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ON THE RELATION BETWEEN THE REPETITION FACTOR AND NUMERICAL STABILITY OF DIRECT QUADRATURE METHODS FOR SECOND KIND VOLTERRA INTEGRAL EQUATIONS*

P. H. M. WOLKENFELT†

Abstract. We consider direct quadrature methods employing quadrature rules which are reducible to linear multistep methods for ordinary differential equations. A simple characterization of both the repetition factor and numerical stability (for small h) is given, which enables us to derive some results with respect to a conjecture of Linz. In particular we show that (i) methods with a repetition factor of one are always numerically stable; (ii) methods with a repetition factor greater than one are not necessarily numerically unstable. Analogous results are derived with respect to the more general notion of an asymptotic repetition factor. We also discuss the concepts of strong stability, absolute stability and relative stability and their (dis)connection with the (asymptotic) repetition factor. Some numerical results are presented as a verification.

Key words. numerical analysis, Volterra integral equations of the second kind, reducible quadrature methods, numerical stability, repetition factor

1. Introduction. Consider the second kind Volterra integral equation

$$(1.1) \quad f(x) = g(x) + \int_0^x K(x, y, f(y)) dy, \quad x \geq 0,$$

where f is the unknown function and where the forcing function g and the kernel K are given smooth functions.

In order to define a discretization of (1.1), let $x_n = nh$ (where h denotes the stepsize) and let $\{w_{nj}\}$ be the weights associated with the quadrature formula

$$(1.2) \quad \int_0^{x_n} \phi(y) dy \cong h \sum_{j=0}^n w_{nj} \phi(x_j), \quad n \geq k.$$

Then a direct quadrature method for (1.1) is given by

$$(1.3) \quad f_n = g(x_n) + h \sum_{j=0}^n w_{nj} K(x_n, x_j, f_j), \quad n \geq k.$$

Here, f_n denotes a numerical approximation to $f(x_n)$ and k depends on the desired accuracy. If the required starting values $f_0, \dots, f_{k-1}$ are known, the values $f_k, f_{k+1}, \dots$ can be computed in a step-by-step fashion. For a detailed discussion of such methods we refer to Baker [1].

It is well known (see e.g. [8], [11], [1], [2]) that the structure of the quadrature weights w_{nj} is important for the stability analysis of the methods (1.3). In this connection, the following notion is relevant.

DEFINITION 1.1. The weights w_{nj} in (1.3) are said to have an (*exact*) *repetition factor* r if r is the smallest positive integer such that $w_{n+r,j} = w_{nj}$ for all $n \geq n_0$ and $n_1 \leq j \leq n - n_2$, where n_0, n_1 and n_2 are independent of n .

A method (1.3) is said to have a repetition factor r if the associated weights w_{nj} have a repetition factor r .

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This paper has been largely motivated by the following conjecture of Linz [8, p. 27] (see also Noble [11]): "We may conjecture that (i) methods with a repetition factor of one tend to be numerically stable, (ii) those with a repetition factor greater than one numerically unstable."

In order to deal with this conjecture in a proper way, it is necessary to have a good understanding of the concept of numerical stability as defined by Linz and Noble. It turns out that numerical stability in the sense of Linz and Noble essentially requires the perturbation sensitivity of the discretization to be "roughly equivalent" to the perturbation sensitivity of the original continuous problem (compare the concept of "strong stability" as discussed by Stetter [13, p. 54]). Their analysis is based on the asymptotic expansion of the global discretization error (see also Kobayasi [6]). An advantage of this approach is its applicability to general equations (1.1), i.e. without any restrictions on the kernel and the forcing function (except for sufficient smoothness). A disadvantage, however, is that the stepsize h should be sufficiently small so that the conclusions need not hold for large values of h . As a consequence, this kind of stability analysis establishes results with regard to the *suitability of a method for general use* (but with small h). On the other hand it is not assumed that h actually tends to zero. This means that the condition for numerical stability in the sense of Linz and Noble is stronger than the condition for zero stability which is necessary for convergence.

To gain insight into the relationship between the repetition factor of the quadrature weights and the stability behavior of the associated direct quadrature method, we consider the class of methods which are reducible to linear multistep methods $\{\rho, \sigma\}$ for ordinary differential equations. For this class we derive some properties of the quadrature weights, and, motivated by these results, we introduce the notion of the *asymptotic repetition factor* as an extension of Definition 1.1.

We shall characterize:

- (i) the exact and asymptotic repetition factor in terms of the location of the essential zeros of the polynomial ρ ;
- (ii) numerical stability in terms of the growth parameters associated with these zeros.

It turns out that with these characterizations, results with regard to the conjecture of Linz can be derived in a rather elegant, and almost straightforward, manner. To be specific, we shall demonstrate that:

- (i) methods with an asymptotic repetition factor of one are always numerically stable in the sense of Linz and Noble;
- (ii) methods with an exact or asymptotic repetition factor greater than one can still be numerically stable;
- (iii) an asymptotic repetition factor of one is necessary and sufficient for strong stability (a concept which we shall define in § 5).

Futhermore, we shall indicate that the stability concept of Linz and Noble is almost identical to the concept of relative stability for small h .

Finally, we present some numerical experiments which serve as an illustration of the theoretical results.

