the principal minors of \( FP \) have any common zeros? Unfortunately, we only showed that this condition is necessary for behavioral controllability, not that it is sufficient. In the next section we try to fix this discrepancy. For this purpose we will characterize a class of time-delay systems for which both conditions turn out to be equivalent. In this way it is possible to derive, at least for this class of systems, a necessary and sufficient condition for behavioral controllability, that is similar to the Hautus-test.

**Remark 4.5** Let \( F_1, F_2 \in \mathcal{E}'(\mathbb{R}) \) be the distributions as defined in Example 4.3, and consider the system represented by the matrix \( P = (F_1 \mid F_2) \in \mathcal{E}'(\mathbb{R})^{1 \times 2} \).

If \( \alpha \) is a rational number, the Fourier transforms \( f_1 = FP_1 \) and \( f_2 = FP_2 \) have infinitely many common zeros, so according to Lemma 3.5, \( B(P) \) is not behaviorally controllable.

If \( \alpha \) is a Liouville number, we do not know whether the system is controllable. In this case, \( f_1 \) and \( f_2 \) have no common zeros, so the necessary condition for controllability is satisfied. However, in Example 4.3 we showed that \( (f_1, f_2)_{PW(C)} \neq PW(C) \), so the sufficient condition for controllability (\( P \) is right-invertible over \( \mathcal{E}'(\mathbb{R}) \)) is violated. The main difficulty is that
despite the absence of common zeros, the zeros of $f_1$ and $f_2$ may become arbitrarily close for large values of $|\omega|$ on the real axis.

If $\alpha$ is an algebraic number, the behavior $B(P)$ is controllable. In Appendix A we will show that in this situation Theorem 4.4 may be used to verify that $\langle f_1, f_2 \rangle_{PW(\mathbb{C})} = PW(\mathbb{C})$, and thus $P$ is right-invertible over $\mathcal{E}(\mathbb{R})$. The same proof applies to any irrational number that is not a Liouville number.

These observations indicate that controllability of the behavior $B(P)$ is not a robust property, because it is sensitive to arbitrary small perturbations of the length $\alpha$ of the time-delay. In practice it is therefore impossible to decide on the controllability of this system. Instead we should try to avoid this undesirable situation, and only consider classes of systems for which controllability is a robust property. The class of time-delay systems that we will study in Section 5 meets this requirement.

5 Controllability of time-delay systems

Since the problem of controllability of AR-convolution systems seems difficult to solve in full generality, we now specialize to a class of delay-differential systems with point delays. Later on we will also incorporate a specific type of distributed time-delays in our framework. First we define the subring $\mathcal{R}$ of $\mathcal{E}(\mathbb{R})$ by

$$\mathcal{R} := \left\{ \sum_{i=1}^{n} a_i \delta_{-\beta_i} \mid n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}, \beta_1, \ldots, \beta_n \geq 0 \right\}. \quad (17)$$

$\mathcal{R}$ is an integral domain, but as $\mathcal{E}(\mathbb{R})$ it is not a unique factorization domain. For a $k$-tuple of incommensurable time-delays $\tau_1, \ldots, \tau_k \in \mathbb{R}^+$, meaning that $\tau_1, \ldots, \tau_k$ are linearly independent over $\mathbb{Q}$, the ring $\mathbb{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$ is a subring of $\mathcal{R}$, where the unit element of $\mathbb{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$ represents the distribution $\delta_0 \in \mathcal{R}$. Conversely, every distribution $F \in \mathcal{R}$ belongs to a ring $\mathbb{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$, after suitable choice of the incommensurable time-delays $\tau_1, \ldots, \tau_k$, i.e.

$$\mathcal{R} = \bigcup_{k \in \mathbb{N}} \bigcup_{\tau_1, \ldots, \tau_k \in \mathbb{R}^+, \text{incommensurable}} \mathbb{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}].$$

Let $\mathcal{R}[\delta_0]$ denote the ring of polynomials in $\delta_0$ with coefficients from $\mathcal{R}$. Again, the integral domain $\mathcal{R}[\delta_0]$ is a subring of $\mathcal{E}(\mathbb{R})$. The convolution operators corresponding to nonzero elements of $\mathcal{R}[\delta_0]$ are delay-differential operators with point delays; according to [6, Proposition 5.12] these operators are always surjective. The behavior $B(P)$ corresponding to a representation matrix $P \in \mathcal{R}[\delta_0]^{n \times m}$ consists of all smooth solutions of an $n$-tuple of delay-differential equations in $m$ variables.

Proposition 5.1 Let $F_1, \ldots, F_N \in \mathcal{R}[\delta_0]$.

(i) If the Fourier transforms $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have no common zeros, then there exists a nonzero $\Phi \in \mathcal{R}$ such that

$$\Phi \in \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]}. \quad (18)$$

(ii) If the Fourier transforms $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have finitely many common zeros (including multiplicities), then there exist $p \in \mathbb{R}[s] \setminus \{0\}$ and $\Phi \in \mathcal{R} \setminus \{0\}$ such that

$$p(\delta_0) \ast \Phi \in \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]}. \quad (19)$$
Proof: Let $F_1, \ldots, F_N \in \mathcal{R}[\delta_0^0]$, and denote the quotient field of $\mathcal{R}$ by $Q := \mathcal{Q}(\mathcal{R})$. Since $\mathcal{Q}[\delta_0^0]$ is a principal ideal domain, the ideal $\langle F_1, \ldots, F_N \rangle_{\mathcal{Q}[\delta_0^0]}$ is principally generated by $G := \gcd_{\mathcal{Q}[\delta_0^0]}(F_1, \ldots, F_N)$. Without loss of generality we assume that $G \in \mathcal{R}[\delta_0^0]$. We will show that in case (i), $G \equiv 1$ or, more precisely, $G = \delta_0$; in case (ii) we show $G = p(\delta_0^0)$ with $p \in \mathbb{R}[s]$.

