A taxonomy of finite automata minimization algorithms

*Citation for published version (APA):*

*Document status and date:*
Published: 01/01/1993

*Document Version:*
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

*Please check the document version of this publication:*
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

*Document status and date:*
Published: 01/01/1993

*Document Version:*
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

*Please check the document version of this publication:*
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

*General rights*
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

*Take down policy*
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
A taxonomy of finite automata
minimization algorithms

by

Bruce W. Watson

ISSN 0926-4515

All rights reserved
editors: prof.dr. J.C.M. Baeten
prof.dr. M. Rem

Computing Science Report 93/44
Eindhoven, January 1995
A taxonomy of finite automata minimization algorithms*

Bruce W. Watson
Faculty of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
e-mail: watson@win.tue.nl
Tel: +31 40 474319
January 24, 1995

Abstract
This paper presents a taxonomy of finite automata minimization algorithms. Brzozowski's elegant minimization algorithm differs from all other known minimization algorithms, and is derived separately. All of the remaining algorithms depend upon computing an equivalence relation on states. We define the equivalence relation, the partition that it induces, and its complement. Additionally, some useful properties are derived. It is shown that the equivalence relation is the greatest fixed point of an equation, providing a useful characterization of the required computation. We derive an upperbound on the number of approximation steps required to compute the fixed point. Algorithms computing the equivalence relation (or the partition, or its complement) are derived systematically in the same framework. The algorithms include Hopcroft's, several algorithms from text-books (including Hopcroft and Ullman's [HU79], Wood's [Wood87], and Aho, Sethi, and Ullman's [ASU86]), and several new algorithms or variants of existing algorithms.

*Third printing.
## Contents

1 Introduction ........................................... 2

2 An algorithm due to Brzozowski ....................... 5

3 Minimization by equivalence of states ................. 6
   3.1 Distinguishability .................................. 7
   3.2 An upper bound on the number of approximation steps .... 8
   3.3 Characterizing the equivalence classes of $E$ ............ 8

4 Algorithms computing $E$, $D$, or $[Q]_E$ .......... 9
   4.1 Computing $D$ and $E$ by layerwise approximations ....... 9
   4.2 Computing $D$, $E$, and $[Q]_E$ by unordered approximation .... 10
   4.3 More efficiently computing $D$ and $E$ by unordered approximation .... 10
   4.4 An algorithm due to Hopcroft and Ullman ............... 11
   4.5 Hopcroft’s algorithm to compute $[Q]_E$ efficiently .... 12
   4.6 Computing $(p, q) \in E$ .......................... 15
   4.7 Computing $E$ by approximation from below ............. 16

5 Conclusions ........................................... 18

A Some basic definitions ................................ 19

B Finite automata ........................................ 20
   B.1 Properties of finite automata ....................... 20
   B.2 Transformations on finite automata .................. 23

References ............................................. 24
1 Introduction

The minimization of deterministic finite automata is a problem that has been studied since the late 1950's. Simply stated, the problem is to find the unique (up to isomorphism) minimal deterministic finite automaton that accepts the same language as a given deterministic finite automaton. Algorithms solving this problem are used in applications ranging from compiler construction to hardware circuit minimization. With such a variety of applications, the number of differing presentations also grew: most text-books present their own variation, while the algorithm with the best running time (Hopcroft's) remains obscure and difficult to understand.

This report presents a taxonomy of finite automata minimization algorithms. The need for a taxonomy is illustrated by the following:

- Most text-book authors claim that their minimization algorithm is directly derived from those presented by Huffman [Huff54] and Moore [Moor56]. Unfortunately, most text-books present vastly differing algorithms (for example, compare [AU92], [ASU86], [HU79], and [Wood87]), and only the algorithms presented by Aho and Ullman and by Wood are directly derived from those originally presented in [Huff54, Moor56].

- While most of the algorithms rely on computing an equivalence relation on states, many of the explanations accompanying the algorithm presentations do not explicitly mention whether the algorithm computed the equivalence relation, the partition (of states) that it induces, or its complement.

- Comparison of the algorithms is further hindered by the vastly differing styles of presentation — sometimes as imperative programs, or as functional programs, but frequently only as a descriptive paragraph.

A related taxonomy of finite automata construction algorithms appears in [Wats93].

All except one of the algorithms rely on determining the set of automaton states which are equivalent\(^1\). The algorithm that does not make use of equivalent states is discussed in Section 2. In Section 3 the definition and some properties of equivalence of states is given. Algorithms that compute equivalent states are presented in Section 4. The main results of the taxonomy are summarized in the conclusions — Section 5. Appendices A and B give the basic definitions required for reading this paper. The definitions related to finite automata are taken from [Wats93]. The minimization algorithm relationships are shown in a "family tree" in Figure 1.

The principal computation in most minimization algorithms is the determination of equivalent (or inequivalent) states — thus yielding an equivalence relation on states. In this paper, we consider the following minimization algorithms:

- Brzozowski's (possibly nondeterministic) finite automaton minimization algorithm as presented in [Brzo62]. This elegant algorithm (Section 2) was originally invented by Brzozowski, and has since been re-invented without credit to Brzozowski. Given a (possibly nondeterministic) finite automaton without \(\epsilon\)-transitions, this algorithm produces the minimal deterministic finite automaton accepting the same language.

- Layerwise computation of equivalence as presented in [Wood87, Moor56, Brau88, Urba89]. This algorithm (Algorithm 4.2) is a straightforward implementation suggested by the approximation sequence arising from the fixed-point definition of equivalence of states.

- Unordered computation of equivalence. This algorithm (Algorithm 4.3, not appearing in the literature) computes the equivalence relation; pairs of states (for consideration of equivalence) are chosen in an arbitrary order.

- Unordered computation of equivalence classes as presented in [ASU86]. This algorithm (Algorithm 4.4) is a modification of the above algorithm computing equivalence of states.

\(^1\)Equivalence of states is defined later.
Figure 1: The family trees of finite automata minimization algorithms. Brzozowski's minimization algorithm is unrelated to the others, and appears as a separate (single vertex) tree. Each algorithm presented in this paper appears as a vertex in this tree. For each algorithm that appears explicitly in this paper, the construction number appears in parentheses (indicating where it appears in this paper). For algorithms that do not appear explicitly, a reference to the section or page number is given. Edges denote a refinement of the solution (and therefore explicit relationships between algorithms). They are labeled with the name of the refinement.
• Improved unordered computation of equivalence. This algorithm (Algorithm 4.5, not appearing in the literature) also computes the equivalence relation in an arbitrary order. The algorithm is a minor improvement over the other unordered algorithm.

• Improved unordered computation of equivalence classes. This algorithm (Algorithm 4.6, not appearing in the literature) is a modification of the above algorithm to compute the equivalence classes of states. This algorithm is used in the derivation of Hopcroft's minimization algorithm.

• Hopcroft and Ullman's algorithm as presented in [HU79J. This algorithm (Algorithm 4.7) computes the inequivalence (distinguishability) relation. Although it is based upon the algorithms of Huffman and Moore [Huff54, Moor56], this algorithm uses some interesting encoding techniques.

• Hopcroft's algorithm as presented in [Hopc71, Grie73J. This algorithm (Algorithm 4.8) is the best known algorithm (in terms of running time analysis) for minimization. As the original presentation by Hopcroft is difficult to understand, the presentation in this paper is based upon the one given by Gries.

• Pointwise computation of equivalence. This algorithm (Algorithm 4.9, not appearing in the literature) computes the equivalence of a given pair of states. It draws upon some non-automata related techniques, such as: structural equivalence of types and memoization of functional programs.

• Computation of equivalence from below (with respect to refinement). This algorithm (Algorithm 4.10, not appearing in the literature) computes the equivalence relation from below. Unlike any of the other known algorithms, the intermediate result of this algorithm can be used to construct a smaller (although not minimal) deterministic finite automaton.
2 An algorithm due to Brzozowski

Most minimization algorithms are applied to a DFA. In the case of a nondeterministic FA, the subset construction is applied first, followed by the minimization algorithm. In this section, we consider the possibility of applying the subset construction (with useless state removal) after an (as yet unknown) algorithm to yield a minimal DFA. We now construct such an algorithm. (The algorithm described in this section can also be used to construct the minimal Complete DFA, by replacing function subsetopt with subset.)

Let \( M_0 = (Q_0, V, T_0, \emptyset, S_0, F_0) \) be the \( \epsilon \)-free FA to be minimized and \( M_2 = (Q_2, V, T_2, \emptyset, S_2, F_2) \) be the minimized DFA such that \( \mathcal{L}_{FA}(M_0) = \mathcal{L}_{FA}(M_2) \) (and of course \( \text{Min}(M_2) \) — see Definition B.19). (For the remainder of this section we make use of \( \text{Minimal} \) (Property B.21) as opposed to \( \text{Min} \).) Since we apply the subset construction last, we have some intermediate finite automaton \( M_1 = (Q_1, V, T_1, \emptyset, S_1, F_1) \) such that \( M_2 = \text{useful}_s \circ \text{subsetopt}(M_1) \). We require that \( M_1 \) is somehow obtained from \( M_0 \), and that \( \mathcal{L}_{FA}(M_2) = \mathcal{L}_{FA}(M_1) = \mathcal{L}_{FA}(M_0) \).

From the definition of \( \text{Minimal}(M_2) \) (Property B.21), we require

\[
(\forall p, q : p \in Q_2 \land q \in Q_2 \land p \neq q : \overline{E}(p) \neq \overline{E}(q)) \land \text{Useful}(M_2)
\]

For all states \( q \in Q_2 \) we have \( q \in \mathcal{P}(Q_1) \) since \( M_2 = \text{useful}_s \circ \text{subsetopt}(M_1) \). Property B.25 of the subset construction gives

\[
(\forall p : p \in Q_2 : \overline{E}(p) = (\cup q : q \in Q_1 \land q \in p : \overline{E}(q))
\]

We need a sufficient condition on \( M_1 \) to ensure \( \text{Minimal}(M_2) \). The following derivation gives such a condition:

\[
\text{Minimal}(M_2) \\
\equiv \{ \text{Definition of } \text{Minimal} \ (\text{Property B.21}) \} \\
(\forall p, q : p \in Q_2 \land q \in Q_2 \land p \neq q : \overline{E}(p) \neq \overline{E}(q)) \land \text{Useful}(M_2) \\
\equiv \{ \text{Property B.25; } M_2 = \text{useful}_s \circ \text{subsetopt}(M_1) \} \\
(\forall p, q : p \in Q_1 \land q \in Q_1 \land p \neq q : \overline{E}(p) \cap \overline{E}(q) = \emptyset) \land \text{Useful}_f(M_1) \\
\equiv \{ \text{Definition of } \text{Det}' \ (\text{Property B.18)} \text{ and } \text{Useful}_s, \text{Useful}_f \ (\text{Remark B.13}) \} \\
\text{Det}'(M_R^R) \land \text{Useful}_s(M_R^R) \\
\equiv \{ \text{Det}'(M) \equiv \text{Det}(M) \} \\
\text{Det}(M_R^R) \land \text{Useful}_s(M_R^R)
\]

The required condition on \( M_1 \) can be established by (writing reversal as a prefix function) \( M_1 = R \circ \text{useful}_s \circ \text{subsetopt} \circ R(M_0) \).