2. Preliminaries. In this paper we restrict our considerations to a special class of quadrature methods. We assume that for $n \geq k$, $0 \leq j \leq n$, the weights w_{nj} in (1.3) can be generated by the recurrence relation

$$(2.1) \quad \sum_{i=0}^k a_i w_{n-i,j} = \begin{cases} 0 & \text{for } j = 0(1)n - k - 1, \\ b_{n-j} & \text{for } j = n - k(1)n, \end{cases}$$

where a_i and b_i ($i = 0(1)k$) are the coefficients of a linear multistep method for ordinary differential equations [7, p. 11]. For the construction of the weights by means of (2.1) we set $w_{nj} = 0$ for $j > \max(n, k - 1)$ and define a set of starting weights $\{w_{nj} | n, j = 0(1)k - 1\}$ (see [15] for details). The quadrature rules generated in this way are called [9] (ρ, σ) -reducible. The direct quadrature method (1.3) employing such quadrature rules is also called (ρ, σ) -reducible. Here, ρ and σ denote the first and second characteristic polynomial associated with the linear multistep method; that is,

$$\rho(\zeta) := \sum_{i=0}^k a_i \zeta^{k-i}, \quad \sigma(\zeta) := \sum_{i=0}^k b_i \zeta^{k-i}.$$

From (2.1) the following property can be derived:

$$(2.2) \quad w_{nj} = \omega_{n-j} \quad \text{for } n - j \geq 0, \quad j \geq k,$$

where the sequence $\{\omega_n\}_{n=0}^\infty$ satisfies

$$(2.3a) \quad \begin{array}{rcl} a_0 \omega_0 & & = b_0, \\ a_0 \omega_1 + a_1 \omega_0 & & = b_1, \\ \vdots & & \vdots \\ a_0 \omega_k + a_1 \omega_{k-1} + \cdots + a_k \omega_0 & = & b_k \end{array}$$

$$(2.3b) \quad a_0 \omega_n + a_1 \omega_{n-1} + \cdots + a_k \omega_{n-k} = 0, \quad n \geq k + 1.$$

We also need the following definitions.

DEFINITION 2.1 (from [13, p. 206]). A polynomial is said to *satisfy the root condition* if it has no zeros outside the closed unit disk and only simple zeros on the unit circle. It is said to *satisfy the strong root condition* if it satisfies the root condition and 1 is its only zero on the unit circle.

DEFINITION 2.2. A nonvanishing zero ζ of a polynomial P is called *essential* if $|\zeta| = 1$ and *nonessential* if $|\zeta| < 1$. A possible vanishing zero of P is called the *trivial* zero of P .

Furthermore, we shall assume throughout this paper that ρ and σ have no common factors and that the method $\{\rho, \sigma\}$ is convergent (that is, $\rho(1) = 0$, $\rho'(1) = \sigma(1)$ and ρ satisfies the root condition).

We remark that the property (2.2) is characteristic for (ρ, σ) -reducible quadrature rules. To be precise, if (2.2) does not hold, then the quadrature rules are *not* (ρ, σ) -reducible. Such a situation occurs for example in the stable Simpson method (Simpson #2 in [11]). For this method $w_{2n, 2n} = \frac{1}{3}$ and $w_{2n-1, 2n-1} = \frac{3}{8}$ and therefore it is not (ρ, σ) -reducible. It turns out that the stable Simpson method is reducible to a 2-cyclic linear multistep method for ODEs.

The above remarks give insight into the structure of (ρ, σ) -reducible quadrature rules. Moreover, they also suggest generalizations of (2.1). A possible extension of (2.1) is to define quadrature rules which are reducible to cyclic linear multistep methods. Occasionally, we shall state some results with respect to such reducible quadrature methods, but for reasons of clarity our results are mainly related to (ρ, σ) -reducible quadrature methods.

3. Properties of the quadrature weights ω_n . We shall derive some properties of the sequence $\{\omega_n\}$ defined in (2.3). This sequence satisfies a homogeneous difference equation with characteristic polynomial ρ . Since, by assumption, ρ satisfies the root condition, the essential zeros of ρ are simple. In order to simplify the presentation of the results and their proofs, we assume subsequently that the nonessential zeros

of ρ are also simple. Without this assumption our results remain true,¹ however, unless the assumption is given explicitly in the statement of the theorems.

First we give the explicit form of the solution of a difference equation with constant coefficients.

LEMMA 3.1. *Let the sequence $\{y_n\}_{n=0}^\infty$ satisfy the difference equation*

$$(3.1) \quad \sum_{i=0}^k a_i y_{n-i} = 0, \quad n \geq k \quad (a_0 \neq 0),$$

with starting values $y_0, \dots, y_{k-1}$. Assume that the characteristic polynomial $\rho(\zeta) = \sum_{i=0}^k a_i \zeta^{k-i}$ has t nonvanishing simple zeros and a zero $\zeta = 0$ of multiplicity m_0 ($m_0 = 0$ is allowed). Furthermore, let the coefficients $\alpha_j^{(i)}$ be defined by

$$(3.2) \quad \sum_{j=0}^{k-1} \alpha_j^{(i)} \zeta^{k-1-j} := \rho(\zeta)/(\zeta - \zeta_i), \quad i = 1, 2, \dots, t.$$

Then the solution $\{y_n\}$ is given by

$$(3.3) \quad y_n = \sum_{i=1}^t \zeta_i^n \Delta_i / \rho'(\zeta_i), \quad n \geq m_0,$$

where

$$(3.4) \quad \Delta_i = \sum_{j=0}^{k-1} \alpha_j^{(i)} y_{k-1-j}.$$

Proof. Proceed along the lines indicated by Henrici [3, p. 238]. $\square$

In view of (3.2) the coefficients $\alpha_j^{(i)}$ can be expressed in ζ_i and the coefficients $a_0, \dots, a_k$; to be specific,

$$(3.5) \quad \alpha_j^{(i)} = \sum_{\nu=0}^j a_\nu \zeta_i^{j-\nu}, \quad i = 1, 2, \dots, t.$$

We now return to the recurrence relation (2.3b). Due to the special structure of the starting values $\omega_1, \dots, \omega_k$ defined by (2.3a), we can prove the following basic result.