For every $i = 1, \ldots, N$ there exist $M_i \in \mathcal{R}[\delta_0^0]$ and $D_i \in \mathcal{R}$ such that

$$F_i = G \cdot \frac{M_i}{D_i}, \quad (i = 1, \ldots, N).$$

Choose incommensurable time-delays $\tau_1, \ldots, \tau_k$, such that the distributions $F_1, \ldots, F_N, G, M_1, \ldots, M_N \in \mathcal{R}[\delta_0^0, \delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$ and $D_1, \ldots, D_N \in \mathcal{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$. We define $\mathcal{R} := \mathcal{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}]$. Then the rings $\mathcal{R}$ and $\mathcal{R}[\delta_0^0]$ are both unique factorization domains.

If $\deg_{\mathcal{Q}[\delta_0^0]}(G) > 0$, and considering $G$ as an element of $\mathcal{R}[\delta_0^0]$, we may assume — without loss of generality — that $G$ is a product of irreducible factors $\Psi \in \mathcal{R}[\delta_0^0]$ of positive degree in $\delta_0^0$. Since for $i = 1, \ldots, N$ we have $D_i \cdot F_i = G \cdot M_i$ and $D_1, \ldots, D_N \in \mathcal{R}$, any irreducible factor $\Psi$ of $G$ is a common factor of $F_1, \ldots, F_N$. In particular, the zeros of $\mathcal{F}\Psi$ are common zeros of $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ (including multiplicities).

Case (i): If $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ do not have any common zeros, we obtain a contradiction; in this situation $\deg_{\mathcal{Q}[\delta_0^0]}(G) = 0$, and considered as an element of $\mathcal{Q}$ we may choose $G = 1$.

Case (ii): If $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have only finitely many common zeros, then $\mathcal{F}G$ has at most finitely many zeros. Therefore, $\mathcal{F}G$ is the product of a polynomial and an exponential function (see e.g. [7, p. 484]). Moreover, $G \in \mathcal{R}[\delta_0^0]$, so $G = p(\delta_0^0) \ast \Upsilon$, with $p \in \mathbb{R}[s]$ and $\Upsilon = \delta_{-\tau}$ for some $\tau \geq 0$. By assumption, $G$ is a product of irreducible factors of positive degree in $\delta_0^0$, so $\Upsilon$ must be a unit of $\mathcal{R}[\delta_0^0]$, i.e. $\Upsilon = \delta_0$, and thus $G = p(\delta_0^0)$.

At this point we know that

$$\gcd_{\mathcal{Q}[\delta_0^0]}(F_1, \ldots, F_N) = p(\delta_0^0),$$

with $p \equiv 1$ in Case (i) and $p \in \mathbb{R}[s]\{0\}$ in Case (ii). Since $\mathcal{Q}[\delta_0^0]$ is a principal ideal domain, there exist $\alpha_1, \ldots, \alpha_N \in \mathcal{R}[\delta_0^0]$ and $\beta_1, \ldots, \beta_N \in \mathcal{R}\{0\}$ such that

$$p(\delta_0^0) = \sum_{i=1}^{N} F_i \cdot \frac{\alpha_i}{\beta_i}. \quad (20)$$

Define $\gamma_i := \prod_{j=1}^{N} \beta_j / \beta_j$, and multiply (20) by $\Phi := \prod_{j=1}^{N} \beta_j \in \mathcal{R}$, then

$$\sum_{i=1}^{N} F_i \cdot \alpha_i \cdot \gamma_i = p(\delta_0^0) \ast \prod_{j=1}^{N} \beta_j = p(\delta_0^0) \ast \Phi,$$

i.e. $p(\delta_0^0) \ast \Phi$ with $\Phi \in \mathcal{R}\{0\}$ is an element of $\langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0^0]}$.

Remark 5.2 Given $F_1, \ldots, F_N \in \mathcal{R}[\delta_0^0]$, there exists a constructive method for the computation of an element $\Phi \in \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0^0]}$ of minimal degree in $\delta_0^0$. The algorithm is based on the idea of pseudo-division.

Let $F, G \in \mathcal{R}[\delta_0^0]$, and let $d$ denote the leading coefficient of $G$, i.e. $d$ is the coefficient of the largest power of $\delta_0^0$ in $G$. Then there is an $\ell \geq 0$ and polynomials $Q, R \in \mathcal{R}[\delta_0^0]$, with $\deg(R) < \deg(G)$, such that

$$d^\ell F = Q \cdot G + R.$$
The factor \( a^\ell \) is introduced to ensure that all polynomials remain in \( \mathcal{R}[\delta_0] \). If \( \ell \) is chosen as small as possible, \( Q \) and \( R \) are determined uniquely, and \( R := \text{prem}(F, G) \) is called the pseudo-remainder after division of \( F \) by \( G \). The polynomials \( Q \) and \( R \) are easily computed using a slight modification of the well known division algorithm (see e.g. [3, p. 297]). Note that \( R \in \langle F, G \rangle_{\mathcal{R}[\delta_0]} \) and \( \gcd Q[\delta_0](F, G) = \gcd Q[\delta_0](G, R) \). For the computation of \( \gcd Q[\delta_0](F, G) \) we may now apply the Euclidean algorithm (see e.g. [3, p. 41]), with usual division with remainder replaced by pseudo-division with remainder. In this way we obtain an element \( H \in \mathcal{R}[\delta_0] \) such that \( H \in \langle F, G \rangle_{\mathcal{R}[\delta_0]} \) and \( H = \gcd Q[\delta_0](F, G) \).