The complete minimization algorithm (for any \( \epsilon \)-free \( M_0 \in \text{FA} \)) is

\[
M_2 = \text{useful}_s \circ \text{subsetopt} \circ R \circ \text{useful}_s \circ \text{subsetopt} \circ R(M_0)
\]

This algorithm was originally given by Brzozowski in [Brz62]. The origin of this algorithm was obscured when Jan van de Snepscheut presented the algorithm in his Ph.D thesis [vdSn85]. In this thesis, the algorithm is attributed to a private communication from Prof. Peremans of the Eindhoven University of Technology. Peremans had originally found the algorithm in an article by Mirkin [Mirk65]. Although Mirkin does cite a paper by Brzozowski [Brz64], it is not clear whether Mirkin's work was influenced by Brzozowski's work on minimization. Jan van de Snepscheut's recent book [vdSn93] describes the algorithm, but provides neither a history nor citations (other than his thesis) for this algorithm.
3 Minimization by equivalence of states

In this subsection, we restrict ourselves to considering minimization of Complete DFA's. This is strictly a notational convenience, as the minimization algorithms can be modified to work for non-Complete DFA's. A Complete minimized DFA will (in general) have one more state (a sink state) than a non-Complete minimized DFA, unless the language of the DFA is \( V' \). Let \( M = (Q, V, T, \emptyset, S, F) \) be a Complete DFA; this particular DFA will be used throughout this section. We also assume that all of the states of \( M \) are start-reachable, that is \( \text{Useful}_u(M) \). Since \( M \) is deterministic and Complete, we will also take the transition relation to be total function \( T \in Q \times V \longrightarrow Q \) instead of \( T \in Q \times V \longrightarrow P(Q) \).

In order to minimize the DFA \( M \), we compute an equivalence relation \( E \subseteq Q \times Q \) defined as:

\[
(p, q) \in E \equiv (L\overline{c}(p) = \overline{L}(q))
\]

Since this is an equivalence relation, we are really interested in unordered pairs of states. It is notationally more convenient to use ordered pairs instead of unordered pairs.

The equivalent states are merged according to equivalence relation \( E \) with the merge transformation.

**Transformation 3.1 (Merging states):** For any equivalence relation \( H \) such that \( H \subseteq E \), the function \( \text{merge} \) can be used to reduce the number of states in the DFA\(^2\). Function \( \text{merge} \) is defined as:

\[
\text{merge}((Q, V, T, \emptyset, \{s\}, F), H) = \begin{cases} T' = \{(p, a, [q]_H) \in T : (p, a, q) \in T \} & \text{let} \\ \end{cases}
\]

The definition of \( \text{merge} \) is independent of the choice of representatives of the equivalence classes. Function \( \text{merge} \) satisfies the property that

\[
\mathcal{L}_F\text{merge}(M, H) = \mathcal{L}_F(M) \land |\text{merge}(M, H)| \leq |M| \land |\text{merge}(M, H)| = |H|
\]

and it preserves Complete, \( \epsilon \)-free, Useful, Det, and Minimal; indeed, \( \text{merge} \) is only defined on \( \epsilon \)-free and deterministic FA's. \( \square \)

In order to compute relation \( E \), we need a property of function \( \overline{L} \).

**Property 3.2 (Function \( \overline{L} \)):** Function \( \overline{L} \) satisfies

\[
\overline{L}(p) = (\cup a : a \in V : \{a\} \cdot \overline{L}(T(p, a)) \cup (\text{if } p \in F \text{ then } \{\epsilon\} \text{ else } \emptyset))
\]

\( \square \)

This allows us to give an alternate (but equivalent) characterization of equivalence of states.

**Definition 3.3 (Equivalence of states):** Equivalence relation \( E \) is the greatest (under refinement) fixed point of the equivalence

\[
(p, q) \in E \equiv (p \in F \equiv q \in F) \land (\forall a : a \in V : (T(p, a), T(q, a)) \in E)
\]

\( \square \)

**Remark 3.4:** The greatest fixed point has the least number of equivalence classes of any such fixed point. \( \square \)

**Remark 3.5:** Any fixed point of the equivalence in Definition 3.3 can be used. In order to minimize the automaton, the greatest fixed point is desired. \( \square \)

\(^2\)When \( H \) is the identity relation on states, function \( \text{merge} \) will not reduce the number of states.
3.1 Distinguishability

Property 3.6 (Approximating \(E\)): We can compute this greatest fixed point with successive approximations. The successive approximations of \(E\) are as follows (for \(k \geq 0\)):

\[(p,q) \in E_{k+1} \equiv (p,q) \in E_k \land (\forall a : a \in V : (T(p,a), T(q,a)) \in E_k)\]

where \(E_0\) is defined by

\[(p,q) \in E_0 \equiv (p \in F \iff q \in F)\]

An equivalent definition of \(E_0\) is \(E_0 = (Q \setminus F)^2 \cup F^2\). We also have the property that \(E_{k+1} \subseteq E_k\) for all \(k \geq 0\).

Remark 3.7: If \(E_k\) is an equivalence relation, then so is \(E_{k+1}\). \(E_0\) is an equivalence relation.

Remark 3.8: An intuitive explanation of \(E_k\) is useful. A pair of states \(p,q\) are said to be \(k\)-equivalent (written \((p,q) \in E_k\)) if and only if there is no string \(w : |w| \leq k\) such that \(w \in \overline{L}(p) \neq w \in \overline{L}(q)\). As a consequence, \(p\) and \(q\) are \(k\)-equivalent if and only if

- they are both final or both non-final, and
- for all \(a \in V\), \(T(p,a)\) and \(T(q,a)\) are \((k - 1)\)-equivalent (by the definitions of \(\overline{L}\) and \(T^*\)).

Remark 3.9: An important property of \(E\) is that it is also the greatest fixed point, under \(\subseteq\) (set containment instead of refinement), of the equivalence in Definition 3.3. As the greatest fixed point, \(E\) can be computed with a \(\subseteq\)-descending sequence of relations, starting with \(Q \times Q\). Such a sequence need not consist only of equivalence relations. There may be more steps in such an approximating sequence than in the \(E_k\) sequence given above. Fortunately, each such step is usually easier to compute than computing \(E_{k+1}\) from \(E_k\). Some algorithms that compute these cheaper (but longer) sequences are given in Sections 4.2-4.5 and 4.7.

All previously known algorithms compute \(E\) by successive approximation from above (with respect to \(\subseteq\)). A new algorithm in Section 4.7 computes \(E\) by successive approximation from below. In that section, the practical importance of this is explained.

3.1 Distinguishability

It is also possible to compute \(E\) by first computing its complement \(D = \neg E\). Relation \(D\) (called the distinguishability relation on states) is defined as:

\[(p,q) \in D \equiv (\overline{L}(p) \neq \overline{L}(q))\]

Definition 3.10 (Distinguishability of states): \(D\) is the least (under \(\subseteq\), set containment\(^3\)) fixed point of an equation

\[(p,q) \in D \equiv (p \in F \neq q \in F) \lor (\exists a : a \in V : (T(p,a), T(q,a)) \in D)\]

\(\square\)

Property 3.11 (Approximating \(D\)): As with equivalence relation \(E\), relation \(D\) can be computed by successive approximations (for \(k \geq 0\))

\[(p,q) \in D_{k+1} \equiv (p,q) \in D_k \lor (\exists a : a \in V : (T(p,a), T(q,a)) \in D_k)\]

with \(D_0 = \neg E_0 = ((Q \setminus F) \times F) \cup (F \times (Q \setminus F))\). For all \(k \geq 0\) we have \(D_k = \neg E_k\). We also have the property that \(D_{k+1} \supseteq D_k\) for \(k \geq 0\).

\(^3\)Here, \(\subseteq\) denotes normal set containment; refinement does not apply since \(D\) is not necessarily an equivalence relation.
Remark 3.12: As with \( E_k \), an intuitive explanation of \( D_k \) is useful. A pair of states \( p, q \) are said to be \( k \)-distinguished (written \( (p, q) \in D_k \)) if and only if there is a string \( w : |w| \leq k \) such that \( w \in \subseteq \mathcal{L}(p) \neq w \in \subseteq \mathcal{L}(q) \). As a consequence, \( p \) and \( q \) are \( k \)-distinguished (some authors say \( k \)- distinguishable) if and only if

- one is final and the other is non-final, or
- there exists \( a \in V \) such that \( T(p, a) \) and \( T(q, a) \) are \( (k - 1) \)-distinguished.

\( \square \)

3.2 An upperbound on the number of approximation steps

We can easily place an upperbound on the number of steps in the computation of \( E \).

Let \( E_j \) be the greatest fixed point of the equation defining \( E \). We have the sequence of approximations (where \( I_Q \) is the identity relation on states):

\[
E_0 \supset E_1 \supset \cdots \supset E_j \supset I_Q
\]

The indices of some of the equivalence relations in the approximation sequence are known: \( \sharp I_Q = |Q| \) and \( \sharp E_0 = 2 \). We can deduce that:

\[
\sharp E_0 < \sharp E_1 < \cdots < \sharp E_j < \sharp I_Q = |Q|
\]

In the case that \( \sharp E_0 = 0 \), we have that \( E_0 \) is the greatest fixed point. In the case that \( \sharp E_1 = 1 \), either all states are final states, or all states are non-final ones; in both cases \( E_0 \) is the greatest fixed point. In the case that \( \sharp E_0 = 2 \), we have \( j + 2 \leq \sharp E_j \). Since \( j + 2 \leq \sharp I_Q = |Q| \) we get

\[
 j \leq |Q| - 2.\quad \text{This gives an upperbound of } \left(|Q|-2\right)\max 0 \text{ steps for the computation (starting at } E_0 \text{) of the greatest fixed point } E_j \text{ (using the approximating sequence given in Property 3.6).}
\]

A consequence of this upperbound is that \( E = E_j \). As we shall see later, this can lead to some efficiency improvements to algorithms computing \( E \). This result is also noted by Wood [Wood87, Lemma 2.4.1]. This upperbound also holds for computing \( D \) and \( [Q]_E \) by approximation.