THEOREM 3.1. *Let the linear multistep method $\{\rho, \sigma\}$ be convergent. Assume that the nonvanishing zeros $\zeta_1 = 1, \zeta_2, \dots, \zeta_t$ of ρ are simple and let m_0 denote the multiplicity of the trivial zero $\zeta = 0$ ($m_0 \geq 0$). Then the solution $\{\omega_n\}$ of (2.3b) with starting values (2.3a) is given by*

$$(3.6) \quad \omega_n = \sum_{i=1}^t \gamma_i \zeta_i^n, \quad n \geq m_0 + 1,$$

where

$$(3.7) \quad \gamma_i = \sigma(\zeta_i)/(\zeta_i \rho'(\zeta_i)) \neq 0, \quad i = 1, 2, \dots, t.$$

Proof. We replace y_n in (3.3) by ω_{n+1} and determine Δ_i . In view of (3.4) $\Delta_i = \sum_{j=0}^{k-1} \alpha_j^{(i)} \omega_{k-j}$. Substitution of (3.5) gives $\Delta_i = a_0 \omega_k + (a_0 \zeta_i + a_1) \omega_{k-1} + \dots + (a_0 \zeta_i^{k-1} + \dots + a_{k-1}) \omega_1$. Collecting powers of ζ_i and using (2.3a) yields

$$\begin{aligned} \Delta_i &= (b_k - a_k \omega_0) + \zeta_i (b_{k-1} - a_{k-1} \omega_0) + \dots + \zeta_i^{k-1} (b_1 - a_1 \omega_0) \\ &= \sum_{j=1}^k b_j \zeta_i^{k-j} - \omega_0 \sum_{j=1}^k a_j \zeta_i^{k-j} = \sigma(\zeta_i) - b_0 \zeta_i^k - \omega_0 \rho(\zeta_i) + a_0 \omega_0 \zeta_i^k = \sigma(\zeta_i), \end{aligned}$$

¹ In this case the proofs need some modification; details can be found in [14].

since $a_0\omega_0 = b_0$ and $\rho(\zeta_i) = 0$. As a result, $\omega_{n+1} = \sum_{i=1}^t \zeta_i^n \sigma(\zeta_i) / \rho'(\zeta_i)$, $n \geq m_0$, and its equivalence with (3.6) is readily seen. Since, by assumption, ρ and σ have no common factor, $\sigma(\zeta_i) \neq 0$, which proves that $\gamma_i \neq 0$. $\square$

Note that $\gamma_1 = \sigma(1)/\rho'(1) = 1$ by virtue of consistency. As a consequence of (3.6) we have

COROLLARY 3.1. *If ρ satisfies the strong root condition, then*

$$(3.8) \quad \lim_{n \rightarrow \infty} \omega_n = 1.$$

In particular, if $\rho(\zeta) = a_0 \zeta^{k-1}(\zeta - 1)$, then

$$(3.9) \quad \omega_n = 1 \quad \text{for all } n \geq k.$$

Property (3.9) holds, for example, for the Adams–Moulton methods (which generate the well-known Gregory quadrature rules). On the other hand, the backward differentiation methods generate a sequence $\{\omega_n\}$ satisfying (3.8).

From (3.6) we can also derive the following periodicity property.

COROLLARY 3.2. *Let the weights ω_n be defined by (2.3). Then*

$$(3.10) \quad \omega_{n+d} = \omega_n \quad \text{for all } n \geq m_0 + 1$$

if and only if the nonvanishing zeros of ρ satisfy $\zeta^d = 1$.

Proof. In view of (3.6), $\omega_{n+d} - \omega_n = \sum_{i=1}^t \gamma_i \zeta_i^n (\zeta_i^d - 1)$ for all $n \geq m_0 + 1$. Since $\gamma_i \neq 0$ for $i = 1, 2, \dots, t$, $\omega_{n+d} - \omega_n = 0$ if and only if $\zeta_i^d = 1$, $i = 1, 2, \dots, t$. $\square$

Obviously, the periodicity of the sequence $\{\omega_n\}$ is lost if ρ has a nonessential zero. We can, however, derive the following asymptotic result.

COROLLARY 3.3. *Let the weights ω_n be defined by (2.3). Then*

$$(3.11) \quad \lim_{n \rightarrow \infty} (\omega_{n+d} - \omega_n) = 0$$

if and only if the essential zeros of ρ satisfy $\zeta^d = 1$.

Proof. Let $\zeta_1, \dots, \zeta_s$ denote the essential zeros of ρ . The weights ω_n are given by (3.6) and can be written as $u_n + v_n$ where $u_n = \sum_{i=1}^s \gamma_i \zeta_i^n$ and where $v_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\lim (\omega_{n+d} - \omega_n) = \lim (u_{n+d} - u_n)$. Using the same argument as in the proof of Cor. 3.2, this limit is zero if and only if $\zeta_i^d = 1$, $i = 1, \dots, s$. $\square$

The properties derived in this section enable us to characterize the repetition factor in terms of the location of the essential zeros of ρ .

4. Characterization of the (asymptotic) repetition factor. In view of Definition 1.1 and property (2.2), the weights w_{nj} of a (ρ, σ) -reducible quadrature method have an exact repetition factor r if and only if r is the smallest positive integer such that $\omega_{n+r} = \omega_n$, $n \geq n_0$. This observation together with Corollary 3.2 yields the following characterization.