For the computation of \( \Phi \) in the proof of Proposition 5.1 we apply this algorithm recursively. First we compute \( H_1 = \gcd Q[\delta_1](F_1, F_2) \), subsequently \( H_2 = \gcd Q[\delta_0](H_1, F_3) \), etc. In this way we obtain \( H = H_{N-1} \in \mathcal{R}[\delta_0^N] \) with \( H \in \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]} \) and \( H = \gcd Q[\delta_0^N](F_1, \ldots, F_N) \). If \( H \in \mathcal{R} \) or \( H = p(\delta_0^N) \Phi \), with \( p \in \mathbb{R}[s] \) and \( \Phi \in \mathcal{R} \setminus \{0\} \), then we have obtained the required element of \( \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]} \). Otherwise, the ideal \( \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]} \) does not contain such an element; consequently the Fourier transforms \( \mathcal{F}F_1, \ldots, \mathcal{F}F_N \) must have common zeros.

Remark 5.3 In the proof of Proposition 5.1 we have to make a transition from the ring \( \mathcal{R} \) to the ring \( \mathcal{R} = \mathbb{R}[\delta_{-\tau_1}, \ldots, \delta_{-\tau_k}] \), where \( \tau_1, \ldots, \tau_k \) is a \( k \)-tuple of incommensurable time-delays. In the proof of Case (i), this step is required to show that \( G = \gcd Q[\delta_0](F_1, \ldots, F_N) \) does not contain a factor of positive degree in \( \delta_0^N \). In Case (ii) a similar factorization argument is used.

Therefore it is necessary to work in the unique factorization domain \( \mathcal{R}[\delta_0^N] \). Although the transition from \( \mathcal{R} \) to \( \mathcal{R} \) is highly sensitive to small perturbations of the lengths of the time-delays, this does not lead to computational problems, because the ring \( \mathcal{R}[\delta_0^N] \) is not involved in the computation of the polynomials \( G, \Phi \) and \( p \) in Proposition 5.1. For this purpose, one may use the method described in Remark 5.2 instead.

Proposition 5.1 has an important implication for the verification of the controllability of a delay system. If \( F_1, \ldots, F_N \) are the principal minors of a matrix over \( \mathcal{R}[\delta_0] \), we want to check whether the Fourier transforms \( \mathcal{F}F_1, \ldots, \mathcal{F}F_N \) satisfy formula (16). On the other hand, a necessary condition for controllability is the absence of common zeros. In that case, Proposition 5.1 implies that the ideal \( \langle F_1, \ldots, F_N \rangle_{\mathcal{R}[\delta_0]} \) contains a nonzero element \( \Phi \in \mathcal{R} \). Obviously, \( \Phi \) is also an element of \( \langle F_1, \ldots, F_N \rangle_{\mathcal{E}(\mathbb{R})} \); in fact we have \( \langle F_1, \ldots, F_N \rangle_{\mathcal{E}(\mathbb{R})} = \langle F_1, \ldots, F_N, \Phi \rangle_{\mathcal{E}(\mathbb{R})} \). The Fourier transform of \( \Phi \) is a sum of exponential functions, and therefore its exponential growth and the location of its zeros are characterized as follows.

Proposition 5.4 Let \( \Phi \in \mathcal{R} \setminus \{0\} \), and denote its Fourier transform \( \mathcal{F}\Phi \) by \( \varphi \). Then there exist \( \alpha > 0, \gamma > 0 \) and \( C > 0 \) such that

(i) All zeros of \( \varphi \) are located in the horizontal strip \( \{ \omega \in \mathbb{C} \mid \text{Im} \omega \leq \alpha \} \).

(ii) \( \forall \omega \in \mathbb{C}, \ |\text{Im} \omega| > \alpha : \ |\varphi(\omega)| \geq C \cdot e^{-\gamma \text{Im} \omega} \).

Proof: Since (ii) implies (i), it suffices to prove (ii). Let \( \Phi = \sum_{j=1}^n a_j \delta_{-b_j} \in \mathcal{R} \), and assume without loss of generality that \( n \geq 1, a_j \neq 0 \) for \( j = 1, \ldots, n \), and \( 0 \leq b_1 < b_2 < \cdots < b_n \). The Fourier transform of \( \Phi \) is given by

\[ \varphi(\omega) = (\mathcal{F}\Phi)(\omega) = \sum_{j=1}^n a_j e^{i\beta_j \omega}. \]
For $\omega = x + iy \in \mathbb{C}$, with $x, y \in \mathbb{R}$, we have

$$|\varphi(x + iy)| = \left| \sum_{j=1}^{n} a_j e^{-\beta_j y} e^{i\delta_j x} \right| \geq |a_1 e^{-\beta_1 y} - \sum_{j=2}^{n} |a_j| e^{-\beta_j y}| \geq e^{-\beta_1 y} \left( |a_1| - \sum_{j=2}^{n} |a_j| e^{(\beta_1 - \beta_j)y} \right).$$

Since $\beta_1 - \beta_j < 0$ for $j = 2, \ldots, n$, there exists $\alpha_1 > 0$ such that for all $x \in \mathbb{R}$ and $y > \alpha_1$: $|\varphi(x + iy)| > \frac{1}{2} |a_1| e^{-\beta_1 y}$. Completely analogously, by taking the term $a_n e^{-\beta_n y}$ apart, and factoring out $e^{-\beta_n y}$, we obtain an $\alpha_2 < 0$ such that for all $x \in \mathbb{R}$ and $y < \alpha_2$: $|\varphi(x + iy)| > \frac{1}{2} |a_n| e^{-\beta_n y}$. The claim follows by choosing $\alpha := \max(|\alpha_1|, |\alpha_2|)$, $C := \frac{1}{2} \min(|a_1|, |a_n|)$, and $\gamma \geq \beta_1$.

