On the degree of approximation of $2\pi$-periodic functions in $C^1$ with positive linear operators of the Jackson type

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by

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Abstract

Let $C^1_{2\pi}$ be the class of real, continuously differentiable, $2\pi$-periodic, functions of a real variable. The positive linear operators of the Jackson type are denoted by $L_{n,p}$ ($n \in \mathbb{N}$), where $p$ is a fixed positive integer. The object of this paper is to study the exact degree of approximation with the operators $L_{n,p}$ for functions in $C^1_{2\pi}$. The value of $\max_x |L_{n,p}(f;x) - f(x)|$ is estimated in terms of $\omega(f';\delta)$, the modulus of continuity of $f'$, with $\delta = \pi/n$. Exact constants of approximation are determined for the operators $L_{n,p}$ ($n \in \mathbb{N}, p \geq 2$) and for the Fejér operators $L_{n,1}$ ($n \in \mathbb{N}$). Furthermore, the limiting behaviour of these constants is investigated as $n \to \infty$ and $p \to \infty$, separately and simultaneously.

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1. Introduction and summary

1.0. The class of real, continuous, $2\pi$-periodic, functions of a real variable is denoted by $C_{2\pi}$. Assume $f \in C_{2\pi}$ and let $p$ be a positive integer. The positive linear operators $L_{n,p}$ are then defined by the relation

\[ L_{n,p}(f;x) = \int_{-\pi}^{\pi} f(x+t)k_{n,p}(t)dt \quad (n \in \mathbb{N}) \]

where

\[ k_{n,p}(t) = A_{n,p}^{-1} \left( \frac{\sin \frac{1}{t}}{\sin \frac{1}{n+1}} \right)^{2p} \]

with $A_{n,p}$ such that $\int_{-\pi}^{\pi} k_{n,p}(t)dt = 1$.

If $p = 1$ we obtain the Fejér operators, while the name of Jackson is associated with $L_{n,2}$. Approximation properties of the operators $L_{n,p}$, in particular those of $L_{n,1}$ and $L_{n,2}$, have been extensively studied; cf., for instance, Butzer and Stark [3], Görlich and Stark [5], Matsuoka [7], Schurer and Steutel [9], where the survey paper [5] deserves special mention because of its wealth of references.

In view of the kind of problems we shall be concerned with in this report, we mention here the following result of Wang Hsing-hua [12], where the exact constant of approximation of functions in $C_{2\pi}$ by the Jackson operators is determined. He obtains

\[ \sup_{x \in [n+1]} \sup_{f \in C_{2\pi}} \max_{x} \left| L_{N,2}(f;x) - f(x) \right| = \frac{3}{2} \quad (N = [\frac{3}{2}] + 1) \]

assuming $f$ to be nonconstant. As usual, $\omega(f;\delta)$ denotes the modulus of continuity of $f$, defined by

\[ \omega(f;\delta) = \sup_{|x-y| \leq \delta} \left| f(x) - f(y) \right| \quad (\delta > 0) . \]

\[ *[a] \] denotes the largest integer not exceeding $a$.  

\[ *) \]
Results analogous to (1.3) for the operators $L_{n,p}$ ($p = 3,4$) and $L_{n,p}$ ($p \geq 5$) may be found in [8] and [9] respectively. A similar problem for the Jackson operators in two variables was solved by Bugaets and Martynyuk [2].

1.1. In [10] we investigated the local degree of approximation of continuously differentiable functions by Bernstein polynomials; a similar analysis was recently carried out for the well-known Meyer-König and Zeller operators (cf. [11]).

This report is concerned with related problems. Here the setting is the class $C^2_{2\pi}$ of real, continuously differentiable, $2\pi$-periodic, functions of a real variable. The degree of approximation is measured in terms of the modulus of continuity of $f'$, denoted by $\omega_1$. In particular, we shall deal with the problem of determining the exact constants of approximation of functions in $C^1_{2\pi}$ by the operators $L_{n,p}$. Furthermore, the limiting behaviour of these constants will be investigated.

More specifically, for $f$ nonconstant and assuming $n \in \mathbb{N}$ and $p \in \mathbb{N}$ fixed, the exact constant of approximation for the Jackson type operator $L_{n,p}$ ($p \geq 2$) is defined by

$$
(1.4) \quad c_{n,p} := n^{-p} \sup \left\{ \frac{|L_{n,p}(f;x) - f(x)|}{\omega_1(f;\pi_n)} : x \in \mathbb{R}, f \in C^1_{2\pi} \right\},
$$

whereas for the Fejér operators the definition reads

$$
(1.5) \quad c_{n,1} := \sup \left\{ \frac{|L_{n,1}(f;x) - f(x)|}{\omega_1(f;\pi_n)} : x \in \mathbb{R}, f \in C^1_{2\pi} \right\}.
$$

The norming is prescribed by asymptotic properties; the Fejér operators $L_{n,1}$ differ in this respect from the operators $L_{n,p}$ ($p \geq 2$). In order to keep the constants $c_{1,p}$ bounded, definition (1.4) in case $n = 1$ is replaced by

$$
(1.6) \quad c_{1,p} := c_{1,2} \quad (p = 3,4,\ldots).
$$

Assuming $p \in \mathbb{N}$ fixed the exact constant of approximation for the sequence of operators $\{L_{n,p}\} (n \in \mathbb{N})$ may then be defined as (cf. (1.3))

$$
(1.7) \quad c^{(p)} := \sup_{n \in \mathbb{N}} c_{n,p}.
$$
Furthermore, the exact constant of approximation for the whole class of operators $L_{n,p}$ ($n \in \mathbb{N}, p \geq 2$) is defined by

\[(1.8) \quad c := \sup_{p \geq 2} c(p).\]

1.2. We now give a brief sketch of the contents of the various sections. Section 2 contains all those preliminary results which will be needed frequently in the sequel. Relevant material on the kernels of the operators $L_{n,p}$ is gathered together from [9], whereas also some inequalities are given that are useful when estimating integrals over these kernels. In section 3 the so-called extremal functions are introduced; just as in the investigation of the Bernstein polynomials and the Meyer-König and Zeller operators, they play a crucial role in determining the constants $c_{n,p}$. The pattern of deducing the extremal functions is in part similar to the procedure given in [10]; a serious complication however is caused by the constraint of periodicity. The material of sections 2 and 3 is then used in section 4 to establish that $c = \frac{2}{\sqrt{\pi}} = 1.12837917$. *) The proof is quite intricate, mainly because a number of different cases of $n$ and $p$ have to be examined separately, each of them needing a different approach. In particular, cases $p = 2$ and $p = 3$ have to be investigated in considerable detail. The numerical values of the constants $c_{n,p}$ for the first few values of $n$ and $p \geq 2$ are given. The exact constant of approximation in the case of the Fejér operators is determined in the last part of section 4; we show that $c(1) = \frac{\pi}{4}$ (cf. (1.7)). Moreover, a table with the values of $c_{n,1}$ $(n = 1(1)10; 15(5)50; 75; 100)$ is given. In section 5 the limiting behaviour of $c_{n,p}$ is considered as $n \to \infty$ and $p \to \infty$, separately and simultaneously (in both orders). For instance, it is proved that

$$\lim_{n \to \infty} \lim_{p \to \infty} c_{n,p} = \lim_{p \to \infty} \lim_{n \to \infty} c_{n,p} = \sqrt{\frac{3}{\pi}} = 0.97720502.$$ 

A separate discussion is devoted to the asymptotic degree of approximation by Fejér operators. It turns out (viz. definitions (1.4) and (1.5)) that their performance is essentially worse than the operators $L_{n,p}$ ($p \geq 2$).

*) Here and elsewhere numbers are rounded to the last digit shown.
The value of \( \lim_{n \to \infty} c_n \) is determined. Section 5 also contains a table with the numerical values of \( c_p := \lim_{n \to \infty} c_{n,p} \) (\( p = 2(1)0; 20(10)50 \)).

2. Preliminary results

2.0. Approximation properties of the operators \( L_{n,p} \) were investigated in [9]. From that paper we need the following four lemmas.

**Lemma 2.1.** If \( v \) and \( \mu \) are positive integers then

\[
S(v, \mu) := \int_{0}^{\infty} \frac{(\sin t)^{2v}}{t^{2\mu}} \, dt = \frac{(-1)^{\mu+v}}{2^{v} (2\mu-1)!} \sum_{j=0}^{v-1} (-1)^j (2^v) (2v-2j)^{2\mu-1} \quad (v \geq \mu),
\]

\[
S(v, \mu+\frac{1}{2}) := \int_{0}^{\infty} \frac{(\sin t)^{2v}}{t^{2\mu+1}} \, dt = \frac{(-1)^{\mu+v+1}}{2^{2v-1} (2\mu)!} \sum_{j=0}^{v-1} (-1)^j (2^v) (2v-2j)^{2\mu} \log(2v-2j) \quad (v > \mu).
\]

**Lemma 2.2.** For any positive integer \( p \) and for \( k = 0, 1, \ldots, 2p-2 \) the following asymptotic equivalence holds

\[
\int_{0}^{\pi} t^k \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2p} \, dt \sim 2^{k+1} \, n^{-2p-k-1} \, S(p, p-k) \quad (n \to \infty).
\]

**Lemma 2.3.** If \( k \) and \( p \) satisfy the conditions of lemma 2.2 then

\[
S(p, p-k) \sim \frac{k+1}{p} \left( \frac{3}{p} \right)^{\frac{k+1}{2}} \Gamma \left( \frac{k+1}{2} \right) \quad (p \to \infty).
\]
**Lemma 2.4.** The coefficients $\mu_{n,p}(n,p)$ in the expansion

\[
\frac{(\sin \frac{nt}{t})^{2p}}{\sin \frac{t}{t}} = \mu_0(n,p) + 2 \sum_{k=1}^{np-p} \mu_k(n,p) \cos kt
\]

are given by

\[
\mu_k(n,p) = \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} \binom{np-p-k-nj-1}{2p-1},
\]

with the usual convention that $\binom{a}{b} = 0$ if $a < b$.

2.1. We proceed by proving a few inequalities that will be used for estimating integrals over the kernel (2.3). These integrals will be encountered in section 4.