THEOREM 4.1. *The weights of a (ρ, σ) -reducible quadrature method have an exact repetition factor r if and only if r is the smallest positive integer such that the nonvanishing zeros of ρ satisfy $\zeta^r = 1$.*

We recall that the polynomial ρ associated with a linear multistep method derived from interpolatory quadrature has the form $\zeta^k - \zeta^{k-r}$ (compare the Adams family ($r = 1$) or the Milne–Simpson family ($r = 2$)). For such methods we have the following result as an immediate consequence of Theorem 4.1.

COROLLARY 4.1. *If $\rho(\zeta) = a_0(\zeta^k - \zeta^{k-r})$ then the weights have an exact repetition factor r .*

We shall now consider the case where ρ has also nonessential zeros. In this case the weights do not have an exact repetition factor r in view of Theorem 4.1. We have seen, however, in Corollary 3.3 that $\omega_{n+d} \approx \omega_n$ for n sufficiently large, if the essential zeros of ρ satisfy $\zeta^d = 1$. In particular, if the weights are computed using finite-precision arithmetic, we have the identity $\omega_{n+d} = \omega_n$ for large n . These observations suggest the following extension of Definition 1.1.

DEFINITION 4.1. The weights w_{nj} in (1.3) are said to have an *asymptotic repetition factor* r if r is the smallest positive integer such that $\lim_{n \rightarrow \infty} (w_{n+r,j} - w_{nj}) = 0$ for all j , $n_1 \leq j \leq n - n_2$, where n_1 and n_2 are independent of n .

With this definition and Corollary 3.3 the following theorem is self-evident.

THEOREM 4.2. *The weights of a (ρ, σ) -reducible quadrature method have an asymptotic repetition factor r if and only if r is the smallest positive integer such that the essential zeros of ρ satisfy $\zeta^r = 1$.*

As an example, the quadrature weights generated by the polynomials $\rho(\zeta) = (\zeta - 1)(\zeta^2 - \zeta + 1)$ and $\rho(\zeta) = (\zeta - 1)(\zeta^2 + 1)(\zeta - \frac{1}{2})$ have an exact repetition factor of 6 and an asymptotic repetition factor of 4, respectively.

As an important special case of Theorems 4.2 and 4.1, we have the following result which we shall use in § 6 in connection with the conjecture of Linz.

COROLLARY 4.2. *The weights of a (ρ, σ) -reducible quadrature method have an asymptotic repetition factor of one if and only if ρ satisfies the strong root condition. In particular, the weights have an exact repetition factor of one if and only if $\rho(\zeta) = a_0 \zeta^{k-1}(\zeta - 1)$.*

5. Characterization of numerical stability (for small h). In the following, numerical stability in the sense of Linz and Noble will be called numerical stability (for small h).

We touched upon the concept of numerical stability (for small h) already in § 1 in connection with the conjecture of Linz. For the sake of completeness we repeat here the stability definitions of both Linz and Noble.

DEFINITION 5.1 (Linz [8, p. 20]). A step-by-step method for (1.1) is *numerically stable* if the error growth is roughly equivalent to that of the solution of the variational equation of (1.1). If there exist some equations for which the error grows much faster than the solution of the variational equation of (1.1), then the method must be considered *numerically unstable*.

DEFINITION 5.2 (Noble [11, p. 25]; see also [1, p. 823]). A step-by-step method for solving a Volterra integral equation is said to be *unstable* if the error in the computed solution has dominant spurious components introduced by the numerical scheme.

We shall now explain how these definitions must be interpreted. For a (ρ, σ) -reducible quadrature method (of order p), the asymptotic expansion of the global discretization error $e(x_n) = f_n - f(x_n)$ assumes the form ([4])

$$(5.1) \quad e(x_n) = h^p \sum_{i=1}^s \zeta_i^n e_p^{(i)}(x) + O(h^{p+1})$$

where $\zeta_1 = 1, \zeta_2, \dots, \zeta_s$ are the essential zeros of ρ and where $e_p^{(i)}(x)$ satisfies

$$(5.2) \quad e_p^{(i)}(x) = g_p^{(i)}(x) + \gamma_i \int_0^x \bar{K}(x, y) e_p^{(i)}(y) dy, \quad i = 1, 2, \dots, s.$$

Here, $\bar{K}(x, y) = (\partial/\partial f)K(x, y, f(y))$ and the quantities γ_i are the so-called *growth parameters* ([3]) defined as

$$(5.3) \quad \gamma_i = \sigma(\zeta_i)/(\zeta_i \rho'(\zeta_i)), \quad i = 1, 2, \dots, s.$$

The functions $g_p^{(i)}(x)$ in (5.2) are related to the (local) quadrature errors and to the errors in the starting values.

The component $e_p^{(1)}(x)$ associated with $\zeta_1 = 1$ is called the *principal error component*. Since $\gamma_1 = 1$, this component satisfies, in view of (5.2), an equation which is identical to the variational equation of the continuous problem (1.1). The remaining components (if any) $e_p^{(2)}(x), \dots, e_p^{(s)}(x)$ associated with $\gamma_2, \dots, \gamma_s$ are called the *spurious error components* introduced by the discretization method. These components satisfy equations (5.2) which are different from the variational equation of (1.1), unless $\gamma_i = 1$. If $|e_p^{(i)}(x)| \gg |e_p^{(1)}(x)|$ for some i ($2 \leq i \leq s$), then $e_p^{(i)}(x)$ is *dominant* and the method is numerically unstable (in the sense of Linz and Noble).