3.3 Characterizing the equivalence classes of \( E \)

It is also practical to compute \( [Q]_E \): the set of equivalence classes of \( E \). In order to characterize partition \( [Q]_E \), we begin our derivation with Definition 3.3, the characterization of \( E \) as the largest equivalence relation (under \( \subseteq \)) such that

\[
(\forall p, q : (p, q) \in E : (p \in F \equiv q \in F) \land (\forall a : a \in V : (T(p, a), T(q, a)) \in E))
\equiv
(\forall p, q, a : (p, q) \in E \land a \in V : (p \in F \equiv q \in F) \land [T(p, a)]_E = [T(q, a)]_E)
\equiv
(\forall Q_0, Q_1, a : Q_0 \subseteq [Q]_E \land Q_1 \subseteq [Q]_E \land a \in V :
(\forall p, q : p \in Q_0 \land q \in Q_0 : (p \in F \equiv q \in F) \land (T(p, a) \in Q_1 \equiv T(q, a) \in Q_1)))
\]

Definition 3.13 (Function \textit{Splittable}): In order to make this quantification more concise, we define

\[
\text{Splittable}(Q_0, Q_1, a) \equiv (\exists p, q : p \in Q_0 \land q \in Q_0 : (T(p, a) \in Q_1 \neq T(q, a) \in Q_1))
\]

\( \square \)

Using \textit{Splittable}, \( [Q]_E \) is the largest partition (under \( \subseteq \)) such that \( [Q]_E \subseteq [Q]_{E_0} \) and

\[
(\forall Q_0, Q_1, a : Q_0 \subseteq [Q]_E \land Q_1 \subseteq [Q]_E \land a \in V : \neg \text{Splittable}(Q_0, Q_1, a))
\]

This characterization will be used in the computation of \( [Q]_E \).
4 Algorithms computing $E$, $D$, or $[Q]_E$

In this section, we consider several algorithms that compute $D$, $E$, or $[Q]_E$. Some of the algorithms are presented in general terms: computing $D$ and $E$. Since only one of $D$ or $E$ is needed (and not both), such a general algorithm would be modified for practical use to compute only one of the two.

4.1 Computing $D$ and $E$ by layerwise approximations

The definition of $E_{k+1}$ in terms of $E_k$ (and likewise for $D$) leads naturally to the following algorithm computing $D$ and $E$ (where variable $k$ is a ghost variable, used only for specifying the invariant)

Algorithm 4.1: 

\[
\begin{align*}
G, H &:= D_0, E_0; \\
G_{old}, H_{old}, k &:= \emptyset, Q \times Q, 0; \\
\{\text{invariant: } G = D_k \land H = E_k \} \\
\text{do } G \neq G_{old} \rightarrow \\
G \neq G\_{old} \land H \neq H\_{old} \rightarrow \\
G_{old}, H_{old} &:= G, H; \\
G &:= (\cup (p, q) : (p, q) \in G_{old} \land (\exists a : a \in V : (T(p, a), T(q, a)) \in G_{old}) : \{(p, q)\}); \\
H &:= (\cup (p, q) : (p, q) \in H_{old} \land (\forall a : a \in V : (T(p, a), T(q, a)) \in H_{old}) : \{(p, q)\}); \\
G &:= \neg H; \\
k &:= k + 1 \\
\text{od}\{G = D \land H = E\}
\end{align*}
\]

This algorithm is said to compute $D$ and $E$ layerwise, since it computes the sequences $D_k$ and $E_k$. The update of $G$ and $H$ in the repetition can be made with another repetition as shown in the program now following.

Algorithm 4.2 (Layerwise computation of $D$ and $E$): 

\[
\begin{align*}
G, H &:= D_0, E_0; \\
G_{old}, H_{old}, k &:= \emptyset, Q \times Q, 0; \\
\{\text{invariant: } G = D_k \land H = E_k \} \\
\text{do } G \neq G_{old} \rightarrow \\
G \neq G\_{old} \land H \neq H\_{old} \rightarrow \\
G_{old}, H_{old} &:= G, H; \\
\text{for } (p, q) : (p, q) \in H_{old} \text{ do } \\
\text{if } (\exists a : a \in V : (T(p, a), T(q, a)) \in G_{old}) \rightarrow G, H := G \cup \{(p, q)\}, H \setminus \{(p, q)\} \\
\| (\forall a : a \in V : (T(p, a), T(q, a)) \in H_{old}) \rightarrow \text{skip} \\
\text{fi} \\
\text{ref} \\
\{G = \neg H\} \\
k &:= k + 1 \\
\text{od}\{G = D \land H = E\}
\end{align*}
\]

The algorithm can be split into two: one computing only $D$, and the other computing only $E$. The algorithm computing only $E$ is essentially the algorithm presented by Wood in [Wood87, pg. 132]. According to Wood, it is based on the work of Moore [Moor56]. Its running times is $O(|Q|^3)$. Brauer uses some encoding techniques to provide an $O(|Q|^2)$ version of this algorithm in [Brau88], while Urbanek improves upon the space requirements of Brauer's version in [Urba89]. None of these variants is given here. The algorithm computing only $D$ does not appear in the literature.

With a little effort this algorithm can be modified to compute $[Q]_E$. 

4.2 Computing $D$, $E$, and $[Q]_E$ by unordered approximation

Instead of computing each $E_k$ (computing $E$ layerwise), we can compute $E$ by considering pairs of states in an arbitrary order (as outlined in Remark 3.9). This is done in the following algorithm (which also computes $D$):

Algorithm 4.3:

$$
\begin{align*}
G, H &:= D_0, E_0; \\
&\text{\{invariant: } G = \neg H \land G \subseteq D \} \\
\text{do } (\exists p, q, a : a \in V \land (p, q) \in H : (T(p, a), T(q, a)) \in G) \rightarrow \\
&\quad \text{let } p, q : (p, q) \in H \land (\exists a : a \in V : (T(p, a), T(q, a)) \in G); \\
&\quad \{(p, q) \in D\} \\
&\quad G, H := G \cup \{(p, q)\}, H \setminus \{(p, q)\} \\
\text{od} \{G = D \land H = E\}
\end{align*}
$$

This algorithm can be split into one computing only $D$, and one computing only $E$. At the end of each iteration step, it may be that $H$ is not an equivalence relation (that is, $H \neq H'$) — see Remark 3.9. A slight modification to this algorithm can be made by adding the following assignment before the od:

$$
H := (\text{MAX}_J J: J \subseteq H \land J = J^* : J); G := \neg H
$$

Addition of this assignment makes the algorithm compute the refinement sequence $E_k$ (see Remark 3.9). This assignment may improve the running time of the algorithm if a cheap method of computing the quantified MAX is used. This algorithm does not appear in the literature.

When we convert the above algorithm to compute $[Q]_E$, the resulting algorithm is the following one, given by Aho, Sethi, and Ullman in [ASU86, Alg. 3.6]:

Algorithm 4.4:

$$
\begin{align*}
P &:= [Q]_{E_0}; \\
&\text{\{invariant: } [Q]_E \subseteq P \subseteq [Q]_{E_0}\} \\
\text{do } (\exists Q_0, Q_1, a : Q_0 \in P \land Q_1 \in P \land a \in V : \text{Splittable}(Q_0, Q_1, a)) \rightarrow \\
&\quad \text{let } Q_0, Q_1, a : \text{Splittable}(Q_0, Q_1, a); \\
&\quad Q_0' := \{p : p \in Q_0 \land T(p, a) \in Q_1\}; \\
&\quad \{\neg \text{Splittable}(Q_0, Q_1, a) \land \neg \text{Splittable}(Q_0, Q_1, a)\} \\
&\quad P := P \setminus \{Q_0\} \cup \{Q_0 \setminus Q_0, Q_0\} \\
\text{od} \\
&\quad \{\forall Q_0, Q_1, a : Q_0 \in P \land Q_1 \in P \land a \in V : \neg \text{Splittable}(Q_0, Q_1, a)\} \\
&\quad \{P = [Q]_E\}
\end{align*}
$$

This algorithm has running time $O(|Q|^2)$.

4.3 More efficiently computing $D$ and $E$ by unordered approximation

We present another algorithm that considers pairs of states in an arbitrary order. This algorithm (which also computes $D$) consists of two nested repetitions:
Algorithm 4.5:

\[G, H := D_0, E_0;\]
{invariant: \( G = \neg H \land G \subseteq D \)}
\[\text{do (3 p, q, a : a \in V \land (p, q) \in H : (T(p, a), T(q, a)) \in G) \rightarrow}
\text{let } p, a : p \in Q \land a \in V \land (3 q : (p, q) \in H : (T(p, a), T(q, a)) \in G);
\text{for } q : (p, q) \in H \land (T(p, a), T(q, a)) \in G \text{ do}
G, H := G \cup \{(p, q)\}, H \setminus \{(p, q)\}
\text{rof}
\text{od}\{G = D \land H = E\}

As with Algorithm 4.3, at the end of each outer iteration step, it may be that \( H \neq H^* \). This can be solved with an assignment to \( H \) as can be done in Algorithm 4.3. This algorithm does not appear in the literature. It can also be modified to compute only \( D \) or only \( E \).