Proposition 5.4 indicates that outside a horizontal strip around the real axis a sum of exponentials (with purely imaginary exponents) satisfies the growth condition of formula (16). So, in order to verify whether $(F_1, \ldots, F_N)_{\mathcal{E}'}(\mathbb{R}) = \mathcal{E}'(\mathbb{R})$, it is sufficient to check whether the Fourier transforms $\mathcal{F}F_1, \ldots, \mathcal{F}F_N, \mathcal{F}\Phi$ satisfy condition (16) inside a horizontal strip around the real axis. For this purpose we introduce an additional assumption, and consider the case where one of the polynomials $F_1, \ldots, F_N \in \mathcal{R}[\delta_0]$ is monic.

**Theorem 5.5** Let $F_1, \ldots, F_N \in \mathcal{R}[\delta_0]$, and assume that

(i) $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have no common zeros,

(ii) $F_1$ is monic in $\delta_0$, i.e. $F_1 = (\delta_0)^n + \sum_{j=0}^{n-1} r_j (\delta_0)^j$, with $r_0, r_1, \ldots, r_{n-1} \in \mathcal{R}$.

Then $(F_1, \ldots, F_N)_{\mathcal{E}'}(\mathbb{R}) = \mathcal{E}'(\mathbb{R})$.

**Proof:** First we apply Proposition 5.1 and choose a nonzero $\Phi \in (F_1, \ldots, F_N)_{\mathcal{E}'}(\mathbb{R}) \cap \mathcal{R}$. Denote the Fourier transforms of $F_1, \ldots, F_N$ by $f_1, \ldots, f_N$, respectively, and let $\varphi := \mathcal{F}\Phi$. According to Proposition 5.4 there exist $\alpha > 0$, $\gamma > 0$ and $C > 0$ such that

$$\forall \omega \in \mathbb{C}, \ |\Im \omega| > \alpha : \ |\varphi(\omega)| \geq C e^{-\gamma |\Im \omega|}.$$ 

Let $F_1 = (\delta_0)^n + \sum_{j=0}^{n-1} r_j (\delta_0)^j$, with $r_0, r_1, \ldots, r_{n-1} \in \mathcal{R}$. Then

$$f_1(\omega) = (\mathcal{F}F_1)(\omega) = (i\omega)^n + \sum_{j=0}^{n-1} (\mathcal{F}r_j)(\omega)(i\omega)^j.$$ 

For $j = 0, 1, \ldots, n-1$, the Fourier transform $\mathcal{F}r_j$ is a sum of exponentials with purely imaginary exponents:

$$(\mathcal{F}r_j)(\omega) = \sum_{k=1}^{L_j} c_{k,j} e^{i\omega \rho_{k,j}},$$

with $c_{k,j} \in \mathbb{R}$ and $\rho_{k,j} \geq 0$ ($j = 0, 1, \ldots, n-1$, $k = 1, \ldots, L_j$); the $\rho_{k,j}$ denote the time-delays occurring in the system. For $\omega \in \mathbb{C}$ with $|\Im \omega| \geq -\alpha$ we have

$$|(\mathcal{F}r_j)(\omega)| \leq \sum_{k=1}^{L_j} |c_{k,j}| e^{-\rho_{k,j}|\Im \omega|} \leq \sum_{k=1}^{L_j} |c_{k,j}| e^{\rho_{k,j} \alpha} =: c_j.$$
Hence, there exists $R > 0$, such that for all $\omega \in \mathbb{C}$, with $\text{Im} \omega \geq -\alpha$ and $|\omega| > R$ we have

$$|f_1(\omega)| \geq |\omega|^n - \sum_{j=0}^{n-1} |(F\tau_j(\omega))||\omega|^j \geq |\omega|^n - \sum_{j=0}^{n-1} c_j |\omega|^j > 1. \quad (22)$$

Next, consider the compact set $V = \{\omega \in \mathbb{C} \mid |\text{Im} \omega| \leq \alpha \text{ and } |\omega| \leq R\}$ (see Figure 1). Since $f_1, \ldots, f_N$ have no common zeros,

$$\varepsilon := \min \left\{ \sum_{i=1}^{N} |f_i(\omega)| \mid \omega \in V \right\},$$

is a well-defined positive number. Define $C_1 := \min(\varepsilon, C, 1)$. We use Figure 1 to show that

$$\forall \omega \in \mathbb{C} : \sum_{i=1}^{N} |f_i(\omega)| + |\varphi(\omega)| \geq C_1 e^{-\gamma |\text{Im} \omega|}. \quad (23)$$

Indeed, if $|\text{Im} \omega| > \alpha$, then $|\varphi(\omega)| \geq C e^{-\gamma |\text{Im} \omega|} \geq C_1 e^{-\gamma |\text{Im} \omega|}$, and for $|\text{Im} \omega| \leq \alpha$ with $|\omega| > R$, we know that $|f_1(\omega)| > 1 \geq C_1$. Finally, for $\omega \in V$ we have $\sum_{i=1}^{N} |f_i(\omega)| \geq \varepsilon \geq C_1$.

According to Theorem 4.4, (23) implies that $(F_1, \ldots, F_N)\mathcal{E}(\mathbb{R}) = (F_1, \ldots, F_N, \Phi)\mathcal{E}(\mathbb{R}) = \mathcal{E}'(\mathbb{R})$.

The proof of Theorem 5.5 guarantees that the phenomenon described in Example 4.3 and Remark 4.5, where zeros became arbitrarily close, but never coincided, cannot occur if one of the polynomials $F_1, \ldots, F_N \in \mathcal{R}[\delta] \subseteq \mathcal{R}[\delta_0]$ is monic in $\delta_0$, because the Fourier transform of a monic polynomial in $\mathcal{R}[\delta_0]$ has finitely many zeros in every horizontal strip $|\text{Im} \omega| \leq \beta$ with $\beta > 0$. Since all zeros of $\varphi = F\Phi$ are located inside the fixed horizontal strip $|\text{Im} \omega| \leq \alpha$, the clustering of zeros for large values of $|\omega|$ is prevented.