From the above explanation we conclude that the values of the growth parameters are crucial for numerical stability of a (ρ, σ) -reducible quadrature method. In order to make this even more transparent we consider the integral equation

$$(5.4) \quad f(x) = g(x) + \lambda \int_0^x \exp(\mu(x-y))f(y) dy,$$

whose solution is given by

$$(5.5) \quad f(x) = g(x) + \lambda \int_0^x \exp((\lambda + \mu)(x-y))g(y) dy.$$

Clearly, the problem (5.4) is well-conditioned with respect to bounded perturbations of g if $\text{Re}(\lambda + \mu)$ is nonpositive. Suppose that for a given method, $\gamma_i \neq 1$ for some i ; then, in view of (5.2) and (5.5), the associated spurious error component $e_p^{(i)}(x)$ is given by

$$(5.6) \quad e_p^{(i)}(x) = g_p^{(i)}(x) + \gamma_i \lambda \int_0^x \exp((\gamma_i \lambda + \mu)(x-y))g_p^{(i)}(y) dy.$$

Since $\gamma_i \neq 1$ one can always choose λ and μ such that $\text{Re}(\lambda + \mu) < 0$ and $\text{Re}(\gamma_i \lambda + \mu) > 0$. As a consequence, the global error has a spurious component $e_p^{(i)}(x)$ which is exponentially increasing in general, whereas the continuous problem (5.4) is well-conditioned.

From the foregoing the following characterization is readily deduced.

THEOREM 5.1. *A reducible quadrature method of the form (1.3) is numerically stable (for small h) (in the sense of Linz and Noble) if each essential zero of ρ has a growth parameter equal to one; the method is weakly stable (for small h) (or numerically unstable in the terminology of Linz and Noble) if there exists at least one essential zero of ρ whose growth parameter is different from one.*

Essentially, this theorem is an equivalent, but more quantitative, definition of the numerical stability concept. We have used the term weak stability rather than numerical instability, because a weakly stable method does not always display an unstable behavior.

We recall that in the numerical treatment of ordinary differential equations a linear multistep method is weakly stable if ρ has an essential zero with $\gamma_i < 0$ (cf. [13, p. 246]). In the context of integral equations however, *weak stability can also occur for positive values of the growth parameters!*

For the expansion (5.1) we also observe that in general the terms ζ_i^n will cause the global error to be nonsmooth at consecutive grid points. This situation cannot occur if $\zeta_1 = 1$ is the only essential zero of ρ . In order to emphasize and distinguish this important feature we give the following definition.

DEFINITION 5.3. A numerically stable reducible quadrature method is called *strongly stable* (for small h) if the associated polynomial ρ satisfies the strong root condition.

Remark 5.1. The terms strong and weak stability are adopted from Henrici [3] and Stetter [13]. Numerical stability (for small h) which is not strong is sometimes called harmless weak stability (cf. [12]).

Remark 5.2. The growth parameters of (ρ, σ) -reducible quadrature methods were defined in (5.3). Since Theorem 5.1 is not restricted to this class of methods, we shall now briefly indicate how the values of the growth parameters can be obtained for more general quadrature methods.

In general, the application of a (step-by-step) direct quadrature method to the test equation $f(x) = 1 + \lambda \int_0^x f(y) dy$ (cf. [2]) is equivalent to the application of an m -cyclic linear multistep method to the ODE test equation $f' = \lambda f$. Let $P(\bar{h}; \zeta)$ ($\bar{h} = mh\lambda$) be the associated characteristic polynomial and let $\zeta_1(0), \dots, \zeta_s(0)$ be the essential zeros of $\rho(\zeta) := P(0; \zeta)$; then the growth parameters γ_i are given by the expansion

$$\zeta_i(\bar{h}) = \zeta_i(0)(1 + \gamma_i \bar{h}) \quad \text{as } h \rightarrow 0.$$

For $m = 1$ the equivalence with (5.3) is well known.

6. Numerical stability versus repetition factor. In § 4 we have characterized the asymptotic repetition factor in terms of the location of the essential zeros of the polynomial ρ , and in § 5 numerical stability was characterized in terms of the growth parameters associated with these zeros. In other words, numerical stability is determined by the rate of change (relative to $\bar{h}$) of the essential zeros and not so much by their location. It is intuitively clear therefore that numerical stability cannot be characterized completely by the repetition factor. We can indicate, however, some connections between the two concepts.

THEOREM 6.1 (Noble [11]). *Step-by-step methods (1.3) with an exact repetition factor of one are numerically stable (for small h).*

With the more general notion of the asymptotic repetition factor introduced in § 4, the above result can be extended.

THEOREM 6.2. *A (ρ, σ) -reducible quadrature method with an asymptotic repetition factor of one is numerically stable (for small h).*

Proof. In view of Corollary 4.2, the polynomial ρ satisfies the strong root condition, or equivalently, $\zeta_1 = 1$ is the only essential zero. Its growth parameter is equal to one by virtue of consistency. Application of Theorem 5.1 yields the result. $\square$

The reverse statements of the theorems above are not true; that is,

THEOREM 6.3. *Methods with an exact or asymptotic repetition factor greater than one can be numerically stable.*

Proof. It is sufficient to consider specific examples. Consider the (ρ, σ) -reducible quadrature method with $\rho(\zeta) = (\zeta^2 - 1)(\zeta - \frac{1}{3})$ and $\sigma(\zeta) = \zeta(\zeta^2 - \frac{2}{3}\zeta + 1)$. In view of Theorem 4.2, the weights have an asymptotic repetition factor of two. The method is numerically stable since the growth parameters associated with the essential zeros $\zeta_1 = 1$ and $\zeta_2 = -1$ are both equal to one. An example of a numerically stable method which has an exact repetition factor of two is obtained by taking $\rho(\zeta) = \zeta^2 - 1$ and $\sigma(\zeta) = \zeta^2 + 1$. $\square$

We emphasize, however, that there exists an *equivalence* between an asymptotic repetition factor of one and strong stability in the sense of Definition 5.3. This important result is given in the following theorem.