**Lemma 2.5.** For $n \in \mathbb{N}$ one has

\[
\frac{\sin t}{n \sin \frac{t}{n}} \geq \exp(-\frac{1}{8}at^2) \quad (0 < t \leq \frac{\pi}{2}),
\]

where

\[
a = \frac{8}{\pi^2} \log(\frac{\pi}{2}) = 0.366039.
\]

**Proof.** As

\[
n \sin \frac{t}{n} \leq (n+1) \sin \frac{t}{n+1} \quad (0 < t \leq \frac{\pi}{2}),
\]

it is sufficient to show that

\[
\frac{\sin t}{t} \geq \exp\left(-\frac{1}{8}at^2\right) \quad (0 < t \leq \frac{\pi}{2}).
\]

Put

\[
f(t) := \frac{\sin t}{t} - \exp\left(-\frac{1}{8}at^2\right).
\]

Expansion in Taylor series shows that $f$ is positive on the interval $(0, \frac{3\pi}{8}]$. Furthermore, $f$ is decreasing on $[\frac{3\pi}{8}, \frac{\pi}{2}]$, while $f(\frac{\pi}{2}) = 0$. This proves the lemma.
In order to investigate the behaviour of the constants $c_3$ as defined in (1.4) we need a slight improvement of lemma 2.5 if $n = 3$. This is given by lemma 2.6.

**Lemma 2.6.**

$$\frac{\sin t}{3 \sin \frac{t}{3}} \geq \exp(-\frac{1}{3}bt^2) \quad (0 < t \leq \frac{\pi}{2}),$$

where

$$b = \frac{8}{\pi^2} \log\left(\frac{3}{2}\right) = 0.328658.$$

**Proof.** Putting $v := \frac{t}{3}$ we have to show that

$$f(v) := 1 - \frac{4}{3} \sin^2 v - \exp(-9bv^2) \geq 0 \quad (0 < v \leq \frac{\pi}{6}).$$

A simple computation shows that $f$ is positive and increasing on $(0, \frac{1}{3}]$. The interval $(\frac{1}{3}, \frac{\pi}{6}]$ is taken care of by noting that $f'$ decreases, i.e. that $f$ is concave there, and $f(\frac{\pi}{6}) = 0$.

**Lemma 2.7.** For $n \in \mathbb{N}$ one has

$$(2.7) \quad \frac{\sin t}{n \sin \frac{t}{n}} \leq \exp(-\frac{1}{6}(1 - \frac{1}{n^2})t^2) \quad (0 < t < \pi).$$

**Proof.** Inequality (2.7) can be rewritten in the form

$$\frac{\sin t}{t} \exp\left(\frac{1}{6}t^2\right) \leq \frac{\sin \frac{t}{n}}{\frac{t}{n}} \exp\left(\frac{1}{6}\frac{t^2}{n^2}\right) \quad (n \in \mathbb{N}).$$

From this it follows that it is sufficient to show that $f(t) := \frac{\sin t}{t} \exp\left(\frac{1}{6}t^2\right)$ is decreasing on $(0, \pi)$. One has

$$f'(t) = \frac{1}{3}t^{-2} \exp\left(\frac{1}{6}t^2\right) (-3 \sin t + 3t \cos t + t^2 \sin t),$$

and the expression between brackets is easily seen to be negative on $(0, \pi)$. 

2.2. The next result will be needed to obtain upper and lower bounds for the integral \( \int_0^\pi t(\cos t)^{2p} \, dt \), which occurs in section 4.1.4.

Lemma 2.8.

\[
\sin t + \frac{1}{6} \sin^3 t + \frac{3}{40} \sin^5 t \leq t \leq \sin t + \frac{1}{6} \sin^3 t + \frac{3}{40} \sin^5 t + \left( \frac{\pi}{2} - \frac{149}{120} \right) \sin t \quad (0 \leq t \leq \frac{\pi}{2}).
\]

Proof. It is well known that

\[
\arcsin x = x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \ldots = \sum_{j=0}^{\infty} a_{2j+1} x^{2j+1} \quad (|x| \leq 1),
\]

with \( a_{2j+1} > 0 \) for all \( j \). Writing \( t := \arcsin x \) it follows that

\[
t = \sum_{j=0}^{\infty} a_{2j+1} (\sin t)^{2j+1} \quad (|t| \leq \frac{\pi}{2})
\]

and also

\[
\frac{\pi}{2} = \sum_{j=0}^{\infty} a_{2j+1}.
\]

As \( a_{2j+1} > 0 \) the left-hand side of inequality (2.8) obviously holds; the right-hand side follows from (2.9) by observing that

\[
t = \sin t + \frac{1}{6} \sin^3 t + \frac{3}{40} \sin^5 t + \sum_{j=3}^{\infty} a_{2j+1} (\sin t)^{2j+1} \leq \\
\leq \sin t + \frac{1}{6} \sin^3 t + \frac{3}{40} \sin^5 t + \sin^7 t \sum_{j=3}^{\infty} a_{2j+1}.
\]

Finally we note that (cf. [6], p. 97)

\[
\int_0^\pi (\sin t)^{2p} \, dt = \int_0^\pi (\cos t)^{2p} \, dt = \frac{\pi(2p)!}{2^{2p+1}(p!)^2},
\]

a result that will be frequently needed in section 4.
3. The extremal functions

3.0. As in (1.1) let

\[ L_{n,p}(f;x) = \int_{-\pi}^{\pi} f(x+t)k_{n,p}(t)dt, \]

where \( k_{n,p} \) is given by (1.2). Assuming \( n \in \mathbb{N} \) and \( p \in \mathbb{N} \) fixed, we shall determine \( d_{n,p} \) defined by

\[ d_{n,p} = \sup \{ |\Delta_{n,p}(f;x)| : x \in \mathbb{R}, f \in F_n \}, \]

where

\[ \Delta_{n,p}(f;x) := L_{n,p}(f;x) - f(x) \]

and \( F := F_n \) is defined by

\[ F = \{ f: [-\pi, \pi] \rightarrow \mathbb{R} ; f \in C \mathbb{L}_{2\pi}, \omega_1(f; \frac{\pi}{n}) \leq 1 \}. \]

3.1. Lemma 3.1.

\[ d_{n,p} = \sup_{f \in F_0} |\Delta_{n,p}f|, \]

where, defining \( \tilde{f} \) by \( \tilde{f}(t) = f(-t), \)

\[ F_0 = \{ f \in F ; \tilde{f} = f, f(0) = 0, f'(t) \geq 0 \text{ for } t \in [0,\pi] \}, \]

and \( \Delta_{n,p}f \) is defined, for \( f \in F_0 \), by

\[ \Delta_{n,p}f = \Delta_{n,p}(f;0) = \int_{-\pi}^{\pi} f(t)k_{n,p}(t)dt. \]

Proof. As for \( x \in \mathbb{R} \) and \( f \in F \) also \( f_x \in F \), where \( f_x \) is defined by \( f_x(t) = f(t+x) \), we have \( L_{n,p}(f;x) = L_{n,p}(f;0) \). Hence it is no restriction to take \( x = 0 \). As \( L_{n,p} \) is linear and \( f - f(0) \in F \) if \( f \in F \), it is no restriction to take \( f(0) = 0 \). Furthermore, as \( k_{n,p} = k_{n,p} \), we have

\[ \Delta_{n,p}\tilde{f} = \Delta_{n,p}f = \Delta_{n,p}\frac{f+f}{2} \]

hence it is no restriction to take \( f \) such that \( f = \tilde{f} \). Finally, it is no restriction to assume that \( \Delta_{n,p}f \geq 0 \), as for \( f \in F \)
we have \(-f \in \mathcal{F}\). It follows that for even \(f \in \mathcal{F}\) with \(f(0) = 0\) we have \(\hat{f} \in \mathcal{F}_0\) and \(\hat{f} \geq f\) if we define \(\hat{f}\) by \(\hat{f}(0) = 0\) and \(\hat{f}'(t) = \max(0,f'(t))\) for \(t \in [0,\pi]\), and by symmetry on \([-\pi,0]\). As \(\hat{f}(t) \geq f(t)\) for all \(t\) it follows from (3.4) that \(\Delta_{\frac{3\pi}{2}} \geq \Delta_{\frac{\pi}{2}} f\). This proves lemma 3.1.

3.2. We now have to maximize

\[
\Delta_{\frac{3\pi}{2}} f = \int_{-\pi}^{\pi} f(t)k_{n,p}(t)dt = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(t)k_{n,p}(t)dt
\]

for \(f \in \mathcal{F}_0\). We first prove two general lemmas.

**Lemma 3.2.** Let \(K\) be a finite, nondecreasing function on \([-\frac{1}{2},\frac{1}{2}]\) and for fixed \(n \in \mathbb{N}\) let \(G_1 := G_{1,n}\) be defined by

\[
G_1 = \{g: [-\frac{1}{2},\frac{1}{2}] \to \mathbb{R}; \tilde{g} = g, g(0) = 0, g' \text{ continuous}, \omega_1(g;\frac{1}{n}) \leq 1\}.
\]

Then

\[
\sup_{g \in G_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)dK(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{g}_1(t)dK(t),
\]

where \(\tilde{g}_1 := \tilde{g}_{1,n}\) is defined by \(\tilde{g}_1(0) = 0\) and

\[
\tilde{g}_1(t) = j + \frac{1}{2} \quad (\frac{j}{n} < t < \frac{j+1}{n}, j = 0, \pm 1, \pm 2, \ldots).
\]

**Proof.** The proof of this lemma involves exactly the same steps as the proof of Theorem 3.1 in [10]. This is apparent if we write the Bernstein polynomial as (cf. [10], formula (1.1))

\[
B_n(f;x) = \int_{0}^{1} f(t)dK(x,t).
\]

**Lemma 3.3.** Let \(K\) be a finite, nondecreasing function on \([-\frac{1}{2},\frac{1}{2}]\) and for fixed \(n \in \mathbb{N}\) let \(G_2 := G_{2,n}\) be defined by

\[
G_2 = \{g: [-\frac{1}{2},\frac{1}{2}] \to \mathbb{R}; \bar{g} = g, g(\frac{1}{2}) = 0, g' \text{ continuous}, \omega_1(g;\frac{1}{n}) \leq 1\}.
\]
Then
\[ \sup_{g \in G_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t) dK(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{g}_2(t) dK(t), \]
where \( \tilde{g}_2 := \tilde{g}_{2,n} \) is defined by \( \tilde{g}_2(\frac{1}{2}) = 0 \) and
\[
\begin{cases} 
\tilde{g}_2'(t) = -j & \left( \frac{2j-1}{4m} < t < \frac{2j+1}{4m}, j = 0, \pm 1, \pm 2, \ldots \right) \text{ if } n = 2m, \\
\tilde{g}_2'(t) = -(j+1) & \left( \frac{j}{2m+1} < t < \frac{j+1}{2m+1}, j = 0, \pm 1, \pm 2, \ldots \right) \text{ if } n = 2m+1.
\end{cases}
\]

Proof. For \( g \in G_2 \) we have, using integration by parts,
\[
(3.6) \quad Dg := \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t) dK(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(t) dK(u) dt.
\]

We first state and prove three propositions.