Combining Theorem 5.5, Lemma 4.1, Lemma 4.2, and Theorem 3.1, and using the fact that a convolution operator $\sigma[P]$, corresponding to a full row rank matrix $P$ over $\mathcal{R}[\delta_0]$, is always surjective (see e.g. [6, Corollary 6.2]), we obtain the main result of this section:
Theorem 5.6 Let $P \in \mathcal{R}[\delta_0]^{n \times m}$, with $\text{rank}(P) = n$. Assume that one of the $n \times n$ minors of $P$ is monic in $\delta_0$. Then the following statements are equivalent:

(i) $\mathcal{B}(P)$ is behaviorally controllable,

(ii) $\mathcal{B}(P)$ has an image representation over $\mathcal{E}'(\mathbb{R})$,

(iii) $P$ is right-invertible over $\mathcal{E}'(\mathbb{R})$,

(iv) $\forall \lambda \in \mathbb{C} : \text{rank}(FP)(\lambda) = n$.

Theorem 5.6 has an important application for time-delay systems written in pseudo-state form:

$$\dot{x}(t) = \sum_{j=1}^{N} A_jx(t-h_j) + B_ju(t-h_j), \quad (24)$$

with $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$, $B_1, \ldots, B_N \in \mathbb{R}^{n \times m}$, time-delays $h_1, \ldots, h_N \geq 0$, and pseudo-state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$ for all $t \in \mathbb{R}$. The corresponding behavior $\mathcal{B}$ is given by

$$\mathcal{B} = \left\{ \left( \begin{array}{c} x \\ u \end{array} \right) \in \mathcal{E}(\mathbb{R})^{n+m} \mid \sigma[P_1]x + \sigma[P_2]u = 0 \right\},$$

with

$$P_1 = \delta_0 I_n + \sum_{j=1}^{N} A_j \delta_{-h_j}, \quad P_2 = \sum_{j=1}^{N} B_j \delta_{-h_j}.$$ 

Since the determinant of $P_1$ is monic in $\delta_0$, Theorem 5.6 (iv) implies that this system is behaviorally controllable if and only if

$$\forall \omega \in \mathbb{C} : \text{rank} \left( i\omega I_n + \sum_{j=1}^{N} A_j e^{ih_j \omega} \left| \sum_{j=1}^{N} B_j e^{ih_j \omega} \right. \right) = n.$$ 

Upon replacing $-i\omega$ by $\lambda$, and changing the sign of the first $n \times n$ block, we obtain the usual condition of spectral controllability (see e.g. [2], [13], and [12]):

$$\forall \lambda \in \mathbb{C} : \text{rank} \left( \lambda I_n - \sum_{j=1}^{N} A_j e^{-h_j \lambda} \left| \sum_{j=1}^{N} B_j e^{-h_j \lambda} \right. \right) = n. \quad (25)$$

Condition (25) may be considered as a generalization of the Hautus-test to time-delay systems with arbitrary point delays. Note however that in the derivation of this result, the structure of the system defining equation (24) was used explicitly.

For time-delay systems with commensurable point delays, it was shown in [16] and [7] that condition (iv) of Theorem 5.6 is necessary and sufficient for behavioral controllability. For these results it is not necessary to assume that one of the full size minors of the system-defining equations is monic in $\delta_0$. The commensurable delays case is simpler because even without an assumption on monicity, the phenomenon studied in Example 4.3 and Remark 4.5 does not occur.

The results of Theorem 5.5 and Theorem 5.6 may be generalized to a slightly larger ring of delay-differential operators, also including a specific type of distributed time-delays. For
this purpose we consider a generalization of the ring $\mathcal{H}$, introduced in [7], that was developed in [9]. Define

$$S := \left\{ \frac{p}{q} \in \mathcal{Q}(\mathbb{R}[\delta_0]) \mid p, q \in \mathbb{R}[\delta_0] \text{ such that } \frac{\mathcal{F}p}{\mathcal{F}q} \in \mathcal{P}W(\mathbb{C}) \right\}. \quad (26)$$

Then $S$ is a subring of $\mathcal{E}'(\mathbb{R})$ because $\frac{\mathcal{F}p}{\mathcal{F}q} \in \mathcal{P}W(\mathbb{C})$. Using a result in [1], it was shown in [9] that the elements of $S$ admit a much simpler representation:

$$S = \left\{ \frac{p}{\delta_{-\beta} \star q} \in \mathcal{E}'(\mathbb{R}) \mid p \in \mathbb{R}[\delta_0], q \in \mathbb{R}[\delta_0], \beta \geq 0 \text{ such that } \frac{\mathcal{F}p}{\mathcal{F}q} \in \mathcal{P}W(\mathbb{C}) \right\}, \quad (27)$$

i.e. every element of $S$ may be written as a quotient $\frac{p}{\delta_{-\beta} \star q}$, where the denominator is the convolution product of a delta distribution $\delta_{-\beta}$ and a polynomial $q \in \mathbb{R}[\delta_0]$. The convolution operators corresponding to distributions in $S$ include delay-differential operators with both point- and distributed time-delays. For example, the distributions considered in Example 4.3 are elements of $S$. According to [6, Corollary 5.15], every convolution operator $\sigma[F]$ with $F \in S \setminus \{0\}$ is surjective.

**Theorem 5.7** Let $F_1, \ldots, F_N \in S$, with $F_j = \frac{p_j}{\delta_{-\beta_j} \star q_j}$ ($j = 1, \ldots, N$), where $p_j \in \mathbb{R}[\delta_0]$, $q_j \in \mathbb{R}[\delta_0]$, and $\beta_j \geq 0$. Assume that

(i) $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have no common zeros,

(ii) $p_1$ is monic in $\delta_0$.

Then $(F_1, \ldots, F_N)_{\mathcal{E}'(\mathbb{R})} = \mathcal{E}'(\mathbb{R})$.