Modifying the above algorithm to compute \([Q]_E\) is particularly interesting; the modified algorithm will be used in Section 4.5 to derive an algorithm (by Hopcroft) which is the best known algorithm for FA minimization. The algorithm is (where variable \( P_{old} \) is used only for the invariant):

Algorithm 4.6:

\[P := [Q]_{E_0};\]
{invariant: \([Q]_E \subseteq P \subseteq [Q]_{E_0}\)}
\[\text{do (3 Q_1, a : Q_1 \in P \land a \in V : (3 Q_0 : Q_0 \in P : \text{Splittable}(Q_0, Q_1, a))) \rightarrow}
\text{let } Q_1, a : (3 Q_0 : Q_0 \in P : \text{Splittable}(Q_0, Q_1, a));
\text{P_{old} := P;}
{invariant: \([Q]_E \subseteq P \subseteq P_{old}\)}
\text{for } Q_0 : Q_0 \in P_{old} \land \text{Splittable}(Q_0, Q_1, a) \text{ do}
Q_0' := \{p : p \in Q_0 \land T(p, a) \in Q_1\};
P := P \setminus \{Q_0\} \cup \{Q_0 \setminus Q_0', Q_0'\}
\text{rof}
\{V Q_0 : Q_0 \in P : \neg \text{Splittable}(Q_0, Q_1, a)\}
\text{od}
\{V Q_1, a : Q_1 \in P \land a \in V : (\forall Q_0 : Q_0 \in P : \neg \text{Splittable}(Q_0, Q_1, a))\}
\{P = [Q]_E\}

The inner repetition "splits" each eligible equivalence class \( Q_0 \) with respect to pair \((Q_1, a)\). (In actuality, some particular \( Q_0 \) will not be split by \((Q_1, a)\) if \(\neg \text{Splittable}(Q_0, Q_1, a)\).)

4.4 An algorithm due to Hopcroft and Ullman

From the definition of \( D \), we see that a pair \((p, q)\) is in \( D \) if and only if \( p \in F \neq q \in F \) or there is some \( a \in V \) such that \((T(p, a), T(q, a)) \in D\). This forms that basis of the algorithm considered in this subsection. With each pair of states \((p, q)\) we associate a set of pairs of states \(L(p, q)\) such that

\[(r, s) \in L(p, q) \Rightarrow ((p, q) \in D \Rightarrow (r, s) \in D)\]

For each pair \((p, q)\) (such that \((p, q) \notin D_0 — p \text{ and } q \text{ are not already known to be distinguished})
we do the following:

- If there is an \( a \in V \) such that we know that \((T(p, a), T(q, a)) \in D\) then \((p, q) \in D\). We add \((p, q)\) to our approximation of \( D \), along with \( L(p, q) \), and for each \((r, s) \in L(p, q)\) add \(L(r, s)\), and for each \((t, u) \in L(r, s)\) add \(L(t, u)\), etc.
If there is no \( a \in V \) such that \( T((p, a), T(q, a)) \in D \) is known to be true, then for all \( b \in V \) we put \( (p, q) \) in the set \( L(T(p, b), T(q, b)) \). If later it turns out that for some \( b \in V \), \( T((p, b), T(q, b)) \in D \), then we will also put \( L(T(p, b), T(q, b)) \) (including \( (p, q) \)) in \( D \).

In our presentation of the algorithm, the invariants given are not sufficient to prove the correctness of the algorithm, but are used to illustrate the method in which the algorithm works. The algorithm is:

```
Algorithm 4.7:

for \((p, q) : (p, q) \in (Q \times Q)\) do
    \( L(p, q) := \emptyset \)
    \( \text{rof} \)
    \( G := D_0 \);  
{invariant: \( G \subseteq D \wedge (\forall p, q : (p, q) \notin D_0 : (\forall r, s : (r, s) \in L(p, q) : (p, q) \in D \Rightarrow (r, s) \in D)) \} \)
for \((p, q) : (p, q) \notin D_0\) do
    if \((\exists a : a \in V : (T(p, a), T(q, a)) \in G) \rightarrow \)
        \( A, B := \{(p, q)\}, \emptyset \);
        \{invariant: \( A \subseteq D \wedge B \subseteq G \wedge A \cap B = \emptyset \) \)
        \( \wedge A \cup B = (\cup p, q : (p, q) \in B : L(p, q)) \}
    if \( A \neq \emptyset \) \rightarrow \)
        let \((r, s) : (r, s) \in A \);
        \( G := G \cup \{(r, s)\} \);
        \( A, B := A \setminus \{(r, s)\}, B \cup \{(r, s)\} \);
        \( A := A \cup (L(r, s) \setminus B) \);
    od
{\( (\forall a : a \in V : (T(p, a), T(q, a)) \notin G) \rightarrow \)
for \( a \in V : T(p, a) \neq T(q, a) \) do
    \( L(T(p, a), T(q, a)) \in D \Rightarrow (p, q) \in D \)
    \( L(T(p, a), T(q, a)) := L(T(p, a), T(q, a)) \cup \{(p, q)\} \)
    \( \text{rof} \)
fi
\( \text{rof} \)
\( G = D \)
```

This algorithm has running time \( O(|Q|^2) \) and is given by Hopcroft and Ullman [HU79, Fig. 3.8]. In [HU79] it is attributed to Huffman [Huff54] and Moore [Moor56]. In their description, Hopcroft and Ullman describe \( L \) as mapping each pair of states to a list of pairs of states. The list data-type is not required here, and a set is used instead.

It is possible to modify the above algorithm to compute \( E \). Such an algorithm does not appear in the literature.

### 4.5 Hopcroft's algorithm to compute \([Q]_E\) efficiently

We now derive an efficient algorithm due to Hopcroft [Hopc71]. This algorithm has also been derived by Gries [Grie73]. This algorithm presently has the best known running time analysis of all DFA minimization algorithms.

We begin with Algorithm 4.6. Recall that the inner repetition "splits" each equivalence class \( Q_0 \) with respect to pair \((Q_1, a)\). An observation (due to Hopcroft) is that once all equivalence classes have been split with respect to a particular \((Q_1, a)\), no equivalence classes need to be split with respect to the same \((Q_1, a)\) on any subsequent iteration step of the outer repetition [Hopc71, pp. 190–191]. [Grie73, Lemma 5]. The observation is simple to prove: the equivalence classes never grown in size, and we need only prove that (for all equivalence classes \( \hat{Q} \)):

\[ \neg\text{Splittable}(\hat{Q}, Q_1, a) \Rightarrow (\forall \hat{Q}_0 : \hat{Q}_0 \subseteq \hat{Q} : \neg\text{Splittable}(\hat{Q}, Q_1, a)) \]
4.5 Hopcroft’s algorithm to compute \([Q]_E\) efficiently

We can use this fact to maintain a set \(L\) of such (equivalence class, alphabet symbol) pairs. We will then split the equivalence classes with respect to elements of \(L\). In the original presentations of this algorithm [Hopc71, Grie73], \(L\) is a list. As this is not necessary, we retain \(L\)’s type as a set.

\[P := [Q]_E;\]
\[L := P \times V;\]
\{invariant: \([Q]_E \subseteq P \subseteq [Q]_E \land L \subseteq (P \times V)\}\]
\[\land \forall Q_0, Q_1, a: Q_0 \in Q \land (Q_1, a) \in L \implies \neg \text{Splittable}(Q_0, Q_1, a) \Rightarrow (P = [Q]_E)\]

\[\text{do } L \neq \emptyset \rightarrow \]
\[\text{let } Q_1, a : (Q_1, a) \in L;\]
\[P_{\text{old}} := P;\]
\[L := L \setminus \{(Q_1, a)\};\]
\{invariant: \([Q]_E \subseteq P \subseteq P_{\text{old}}\}\]
\[\text{for } Q_0 : Q_0 \in P_{\text{old}} \land \text{Splittable}(Q_0, Q_1, a) \text{ do}\]
\[Q_0' := \{p : p \in Q_0 \land T(p, a) \in Q_1\};\]
\[P := P \setminus \{Q_0\} \cup \{Q_0 \setminus Q_0', Q_0';\};\]
\[\text{for } b : b \in V \text{ do}\]
\[\text{if } (Q_0, b) \in L \rightarrow L := L \setminus \{(Q_0, b)\} \cup \{(Q_0', b), (Q_0 \setminus Q_0', b)\}\]
\[\text{fi}\]
\[\text{for } b : b \in V \text{ do}\]
\[\text{if } (Q_0, b) \notin L \rightarrow L := L \cup \{(Q_0', b), (Q_0 \setminus Q_0', b)\}\]
\[\text{fi}\]
\[\text{for } b : b \in V \text{ do}\]
\[\text{if } (Q_0, b) \notin L \rightarrow L := L \cup \{(Q_0', b), (Q_0 \setminus Q_0', b)\}\]
\[\text{fi}\]
\[\text{od}\]
\{\forall Q_0 : Q_0 \in P : \neg \text{Splittable}(Q_0, Q_1, a)\}\]

The innermost update of \(L\) is intentionally clumsy and will be used to arrive at the algorithm given by Hopcroft and Gries. In the update of set \(L\), if \((Q_0, b) \in L\) (for some \(b \in V\)) and \(Q_0\) has been split into \(Q_0 \setminus Q_0'\) and \(Q_0'\) then \((Q_0, b)\) is replaced (in \(L\)) by \((Q_0 \setminus Q_0', b)\) and \((Q_0', b)\).

Another observation due to Hopcroft is that splitting an equivalence class with respect to any two of \((Q_0, b), (Q_0', b),\) and \((Q_0 \setminus Q_0', b)\) is the same as splitting the equivalence class with respect to all three [Hopc71, pp. 190–191], [Grie73, Lemma 6]. This is shown in the following intermezzo.