**Proposition (i).** It is no restriction to take \( g \) concave, i.e. to take \( g' \) nonincreasing.

Proof. For \( g \in G_2 \) we define an even function \( \check{g} \) by \( \check{g}(\frac{1}{2}) = 0 \), and
\[
\check{g}'(t) = \sup_{t \leq s \leq 0} g'(s) \quad (-\frac{1}{2} \leq t \leq 0).
\]
It is easily verified that \( \check{g}' \) is nonincreasing and that \( \check{g} \in G_2 \). As \( \check{g}'(-t) - g'(-t) = g'(t) - \check{g}'(t) \geq 0 \) on \([0, \frac{1}{2}]\) and \( \int_{-\frac{1}{2}}^{\frac{1}{2}} dK(u) \) is nonincreasing, it follows from (3.6) that \( D\check{g} \geq Dg \).

**Proposition (ii).** Let \( G_2^* := \{ g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R} ; \ g = g, g(\frac{1}{2}) = 0, g' \) nonincreasing, \( g' \) continuous except for finitely many jumps, \( \omega_1(g; \frac{1}{n}) \leq 1 \} \).

Then
\[
\sup_{g \in G_2} Dg = \sup_{g \in G_2^*} Dg.
\]
Proof. By proposition (i) $g'$ may be taken to be nonincreasing. Furthermore any $g'$ with $g \in G^*_2$ is the pointwise limit of functions $g'_n$ with $g_n \in G_2$ and having the same $\omega_1$ value. By (3.6) this proves proposition (ii).

Proposition (iii). It is no restriction to assume that $g \in G^*_2$ satisfies

\[(3.7) \quad g'(t) = g'(t - \frac{1}{n}) - 1 \quad (\frac{-1}{n} + \frac{1}{n} \leq t \leq \frac{1}{n}).\]

Proof. If for $g \in G^*_2$ condition (3.7) is violated anywhere for $t = t_0$ with $t_0 \in [-\frac{1}{n} + \frac{1}{2n}, \frac{1}{2n})$, then $g$ can be replaced by $g_0 \in G^*_2$ as indicated in figure 3.1 below, where the graphs of $g'$ and $g'_0$ are shown. Here $g'_0$ is obtained from $g'$ for $t < 0$ as follows:

$$g'_0(t) = \begin{cases} 
  g'(t_0) + 1 & (t \leq t_0 - \frac{1}{n} \text{ and } g'(t) < g'(t_0) + 1) \\
  g'(t) & \text{(otherwise)}, 
\end{cases}$$

and by symmetry for $t > 0$.

![Figure 3.1.](attachment:image.png)
Clearly, $Dg_0 \geq Dg$ (cf. the proof of proposition (i)), and $g_0 \in G^*_2$ if $g \in G^*_2$. Hence we may restrict our attention to functions $g$ satisfying (3.7), with the possible exception of the point $t = \frac{1}{2n}$, which does not affect the value of $Dg$. This proves the proposition.

Now let $g \in G^*_2$ satisfy (3.7) and let $g'(t) = \frac{1}{2}(g'(t-0) + g'(t+0))$ be defined for $t \in [-\frac{1}{2}, \frac{1}{2}]$; redefining $g'$ in this sense at discontinuity points does not affect (3.7). Then, as we have $g'(0) = 0$ and $g'(-\frac{1}{2n}) = -g'(-\frac{1}{2n}) = \frac{1}{2}$, it follows in view of (3.7) that for $j = 0, 1, \ldots, n$ we have

$$
(3.8) \quad g'(-\frac{i}{n} + \frac{j}{n}) = \frac{n}{2} - j.
$$

We now replace $g$ by $\tilde{g} \in G^*_2$ obtained by joining the straight lines tangent to the graph of $g$ at the points $(-\frac{i}{n} + \frac{j}{n}, g(-\frac{i}{n} + \frac{j}{n}))$, i.e. with tangents given by (3.8). As $g$ is concave we have $\tilde{g} \geq g$ and hence $D\tilde{g} \geq Dg$. Finally we show that $\tilde{g} \in G^*_2$. Writing $\gamma_j = g(-\frac{i}{n} + \frac{j}{n})$ for $g \in G^*_2$ satisfying (3.8), we have

$$
\gamma_{j+1} = 2\gamma_j + \gamma_{j-1} - \frac{1}{n},
$$

with $\gamma_0 = \gamma_n = 0$. It follows that all functions in $G^*_2$ satisfying (3.8) have graphs that pass through the points $(-\frac{i}{n} + \frac{j}{n}, \frac{1}{2}(1-\frac{j}{n}))$ for $j = 0, 1, \ldots, n$.

Thus the graph of $\tilde{g}$ passes through these points, and hence $\tilde{g}$ is identical with the function $g_2$ as defined in (3.5). Clearly, from the previous propositions and the construction of $\tilde{g}_2$ it follows that $D\tilde{g}_2 \geq Dg$ for $g \in G^*_2$. This proves lemma 3.3.

A sketch of the functions $\tilde{g}_1$ and $\tilde{g}_2$ with their derivatives is given in figure 3.2 below for $n = 4$. 

---

Figure 3.2.
3.3. We are now in a position to prove the main result of this section.

Theorem 3.1. Let $d_{n,p}$ be defined as in (3.1). Then

$$
(3.9) \quad d_{n,p} = \int_{-\pi}^{\pi} \tilde{f}_n(t) k_{n,p}(t) dt,
$$

where $\tilde{f}_n$ is defined by $\tilde{f}_n(0) = 0$, $\tilde{f}_n$ is even, and

$$
(3.10) \quad \tilde{f}_{2m}'(t) = \begin{cases} 
  j + \frac{1}{2} & (\frac{j\pi}{2m} < t < \frac{(j+1)\pi}{2m}, j = 0,1,\ldots,m-1) \\
  2m - j & (\frac{\pi}{2} < \frac{(2j-1)\pi}{4m} < t < \frac{(2j+1)\pi}{4m} < \pi, j = m,m+1,\ldots,2m),
\end{cases} \quad (m = 1,2,\ldots)
$$

$$
(3.11) \quad \tilde{f}_{2m+1}'(t) = \begin{cases} 
  j + \frac{1}{2} & (\frac{j\pi}{2m+1} < t < \frac{(j+1)\pi}{2m+1}, j = 0,1,\ldots,m) \\
  2m - j + \frac{1}{2} & (\frac{\pi}{2} < \frac{(2j-1)\pi}{2m+1} < t < \frac{(2j+1)\pi}{2m+1} < \pi, j = m+1,m+2,\ldots,2m).
\end{cases} \quad (m = 0,1,\ldots)
$$

Proof. The function $\tilde{f}_n^\prime$, except for a linear transformation, consists of the functions $\tilde{g}_1,n$ and $\tilde{g}_2,n$ put together. To be precise we have

$$
\tilde{f}_n(t) = \begin{cases} 
  \pi \tilde{g}_1,n(t) & (0 \leq t \leq \frac{\pi}{2}) \\
  \tilde{f}_n(\frac{\pi}{2}) + \pi \tilde{g}_2,n(\frac{\pi}{2} - 1) & (\frac{\pi}{2} \leq t \leq \pi)
\end{cases}
$$

for $t \in [0,\pi]$, and by symmetry elsewhere. One easily verifies that (by good luck) the jumps at (or close to) $t = \frac{\pi}{2}$ of $\tilde{f}_n^\prime$ are such that $\omega_1(\tilde{f}_n^\prime; \frac{\pi}{2}) = 1$, and hence that $\tilde{f}_n^\prime$ is the pointwise limit of derivatives of functions in $F_0$ (cf. (3.3)). Finally we have

$$
\sup_{f \in F_0} \left\{ \int_{0}^{\pi} f(t) k_{n,p}(t) dt \right\} \leq \sup_{f_1 \in F_1} \left\{ \int_{0}^{\pi} f_1(t) k_{n,p}(t) dt + f_1(\frac{\pi}{2}) \right\} \leq \int_{0}^{\pi} k_{n,p}(t) dt = \sum_{m=0}^{\infty} \left( \int_{2m\pi}^{(2m+1)\pi} k_{n,p}(t) dt \right).
$$
This proves the theorem. \(\blacksquare\)

**Corollary.** The (extremal) functions \(\tilde{f}_n\) are given by

\[
(3.12) \quad \tilde{f}_{2m}(t) = \begin{cases} 
\frac{1}{2}\pi & \text{if } |t| \leq \frac{\pi}{2m}, \\
\frac{1}{2}\pi + \sum_{j=1}^{m-1} (|t| - \frac{\pi}{2m}) + \frac{1}{2}(|t| - \frac{\pi}{2}) - \sum_{j=m}^{2m-1} (|t| - \frac{(2j+1)\pi}{4m}) & \text{if } |t| > \frac{\pi}{2m},
\end{cases}
\]

\( (m = 1, 2, \ldots) \)

\[
(3.13) \quad \tilde{f}_{2m+1}(t) = \begin{cases} 
\frac{1}{2}|t| + \sum_{j=1}^{m} (|t| - \frac{j\pi}{2m+1}) + \frac{2m}{2m+1} & \text{if } |t| \leq \frac{\pi}{2m+1}, \\
\frac{1}{2}|t| - \sum_{j=m+1}^{2m} (|t| - \frac{j\pi}{2m+1}) & \text{if } |t| > \frac{\pi}{2m+1},
\end{cases}
\]

\( (m = 0, 1, 2, \ldots) \),

where \(a_+ := \max(0, a)\).

A sketch of \(\tilde{f}_n\) together with its derivative is given in figure 3.3 below for \(n = 4\) and \(n = 5\) respectively. (Because of symmetry the graph of \(\tilde{f}_n\) is only shown on \([0, \pi]\).)
4. The exact constants of approximation $c_{n,p}$ for the operators $L_{n,p}$

4.1. Case $p \geq 2$. Assuming $n \in \mathbb{N}$ and $p \geq 2$ fixed, the exact constant of approximation $c_{n,p}$ was defined in the introductory section by (1.4). Both in (1.4) and in the definition of $d_{n,p}$ in (3.1) it is not an essential restriction to take $\omega_1$ in fact equal to one. It now follows from theorem 3.1 that

$$c_{n,p} = \frac{n \sqrt{p}}{n \sqrt{p}} d_{n,p} = \frac{n \sqrt{p}}{n \sqrt{p}} \int_{-\pi}^{\pi} \tilde{f}_{n}(t) k_{n,p}(t) dt \quad (n, p = 2, 3, \ldots)$$

and, according to definition (1.6)

$$c_{1,p} := c_{1,2} \quad (p = 3, 4, \ldots).$$

By means of (3.13) and (1.2) it immediately follows that

$$c_{1,p} = c_{1,2} = \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \frac{1}{4} |t| dt = \frac{1}{4} \pi \sqrt{2} = 1.110721.$$
The main object of this section is to determine the exact constant of approximation for the whole class of operators $L_{n,p}$ ($n \in \mathbb{N}, p \geq 2$), i.e. to determine the value of

$$c := \sup_{p \geq 2} \sup_{n \in \mathbb{N}} c_{n,p}.$$ 

Solving this problem turns out to be a quite cumbersome task; a good many particular cases have to be considered. The analysis is split up in several subsections.