**Proof:** Since for every $j = 1, \ldots, N$ the Fourier transform $\mathcal{F}q_j$ has finitely many zeros, condition (i) implies that $\mathcal{F}p_1, \ldots, \mathcal{F}p_N$ have at most finitely many common zeros. According to Proposition 5.1 (ii) there exist $p \in \mathbb{R}[s] \setminus \{0\}$ and $\Phi \in \mathbb{R} \setminus \{0\}$ such that

$$p(\delta_0) \ast \Phi \in \mathbb{R}[p_1, \ldots, p_N] \mathbb{R}[\delta_0] \subset (F_1, \ldots, F_N)_{\mathcal{E}'(\mathbb{R})}.$$
Corollary 5.8 Let $P \in S^{n \times m}$ with $\text{rank}(P) = n$, and assume that the numerator of one of the $n \times n$ minors of $P$ is monic in $\delta_0$. Then $B(P)$ is behaviorally controllable if and only if

$$\forall \lambda \in \mathbb{C} : \text{rank}(\mathcal{F}P)(\lambda) = n.$$ 

Example 5.9 Consider the delay system with pseudo-state representation

$$\dot{x}(t) = x(t - h_1) + \int_{-h_3}^{-h_2} u(t + \tau) \, d\tau,$$

with $h_1 > 0$ and $h_3 > h_2 > 0$. The corresponding behavior is given by

$$B = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{E}(\mathbb{R})^2 \mid \sigma[F_1]x + \sigma[F_2]u = 0 \right\},$$

where $F_1 = \delta_0^t + \delta_{-h_1}$ and $F_2 = 1_{[-h_3,-h_2]}$. Alternatively, the distribution $F_2$ may be written as $F_2 = \frac{\delta_{-h_3} - \delta_{-h_2}}{\delta_0^t}$, so $F_2 \in S$. Since $F_1$ is monic in $\delta_0$, the behavior $B$ is controllable if and only if $\mathcal{F}F_1$ and $\mathcal{F}F_2$ have no common zeros. We start by computing both Fourier transforms:

$$(\mathcal{F}F_1)(\omega) = i\omega + e^{i\omega h_1},$$

$$(\mathcal{F}F_2)(\omega) = \frac{e^{i\omega h_3} - e^{i\omega h_2}}{i\omega}.$$ 

The zeros of $\mathcal{F}F_2$ are given by $\omega_k = \frac{2k\pi}{h_3 - h_2}$ for $k \in \mathbb{Z}\setminus\{0\}$; therefore we have to verify whether

$$(\mathcal{F}F_1)(\omega_k) = \frac{2k\pi i}{h_3 - h_2} + e^{2k\pi i \frac{h_1}{h_3 - h_2}} \neq 0 \quad \text{for all } k \in \mathbb{Z}\setminus\{0\}.$$ 

Since $e^{2k\pi i \frac{h_1}{h_3 - h_2}} = 1$, the only possibility for $\omega_k$ to be a zero of $\mathcal{F}F_1$ occurs when $h_3 - h_2 = \pm 2k\pi$. If $h_3 - h_2 = 2k\pi$, then $(\mathcal{F}F_1)(\omega_k) = i + e^{ih_1}$, and if $h_3 - h_2 = -2k\pi$, then $(\mathcal{F}F_1)(\omega_k) = -i + e^{-ih_1}$. In either case, $(\mathcal{F}F_1)(\omega_k) = 0$ for $h_1 = (2\ell - \frac{1}{2})\pi$ ($\ell \in \mathbb{Z}$). We conclude that the system (28) is not behaviorally controllable if there exist $k, \ell \in \mathbb{Z}$ such that

$$h_1 = (2\ell - \frac{1}{2})\pi \quad \text{and} \quad h_3 - h_2 = 2k\pi.$$ 

For all other values of the time-delays $h_1, h_2, h_3$, the system is behaviorally controllable.

In this section we have shown that for a large class of time-delay systems behavioral controllability is equivalent with a pointwise full rank condition. The considerations that were used to derive this condition, may (in principle) also be applied for its verification. In this way, a constructive method for testing behavioral controllability of time-delay systems is obtained. Below we will describe this algorithm for systems over the ring $\mathcal{R}[\delta_0^t]$, considered in Theorem 5.6. The modification needed to incorporate systems over the ring $\mathcal{S}$ (see Corollary 5.8) is straightforward and therefore omitted.

Let $P \in \mathcal{R}[\delta_0^t]^{n \times m}$, with $\text{rank}(P) = n$, and denote its $n \times n$ minors by $F_1, \ldots, F_N$. Assume that $F_1$ is monic in $\delta_0^t$. We want to verify whether the corresponding Fourier transforms $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ are devoid of common zeros. For this, we first compute $G = \gcd_{\mathcal{R}[\delta_0^t]}(F_1, \ldots, F_N)$, with $G \in \mathcal{R}[\delta_0^t]$, using the pseudo-division method described in Remark 5.2. If $G \not\in \mathcal{R}$, then $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have common zeros and the system is not controllable. Otherwise, we find a nonzero $G \in \{F_1, \ldots, F_N\} \cap \mathcal{R}$. According to Proposition 5.4, the zeros of $\mathcal{F}G$ are located in the horizontal strip $\{\omega \in \mathbb{C} \mid \text{Im} \omega \leq \alpha\}$ around the real axis. The value of $\alpha$ may be computed with the same techniques that were used in the proof of Proposition 5.4. Next, the Fourier transform $\mathcal{F}F_1$ of the monic polynomial $F_1$ is used to
compute the value $R$ of the radius of the circle, depicted in Figure 1. First, formula (21) is applied for the computation of the constants $c_j$ (note that for this purpose the value of $\alpha$ has to be known), subsequently $R$ is obtained according to formula (22). At this point we know that if $\mathcal{F}F_1, \ldots, \mathcal{F}F_N$ have common zeros, they are located in the compact set $V = \{ \omega \in \mathbb{C} \mid |\text{Im}\omega| \leq \alpha \text{ and } |\omega| \leq R \}$ (see Figure 1). Therefore numerical techniques may be applied to test the presence of common zeros.