We only prove that: if an equivalence class \(\hat{Q}\) has been split with respect to \((Q_0, b)\) and \((Q_0', b)\), then it need not be split with respect to \((Q_0 \setminus Q_0', b)\). The two remaining cases can be proven analogously.

\[\neg \text{Splittable}(\hat{Q}, Q_0, b) \land \neg \text{Splittable}(\hat{Q}, Q_0', b)\]
\[\equiv\quad\{\text{De Morgan}\}\]
\[\neg(\text{Splittable}(\hat{Q}, Q_0, b) \lor \text{Splittable}(\hat{Q}, Q_0', b))\]
\[\equiv\quad\{\text{Definition of Splittable}\}\]
\[\neg((\exists p, q : p, q \in \hat{Q} : T(p, b) \in Q_0 \neq T(q, b) \in Q_0)\]
\[\lor ((\exists p, q : p, q \in \hat{Q} : T(p, b) \in Q_0 \lor T(q, b) \in Q_0))\]
\[\Rightarrow\quad\{\text{Combine existential quantifiers}\}\]
\[\neg((\exists p, q : p, q \in \hat{Q} : T(p, b) \in Q_0 \neq T(q, b) \in Q_0) \lor (T(p, b) \in Q_0 \neq T(q, b) \in Q_0))\]
\[\Rightarrow\quad\{Q_0 \subseteq Q_0\}\]
\[\neg((\exists p, q : p, q \in \hat{Q} : T(p, b) \in Q_0 \land Q_0 \neq T(q, b) \in Q_0 \lor Q_0 \setminus Q_0' \neq T(q, b) \in Q_0)\]
\[\Rightarrow\quad\{\text{Definition of Splittable}\}\]
\[\neg \text{Splittable}(\hat{Q}, Q_0 \setminus Q_0', b)\]

The two remaining cases can be proven analogously.
For efficiency reasons we therefore choose the smallest two of the three (comparing \(|Q_0|, |Q'_0|, \) and \(|Q_0 \setminus Q'_0|\)) in the update of set \(L\). If \((Q_0, b) \not\in L\), then splitting has already been done with respect to \((Q_0, b)\) and we add either \((Q'_0, b)\) or \((Q_0 \setminus Q'_0, b)\) (whichever is smallest) to \(L\). On the other hand, if \((Q_0, b) \in L\), then splitting has not yet been done and we remove \((Q_0, b)\) from \(L\) and add \((Q'_0, b)\) and \((Q_0 \setminus Q'_0, b)\) instead.

Lastly, we observe that by starting with \(P = [Q]_E = \{Q \setminus F, F\}\) we have already split \(Q\). As a result, we need only split with respect to either \((Q \setminus F, b)\) or \((F, b)\) (for all \(b \in V\)) [Hopc71, pp. 190–191], [Grie73, Lemma 7].

This gives the algorithm\(^4\):

\begin{algorithm}
\textbf{Algorithm 4.8 (Hopcroft)}:
\begin{align*}
P &:= [Q]_E; \\
L &:= (\text{if } (|F| \leq |Q \setminus F|) \text{ then } \{F\} \text{ else } (Q \setminus F) \times V); \\
\{\text{invariant: } [Q]_E \subseteq P \subseteq [Q]_E \wedge L \subseteq (P \times V) \wedge (\forall Q_0, Q_1, a: Q_0 \in Q \wedge (Q_1, a) \in L : \neg Splittable(Q_0, Q_1, a)) \Rightarrow (P = [Q]_E)\} \\
\text{do } L \neq \emptyset \rightarrow \\
\quad \text{let } Q_1, a: (Q_1, a) \in L; \\
\quad L \text{old} := P; \\
\quad L := L \setminus \{(Q_1, a)\}; \\
\quad \{\text{invariant: } [Q]_E \subseteq P \subseteq P_\text{old}\} \\
\text{for } Q_0 : Q_0 \in P_\text{old} \wedge Splittable(Q_0, Q_1, a) \text{ do} \\
\quad Q'_0 := \{p : p \in Q_0 \wedge T(p, a) \in Q_1\}; \\
\quad P := P \setminus \{Q_0\} \cup \{Q_0 \setminus Q'_0, Q'_0\}; \\
\text{for } b : b \in V \text{ do} \\
\quad \text{if } (Q_0, b) \in L \rightarrow L := L \setminus \{(Q_0, b)\} \cup \{(Q'_0, b), (Q_0 \setminus Q'_0, b)\} \\
\quad \mid (Q_0, b) \not\in L \rightarrow \\
\quad \quad L := L \cup (\text{if } (|Q'_0| \leq |Q_0 \setminus Q'_0|) \text{ then } \{(Q'_0, b)\} \text{ else } \{(Q_0 \setminus Q'_0, b)\} \times V) \\
\quad \text{fi} \\
\text{rof} \\
\quad \{\forall Q_0 : Q_0 \in P : \neg Splittable(Q_0, Q_1, a)\} \\
\text{od}(P = [Q]_E)
\end{align*}
\end{algorithm}

Unfortunately, the running time analysis of this algorithm is complicated and is not discussed here. It is shown by both Hopcroft and Gries that it is \(O(|Q| \log |Q|)\), [Grie73, Hopc71].

\(^4\)Part of the invariant has been omitted, being rather complicated to derive.
4.6 Computing \((p, q) \in E\)

From the problem of deciding the structural equivalence of two types, it is known that equivalence of two states can be computed recursively by turning the fixed point definition of \(E\) into a functional program. If the (unmodified) definition were to be used directly as a functional program, there is the possibility of non-termination. In order for the functional program to work, it takes a third parameter along with the two states.

The following program, similar to the one presented in [t-Ei91], computes relation \(E\) pointwise; an invocation \(\text{equiv}(p, q, \emptyset)\) determines whether states \(p\) and \(q\) are equivalent. It assumes that two states are equivalent (by placing the pair of states in \(S\), the third parameter) until shown otherwise.

```plaintext
function equiv(p, q, S) is
    if \(\{p, q\} \in S\) → eq := true
    | \{p, q\} \not\in S →
        eq := (p \in F \equiv q \in F);
        eq := eq \land (\forall a : a \in V : equiv(T(p, a), T(q, a), S \cup \{\{p, q\}\}))
    fi;
    return eq
```

The \(\forall\) quantification can be implemented using a repetition.

```plaintext
function equiv(p, q, S) is
    if \(\{p, q\} \in S\) → eq := true
    | \{p, q\} \not\in S →
        eq := (p \in F \equiv q \in F);
        for a : a \in V do
            eq := eq \land equiv(T(p, a), T(q, a), S \cup \{\{p, q\}\})
        rof
    fi;
    return eq
```

The correctness of this program is shown in [t-Ei91]. Naturally, the guard \(eq\) can be used in the repetition (to terminate the repetition when \(eq \equiv \text{false}\)) in a practical implementation. This optimization is omitted here for clarity.

There are a number of methods for making this program more efficient. From Section 3.2 recall that \(E = E_{\{(Q-2)\max 0}}\). We add a parameter \(k\) to function \(\text{equiv}\) such that an invocation \(\text{equiv}(p, q, \emptyset, k)\) returns \((p, q) \in E_k\) as its result. The recursion depth is bounded by \((|Q| - 2)\max 0\). The new function is

```plaintext
function equiv(p, q, S, k) is
    if \(k = 0\) → eq := (p \in F \equiv q \in F)
    | \(k \not= 0\) →
        \(\forall \{p, q\} \in S\) → eq := true
    | \(k \not= 0\) →
        \(\forall \{p, q\} \not\in S\) →
            eq := (p \in F \equiv q \in F);
            for a : a \in V do
                eq := eq \land equiv(T(p, a), T(q, a), S \cup \{\{p, q\}\}, k - 1)
            rof
        fi;
    return eq
```

The third parameter $S$ is made a global variable, improving the efficiency of this algorithm in practice. As a result, $equiv$ is no longer a functional program in the sense that it now makes use of a global variable. The correctness of this transformation is shown in [t-E91]. We assume that $S$ is initialized to 0. When $S = 0$, an invocation $equiv(p, q, (|Q| - 2) \text{max } 0)$ returns $(p, q) \in E$; after such an invocation $S = 0$.

**Algorithm 4.9 (Pointwise computation of $E$):**

```plaintext
function equiv(p, q, k) is
  if $k = 0$ then $eq := (p \in F \equiv q \in F)$
  $\mid k \neq 0 \land \{p, q\} \in S$ then $eq := true$
  $\mid k \neq 0 \land \{p, q\} \notin S$ then
    $eq := (p \in F \equiv q \in F)$;
    $S := S \cup \{\{p, q\}\}$;
  for $a : a \in V$ do
    $eq := eq \land equiv(T(p, a), T(q, a), k - 1)$
  endfor;
  $S := S \setminus \{\{p, q\}\}$
  return $eq$
endfunction
```

The procedure $equiv$ can be memoized to further improve the running time in practice.

This algorithm does not appear in the literature.

### 4.7 Computing $E$ by approximation from below

This latest version of function $equiv$ can be used to compute $E$ and $D$ (assuming $I_Q$ is the identity relation on states, and $S$ is the global variable used in Algorithm 4.9):

**Algorithm 4.10 (Computing $E$ from below):**

```plaintext
S, G, H := 0, 0, I_Q;
{invariant: $G \cup H \subseteq (Q \times Q) \land G \subseteq D \land H \subseteq E$}
do ($G \cup H) \neq Q \times Q$ then
  let $p, q : (p, q) \in ((Q \times Q) \setminus (G \cup H))$;
  if $equiv(p, q, (|Q| - 2) \text{max } 0)$ then $H := H \cup \{(p, q)\}$
  else $G := G \cup \{(p, q)\}$
enddo
return $G = D \land H = E$
```

Further efficiency improvements can be made as follows:

- We change the initialization of $G$ to $G := ((Q \setminus F) \times F) \cup (F \times (Q \setminus F))$.
- We make use of the fact that $E = E^*$; obviously $E$ is symmetrical, halving the required amount of computation. $H$ can be updated at each iteration step by $H := H^*$ (provided the data-structures in the implementation are such that *-closure is easily implemented).
- Make use of the facts that
  
  $$(p, q) \notin E \Rightarrow (\forall r, s : r \in Q \land s \in Q \land \exists w : w \in V^* : T^*(r, w) = p \land T^*(s, w) = q) : ((r, s) \notin E)$$
  $$(p, q) \in E \Rightarrow (\forall w : w \in V^* : (T^*(p, w), T^*(q, w)) \in E)$$

The first (respectively second) implication states that if $p, q$ are two distinguished (respectively equivalent) states, and $r, s$ are two states such that there is $w \in V^*$ and $T(r, w) = p \land T(s, w) = q$ (respectively $T(p, w) = r \land T(q, w) = s$), then $r, s$ are also distinguished (respectively equivalent).
This algorithm has worse running time than the $O(|Q| \log |Q|)$ of Hopcroft's algorithm [Hopc71, Grie73]. This algorithm has a significant advantage over all of the known algorithms: although function $equiv$ computes $E$ pointwise from above (with respect to $\subseteq$, refinement), the main program computes $E$ from below (with respect to $\subseteq$, normal set inclusion$^5$). As such, any intermediate result $H$ in the computation of $E$ is usable in (at least partially) reducing the size of an automaton; all of the other algorithms presented have unusable intermediate results. This property has use in reducing the size of automata when the running time of the minimization algorithm is restricted for some reason (for example, in real-time applications).