4.1.0. Defining

(4.3) $\tilde{f}_n(t) = \frac{1}{2} |t| + h_n(t),$

we conclude from (4.1) that

(4.4) $c_{n,p} = S_1(n,p) + S_2(n,p),$

where

$$S_1(n,p) = n \sqrt{p} \int_{-\pi}^{\pi} \frac{1}{2} |t| k_{n,p}(t) dt,$$

$$S_2(n,p) = n \sqrt{p} \int_{-\pi}^{\pi} h_n(t) k_{n,p}(t) dt.$$

As will become apparent $S_1(n,p)$ is by far the main contributor to the constant $c_{n,p},$ whereas $S_2(n,p)$ rapidly becomes very small when $p$ increases (cf. table 4.1 of section 4.1.11). Because of this phenomenon partition (4.4) is made. We proceed by investigating $S_1(n,p)$ and $S_2(n,p)$ separately.

Using (1.2) we easily find that

(4.5) $S_1(n,p) = \frac{\frac{n \pi}{2}}{p} \int_{0}^{\frac{\pi}{2}} \frac{\left( \sin \frac{t}{n} \right)^2}{\sqrt{n}} dt = \frac{n \pi \sqrt{p}}{2} \int_{0}^{\frac{\pi}{2}} \frac{\left( \sin \frac{t}{\sqrt{p}} \right)^2}{n \sin \frac{t}{n}} dt = \frac{T_1(n,p)}{N(n,p)}.$
Taking into account lemma 2.4 we note that the denominator in the right-hand side of (4.5) is equal to

\[ N(n, p) = \frac{\pi \mu_0(n, p) \sqrt{p}}{2n^{2p-1}}, \]

where \( \mu_0(n, p) \) is given by (2.4). Formula (4.6) will be used in the sequel for small values of \( p \). We also need a lower bound for \( N(n, p) \).

4.1.1. Lemma 4.1.

\[ N(n, p) \geq \frac{1}{\sqrt{\pi a}} \operatorname{erf}(\frac{\pi}{2} \sqrt{pa}) \quad (n \in \mathbb{N}, \ p \in \mathbb{N}), \]

where (cf. (2.5))

\[ a = \frac{8}{\pi^2} \log(\frac{\pi}{2}) = 0.366039. \]

Proof. In view of (4.5) we have

\[ N(n, p) = \int_0^{\pi \sqrt{p}} \left( \frac{\sin t}{n \sin \frac{t}{\sqrt{p}}} \right)^{2p} \, dt \geq \int_0^{\pi \sqrt{p}} \left( \frac{\sin t}{\sqrt{p}} \right)^{2p} \, dt. \]

An application of lemma 2.5 gives

\[ \int_0^{\pi \sqrt{p}} \left( \frac{\sin t}{\sqrt{p}} \right)^{2p} \, dt \geq \int_0^{\pi \sqrt{p}} \exp(-at^2) \, dt = \frac{\pi \sqrt{p}}{2} \]

\[ = \frac{1}{\sqrt{a}} \frac{\sqrt{\pi}}{2} \int_0^{\pi \sqrt{p}} \exp(-v^2) \, dv = \frac{1}{\sqrt{a}} \operatorname{erf}(\frac{\pi}{2} \sqrt{pa}). \]

This lower bound for \( N(n, p) \) is increasing with \( p \). Numerical values can be obtained from Abramowitz-Stegun ([1], p. 311).
4.1.2. The next result furnishes an upper bound for the expression $T_1(n,p)$ occurring in (4.5).

Lemma 4.2.

(4.8) \[ T_1(n,p) \leq \frac{3}{2(1 - \frac{1}{n^2})} \{1 - \exp\left(-\frac{1}{3}(1 - \frac{1}{n^2})^2 p\right)\} + \frac{p}{p - 1} \pi^2 \frac{1}{2}(2p + 1) \quad (n,p = 2, 3, \ldots). \]

Proof. Assuming $n \geq 2$, $p \geq 2$ and taking into account (4.5) one has

\[ T_1(n,p) = \int_0^{\pi\sqrt{p}} t \left(\frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{\sqrt{p}}}\right)^{2p} \frac{n\pi\sqrt{p}}{2} dt + \int_{\pi\sqrt{p}}^{\infty} t \left(\frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{\sqrt{p}}}\right)^{2p} \frac{n\pi\sqrt{p}}{2} dt = \int_0^{\pi\sqrt{p}} + R. \]

By means of the inequality

(4.9) \[ \sin x \geq \frac{2}{\pi} x \quad (0 \leq x \leq \frac{\pi}{2}), \]

a crude estimate for $R$ can be deduced as follows.

(4.10) \[ R = \int_0^{\pi\sqrt{p}} \frac{n\pi\sqrt{p}}{2} - \frac{n\pi}{2} \frac{t}{\pi} \left(\frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{\sqrt{p}}}\right)^{2p} dt \leq p \left(\frac{\pi}{2}\right)^{2p} \int_0^{\pi\sqrt{p}} \frac{n\pi}{2} \frac{t}{\pi} - 2p + 1 dt < p \left(\frac{\pi}{2}\right)^{2p} \int_0^{\infty} \frac{t}{\pi} - 2p + 1 dt = \frac{p}{p - 1} \pi^2 \frac{1}{2}(2p + 1). \]

Moreover, using lemma 2.7 one has

\[ \frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{\sqrt{p}}} \leq \exp\left(-\frac{1}{6}(1 - \frac{1}{n^2})^2 p\right) \quad (0 < t \leq \pi\sqrt{p}) \]

and hence
This, together with (4.10), establishes lemma 4.2.

4.1.3. We proceed with an estimate for $S_2(n,p)$, the second contributor to the constant $c_{n,p}$ (cf. (4.4)). Taking into account that $h_n(t) \equiv 0$ if $|t| \leq \frac{p}{n}$, and using (1.2) we have

\[
S_2(n,p) = \frac{\pi}{\sqrt{p}} \int_0^\pi \left( \frac{\sin \frac{t}{n}}{\sqrt{p}} \right)^{2p} dt = \frac{\pi}{\sqrt{p}} \int_0^\pi t \exp\left(-\frac{1}{3} \left(1 - \frac{1}{n^2}\right)t^2\right) dt = \frac{3}{2(1 - \frac{1}{n^2})} \{1 - \exp\left(-\frac{1}{3} \left(1 - \frac{1}{n^2}\right)^2\right)\}.
\]

Lemma 4.3.

\[
T_2(n,p) = \frac{p}{\pi} \left( 1 - \frac{2}{n^2} \right) \int_0^\pi \left( \frac{\sin \frac{t}{n}}{\sqrt{p}} \right)^{2p} dt = \frac{p}{\pi} \left( 1 - \frac{2}{n^2} \right) \int_0^\pi \left( \frac{\sin \frac{t}{n}}{n \sin \frac{t}{n}} \right)^{2p} dt
\]

Proof. Let $n \geq 2$ be fixed. The function $h_n$ is approximated by a polynomial $q$ of low degree such that $h_n(t) \leq q(t)$ for $t \geq 0$. In order to be able to perform the resulting integration easily when $h_n$ is replaced by $q$ in the expression for $T_2(n,p)$, we choose a polynomial of odd degree. One simply verifies that $q(t) = \frac{4n^2}{27\pi^2} t^3$ will do, i.e. that

\[
h_n(t) \leq \frac{4n^2}{27\pi^2} t^3 \quad (t \geq 0).
\]
Taking into account (4.11) and using (4.13) one has

\( T_2(n,p) \leq \frac{32}{27\pi^2} \left\{ \int_{\frac{\pi}{2}}^{\pi} \left( \frac{\sin \frac{t}{n \sin \frac{t}{n}}}{\sin \frac{t}{n}} \right)^{2p} dt + \int_{\pi}^{\frac{n\pi}{2}} \left( \frac{\sin \frac{t}{n \sin \frac{t}{n}}}{\sin \frac{t}{n}} \right)^{2p} dt \right\} =

= \frac{32}{27\pi^2} \left\{ \frac{1}{p} \int_{\frac{\pi}{2}}^{\pi \sqrt{p}} t^3 \left( \frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{n \sqrt{p}}} \right)^{2p} dt + \int_{\pi}^{\frac{n\pi}{2}} \left( \frac{\sin \frac{t}{n \sin \frac{t}{n}}}{\sin \frac{t}{n}} \right)^{2p} dt \right\}.

An estimate similar to (4.10) takes care of the second integral in the right-hand side of (4.14); this gives rise to the contribution \( \frac{\pi^2 p}{27(p-2)} 2^{-2p+4} \) in (4.12). The other integral in (4.14) may be handled by lemma 2.7 as follows.

\[
\int_{\frac{\pi}{2}}^{\pi \sqrt{p}} t^3 \left( \frac{\sin \frac{t}{\sqrt{p}}}{n \sin \frac{t}{n \sqrt{p}}} \right)^{2p} dt \leq \int_{\frac{\pi}{2 \sqrt{p}}}^{\pi} t \exp \left( -\frac{1}{3} \left( 1 - \frac{1}{2} \right) t^2 \right) dt =

= \frac{1}{4} \int_{\frac{\pi}{2 \sqrt{p}}}^{\pi} t \exp \left( -\frac{1}{3} \left( 1 - \frac{1}{2} \right) t^2 \right) dt < \frac{1}{4} \int_{\frac{\pi}{2 \sqrt{p}}}^{\infty} t \exp \left( -\frac{1}{3} \left( 1 - \frac{1}{2} \right) t \right) dt.
\]

Performing the resulting integration one arrives at (4.12).

Lemmas 4.1, 4.2 and 4.3 will be used in the sequel to obtain upper bounds for the constants \( c_{n,p} \) if \( n \) and \( p \) are not too small, say \( \geq 5 \). The estimates are rather poor for small values of \( n \) and \( p \) (for instance when \( p = 2 \) or \( p = 3 \)). In that case a different approach will be needed.