THEOREM 6.4. *A (ρ, σ) -reducible quadrature method for solving second kind Volterra integral equations is strongly stable (for small h) if and only if the quadrature weights have an asymptotic repetition factor of one.*

Proof. See the proof of Theorem 6.2. $\square$

We remark that Theorem 6.4 does not hold if the asymptotic repetition factor is replaced by an exact repetition factor. This clearly shows the relevance of the former notion.

We conjecture that for more general quadrature methods (e.g., methods which are reducible to cyclic linear multistep methods) a result analogous to that of Theorem 6.4 can be derived. Such a derivation, however, is beyond the scope of this paper.

7. Absolute and relative stability. McKee and Brunner [10] have interpreted the stability concept of Linz and Noble in a different way. With reference to the test equation (cf. [2])

$$(7.1) \quad f(x) = 1 + \lambda \int_0^x f(y) dy, \quad \lambda < 0,$$

(whose solution $f(x) = \exp(\lambda x)$ decays to zero as $x \rightarrow \infty$), they give the following definition.

DEFINITION 7.1 (from [10]). A method for (1.1) is called *numerically stable* if when applied to (7.1) the discretized solution f_n tends to zero as $n \rightarrow \infty$ for some fixed h .

Note that this definition is reminiscent of the definition of absolute stability in the numerical treatment of ODEs.

With this definition of numerical stability, McKee and Brunner give the following example to demonstrate that the conjecture of Linz is incorrect. They consider the (second order) method generated by the quadrature weights

$$(7.2) \quad W_0 = \frac{1}{6} \begin{bmatrix} 0 & & & & & & & & \\ 3 & 3 & & & & & & & \\ 4 & 4 & 4 & & & & & & \\ 3 & 6 & 6 & 3 & & & & & \\ 4 & 4 & 8 & 4 & 4 & & & & \\ & & \dots & & & & & & \\ 3 & 6 & 6 & 6 & 6 & 6 & \dots & 6 & 3 \\ 4 & 4 & 8 & 4 & 8 & 4 & \dots & 8 & 4 & 4 \end{bmatrix}.$$

The weights in (7.2) are not (ρ, σ) -reducible, but are reducible to a 2-cyclic linear multistep method. Clearly, W_0 has a repetition factor of two. Furthermore, McKee and Brunner show that the method has a nonvanishing interval of absolute stability of the form $(-\alpha, 0)$, and therefore (7.2) is numerically stable in the sense of Definition 7.1.

We recall from § 1 that the asymptotic analysis of Linz and Noble is applicable to general second kind Volterra equations. Since the stability Definition 7.1 refers to one special test equation, it will be clear that numerical stability in the sense of Definition 7.1 (i.e. absolute stability) is only a necessary condition for numerical stability

in the sense of Linz and Noble. To demonstrate this we have determined (cf. Remark 5.2) the values of the growth parameters γ_i of the method (7.2) and obtained $\gamma_1 = 1$ and $\gamma_2 = \frac{1}{6}$. Therefore, in view of Theorem 5.1, the method (7.2) having repetition factor two is numerically unstable in the sense of Definitions 5.1 and 5.2. In § 8 we shall demonstrate the unstable behavior of the method (7.2) when applied to an equation different from (7.1).

Keech [5] employs essentially the same stability definition as McKee and Brunner, and gives the following example:

$$(7.3) \quad W_1 = \frac{1}{2} \begin{bmatrix} 0 & & & & & & & & & & \\ 1 & 1 & & & & & & & & & \\ 2 & 0 & 2 & & & & & & & & \\ 0 & 4 & 1 & 1 & & & & & & & \\ 2 & 0 & 4 & 0 & 2 & & & & & & \\ & & \dots & & & & & & & & \\ 0 & 4 & 0 & 4 & \dots & 0 & 4 & 1 & 1 & & \\ 2 & 0 & 4 & 0 & \dots & 4 & 0 & 4 & 0 & 2 & \\ & & \dots & & & & & & & & \end{bmatrix}.$$

His method is reducible to a 2-cyclic linear multistep method. It has repetition factor two, and Keech shows that the interval of absolute stability is $(-2, 0)$. Therefore, the method (7.3) is numerically stable in the sense of Definition 7.1. It turns out that the growth parameters associated with (7.3) are both equal to one, so that the method of Keech is also numerically stable in the sense of Linz and Noble. Clearly, the method is *not* strongly stable.

Instead of looking at absolute stability as was done by McKee and Brunner and by Keech, we can also adopt the concept of relative stability of a method with respect to (7.1) with $\lambda \in \mathbb{R}$. That is, we require all roots of $P(\bar{h}; \zeta) = 0$ (see Remark 5.2) to satisfy

$$(7.4) \quad |\zeta_i(\bar{h})| \leq |\zeta_1(\bar{h})|, \quad i = 2, 3, \dots,$$

where $\zeta_1(\bar{h})$ corresponds to the principal root (i.e. $\zeta_1(\bar{h}) = \exp(\bar{h}) + O(\bar{h}^{p+1})$ for a method of order p). For the weakly stable method (7.2), the interval of relative stability has the form $(0, \beta)$, $\beta > 0$, whereas for the stable method (7.3), this interval is approximately $(-\frac{3}{4}, \frac{5}{4})$. It is known (see [12]) that the existence of an interval of relative stability of the form $(-\alpha, \beta)$, $\alpha, \beta > 0$, implies that all growth parameters associated with the essential zeros are equal to one. This yields

THEOREM 7.1. *A reducible quadrature method is numerically stable (for small h) if there exists an interval $(-\alpha, \beta)$, $\alpha, \beta > 0$, such that the method is relatively stable for all $\bar{h} \in (-\alpha, \beta)$.*

If a method is numerically stable and not relatively stable for $\bar{h} \in (-\alpha, \beta)$, then the violation of (7.4) for some i is caused only by the $\bar{h}^2$ or higher order terms in the expansion of the essential zero $\zeta_i(\bar{h})$. Therefore for small h , the existence of an interval of relative stability of the form $(-\alpha, \beta)$ is also “almost” necessary for numerical stability in the sense of Linz and Noble.