$^5$This is set inclusion, as opposed to refinement, since the intermediate result $H$ may not be an equivalence relation during the computation.
5 Conclusions

The conclusions about minimization algorithms are:

- A derivation of Brzozowski's minimization algorithm was presented. This derivation proved to be easier to understand than either the original derivation (by Brzozowski), or the derivation given by van de Snepscheut. A brief history of the minimization algorithm was presented, hopefully resolving some misattributions of its discovery.

- The definition of equivalence (relation \(E\)) and distinguishability (relation \(D\)) as fixed points of certain equations proved easier to understand than many text-book presentations.

- The fixed point characterization of \(E\) made it particularly easy to calculate an upperbound on the number of approximation steps required to compute \(E\) (or \(D\)). This upperbound later proved useful in determining the running time of some of the algorithms, and also in making efficiency improvements to the pointwise algorithm.

- The definition of \(E\) as a greatest fixed point helped to identify the fact that all of the (previously) known algorithm computed \(E\) from above (with respect to refinement). As such, all of these algorithms have intermediate results that are not usable in minimizing the finite automaton.

- We successfully presented all of the well-known text-book algorithms in the same framework. Most of them were shown to be essentially the same, with minor differences in their loop structures. One exception was Hopcroft and Ullman's algorithm [HU79], which has a distinctly different loop structure. The presentation of that algorithm (with invariants) in this paper is arguably easier to understand than the original presentation. Our presentation highlights the fact that the main data-structure in the algorithm need not be a list — a set suffices.

- Hopcroft's minimization algorithm [Hopc71] was originally presented in a style that is not very understandable. As with Gries's paper [Grie73], we strive to derive this algorithm in a clear and precise manner. The presentation in this paper highlights two important facts: the beginning point for the derivation of this algorithm is one of the easily understood straightforward algorithms; and, the use of a list data-structure in both Hopcroft's and Gries's presentation of this algorithm is not necessary — a set can be used instead.

- This paper presented several new minimization algorithms, many of which were variations on the well-known algorithms. Two of the new algorithms (presented in Sections 4.6 and 4.7) are not derived from any of the well-known algorithms, and are significant in their own right.
  - An algorithm was presented that computes the relation \(E\) in a pointwise manner. This algorithm was refined from an algorithm used to determine the structural equivalence of types. Several techniques played important roles in the refinement:
    * the upperbound on the number of steps required to compute \(E\) was used to improve the algorithm by limiting the number of pairs of states that need to be considered in computing \(E\) pointwise;
    * memoization of the functional-program portion of the algorithm was used to reduce the amount of redundant computation.
  - A new algorithm was presented, that computes \(E\) from below. This algorithm makes use of the pointwise computation of \(E\) to construct and refine an approximation of \(E\). Since the computation is from below, the intermediate results of this algorithm are usable in (at least partially) reducing the size of the DFA. This can be useful in applications where the amount of time available for minimization of the DFA is limited (as in real-time applications). In contrast, all of the (previously) known algorithms have unusable intermediate results.
A  Some basic definitions

Convention A.1 (Powerset): For any set $A$ we use $\mathcal{P}(A)$ to denote the set of all subsets of $A$. $\mathcal{P}(A)$ is called the powerset of $A$; it is sometimes written $2^A$. □

Convention A.2 (Sets of functions): For sets $A$ and $B$, $A \rightarrow B$ denotes the set of all total functions from $A$ to $B$, while $A \rightarrow \leftarrow B$ denotes the set of all partial functions from $A$ to $B$. □

Remark A.3: For sets $A, B$ and relation $C \subseteq A \times B$ we can interpret $C$ as a function $C \in A \rightarrow \mathcal{P}(B)$. □

Convention A.4 (Tuple projection): For an $n$-tuple $t = (x_1, x_2, \ldots, x_n)$ we use the notation $\pi_i(t)$ ($1 \leq i \leq n$) to denote tuple element $x_i$; we use the notation $\overline{\pi}_i(t)$ ($1 \leq i \leq n$) to denote the $(n - 1)$-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Both $\pi$ and $\overline{\pi}$ extend naturally to sets of tuples. □

Convention A.5 (Relation composition): Given sets $A, B, C$ (not necessarily different) and two relations, $E \subseteq A \times B$ and $F \subseteq B \times C$, we define relation composition (infix operator $\circ$) as:

$$E \circ F = \{ (a, c) : (\exists b : (a, b) \in E \land (b, c) \in F) \}$$

□

Convention A.6 (Equivalence classes of an equivalence relation): For any equivalence relation $E$ on set $A$ we denote the set of equivalence classes of $E$ by $[A]_E$; that is

$$[A]_E = \{ [a]_E : a \in A \}$$

Set $[A]_E$ is also called the partition of $A$ induced by $E$. □

Definition A.7 (Index of an equivalence class): For equivalence relation $E$ on set $A$, define $\sharp E = |[A]_E|$. $\sharp E$ is called the index of $E$. □

Definition A.8 (Alphabet): An alphabet is a non-empty set of finite size. □

Definition A.9 (Refinement of an equivalence relation): For equivalence relations $E$ and $E'$ (on set $A$), $E$ is a refinement of $E'$ if and only if $E \subseteq E'$. □

Definition A.10 (Refinement ($\subseteq$) relation on partitions): For equivalence relations $E$ and $E'$ (on set $A$), $[A]_E$ is said to be a refinement of $[A]_{E'}$ (written $[A]_E \subseteq [A]_{E'}$) if and only if $E \subseteq E'$. An equivalent statement is that $[A]_E \subseteq [A]_{E'}$ if and only if every equivalence class (of $A$) under $E$ is entirely contained in some equivalence class (of $A$) under $E'$. □

Definition A.11 (Tuple and relation reversal): For an $n$-tuple $(x_1, x_2, \ldots, x_n)$ define reversal as (postfix and superscript) function $R$:

$$(x_1, x_2, \ldots, x_n)^R = (x_n, \ldots, x_2, x_1)$$

Given a set $A$ of tuples, we define $A^R = \{ x^R : x \in A \}$. □
In this section we define finite automata, some of their properties, and some transformations on finite automata. Most of these definitions are taken directly from [Wats93].

Definition B.1 (Finite automaton): A finite automaton (an FA) is a 6-tuple \((Q, V, T, E, S, F)\) where

- \(Q\) is a finite set of states,
- \(V\) is an alphabet,
- \(T \subseteq \mathcal{P}(Q \times V \times Q)\) is a transition relation,
- \(E \subseteq \mathcal{P}(Q \times Q)\) is an \(\epsilon\)-transition relation
- \(S \subseteq Q\) is a set of start states, and
- \(F \subseteq Q\) is a set of final states.

The definitions of an alphabet and function \(\mathcal{P}\) are in Definition A.8 and Convention A.1 respectively.

Remark B.2: We will take some liberty in our interpretation of the signatures of the transition relations. For example, we also use the signatures \(T \subseteq V \to \mathcal{P}(Q \times Q), T \subseteq Q \times Q \to \mathcal{P}(V), T \subseteq Q \times V \to \mathcal{P}(Q), \) and \(T \subseteq Q \to \mathcal{P}(Q)\). Each of these works as \(T(p) = \{(a, q) : (p, a, q) \in T\}\). The signature that is used will be clear from the context. See Remark A.3. The definition of \(\to\) appears in Convention A.2.

Since we only consider finite automata in this paper, we will frequently simply use the term automata.

B.1 Properties of finite automata

In this subsection we define some properties of finite automata. To make these definitions more concise, we introduce particular finite automata \(M = (Q, V, T, E, S, F)\), \(M_0 = (Q_0, V_0, T_0, E_0, S_0, F_0)\), and \(M_1 = (Q_1, V_1, T_1, E_1, S_1, F_1)\).

Definition B.3 (Size of an FA): Define the size of an FA as \(|M| = |Q|\).

Definition B.4 (Isomorphism \((\cong)\) of FA's): We define isomorphism \((\cong)\) as an equivalence relation on FA's. \(M_0\) and \(M_1\) are isomorphic (written \(M_0 \cong M_1\)) if and only if \(V_0 = V_1\) and there exists a bijection \(g \in Q_0 \to Q_1\) such that

- \(T_1 = \{(g(p), a, g(q)) : (p, a, q) \in T_0\}\),
- \(E_1 = \{(g(p), g(q)) : (p, q) \in E_0\}\),
- \(S_1 = \{g(s) : s \in S_0\}\), and
- \(F_1 = \{g(f) : f \in F_0\}\).

Definition B.5 (Extending the transition relation \(T\)): We extend transition relation \(T \subseteq V \to \mathcal{P}(Q \times Q)\) to \(T^* \subseteq V^* \to \mathcal{P}(Q \times Q)\) as follows:

\[T^*(\epsilon) = E^*\]
B.1 Properties of finite automata

and (for \( a \in V, w \in V^* \))

\[
T^*(aw) = E^* \circ T(a) \circ T^*(w)
\]

Operator \( \circ \) (composition) is defined in Convention A.5. This definition could also have been presented symmetrically. □

Remark B.6: We also sometimes use the signature \( T^* \in Q \times Q \rightarrow \mathcal{P}(V^*) \). □

Definition B.7 (Left and right languages): The left language of a state (in \( M \)) is given by function \( \mathcal{L}_M \in Q \rightarrow \mathcal{P}(V^*) \), where

\[
\mathcal{L}_M(q) = (\cup s : s \in S : T^*(s, q))
\]

The right language of a state (in \( M \)) is given by function \( \overline{\mathcal{L}}_M \in Q \rightarrow \mathcal{P}(V^*) \), where

\[
\overline{\mathcal{L}}_M(q) = (\cup f : f \in F : T^*(q, f))
\]

The subscript \( M \) is usually dropped when no ambiguity can arise. □

Definition B.8 (Language of an FA): The language of a finite automaton (with alphabet \( V \)) is given by the function \( \mathcal{L}_{FA} \in FA \rightarrow \mathcal{P}(V^*) \) defined as:

\[
\mathcal{L}_{FA}(M) = (\cup s, f : s \in S \land f \in F : T^*(s, f))
\]

Definition B.9 (Complete): A Complete finite automaton is one satisfying the following:

\[
\text{Complete}(M) \equiv (\forall q, a : q \in Q \land a \in V : T(q, a) \neq \emptyset)
\]

Definition B.10 (\( \epsilon \)-free): Automaton \( M \) is \( \epsilon \)-free if and only if \( E = \emptyset \). □

Definition B.11 (Start-useful automaton): A Useful\(_s\) finite automaton is defined as follows:

\[
\text{Useful}_s(M) \equiv (\forall q : q \in Q : \overline{\mathcal{L}}(q) \neq \emptyset)
\]

Definition B.12 (Final-useful automaton): A Useful\(_f\) finite automaton is defined as:

\[
\text{Useful}_f(M) \equiv (\forall q : q \in Q : \mathcal{L}(q) \neq \emptyset)
\]

Remark B.13: \( \text{Useful}_s \) and \( \text{Useful}_f \) are closely related by FA reversal (to be presented in Transformation B.22). For all \( M \in FA \) we have \( \text{Useful}_f(M) \equiv \text{Useful}_s(M^R) \). □

Definition B.14 (Useful automaton): A Useful finite automaton is one with only reachable states:

\[
\text{Useful}(M) \equiv \text{Useful}_s(M) \land \text{Useful}_f(M)
\]

Property B.15 (Deterministic finite automaton): A finite automaton \( M \) is deterministic if and only if
• it does not have multiple start states,
• it is $\epsilon$-free, and
• transition function $T \in Q \times V \rightarrow \mathcal{P}(Q)$ does not map pairs in $Q \times V$ to multiple states.