The previous algorithm has a shortcoming, due to the algebraic computation of the greatest common divisor (gcd) of $F_1, \ldots, F_N \in \mathbb{R}[\omega]$ with the pseudo-division algorithm. Although the gcd is obtained by exact computation, the outcome may be sensitive for small perturbations of the lengths of the time-delays occurring in $F_1, \ldots, F_N$. More research seems necessary to overcome this difficulty and to construct an alternative algorithm for testing on common zeros. So the main contribution of this section is not the previous algorithm but the result of Theorem 5.5. It shows that for $F_1, \ldots, F_N \in \mathbb{R}[\omega]$, the absence of common zeros of the Fourier transforms, in combination with a monicity assumption, is sufficient to guarantee that $\langle F_1, \ldots, F_N \rangle_{C'(\mathbb{R})} = C'(\mathbb{R})$, because the clustering of zeros, like in Example 4.3, is prevented.

### 6 Concluding remarks

In this paper we have derived a necessary and a sufficient condition for behavioral controllability of autoregressive convolution systems. Whereas right-invertibility of the system representation matrix $P$ over the ring $C'(\mathbb{R})$ is sufficient for controllability, a pointwise full row rank condition on its Fourier transform $\mathcal{F}P$ was shown to be necessary. For time-delay systems with point delays, written in pseudo-state form, both conditions were proven to be equivalent. In this situation, the notions of spectral controllability and behavioral controllability are the same.

More research is necessary for the development of a reliable algorithm to test the pointwise full rank condition. Furthermore, the extension of these results to systems with more general signal spaces like $C_0(\mathbb{R})$ or $L^p_{\text{loc}} (p \geq 1)$ remains an open problem. The notion of translatable Fréchet spaces, as described in [6], seems a useful tool for the solution of this problem.

### Appendix A

In Example 4.3 and Remark 4.5 we considered the distributions $F_1 = \frac{1}{3} \cdot 1_{[-2,0]}$ and $F_2 = \frac{1}{2} \cdot 1_{[-2\alpha,0]}$, with Fourier transforms $f_1(\omega) = \frac{e^{i\omega \sin(\omega)}}{\omega}$ and $f_2(\omega) = \frac{e^{i\alpha \omega \sin(\alpha \omega)}}{\omega}$, respectively. In this appendix we will show that if $\alpha$ is an algebraic number, then $\langle f_1, f_2 \rangle_{PW(C)} = PW(C)$. The proof is based on the application of Theorem 4.4 in combination with Liouville's Theorem (see e.g. [17, p. 1]).

**Theorem A.1** (Liouville) Let $\alpha$ be an algebraic number of degree $n > 1$. Then there exists a number $c > 0$, only depending on $\alpha$, such that for all rational numbers $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$:

$$|\alpha - \frac{p}{q}| > \frac{c}{q^n}. \quad (29)$$

**Lemma A.2** Let $\alpha$ be an algebraic number of degree $n > 1$. Then there exists a number $c > 0$, only depending on $\alpha$, such that for all $q \in \mathbb{Z} \setminus \{0\}$:

$$|\sin(\alpha q \pi)| > \frac{2c}{|q|^{n-1}}. \quad (30)$$
Proof: Let \( c > 0 \) be such that formula (29) is satisfied. Let \( q \in \mathbb{Z}\setminus\{0\} \), and choose \( p \in \mathbb{Z} \), such that \( |\alpha - \frac{p}{q}| < \frac{1}{2q} \). Then \( |(\alpha - \frac{p}{q})q\pi| < \frac{1}{2}\pi \), and thus

\[
|\sin(q\pi)| = |\sin((\alpha - \frac{p}{q})q\pi + p\pi)| = |\sin((\alpha - \frac{p}{q})q\pi)| \geq \frac{2}{\pi} |\alpha - \frac{p}{q}| |q|\pi > \frac{2c}{|q||q|^n |q|\pi} = \frac{2c}{|q|^{n-1}}.
\]

Lemma A.3 Let \( \alpha \) be an algebraic number of degree \( n > 1 \). Then there exists a number \( c > 0 \) such that

\[
\forall x \in \mathbb{R}, \: |x| \geq \pi : \: |\sin(x)| + |\sin(ax)| > \frac{c}{(1 + |x|)^{n-1}}.
\]  

Proof: Denote \( g_1(x) = \sin(x) \) and \( g_2(x) = \sin(ax) \). Let \( c_1 \) be such that (30) is satisfied. If \( x_0 \neq 0 \) is a zero of \( g_1 \), i.e. \( x_0 = k\pi \) for some \( k \in \mathbb{Z}\setminus\{0\} \), then

\[
|g_2(x_0)| = |\sin(\alpha k\pi)| > \frac{2c_1}{|k|^{n-1}} > \frac{2c_1\pi^{n-1}}{(1 + |k\pi|)^{n-1}} = \frac{2c_1\pi^{n-1}}{(1 + |x_0|)^{n-1}}.
\]

Similarly, let \( c_2 \) be such that (30) is satisfied with \( \alpha \) replaced by \( \frac{1}{\alpha} \). If \( x_1 \neq 0 \) is a zero of \( g_2 \), then \( x_1 = \frac{k\pi}{\alpha} \) for some \( k \in \mathbb{Z}\setminus\{0\} \), and

\[
|g_1(x_1)| = |\sin(\frac{1}{\alpha} k\pi)| > \frac{2c_2}{|k|^{n-1}} > \frac{2c_2(\frac{\pi}{|\alpha|})^{n-1}}{(1 + |\frac{k\pi}{\alpha}|)^{n-1}} = \frac{2c_2(\frac{\pi}{|\alpha|})^{n-1}}{(1 + |x_1|)^{n-1}}.
\]

Define \( c := \min(2c_1\pi^{n-1}, 2c_2(\frac{\pi}{|\alpha|})^{n-1}) \). Then (31) is satisfied for all \( |x| \geq \pi \) that are zeros of either \( g_1 \) or \( g_2 \).