8. Numerical illustration. In this section we present numerical results partly as an illustration of the various stability concepts discussed in § 5 and partly as a verification of our theoretical results.

For our experiments we have constructed the following quadrature method parameterized by γ ($\gamma \neq 0, 1$):

$$(8.1) \quad W_2(\gamma) = \frac{1}{2} \begin{bmatrix} 0 & & & & & & \\ 1 & & 1 & & & & 0 \\ 1+\gamma & 2-2\gamma & 1+\gamma & & & & \\ 1 & 2+\gamma & 2-2\gamma & 1+\gamma & & & \\ 1+\gamma & 2-2\gamma & 2+2\gamma & 2-2\gamma & 1+\gamma & & \\ 1 & 2+\gamma & 2-2\gamma & 2+2\gamma & 2-2\gamma & 1+\gamma & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The quadrature weights in (8.1) are reducible to the linear multistep method $\{\rho, \sigma\}$ with $\rho(\zeta) = \zeta^2 - 1$ and $\sigma(\zeta) = \zeta^2(1+\gamma)/2 + \zeta(1-\gamma) + (1+\gamma)/2$. In view of Corollary 4.1 the weights have an exact repetition factor of two. Furthermore, the method is second order convergent and the growth parameters associated with $\zeta_1 = 1$ and $\zeta_2 = -1$ are $\gamma_1 = 1$ and $\gamma_2 = \gamma$, respectively. The method has an interval of absolute stability $(-\infty, 0)$ if $\gamma > 0$. It has an interval of relative stability $(0, \infty)$ if $\gamma < 1$, and $(-\infty, 0)$ if $\gamma > 1$. Since $\gamma \neq 1$, the method is weakly stable in view of Theorem 5.1.

We have solved the integral equation of the form (5.4)

$$(8.2) \quad f(x) = 1 + \lambda \int_0^x \exp(\mu(x-y))f(y) dy,$$

whose exact solution is, in view of (5.5), given by

$$(8.3) \quad f(x) = (\mu + \lambda \exp(\lambda + \mu)x)/(\lambda + \mu).$$

We took the values $(\lambda, \mu) = (1, -2)$ and $(\lambda, \mu) = (-2, 1)$. Seven different methods were used, to be specific: the method of McKee and Brunner given by (7.2) and denoted by W_0 ; the method of Keech given by (7.3) and denoted by W_1 ; the method (8.1) with $\gamma = \frac{1}{6}$ and $\gamma = 3$ and denoted by $W_2(\frac{1}{6})$ and $W_2(3)$, respectively; the methods employing quadrature weights which are reducible to the second and third order backward differentiation method and to the third order Adams–Moulton method and denoted by BD_2 , BD_3 and AM_3 , respectively. (Note that the method AM_3 is identical to the quadrature method employing the third order Newton–Gregory quadrature rules.)

Since the polynomial ρ associated with the methods BD_2 , BD_3 and AM_3 satisfies the strong root condition, these methods have an asymptotic repetition factor of one (cf. Corollary 4.2) and are strongly stable (cf. Theorem 6.4). The method W_1 has an exact repetition factor of two and is stable but not strongly stable. The remaining methods also have an exact repetition factor of two but are weakly stable (see Theorem 5.1). In view of the values of the growth parameters, we expect that for $(\lambda, \mu) = (1, -2)$ all methods except $W_2(3)$ yield stable results, whereas for $(\lambda, \mu) = (-2, 1)$ the methods W_0 and $W_2(\frac{1}{6})$ are expected to behave unstably.

To demonstrate clearly this unstable behavior of some of the methods, it is necessary to integrate over a rather long time interval. We have integrated the problem (8.2) on the interval $[0, 25]$ with stepsizes $h = 0.1$ and $h = 0.05$. In Tables 8.1 and 8.2 we have listed the true error only for $h = 0.05$ (the results for $h = 0.1$ show the same behavior).

From these tables we conclude that, dependent on the values λ and μ , the methods with a growth parameter different from one (that is, W_0 , $W_2(\frac{1}{6})$ and $W_2(3)$) are unstable, whereas the strongly stable methods (BD_2 , BD_3 and AM_3) and the stable method of

TABLE 8.1
True error for (8.2) with $\lambda = 1, \mu = -2$ and $h = 0.05$.

x	W_0	$W_2(\frac{1}{6})$	$W_2(3)$	W_1	BD_2	BD_3	AM_3
5.0	$-3.0_{10^{-3}}$	$-2.5_{10^{-3}}$	$-6.7_{10^{-2}}$	$-6.6_{10^{-3}}$	$-6.2_{10^{-3}}$	$-4.4_{10^{-4}}$	$-8.0_{10^{-5}}$
15.0	$-3.0_{10^{-3}}$	$-2.5_{10^{-3}}$	$-1.3_{10^{+3}}$	$-6.7_{10^{-3}}$	$-6.2_{10^{-3}}$	$-4.4_{10^{-4}}$	$-8.0_{10^{-5}}$
25.0	$-3.0_{10^{-3}}$	$-2.5_{10^{-3}}$	$-3.1_{10^{+7}}$	$-6.7_{10^{-3}}$	$-6.2_{10^{-3}}$	$-4.4_{10^{-4}}$	$-8.0_{10^{-5}}$