4.1.4. Case \( n = 2, p \geq 2 \). We next consider the behaviour of the sequence \( \{c_{2,p}\} \). According to formulae (4.1), (3.12), (4.3) and (2.10) we have
\[
(4.15) \quad c_{2,p} = 2\sqrt{p} \int_{-\pi}^{\pi} \frac{1}{2} f_{2,p}(t) k_{2,p}(t) dt =
\]
\[
= \frac{2^{2p+2}(p!)^2}{\pi(2p)!} \left\{ \int_{0}^{\frac{\pi}{2}} t(\cos t)^{2p} dt + \int_{0}^{\frac{\pi}{2}} h_2(2t)(\cos t)^{2p} dt \right\},
\]

where

\[
(4.16) \quad h_2(2t) = \begin{cases} 
0 & (0 \leq t \leq \frac{\pi}{4}) \\
\frac{\pi}{4} - t & \left(\frac{\pi}{4} \leq t \leq \frac{3\pi}{8} \right) \\
-t + \frac{\pi}{2} & \left(\frac{3\pi}{8} \leq t \leq \frac{\pi}{2} \right).
\end{cases}
\]

**Lemma 4.4.** The sequence \( \{c_{2,p}\}_{p=1}^{\infty} \) is increasing and

\[
(4.17) \quad \lim_{p \to \infty} c_{2,p} = \frac{2}{\sqrt{\pi}} = 1.12837917 .
\]

**Proof.** In view of (4.15) the first assertion of the lemma amounts to proving that

\[
(4.18) \quad (2p+1)\sqrt{p} \left\{ \int_{0}^{\frac{\pi}{2}} t(\cos t)^{2p} dt + \int_{0}^{\frac{\pi}{2}} h_2(2t)(\cos t)^{2p} dt \right\} <
\]
\[
< 2(p+1)^2 \left\{ \int_{0}^{\frac{\pi}{2}} t(\cos t)^{2p+2} dt + \int_{0}^{\frac{\pi}{2}} h_2(2t)(\cos t)^{2p+2} dt \right\}.
\]

Using lemma 2.8 one obtains

\[
(4.19) \quad \int_{0}^{\frac{\pi}{2}} t(\cos t)^{2p} dt \leq \int_{0}^{\frac{\pi}{2}} \left\{ \sin t + \frac{1}{6}\sin^3 t + \frac{3}{40}\sin^5 t + \frac{1}{120}(\pi - \frac{149}{6})\sin^7 t \right\} (\cos t)^{2p} dt =
\]
\[
= \frac{1}{2p+1} + \frac{1}{6}\left( \frac{1}{2p+1} - \frac{1}{2p+3} \right) + \frac{3}{40}\left( \frac{1}{2p+1} - \frac{2}{2p+3} + \frac{1}{2p+5} \right) + \frac{\pi}{2} \left( \frac{149}{120} \right)\left( \frac{1}{2p+1} - \frac{3}{2p+3} + \frac{3}{2p+5} - \frac{1}{2p+7} \right).
\]
Again using lemma 2.8 the integral $\int_0^{\pi/2} t (\cos t)^{2p+2} \, dt$ can be treated in the same fashion to obtain a lower bound. Furthermore, applying lemma 2.7 if $n = 2$ we get in view of (4.16)

\begin{equation}
(4.20) \quad \int_0^{\pi/2} h_2(2t)(\cos t)^{2p} \, dt = \int_0^{\pi/4} h_2(2t)(\cos t)^{2p} \, dt \leq
\end{equation}

\begin{align*}
&\leq \int_{\pi/4}^{\pi/2} (t - \pi/4) \exp(-pt^2) \, dt \leq \int_{\pi/4}^{\infty} (t - \pi/4) \exp(-pt^2) \, dt = \\
&= \frac{1}{2p} \exp(-\frac{p\pi^2}{16}) - \frac{\pi}{4\sqrt{p}} \int_{\infty}^{\frac{\pi\sqrt{p}}{4}} \exp(-u^2) \, du.
\end{align*}

According to [1], p. 298 one has

\begin{equation}
\frac{1}{x + \sqrt{x^2 + 2}} < \exp(x^2) \int_{x}^{\infty} \exp(-u^2) \, du \leq \frac{1}{x + \sqrt{x^2 + 4}} \quad (x \geq 0).
\end{equation}

Using the left-hand side of this inequality in (4.20) one easily verifies that

\begin{equation}
(4.21) \quad \int_0^{\pi/2} h_2(2t)(\cos t)^{2p} \, dt < \frac{4}{\pi p} \exp(-\frac{p\pi^2}{16}).
\end{equation}

Estimate (4.19) and a lower bound for $\int_0^{\pi/2} t (\cos t)^{2p+2} \, dt$, together with (4.21), will be used in (4.18). Accordingly, it is sufficient to prove that
An elementary, but tedious, computation shows that this is indeed the case for $p \geq 8$; we omit the details. The constants $c_{2,p}$ ($p = 2, 3, \ldots, 8$) can be evaluated explicitly (cf. table 4.1 of section 4.1.11). Taking these data into account it follows that the monotonicity holds for the whole sequence $\{c_{2,p}\}$. This proves the first part of the lemma.

As for assertion (4.17), it is clear from (4.15) and (4.21) that the limiting behaviour of $\{c_{2,p}\}$ is governed by

\begin{equation}
\frac{2^{2p+2}(p!)^2}{\pi(2p+1)} \int_0^\frac{\pi}{2} (\cos t)^{2p} dt \quad (p \to \infty),
\end{equation}

We have

\[
p \int_0^{\frac{\pi}{2}} (\cos t)^{2p} dt = \frac{\pi\sqrt{p}}{2} u(\cos u)^{2p} \frac{du}{\sqrt{p}} \sim \int_0^\infty u \exp(-u^2) du = \frac{1}{2} \quad (p \to \infty).
\]

This, together with an application of Stirling's formula in (4.22) proves (4.17).

The remaining part of this section will be devoted to showing that

\[
\sup_{p \geq 2} \sup_{n \in \mathbb{N}} c_{n,p} = \frac{2}{\sqrt{\pi}}.
\]

4.1.5. Case $n \geq 4$, $p \geq 5$. Here we shall be concerned with estimating the exact constants of approximation $c_{n,p}$ if $n \geq 4$ and $p \geq 5$. In view of (4.7) and using [1], p. 311 we find that

\begin{equation}
N(n,p) \geq \frac{\sqrt{\pi}}{a} \text{erf}\left(\frac{\pi}{2} \sqrt{5a}\right) > 1.4579 \quad (n \in \mathbb{N}, \ p \geq 5).
\end{equation}
An application of lemma 4.2 with \( n = 4, \ p = 5 \) yields

\[(4.24) \quad T_1(4,5) \leq \frac{8}{5}(1 - \exp(-\frac{25\pi^2}{16})) + 5\pi^2 - 13 < \frac{8}{5} + 5\pi^2 - 13 < 1.6061 .\]

The last estimate, taking into account formula (4.8), holds for all \( n \geq 4 \) and \( p \geq 5 \).

Finally, if we use lemma 4.3 in case \( n = 4, \ p = 5 \), it follows that for all \( n \geq 4 \) and \( p \geq 5 \) one has

\[(4.25) \quad T_2(n,p) \leq \left( \frac{64}{135} + \frac{4096}{3375\pi} \right) \exp(-\frac{25\pi^2}{64}) + \frac{5\pi^2}{3184} < 0.0222 .\]

Using formulae (4.4), (4.5), (4.11) we obtain from (4.23), (4.24), (4.25) that

\[c_{n,p} < 1.1169 < \frac{2}{\sqrt{\pi}} \quad (n \geq 4, \ p \geq 5) .\]

4.1.6. Case \( n = 3, \ p \geq 5 \). When we try to show that \( c_{3,p} < \frac{2}{\sqrt{\pi}} \) with the help of formulae (4.7), (4.8) and (4.12), it turns out that these are not quite adequate in this case. A small modification is needed: we slightly sharpen the estimate (4.7) for \( N(n,p) \) in case \( n = 3 \). This can be accomplished as follows. Using lemma 2.6 and (4.5) we have

\[N(3,p) = \int_0^{3\sqrt{p}/2} \left( \frac{3}{2} \sin \frac{\frac{t}{\sqrt{p}}}{3} \right)^{2p} dt > \int_0^{\pi \sqrt{p}/2} \left( \frac{\frac{t}{\sqrt{p}}}{3} \right)^{2p} dt \geq \]

\[\geq \int_0^{\pi \sqrt{p}/2} \exp(-bt^2) dt = \frac{1}{2} \sqrt{\frac{\pi}{b}} \ \text{erf}(\frac{\pi \sqrt{p}}{2}) ,\]

where \( b = \frac{8}{\pi^2} \log(\frac{3}{2}) = 0.328658 \).

Consequently, for all \( p \geq 5 \) one has (cf. [1], p. 311)

\[(4.26) \quad N(3,p) > 1.5386 .\]
Lemmas 4.2 and 4.3 yield
\[ T_1(3,p) \leq \frac{27}{16} + 5\pi^2 2^{-13} < 1.6936 \quad (p \geq 5), \]
\[ T_2(3,p) \leq \left( \frac{1 + 27}{20\pi^2} \right) \exp \left( -\frac{10\pi^2}{27} \right) + \frac{5\pi^2}{5184} < 0.0260 \quad (p \geq 5). \]

In view of (4.26) one obtains
\[ c_{3,p} < 1.1177 < \frac{2}{\sqrt{\pi}} \quad (p \geq 5). \]

4.1.7. Case \( n \geq 3, p = 4 \). Next we shall deal with the constants \( c_{n,4} \). Also in this case it turns out that a mere application of lemmas 4.1, 4.2 and 4.3 is not accurate enough to ensure that \( c_{n,4} < \frac{2}{\sqrt{\pi}} \). However, inequality (4.7) of lemma 4.1 can easily be replaced by a sharper bound as follows.