If \( x \) is not a zero of \( g_1 \) or \( g_2 \), there exist \( x_0, x_1 \in \mathbb{R} \), such that \( x_0 \) is the largest zero of \( g_1 \) or \( g_2 \), smaller than \( x \), and \( x_1 \) is the smallest zero of \( g_1 \) or \( g_2 \), larger than \( x \). On the interval \([x_0, x_1]\), the functions \( |g_1| \) and \( |g_2| \) are both convex, and so is their sum \( |g_1| + |g_2| \). On the other hand, the function \( h(\xi) = (1 + \xi)^{n-1} \) is concave on the interval \([x_0, x_1]\), and furthermore \( |g_1(x_0)| + |g_2(x_0)| > h(x_0) \) and \( |g_1(x_1)| + |g_2(x_1)| > h(x_1) \). We conclude that \( |g_1(\xi)| + |g_2(\xi)| > h(\xi) \) for any \( \xi \in [x_0, x_1] \), so in particular

\[
|g_1(x)| + |g_2(x)| > h(x) = \frac{c}{(1 + |x|)^{n-1}}.
\]

Next we use Lemma A.3 to show that the Fourier transforms of \( F_1 \) and \( F_2 \) satisfy the condition of Theorem 4.4. For this purpose we note that for any \( \beta \in \mathbb{R} \), and for all \( \omega = x + iy \in \mathbb{C}, \) with \( x, y \in \mathbb{R} \):

\[
\sin(\beta \omega) = \sin(\beta x + iy) = \sin(\beta x) \cosh(\beta y) + i \cos(\beta x) \sinh(\alpha y).
\]  

Proposition A.4 Let \( \alpha \) be an algebraic number of degree \( n > 1 \). Then there exists \( c > 0 \), such that for all \( \omega \in \mathbb{C} \):

\[
\frac{|\sin(\omega)|}{|\omega|} + \frac{|\sin(\alpha \omega)|}{|\omega|} > \frac{c}{(1 + |\omega|)^n}.
\]

Proof: To prove the claim, we divide the complex plane in three parts.
1. \( V_1 := \{ \omega \in \mathbb{C} \mid |\text{Re}\omega| \leq \pi \text{ and } |\text{Im}\omega| \leq \pi \} \). Since \( \frac{\sin(\omega)}{\omega} \) and \( \frac{\sin(\alpha \omega)}{\omega} \) do not have any common zeros, and \( V_1 \) is a compact set, \( M_1 := \min\{\frac{\sin(\omega)}{|\omega|} + \frac{\sin(\alpha \omega)}{|\omega|} \mid \omega \in V_1\} \) is a well-defined positive number. Obviously, \( \frac{\sin(\omega)}{|\omega|} + \frac{\sin(\alpha \omega)}{|\omega|} \geq M_1 \) for all \( \omega \in V_1 \).

2. \( V_2 := \{ \omega \in \mathbb{C} \mid |\text{Re}\omega| > \pi \} \). Choose \( c_1 > 0 \) such that (31) is satisfied for all \( x \in \mathbb{R} \) with \( |x| > \pi \). Application of (32) and (31) yields for every \( \omega \in V_2 \):

\[
\frac{|\sin(\omega)|}{|\omega|} + \frac{|\sin(\alpha \omega)|}{|\omega|} \geq \frac{1}{|\omega|} \left( |\sin(\text{Re}\omega)| + |\sin(\alpha \text{Re}\omega)| \right) \\
\geq \frac{c_1}{1 + |\omega|} \cdot \left( 1 + |\text{Re}\omega| \right)^{n-1} \geq \frac{c_1}{(1 + |\omega|)^n}.
\]

3. \( V_3 := \{ \omega \in \mathbb{C} \mid |\text{Re}\omega| \leq \pi \text{ and } |\text{Im}\omega| > \pi \} \). Define \( M_2 := \sinh(|\alpha \pi|) \). Using (32) we find for all \( \omega = x + iy \in V_3 \):

\[
\frac{|\sin(\omega)|}{|\omega|} + \frac{|\sin(\alpha \omega)|}{|\omega|} \geq \frac{|\sin(\omega)|}{|\omega|} > \frac{1}{1 + |\omega|} \cdot |\sin(\alpha x) \cosh(\alpha y) + i \cos(\alpha x) \sinh(\alpha y)| \\
\geq \frac{1}{1 + |\omega|} |\sin(\alpha x) M_2 + i \cos(\alpha x) M_2| = M_2 \frac{1}{1 + |\omega|}.
\]

We conclude that \( c := \frac{1}{2} \min(c_1, M_1, M_2) \) satisfies the claim.

Proposition A.4 and Theorem 4.4 imply that if \( \alpha \) is an algebraic number, then the ideal generated by \( g_1, g_2 \in PW(\mathbb{C}) \), with \( g_1(\omega) = \frac{\sin(\omega)}{\omega} \) and \( g_2(\omega) = \frac{\sin(\alpha \omega)}{\omega} \), is the whole ring \( PW(\mathbb{C}) \). Since \( f_1(\omega) = e^{i\omega} g_1(\omega) \) and \( f_2(\omega) = e^{i\alpha \omega} g_2(\omega) \), and \( e^{i\omega} \) and \( e^{i\alpha \omega} \) are units of \( PW(\mathbb{C}) \), we conclude that \( (f_1, f_2)_{PW(\mathbb{C})} = PW(\mathbb{C}) \).

References