TABLE 8.2
True error for (8.2) with $\lambda = -2, \mu = 1$ and $h = 0.05$.

x	W_0	$W_2(\frac{1}{6})$	$W_2(3)$	W_1	BD_2	BD_3	AM_3
5.0	$1.5_{10^{-2}}$	$5.3_{10^{-4}}$	$-3.0_{10^{-3}}$	$-1.2_{10^{-3}}$	$-1.2_{10^{-3}}$	$2.9_{10^{-5}}$	$5.7_{10^{-6}}$
15.0	$1.2_{10^{+1}}$	$7.7_{10^{-1}}$	$-4.1_{10^{-3}}$	$-1.7_{10^{-3}}$	$-1.7_{10^{-3}}$	$6.6_{10^{-5}}$	$1.1_{10^{-5}}$
25.0	$9.5_{10^{+3}}$	$6.0_{10^{+2}}$	$-4.1_{10^{-3}}$	$-1.7_{10^{-3}}$	$-1.7_{10^{-3}}$	$6.6_{10^{-5}}$	$1.1_{10^{-5}}$

Keech (W_1) yield stable results for both problems. Although it was not included in the tables of results, we also noticed that for the stable method of Keech the true error changes sign at every mesh point, whereas the strongly stable methods yield a smooth global error. Clearly, the numerical results are in full agreement with the theory.

In order to eliminate the effect of the quadrature error (which may be quite large when solving (8.2) with $\lambda = -2$ and $\mu = 1$), we have also investigated the effect of an isolated perturbation (see [2]). The methods were applied to (8.2) yielding values f_n ; next the value of f_1 was perturbed by an amount of 0.01, and the method was applied once again yielding perturbed values $\tilde{f}_n$. In Tables 8.3 and 8.4 we have listed the difference $|\tilde{f}_n - f_n|$ at some meshpoints. The tables show that for both problems the perturbation is damped by the strongly stable and stable methods, whereas it is amplified by the weakly stable methods (dependent on the values of λ and μ). We remark that for the method of Keech we have perturbed f_2 instead of f_1 , since it can be seen from (7.3) that a perturbation of f_1 has no effect on the even-numbered gridpoints which are displayed in the tables.

TABLE 8.3
Effect of an isolated perturbation ($\lambda = 1, \mu = -2; h = 0.1$).

x	W_0	$W_2(\frac{1}{6})$	$W_2(3)$	W_1	BD_2	BD_3	AM_3
5.0	$4.7_{10^{-6}}$	$5.6_{10^{-6}}$	$2.7_{10^{-1}}$	$1.0_{10^{-5}}$	$4.4_{10^{-6}}$	$8.8_{10^{-6}}$	$6.3_{10^{-6}}$
15.0	$2.2_{10^{-10}}$	$2.6_{10^{-10}}$	$9.4_{10^{+3}}$	$4.8_{10^{-10}}$	$2.1_{10^{-10}}$	$4.0_{10^{-10}}$	$2.9_{10^{-10}}$
25.0	$1.4_{10^{-14}}$	$1.4_{10^{-14}}$	$3.3_{10^{+8}}$	$1.4_{10^{-14}}$	$1.4_{10^{-14}}$	$1.4_{10^{-14}}$	$1.4_{10^{-14}}$

TABLE 8.4
Effect of an isolated perturbation ($\lambda = -2, \mu = 1; h = 0.1$).

x	W_0	$W_2(\frac{1}{6})$	$W_2(3)$	W_1	BD_2	BD_3	AM_3
5.0	$8.5_{10^{-3}}$	$8.3_{10^{-3}}$	$-6.7_{10^{-6}}$	$4.4_{10^{-5}}$	$1.3_{10^{-5}}$	$3.7_{10^{-5}}$	$1.9_{10^{-5}}$
15.0	$6.7_{10^{+0}}$	$6.5_{10^{+0}}$	$-1.5_{10^{-10}}$	$1.5_{10^{-9}}$	$4.1_{10^{-10}}$	$1.8_{10^{-9}}$	$8.9_{10^{-10}}$
25.0	$5.2_{10^{+3}}$	$5.1_{10^{+3}}$	0	$5.0_{10^{-14}}$	0	$6.4_{10^{-14}}$	$5.0_{10^{-14}}$

All calculations were performed on a CDC-CYBER 750 computer system using single precision (60-bit wordlength with a 48-bit mantissa).

9. Concluding remarks. Motivated by a conjecture of Linz, we have investigated for a special class of quadrature methods the relationship between the (asymptotic) repetition factor and numerical stability. We have shown that the methods with an (asymptotic) repetition factor of one are strongly stable, which implies numerical stability in the sense of Linze and Noble. However, if a method has a repetition factor greater than one, we *need additional information* in order to determine whether that method is numerically stable or not. To be specific, we have to check that the values of the growth parameters associated with the essential zeros of the polynomial p are equal to one. In general, these values can be determined from the stability polynomial associated with the method when applied to the test equation (7.1), and in this connection the analysis of Baker and Keech [2] can be used, although that analysis was developed for a different type of stability.

On the other hand, it is the rule rather than the exception that for a nonartificially constructed method, the growth parameters associated with the essential zeros ($\neq 1$) are different from one (see e.g. [13, p. 247]), so that we share the general opinion that methods with an (asymptotic) repetition factor greater than one should not be generally employed for the solution of second kind Volterra integral equations.

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