Taking into account (4.6) and (2.4) we have
\[ (4.27) \quad N(n,4) = \frac{\pi(151n^7 + 70n^5 + 49n^3 + 45n)}{315n^7} > 1.5059 \quad (n \in \mathbb{N}). \]

Lemmas 4.2 and 4.3 supply the following bounds for \( T_1(n,4) \) and \( T_2(n,4) \).
\[ T_1(n,4) \leq \frac{25}{16}(1 - \exp(-\frac{32\pi^2}{25})) + \frac{\pi^2 - 7}{3} < 1.5882 \quad (n \geq 5), \]
\[ T_2(n,4) \leq \frac{25}{34} + \frac{625}{432\pi^2} \exp(-\frac{8\pi^2}{25}) + \frac{\pi^2}{216} < 0.0716 \quad (n \geq 5). \]

As a consequence of the above results one finds
\[ c_{n,4} < 1.1022 < \frac{2}{\sqrt{\pi}} \quad (n \geq 5). \]

The constants \( c_{3,4} \) and \( c_{4,4} \) will be taken care of by computing these numbers explicitly (cf. table 4.1 of section 4.1.11). Of course, \( N(3,4) \) and \( N(4,4) \) can be evaluated by using (4.27), and in these cases the bound given in (4.27) can be improved. This, together with lemma 4.2 and a small modification of lemma 4.3, also shows that both constants \( c_{3,4} \) and \( c_{4,4} \) are smaller than \( \frac{2}{\sqrt{\pi}} \).
4.1.8. Up to now we have been working with the estimates of lemmas 4.1, 4.2 and 4.3. In case both \( n \) and \( p \) are not too small, they apparently give satisfactory results. It also turned out that when \( n \) or/and \( p \) is small (for instance, \( n = 3 \) or \( p = 4 \)), then some modifications were needed. However, in case \( p = 2 \) or \( p = 3 \) the aforementioned lemmas do not work; the bounds they supply are not accurate enough to show that 
\[
\frac{c_{n,2}}{\sqrt{\pi}} < \frac{2}{\sqrt{\pi}}, \quad \frac{c_{n,3}}{\sqrt{\pi}} < \frac{2}{\sqrt{\pi}}
\]
\((n \in \mathbb{N})\). Consequently, a different approach will be needed with which we shall now be concerned. We first note that it is easy to obtain good bounds on \( N(n,2) \) and \( N(n,3) \). In view of (4.6) and (2.4) we have for \( n \in \mathbb{N} \)

\[
N(n,2) = \frac{\pi \sqrt{2(2n^3+n)}}{6n^3} > \frac{1}{3} \pi \sqrt{2} > 1.4809 ,
\]

\[
N(n,3) = \frac{\pi \sqrt{3(11n^5+5n^3+4n)}}{40n^5} > \frac{1}{40} \pi \sqrt{3} > 1.4963 .
\]

We now consider the integral 
\[
\int_0^{\frac{\pi}{2}} t \left( \frac{\sin t}{(n+1)\sin \frac{t}{n+1}} \right)^{2p} dt ,
\]
which, apart from a factor \( p \), is identical with \( T_1(n,p) \) (cf. (4.5)). Using (2.6) we deduce that

\[
I_{n+1,p} := \int_0^{\frac{(n+1)\pi}{2}} t \left( \frac{\sin t}{(n+1)\sin \frac{t}{n+1}} \right)^{2p} dt \leq \int_0^{\frac{\pi}{2}} t \left( \frac{\sin t}{n\sin \frac{t}{n}} \right)^{2p} dt + \int_{\frac{\pi}{2}}^{\frac{(n+1)\pi}{2}} \frac{t}{(n+1)\sin \frac{t}{n+1}} 2p dt + \int_{\frac{\pi}{2}}^{\frac{(n+1)\pi}{2}} t \left( \frac{\sin t}{(n+1)\sin \frac{t}{n+1}} \right)^{2p} dt + \int_{\frac{(n+1)\pi}{2}}^{\frac{(n+1)\pi}{2}} \frac{t}{(n+1)\sin \left( \frac{n\pi}{2(n+1)} - \frac{t}{2} \right)} 2p dt =
\]

\[
= I_ {n, p} + \frac{\pi^2 (2p)!}{2^{2p+2} (p!)^2} (n+1)^{-2p+1} (\sin \left( \frac{n\pi}{2(n+1)} \right))^{-2p} ,
\]

where we have used (2.10).
Repeated application of (4.30) for a fixed $n_0 \in \mathbb{N}$ gives

\[(4.31) \quad I_{n_0+s,p} \leq I_{n_0,p} + \frac{\pi^2 (2p)!}{2^{2p} (p!)^2} \sum_{j=n_0+1}^{n_0+s} \left( \sin\left(\frac{(j-1)\pi}{2}\right) \right)^{2p-2} \left( \sin\left(\frac{\pi}{2(n_0+1)}\right) \right)^{-2p} \sum_{j=n_0+1}^{\infty} \frac{1}{j^{2p+1}}.\]

A similar procedure will be used to obtain an estimate for the integral

\[(4.32) \quad \int_{\pi/2}^{n\pi/2} nh_n \left( \frac{2t}{n} \right) \left( \frac{\sin t}{\sin \frac{\pi}{n}} \right)^{2p} \left( \sin \frac{\pi}{n} \right)^{2p} dt,
\]

which, apart from a factor $p$, is identical with $T_2(n,p)$ (cf. (4.11)).

In order to do this we need the following lemma.

**Lemma 4.5.** Let the function $h_n$ be defined by (4.3). Then for $n \in \mathbb{N}$ one has

\[(4.33) \quad nh_n \left( \frac{2t}{n} \right) \leq 1.04 \frac{n^2 \pi}{4} \sin^2 \frac{t}{n} \quad (|t| \leq \frac{n\pi}{2}).\]

**Proof.** As $h_n \left( \frac{2t}{n} \right)$ is an even function that is identically zero on $[0, \pi/2]$, it is sufficient to prove the inequality for the range $\pi/2 \leq t \leq \frac{n\pi}{2}$. Owing to the definition of $\tilde{f}_n$ as given by the formulae (3.12) and (3.13) the cases $n = 2m$ and $n = 2m+1$ must be considered separately. We first assume that $n = 2m$. It is then easy to verify (cf. figure 4.1) that on the interval $[\frac{\pi}{2}, \frac{m\pi}{2}]$ one has

\[2m h_{2m} \left( \frac{\xi}{m} \right) < \frac{2}{\pi} \frac{\xi^2}{\pi} \leq m^2 \pi \sin^2 \frac{\xi}{2m}.\]

This establishes inequality (4.33) for the range $\pi/2 \leq \xi \leq \frac{m\pi}{2}$. If $\frac{m\pi}{2} \leq \xi \leq m\pi$ the function $2m h_{2m} \left( \frac{\xi}{m} \right)$ is again approximated by a quadratic function. Actually, one has

\[2m h_{2m} \left( \frac{\xi}{m} \right) < m^2 \pi - \frac{2}{\pi} (m\pi - \xi)^2 \quad \left( \frac{m\pi}{2} \leq \xi \leq m\pi \right).\]
Now we determine $\lambda \geq 1$ such that
\[ m^2 \pi - \frac{2}{\pi} (m \pi - t)^2 \leq \lambda m^2 \pi \sin^2 \frac{t}{2m} \quad \text{for} \quad \left(\frac{m \pi}{2} \leq t \leq m \pi\right). \]

Dividing through by $\frac{m^2}{\pi}$, we can write this relation as
\[ \pi^2 - 8\left(\frac{\pi}{2} - \frac{t}{2m}\right)^2 \leq \lambda \pi^2 \sin^2 \frac{t}{2m}. \]

If we put $\alpha := \frac{\pi}{2} - \frac{t}{2m}$, then $0 \leq \alpha \leq \frac{\pi}{4}$ and $\lambda$ must be chosen such that the inequality
\[ \pi^2 - 8\alpha^2 \leq \lambda \pi^2 \cos^2 \alpha \quad (0 \leq \alpha \leq \frac{\pi}{4}) \]
holds. In order to obtain a value of $\lambda$ as small as possible we take $\lambda = \lambda^*$ with
\[ \lambda^* = \max_{0 \leq \alpha \leq \frac{\pi}{4}} \frac{\pi^2 - 8\alpha^2}{\pi^2 \cos^2 \alpha}. \]

The expression $\frac{\pi^2 - 8\alpha^2}{\pi^2 \cos^2 \alpha}$ has a unique maximum on $[0, \frac{\pi}{4}]$, which is attained at $\alpha = 0.5957$. Consequently $\lambda^* = 1.0397 < 1.04$.

This proves the asserted inequality (4.33) if $n = 2m$. When $n = 2m + 1$, one can proceed in exactly the same way, i.e., the same functions $\frac{2}{\pi} t^2$ and $m^2 \pi - \frac{2}{\pi} (m \pi - t)^2$ can be used to approximate $\text{nh}_{\frac{2n}{n}} (\frac{2t}{n})$ on the intervals $[\frac{\pi}{2}, (2m+1)\pi]$ and $[(2m+1)\pi, m\pi]$ respectively. Finally, the interval $[m\pi, (m+\frac{1}{2})\pi]$ is taken care of by noting that if $n$ is odd $\text{nh}_{\frac{2n}{n}} (\frac{2t}{n})$ is constant there (cf. figure 4.1), whereas $1.04 \frac{n \pi}{4} \sin^2 \frac{t}{n}$ is still increasing on this interval. This completely proves the lemma.

We note that the factor 1.04 in (4.33) is certainly not best possible; however, it cannot be replaced by 1.

In order to estimate the integral (4.32) it will be convenient to have the graphs of the function $\text{nh}_{\frac{2n}{n}} (\frac{2t}{n})$ available for the first few values of $n$. For this we refer to figure 4.1 below, in the construction of which formulae (3.12), (3.13) have been used.
Using lemma 4.5 and taking into account (2.6), (2.10) and (4.9) we have for \( n \geq 5 \)

\[
(4.34) \quad J_{n,p} := \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} nh_n \left( \frac{2t}{n} \right) \left( \frac{\sin t}{n \sin \frac{t}{n}} \right)^{2p} \ dt = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3\pi}{2} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3\pi}{2} 5h_n \left( \frac{2t}{5} \right) \left( \frac{\sin t}{5 \sin \frac{t}{5}} \right)^{2p} \ dt + 0.26\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin t}{(n \sin \frac{t}{n})^{2p-2}} <
\]
Inequalities (4.29), (4.31) and (4.34) will be used to dispose of the case \( p = 3 \) and \( n \) sufficiently large. This is the subject of the next section.

4.1.9. Case \( n \geq 3, p = 3 \). In section 4.1.11 we shall show, by way of working an example in detail, how the constants \( c_{n,p} \) may be computed explicitly. This can be accomplished by hand when both the values of \( n \) and \( p \) are small. Proceeding in that fashion it can be shown that we have

\[
c_{5,3} = S_1(5,3) + S_2(5,3),
\]

where

\[
S_1(5,3) = \frac{5\sqrt{3}}{4} - \frac{10\sqrt{3}}{1751\pi} (1686 + \frac{1246}{9} + \frac{666}{25} + \frac{246}{49} + \frac{56}{81} + \frac{6}{121}) = 0.955187,
\]

\[
S_2(5,3) = 2\pi\sqrt{3} - \frac{20\sqrt{3}}{1751\pi} (2\cos\frac{\pi}{5} + \cos\frac{\pi}{3} - 1)(1686 - \frac{1246}{9} - \frac{246}{49} + \frac{56}{81} + \frac{6}{121}) = 0.017208.
\]

Accordingly \( c_{5,3} = 0.972395 \).