Formally,

$$\text{Det}(M) \equiv (|S| \leq 1 \land \epsilon\text{-free}(E) \land (\forall q, a : q \in Q \land a \in V : |T(q, a)| \leq 1))$$

Definition B.16 (Deterministic FA's): DFA denotes the set of all deterministic finite automata. We call $FA \setminus DFA$ the set of nondeterministic finite automata.

Convention B.17 (Transition function of a DFA): For $(Q, V, T, \emptyset, S, F) \in DFA$ we can consider the transition function to have signature $T \in Q \times V \rightarrow Q$. (A definition of $\rightarrow$ appears in Convention A.2.) The transition function is total if and only if the DFA is Complete.

Property B.18 (Weakly deterministic automaton): Some authors use a definition of a deterministic automaton that is weaker than $\text{Det}$; it uses left languages and is defined as follows:

$$\text{Det}'(M) \equiv (\forall q_0, q_1 : q_0 \in Q \land q_1 \in Q \land q_0 \neq q_1 : \overrightarrow{L}(q_0) \cap \overrightarrow{L}(q_1) = \emptyset)$$

$\text{Det}(M) \Rightarrow \text{Det}'(M)$ is easily proved.

Definition B.19 (Minimality of a DFA): An $M \in DFA$ is minimal as follows:

$$\text{Min}(M) \equiv (\forall M' : M' \in DFA \land \mathcal{L}_{FA}(M) = \mathcal{L}_{FA}(M') : |M| \leq |M'|)$$

Predicate $\text{Min}$ is defined only on DFA's. Some definitions are simpler if we define a minimal, but still Complete, DFA as follows:

$$\text{Min}_{C}(M) \equiv (\forall M' : M' \in DFA \land \text{Complete}(M') \land \mathcal{L}_{FA}(M) = \mathcal{L}_{FA}(M') : |M| \leq |M'|)$$

Predicate $\text{Min}_{C}$ is defined only on Complete DFA's.

Property B.20 (Minimality of a DFA): An $M$, such that $\text{Min}(M)$, is the unique (modulo $\equiv$) minimal DFA, due to the Myhill-Nerode theorem. Introductory presentations of the theorem appear in [HU79, Wats93].

Property B.21 (An alternate definition of minimality of a DFA): For the purposes of minimizing a DFA, we use the definition (defined only on DFA's):

$$\text{Minimal}(Q, V, T, \emptyset, S, F) \equiv (\forall q_0, q_1 : q_0 \in Q \land q_1 \in Q \land q_0 \neq q_1 : \overrightarrow{L}(q_0) \neq \overrightarrow{L}(q_1))$$

$$\land \text{Useful}(Q, V, T, \emptyset, S, F)$$

We have the property that (for all $M \in DFA$) $\text{Minimal}(M) \equiv \text{Min}(M)$. It is easy to prove that $\text{Min}(M) \Rightarrow \text{Minimal}(M)$. The reverse direction follows from the Myhill-Nerode theorem.

A similar definition that relates to $\text{Min}_{C}$ is (also defined only on DFA's):

$$\text{Minimal}_{C}(Q, V, T, \emptyset, S, F) \equiv (\forall q_0, q_1 : q_0 \in Q \land q_1 \in Q \land q_0 \neq q_1 : \overrightarrow{L}(q_0) \neq \overrightarrow{L}(q_1))$$

$$\land \text{Useful}_{C}(Q, V, T, \emptyset, S, F)$$

We have the property that (for all $M \in DFA$ such that Complete$(M)$) $\text{Minimal}_{C}(M) \equiv \text{Min}_{C}(M)$. The contrapositive of $\text{Min}_{C}(M) \Rightarrow \text{Minimal}_{C}(M)$ is easily proved, and the reverse direction also follows from the Myhill-Nerode theorem.
B.2 Transformations on finite automata

Transformation B.22 (FA reversal): FA reversal is given by postfix (superscript) function \( R \in FA \rightarrow FA \), defined as:

\[
\]

Function \( R \) satisfies

\[
(\forall M : M \in FA : (L_{FA}(M))^R = L_{FA}(M^R)).
\]

Transformation B.23 (Removing start state unreachable states): Transformation \( \text{useful} \in FA \rightarrow FA \) removes those states that are not start-reachable:

\[
\text{useful}(Q, V, T, E, S, F) = \text{let } U = SReachable(Q, V, T, E, S, F) \text{ in } (U, V, T \cap (U \times V \times U), E \cap (U \times U), S \cup U, F \cap U) \text{ end}
\]

Function \( \text{useful} \) satisfies

\[
(\forall M : M \in FA : \text{Useful}(\text{useful}(M)) \land L_{FA}(\text{useful}(M)) = L_{FA}(M))
\]

Transformation B.24 (Subset construction): The function \( \text{subset} \) transforms an \( \epsilon \)-free FA into a DFA (in the let clause \( T' \in \mathcal{P}(Q) \times V \rightarrow \mathcal{P}(\mathcal{P}(Q)) \))

\[
\text{subset}(Q, V, T, \emptyset, S, F) = \text{let } T'(U, a) = \{(U \cup q : q \in U : T(q, a))\}
\]

\[
F' = \{U : U \in \mathcal{P}(Q) \land U \cap F \neq \emptyset\}
\]

\[
(\mathcal{P}(Q), V, T', \emptyset, \{S\}, F') \text{ end}
\]

In addition to the obvious property that (for all \( M \in FA \)) \( L_{FA}(\text{subset}(M)) = L_{FA}(M) \), function \( \text{subset} \) satisfies

\[
(\forall M : M \in FA \land \epsilon\text{-free}(M) : \text{Det}(\text{subset}(M)) \land \text{Complete}(\text{subset}(M)))
\]

It is also known as the “powerset” construction. \( \Box \)

Property B.25 (Subset construction): Let \( M_0 = (Q_0, V, T_0, \emptyset, S_0, F_0) \) and \( M_1 = \text{subset}(M_0) \) be finite automata. By the subset construction, the state set of \( M_1 \) is \( \mathcal{P}(Q_0) \). We have the following property:

\[
(\forall p : p \in \mathcal{P}(Q_0) : \overline{L}_{M_1}(p) = (\cup q : q \in p : \overline{L}_{M_0}(q)))
\]

\( \Box \)

Definition B.26 (Optimized subset construction): The function \( \text{subsetopt} \) transforms an \( \epsilon \)-free FA into a DFA. This function is an optimized version of \( \text{subset} \).

\[
\text{subsetopt}(Q, V, T, \emptyset, S, F) = \text{let } T'(U, a) = \{(U \cup q : q \in U : T(q, a))\}
\]

\[
Q' = \mathcal{P}(Q) \setminus \{\emptyset\}
\]

\[
F' = \{U : U \in \mathcal{P}(Q) \land U \cap F \neq \emptyset\}
\]

\[
(Q', V, T' \cap (Q' \times V \times Q'), \emptyset, \{S\}, F') \text{ end}
\]

In addition to the property that (for all \( M \in FA \)) \( L_{FA}(\text{subsetopt}(M)) = L_{FA}(M) \), function \( \text{subsetopt} \) satisfies

\[
(\forall M : M \in FA \land \epsilon\text{-free}(M) : \text{Det}(\text{subsetopt}(M)))
\]

\( \Box \)
References


In this series appeared:

<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>91/02</td>
<td>R.P. Nederpelt, H.C.M. de Swart</td>
<td>Implication. A survey of the different logical analyses &quot;if...then...&quot;, p. 26.</td>
</tr>
<tr>
<td>91/03</td>
<td>J.P. Katoen, L.A.M. Schoenmakers</td>
<td>Parallel Programs for the Recognition of $P$-invariant Segments, p. 16.</td>
</tr>
<tr>
<td>91/05</td>
<td>D. de Reus</td>
<td>An Implementation Model for GOOD, p. 18.</td>
</tr>
<tr>
<td>91/06</td>
<td>K.M. van Hee</td>
<td>SPECIFICATIEMETHODEN, een overzicht, p. 20.</td>
</tr>
<tr>
<td>91/07</td>
<td>E. Poll</td>
<td>CPO-models for second order lambda calculus with recursive types and subtyping, p. 49.</td>
</tr>
<tr>
<td>91/12</td>
<td>E. van der Sluis</td>
<td>A parallel local search algorithm for the travelling salesman problem, p. 12.</td>
</tr>
<tr>
<td>91/14</td>
<td>P. Lemmens</td>
<td>The PDB Hypermedia Package. Why and how it was built, p. 63.</td>
</tr>
<tr>
<td>91/16</td>
<td>A.J.J.M. Marcelis</td>
<td>An example of proving attribute grammars correct: the representation of arithmetical expressions by DAGs, p. 25.</td>
</tr>
<tr>
<td>Page</td>
<td>Authors</td>
<td>Title</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>91/18</td>
<td>Rik van Geldrop</td>
<td>Transformational Query Solving, p. 35.</td>
</tr>
<tr>
<td>91/19</td>
<td>Erik Poll</td>
<td>Some categorical properties for a model for second order lambda calculus with subtyping, p. 21.</td>
</tr>
<tr>
<td>91/23</td>
<td>K.M. van Hee, L.J. Somers, M. Voorhoeve</td>
<td>Z and high level Petri nets, p. 16.</td>
</tr>
<tr>
<td>91/24</td>
<td>A.T.M. Aerts, D. de Reus</td>
<td>Formal semantics for BRM with examples, p. 25.</td>
</tr>
<tr>
<td>91/25</td>
<td>P. Zhou, J. Hooman, R. Kuiper</td>
<td>A compositional proof system for real-time systems based on explicit clock temporal logic: soundness and completeness, p. 52.</td>
</tr>
<tr>
<td>91/27</td>
<td>F. de Boer, C. Palamidessi</td>
<td>Embedding as a tool for language comparison: On the CSP hierarchy, p. 17.</td>
</tr>
<tr>
<td>91/28</td>
<td>F. de Boer</td>
<td>A compositional proof system for dynamic process creation, p. 24.</td>
</tr>
<tr>
<td>91/30</td>
<td>J.C.M. Baeten, F.W. Vaandrager</td>
<td>An Algebra for Process Creation, p. 29.</td>
</tr>
<tr>
<td>91/31</td>
<td>H. ten Eikelder</td>
<td>Some algorithms to decide the equivalence of recursive types, p. 26.</td>
</tr>
<tr>
<td>91/33</td>
<td>W. v.d. Aalst</td>
<td>The modelling and analysis of queueing systems with QNM-ExSpect, p. 23.</td>
</tr>
<tr>
<td>91/34</td>
<td>J. Coenen</td>
<td>Specifying fault tolerant programs in deontic logic, p. 15.</td>
</tr>
</tbody>
</table>
91/35 F.S. de Boer  
J.W. Klop  
C. Palamidessi  

Asynchronous communication in process algebra, p. 20.

92/01 J. Coenen  
J. Zwiers  
W.-P. de Roever  

A note on compositional refinement, p. 27.

92/02 J. Coenen  
J. Hooman  

A compositional semantics for fault tolerant real-time systems, p. 18.

92/03 J.C.M. Baeten  
J.A. Bergstra  

Real space process algebra, p. 42.

92/04 J.P.H.W.v.d.Eijnde  

Program derivation in acyclic graphs and related problems, p. 90.

92/05 J.P.H.W.v.d.Eijnde  

Conservative fixpoint functions on a graph, p. 25.

92/06 J.C.M. Baeten  
J.A. Bergstra  

Discrete time process algebra, p.45.

92/07 R.P. Nederpelt  

The fine-structure of lambda calculus, p. 110.

92/08 R.P. Nederpelt  
F. Kamareddine  

On stepwise explicit substitution, p. 30.

92/09 R.C. Backhouse  


92/10 P.M.P. Rambags  

Composition and decomposition in a CPN model, p. 55.

92/11 R.C. Backhouse  
J.S.C.P.v.d.Woude  

Demonic operators and monotype factors, p. 29.

92/12 F. Kamareddine  


92/13 F. Kamareddine  

Set theory and nominalisation, Part II, p.22.

92/14 J.C.M. Baeten  

The total order assumption, p. 10.

92/15 F. Kamareddine  

A system at the cross-roads of functional and logic programming, p.36.

92/16 R.R. Seljée  

Integrity checking in deductive databases; an exposition, p.32.

92/17 W.M.P. van der Aalst  

Interval timed coloured Petri nets and their analysis, p. 20.

92/18 R.Nederpelt  
F. Kamareddine  

A unified approach to Type Theory through a refined lambda-calculus, p. 30.

92/19 J.C.M.Baeten  
J.A.Bergstra  
S.A.Smolka  

Axiomatizing Probabilistic Processes: ACP with Generative Probabilities, p. 36.

92/20 F.Kamareddine  

Are Types for Natural Language? P. 32.
<table>
<thead>
<tr>
<th>Number</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>92/21</td>
<td>F.Kamareddine</td>
<td>Non well-foundedness and type freeness can unify the interpretation of functional application, p. 16.</td>
</tr>
<tr>
<td>92/22</td>
<td>R. Nederpelt, F.Kamareddine</td>
<td>A useful lambda notation, p. 17.</td>
</tr>
<tr>
<td>92/23</td>
<td>F.Kamareddine, E.Klein</td>
<td>Nominalization, Predication and Type Containment, p. 40.</td>
</tr>
<tr>
<td>92/24</td>
<td>M.Codish, D.Dams, Eyal Yardeni</td>
<td>Bottom-up Abstract Interpretation of Logic Programs, p. 33.</td>
</tr>
<tr>
<td>92/25</td>
<td>E.Poli</td>
<td>A Programming Logic for Fo, p. 15.</td>
</tr>
<tr>
<td>93/01</td>
<td>R. van Geldrop</td>
<td>Deriving the Aho-Corasick algorithms: a case study into the synergy of programming methods, p. 36.</td>
</tr>
<tr>
<td>93/02</td>
<td>T. Verhoeff</td>
<td>A continuous version of the Prisoner’s Dilemma, p. 17</td>
</tr>
<tr>
<td>93/03</td>
<td>T. Verhoeff</td>
<td>Quicksort for linked lists, p. 8.</td>
</tr>
<tr>
<td>93/04</td>
<td>E.H.L. Aarts, J.H.M. Korst, P.J. Zwietering</td>
<td>Deterministic and randomized local search, p. 78.</td>
</tr>
<tr>
<td>93/05</td>
<td>J.C.M. Baeten, C. Verhoef</td>
<td>A congruence theorem for structured operational semantics with predicates, p. 18.</td>
</tr>
<tr>
<td>93/06</td>
<td>J.P. Veltkamp</td>
<td>On the unavoidability of metastable behaviour, p. 29</td>
</tr>
<tr>
<td>93/07</td>
<td>P.D. Moerland</td>
<td>Exercises in Multiprogramming, p. 97</td>
</tr>
<tr>
<td>93/08</td>
<td>J. Verhoosel</td>
<td>A Formal Deterministic Scheduling Model for Hard Real-Time Executions in DEDOS, p. 32.</td>
</tr>
<tr>
<td>93/10</td>
<td>K.M. van Hee</td>
<td>Systems Engineering: a Formal Approach Part II: Frameworks, p. 44.</td>
</tr>
<tr>
<td>ID</td>
<td>Authors</td>
<td>Title</td>
</tr>
<tr>
<td>------</td>
<td>----------------------------------------------</td>
<td>-----------------------------------------------------------------------</td>
</tr>
<tr>
<td>93/16</td>
<td>H. Schepers, J. Hooman</td>
<td>A Trace-Based Compositional Proof Theory for Fault Tolerant Distributed Systems, p. 27</td>
</tr>
<tr>
<td>93/17</td>
<td>D. Alstein, P. van der Stok</td>
<td>Hard Real-Time Reliable Multicast in the DEDOS system, p. 19.</td>
</tr>
<tr>
<td>93/18</td>
<td>C. Verhoef</td>
<td>A congruence theorem for structured operational semantics with predicates and negative premises, p. 22.</td>
</tr>
<tr>
<td>93/19</td>
<td>G-J. Houben</td>
<td>The Design of an Online Help Facility for ExSpect, p. 21.</td>
</tr>
<tr>
<td>93/22</td>
<td>E. Poll</td>
<td>A Typechecker for Bijective Pure Type Systems, p. 28.</td>
</tr>
<tr>
<td>93/23</td>
<td>E. de Kogel</td>
<td>Relational Algebra and Equational Proofs, p. 23.</td>
</tr>
<tr>
<td>93/24</td>
<td>E. Poll and Paula Severi</td>
<td>Pure Type Systems with Definitions, p. 38.</td>
</tr>
<tr>
<td>93/26</td>
<td>W.M.P. van der Aalst</td>
<td>Multi-dimensional Petri nets, p. 25.</td>
</tr>
<tr>
<td>93/27</td>
<td>T. Kloks and D. Kratsch</td>
<td>Finding all minimal separators of a graph, p. 11.</td>
</tr>
<tr>
<td>93/28</td>
<td>F. Kamareddine and R. Nederpelt</td>
<td>A Semantics for a fine (\lambda)-calculus with de Bruijn indices, p. 49.</td>
</tr>
<tr>
<td>93/29</td>
<td>R. Post and P. De Bra</td>
<td>GOLD, a Graph Oriented Language for Databases, p. 42.</td>
</tr>
<tr>
<td>93/30</td>
<td>J. Deogun, T. Kloks, D. Kratsch, H. Müller</td>
<td>On Vertex Ranking for Permutation and Other Graphs, p. 11.</td>
</tr>
<tr>
<td>93/31</td>
<td>W. Körver</td>
<td>Derivation of delay insensitive and speed independent CMOS circuits, using directed commands and production rule sets, p. 40.</td>
</tr>
<tr>
<td>93/33</td>
<td>L. Loyens and J. Moonen</td>
<td>ILIAS, a sequential language for parallel matrix computations, p. 20.</td>
</tr>
</tbody>
</table>
93/34 J.C.M. Baeten and J.A. Bergstra

Real Time Process Algebra with Infinitesimals, p.39.

93/35 W. Ferrer and P. Severi

Abstract Reduction and Topology, p. 28.

93/36 J.C.M. Baeten and J.A. Bergstra

Non Interleaving Process Algebra, p. 17.

93/37 J. Brunekreef J-P. Katoen R. Koymans S. Mauw

Design and Analysis of Dynamic Leader Election Protocols in Broadcast Networks, p. 73.

93/38 C. Verhoef

A general conservative extension theorem in process algebra, p. 17.

93/39 W.P.M. Nuijten E.H.L. Aarts D.A.A. van Erp Taalman Kip K.M. van Hee

Job Shop Scheduling by Constraint Satisfaction, p. 22.

93/40 P.D.V. van der Stok M.M.M.P.J. Claessen D. Alstein


93/41 A. Bijlsma

Temporal operators viewed as predicate transformers, p. 11.

93/42 P.M.P. Rambags

Automatic Verification of Regular Protocols in P/T Nets, p. 23.

93/43 B.W. Watson

A taxonomy of finite automata construction algorithms, p. 87.