Using these data and taking into account (4.29) and the definitions of the integrals \( I_{n,p} \) and \( J_{n,p} \) respectively, we obtain

\[
I_{5,3} = 0.485382, \quad J_{5,3} = 0.008744.
\]
Applying (4.31) and (4.34) in the case \( p = 3 \) yields for \( n > 5 \)

\[
I_{n,3} \leq I_{5,3} + \frac{5\pi^2}{64} \left( \sin \frac{5\pi}{12} \right)^{-6} \sum_{j=6}^{\infty} j^{-5} < 0.485635 ,
\]

\[
J_{n,3} \leq J_{5,3} + \frac{1.3\pi^2}{32} \sum_{j=3}^{\infty} j^{-4} < 0.016693 .
\]

From these results and (4.29) it follows that

\[
S_1(n,3) < 0.9737 , \quad S_2(n,3) < 0.0335 \quad (n > 5)
\]

and hence

\[
c_{n,3} < 1.0072 < \frac{2}{\sqrt{\pi}} \quad (n > 5).
\]

The constants \( c_{3,3}, c_{4,3} \) will be computed explicitly; their values are contained in table 4.1 of section 4.1.11.

4.1.10. Case \( n \geq 3, p = 2 \). The case \( p = 2 \) is treated in a similar way as case \( p = 3 \). However, because inequalities (4.31) and (4.34) give a better performance for \( p = 3 \) than for \( p = 2 \), the value of \( n_0 \) in (4.31) must be taken greater than 5, and inequality (4.34) must be slightly modified.

In quite an analogous way as (4.34) was derived, we deduce that for a fixed \( n_0 \in \mathbb{N} \) there holds

\[
J_{n,p} \leq J_{n_0,p} + \frac{0.26\pi^2 (2p)!}{2^{2p+1} (p!)^2} \sum_{j=\left\lceil \frac{n_0+1}{2} \right\rceil}^{\infty} j^{-2p+2} \quad (n > n_0).
\]

In order to obtain satisfactory results from (4.31) and this formula in case \( p = 2 \) (i.e., to show that from a certain index on we have \( c_{n,2} < \frac{2}{\sqrt{\pi}} \)), the number \( n_0 \) in (4.31) and (4.35) cannot be chosen too small. Numerical computations show that the choice \( n_0 = 19 \) will do. As a consequence the constants \( c_{n,2} (n = 2, 3, \ldots, 19) \) must be computed explicitly (cf. section 4.1.11). An application of (4.31) and (4.35) in case \( p = 2 \) and \( n_0 = 19 \) implies that \( I_{19,2} \) and \( J_{19,2} \) must be known. In view of their definitions this means that \( S_1(19,2) \) and \( S_2(19,2) \) (cf. (4.4)) must be available. This is accomplished by computing \( c_{19,2} \) along the lines of section 4.1.11. We obtain (cf. table 4.1)
In view of (4.28) one has
\[ I_{19,2} = 0.696484, \quad J_{19,2} = 0.085035. \]
Applying (4.31) and (4.35) if \( n_0 = 19, p = 2 \) gives for \( n > 19 \)
\[
I_{n,2} \leq I_{19,2} + \frac{3\pi^2}{32} \left( \sin \frac{19\pi}{40} \right)^4 \sum_{j=20}^{\infty} j^{-3} < 0.697716,
\]
\[
J_{n,2} \leq J_{19,2} + \frac{0.39\pi^2}{8} \sum_{j=10}^{\infty} j^{-2} < 0.135636.
\]
Using these results and taking into account (4.28) one has
\[ S_1(n,2) < 0.9423, \quad S_2(n,2) < 0.1832 \quad (n > 19) \]
and hence
\[ c_{n,2} < 1.1255 < \frac{2}{\sqrt{\pi}} \quad (n > 19). \]

4.1.11. What remains to be done is the explicit computation of the values of a few particular exact constants of approximation, because these could not be estimated adequately using the various results of the preceding sections. In particular, we have to compute \( c_{2,p} \) (\( p = 2,3,\ldots,8 \)) (cf. section 4.1.4), \( c_{3,4} \) and \( c_{4,4} \) (cf. section 4.1.7), \( c_{3,3} \) and \( c_{4,3} \) (cf. section 4.1.9) and \( c_{n,2} \) (\( n = 2,3,\ldots,19 \)) (cf. section 4.1.10). In what follows we shall first work a specific example in detail and, using the exhibited pattern of computation, sketch how the numbers \( c_{n,p} \) may be evaluated in general.

Let \( n = 3 \) and \( p = 2 \), i.e. we shall compute \( c_{3,2} \). In view of (4.4) the computation is split up in two parts. An application of lemma 2.4 gives
\[ (4.36) \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 = 19 + 32 \cos t + 20 \cos 2t + 8 \cos 3t + 2 \cos 4t. \]

In view of (1.2) one has \( A_{3,2} = 38\pi. \) According to (4.4)
\[ S_1(3,2) = \frac{3\sqrt{2}}{A_{3,2}} \int_{-\pi}^{\pi} \left| t \right| \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 dt = \frac{3\sqrt{2}}{38\pi} \int_{0}^{\pi} t \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 dt . \]

As
\[ \int_{0}^{\pi} t \cos kt dt = \frac{(-1)^{k-1}}{k^2} \quad (k \in \mathbb{N}) , \]
it easily follows from (4.36), (4.37) and (4.38) that we have
\[ S_1(3,2) = \frac{3\sqrt{2}}{4\pi} \left\{ \pi^2 - \frac{1184}{171} \right\} = 0.994949 . \]

By (3.13) and (4.3) the function \( h_3 \) is defined as follows.
\[
 h_3(t) = \begin{cases} 
 0 & (0 \leq t \leq \frac{\pi}{3}) \\
 t - \frac{\pi}{3} & (\frac{\pi}{3} \leq t \leq \frac{2\pi}{3}) \\
 \frac{\pi}{3} & (\frac{2\pi}{3} \leq t \leq \pi) .
\end{cases}
\]

Consequently,
\[
 S_2(3,2) = \frac{3\sqrt{2}}{A_{3,2}} \int_{-\pi}^{\pi} h_3(t) \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 dt = 
\]
\[
 = \frac{3\sqrt{2}}{19\pi} \left\{ \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left( t - \frac{\pi}{3} \right) \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 dt + \int_{\frac{2\pi}{3}}^{\pi} \frac{\pi}{3} \left( \frac{\sin \frac{3}{2}t}{\sin \frac{1}{2}t} \right)^4 dt \right\} .
\]

Performing the integrations in such a way that the upper limit of integration is taken to be \( \pi \), and observing that
\[
 \int t \cos kt dt = \frac{1}{k} t \sin kt + \frac{1}{k^2} \cos kt + C \quad (k \neq 0) ,
\]
one has in view of (4.36)
\[(4.40) \quad S_2(3,2) = \frac{3\sqrt{2}}{19\pi} \left\{ \int_{\frac{\pi}{3}}^{\pi} \left( t - \frac{\pi}{3} \right) \frac{\sin^4 \frac{3t}{2}}{\sin \frac{1}{2}t} \, dt + \int_{\frac{2\pi}{3}}^{\frac{2\pi}{3}} (-t + \frac{2\pi}{3}) \frac{\sin^4 \frac{3t}{2}}{\sin \frac{1}{2}t} \, dt \right\} = \]

\[= \frac{3\sqrt{2}}{19\pi} \left\{ \frac{19}{2} \left( t - \frac{\pi}{3} \right)^2 \int_{\frac{\pi}{3}}^{\pi} \frac{\cos^2 t}{\sin \frac{1}{2}t} \, dt + \frac{19}{2} (-t + \frac{2\pi}{3})^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{\cos^2 t}{\sin \frac{1}{2}t} \, dt \right\} + \]

\[+ \frac{3\sqrt{2}}{19\pi} \left\{ 32 \cos t + \frac{20}{4} \cos 2t + \frac{8}{9} \cos 3t + \frac{2}{16} \cos 4t \right\} \left( \int_{\frac{\pi}{3}}^{\pi} + \int_{\frac{2\pi}{3}}^{\pi} \right) = \]

\[= \frac{1}{2} \pi \sqrt{2} - \frac{272\sqrt{2}}{57\pi} = 0.073318. \]

By (4.39) and (4.40)

\[c_{3,2} = S_1(3,2) + S_2(3,2) = \frac{5}{4} \pi \sqrt{2} - \frac{568\sqrt{2}}{57\pi} = 1.067817. \]

From the details given in this example it will be clear how the numbers \(c_{n,p}\) can be computed in general. Basic to the computation is lemma 2.4, together with formulae (3.12) and (3.13). To simplify the integrations necessary to find \(S_2(n,p)\), we take the upper limit of integration always equal to \(\pi\).

Using (4.38) and (4.4) one has

\[(4.41) \quad S_1(n,p) = \frac{n\sqrt{p}}{A_{n,p}} \int_{-\pi}^{\pi} \frac{\sin \frac{1}{2}t}{\sin \frac{1}{2}t} \, dt = \]

\[= \frac{n\sqrt{p}}{2\pi \mu_0} \int_{0}^{\pi} t \left\{ \mu_0 + 2 \sum_{k=1}^{\frac{np-p}{2}} \mu_k \cos kt \right\} \, dt = \]

\[= \frac{n\sqrt{p}}{4\pi} \left\{ 2 - \frac{2}{\mu_0} \sum_{k=1}^{\frac{np-p+1}{2}} \frac{\mu_{2k-1}}{\mu_0(2k-1)^2} \right\}, \]

where \(\mu_k := \mu_k(n,p)\).
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</table>

Table 4.1
In the case $n = 2m + 1$ a computation analogous to that performed in (4.40) shows that we have

$$S_2(n, p) = \frac{1}{n} m^2 \sqrt{p} + \frac{2n\sqrt{p}}{\pi} \sum_{j=1}^{m} \left[ \sum_{k=1}^{n} \left( \frac{\cos(kt)}{k^2} \right) \right] \frac{(m+j)^\pi}{n}.$$

By use of (2.4) both expressions (4.41) and (4.42) can be evaluated. When $n = 2m$ a similar expression for $S_2(n, p)$ can be derived, which we refrain from giving here.

By means of these formulae the exact constants of approximation $c_{n, p}$ may be computed for any values of $n$ and $p$, although the amount of computational work involved grows quite rapidly. For small values of both $n$ and $p$ it can be done by hand, cf. the example of $c_{3, 2}$ in this section and the value of $c_{5, 3}$ as given in section 4.1.9.

We include here a table containing the numerical values of the constants $c_{n, p}$ that were mentioned in the beginning of this section. These data were computed on the Burroughs 7700 of the Computing Centre of the Eindhoven University of Technology.

4.1.12. Taking into account (4.17) and the estimates for the constants $c_{n, p}$ in sections 4.1.5, 4.1.6, 4.1.7, 4.1.9, 4.1.10, together with the contents of table 4.1 and formula (4.2), we have the following theorem and corollary.

**Theorem 4.1.** Let $c_{n, p}$ be the exact constant of approximation for the operator $L_{n, p}$ as defined in (1.4). Then

$$\lim_{p \to \infty} c_{2, p} = \frac{2}{\sqrt{\pi}} = 1.12837917.$$

**Corollary.** Let $f \in C^{1}_{2\pi}$ and let $\omega_1(f; \delta) := \omega(f'; \delta)$ be the modulus of continuity of $f'$, then for $n \in \mathbb{N}$ and $p = 2, 3, \ldots$ one has

$$\max_{x} |L_{n,p}(f; x) - f(x)| \leq \frac{2}{\sqrt{\pi}} \frac{1}{n^p} \omega_1(f; \frac{\pi}{n}) ,$$

where the value $\frac{2}{\sqrt{\pi}}$ is best possible in (4.43).
4.2. Case $p = 1$. Taking into account definition (1.5) and the results on the extremal functions of section 3, we have (cf. formulae (3.1), (3.2) and theorem 3.1)

$$c_{n,1} = \int_{-\pi}^{\pi} \frac{f_n(t)}{\sin \frac{\pi t}{n}} k_{n,1}(t) \, dt \quad (n \in \mathbb{N}),$$

where according to (1.2) and lemma 2.4

$$k_{n,1}(t) = \frac{1}{2\pi n} \left( \frac{\sin \frac{\pi t}{n}}{\sin \frac{\pi}{t}} \right)^2.$$

We write (cf. (4.3) and (4.4))

$$c_{n,1} = S_1(n,1) + S_2(n,1),$$

where

$$S_1(n,1) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} |t| \left( \frac{\sin \frac{\pi t}{n}}{\sin \frac{\pi}{t}} \right)^2 \, dt,$$

$$S_2(n,1) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} h_n(t) \left( \frac{\sin \frac{\pi t}{n}}{\sin \frac{\pi}{t}} \right)^2 \, dt.$$

By lemma 2.4 it is easily shown that $S_1(n,1)$ decreases monotonically to zero when $n \to \infty$. In contrast to the case $p \geq 2$ here the term $S_2(n,1)$ is the main contributor to $c_{n,1}$ when $n$ becomes large (cf. table 4.2). In order to determine $c(1) = \sup_{n \in \mathbb{N}} c_{n,1}$ we need an upper bound for $S_2(n,1)$. Taking into account that $h_n(t) \equiv 0$ if $|t| \leq \frac{\pi}{n}$ one has for $n \in \mathbb{N}$

$$S_2(n,1) \leq \frac{1}{\pi n} \int_{\frac{\pi}{n}}^{\pi} h_n(t) \left( \frac{\sin \frac{\pi t}{n}}{\sin \frac{\pi}{t}} \right)^2 \, dt \leq \frac{2}{\pi n} \int_{\frac{\pi}{n}}^{\frac{\pi n}{2}} h_n(t) \left( \frac{\sin \frac{\pi t}{n}}{\sin \frac{\pi}{t}} \right)^2 \, dt \leq 0.52 \frac{\pi}{n} \int_{\frac{\pi}{2}}^{\frac{\pi n}{2}} \sin^2 t \, dt < 0.13 \pi = 0.4084,$$

by an application of lemma 4.5.
Furthermore, one easily verifies that

\[(4.45) \quad c_{1,1} = S_1(1,1) = \frac{\pi}{4} = 0.78539816.\]

The constants \(c_{n,1}\) can be computed by use of formulae that, apart from a factor \(n^2\), are identical with (4.41) and (4.42). Again, if \(n\) is even a slight modification of (4.42) is needed. Table 4.2 contains the numerical values of the constants \(c_{n,1}\) \((n = 1(1)10; 15(5)50; 75; 100)\), together with the corresponding \(S_1(n,1)\) and \(S_2(n,1)\).

In particular we have

\[(4.46) \quad S_1(3,1) = \frac{\pi}{4} - \frac{4}{3\pi} = 0.3610.\]

As \(S_1(n,1)\) is decreasing formulae (4.44) and (4.46), together with (4.45), imply the following theorem.

**Theorem 4.2.** Let \(c_{n,1}\) be the exact constant of approximation for the operator \(L_{n,1}\) as defined in (1.5). Then

\[c^{(1)} := \sup_{n \in \mathbb{N}} c_{n,1} = c_{1,1} = \frac{\pi}{4} = 0.78539816.\]

**Corollary.** Let \(f \in C_{2\pi}\) and let \(\omega_1(f;\delta) := \omega(f';\delta)\) be the modulus of continuity of \(f'\), then for \(n \in \mathbb{N}\) one has for the Fejér operators

\[(4.47) \quad \max_x |L_{n,1}(f;x) - f(x)| \leq \frac{\pi}{4} \omega_1(f;\frac{\pi}{n}),\]

where the value \(\frac{\pi}{4}\) is best possible in (4.47).
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<tr>
<th>n</th>
<th>$S_1(n, l)$</th>
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<th>$c_{n, l}$</th>
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Table 4.2
5. The limiting behaviour of the constants $c_{n,p}$

5.0. In this section we investigate the limiting behaviour as $n \to \infty$ or/and $p \to \infty$ of the exact constants of approximation $c_{n,p}$. It turns out that there are four cases to be considered, viz. $n \to \infty$, $p \geq 2$; $n \geq 2$, $p \to \infty$; $n \to \infty$, $p \to \infty$ and $n \to \infty$, $p = 1$, the last case corresponding to the Fejér operators which have a degree of approximation different from those of the Jackson type.

5.1. Case $n \to \infty$, $p \geq 2$. Let $d_{n,p}$ be given by (3.9), i.e. let

$$d_{n,p} = \int_{-\pi}^{\pi} \tilde{f}_n(t) k_{n,p}(t) dt \quad (p \geq 2).$$

As a guide to norming we regard $k_{n,p}$ as the probability density of a random variable (r.v.) $T_{n,p}$. For the expectation $E_{n,p}$ and variance $\text{var}_{n,p}$ we have

$$E_{n,p} = 0, \quad \text{var}_{n,p} = E_{n,p}^2 = \int_{-\pi}^{\pi} t^2 k_{n,p}(t) dt.$$

By lemmas 2.2 and 2.3 it is easily verified that

$$\text{var}_{n,p} \sim 6p^{-1}n^{-2} \quad (n \to \infty, p \to \infty).$$

Denoting the probability density of a r.v. $X$ by $g_X$ we generally have for $a > 0$ (cf. [4], p. 45)

$$g_{aX}(t) = \frac{1}{a} g_X\left(\frac{t}{a}\right),$$

and therefore, letting $n \to \infty$,

$$g_{\frac{2}{n}T_{n,p}}(t) = \frac{2}{n} g_{\frac{2t}{n}} = \frac{2}{n} k_{n,p}\left(\frac{2t}{n}\right) \to \left(\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt\right)^{-1} \left(\frac{\sin t}{t}\right)^2 =: g_p(t).$$
It follows by dominated convergence ($\hat{f}_n(t) \leq a|t| + bnt^2$; cf. (5.6)) that for $n \to \infty$ one has

$$\frac{n\pi}{2} \int_{-\pi}^{\pi} \tilde{f}_n(t)k_{n,p}(t)dt = \frac{n\pi}{2} \int_{-\frac{n\pi}{2}}^{\frac{n\pi}{2}} \tilde{f}_n\left(\frac{2t}{n}\right)k_{n,p}\left(\frac{2t}{n}\right)dt \to \int_{-\infty}^{\infty} f^*(t)g_p(t)dt,$$

where (cf. formulae (3.12) and (3.13))

$$f^*(t) := \lim_{n \to \infty} \frac{n}{2} \tilde{f}_n\left(\frac{2t}{n}\right) = \frac{1}{2}|t| + \sum_{j=1}^{\infty} \left(|t| - \frac{\pi j}{2}\right)^+.$$

Summing up we have the following theorem.

**Theorem 5.1.** For $p \geq 2$

$$c := \lim_{n \to \infty} c_{n,p} = \lim_{n \to \infty} \sqrt{n^p} \int_{-\pi}^{\pi} \tilde{f}_n(t)k_{n,p}(t)dt = \sqrt{p} \int_{-\infty}^{\infty} |t|g_p(t)dt + 2\sqrt{p} \sum_{j=1}^{\infty} \int_{\pi/2}^{\pi j} (t - \frac{\pi j}{2})g_p(t)dt,$$

where $g_p$ is defined as in (5.1).

**Remark.** On account of lemma 2.1, theorem 5.1 may also be written in the form

$$c_p = a + b_p,$$

where

$$a_p = \sqrt{p} \frac{S(p,p-\frac{1}{p})}{S(p,p)},$$

$$b_p = \frac{2\sqrt{p}}{S(p,p)} \sum_{j=1}^{\infty} \int_{0}^{\infty} (t - \frac{i\pi}{2})^+ \left(\frac{\sin t}{t}\right)^{2p} dt.$$

The computation of $a_p$ causes no difficulties. For small values of $p$ the integrals $S(p,p)$ and $S(p,p-\frac{1}{p})$ can be evaluated by (2.1) and (2.2) respectively, whereas for large values of $p$ it is preferable to use numerical integration. Expression (5.3) for $b_p$ is suitable for numerical integration.
when $p$ is not too small, $p \geq 4$, say. If $p = 2$ or $p = 3$ however, the right-hand side of (5.3) converges too slowly. To speed up the convergence we proceed as follows.

$$b_p = \frac{2\sqrt{p}}{S(p,p)} \int_0^\infty f(t) \left(\frac{\sin t}{t}\right)^{2p} \, dt,$$

where

$$f(t) = \sum_{j=1}^\infty \left(t - \frac{j\pi}{2}\right)^+ \quad (t \geq 0).$$

It is easily verified that

$$f(t) = \frac{1}{\pi} (t - \frac{\pi}{4})^2 - g(t),$$

where $g$ is a $\frac{\pi}{2}$-periodic function and

$$g(t) = \frac{1}{\pi} (t - \frac{\pi}{4})^2 \quad (0 \leq t \leq \frac{\pi}{2}).$$

Hence

$$b_p = \frac{2\sqrt{p}}{S(p,p)} \left\{ \int_0^\infty \frac{1}{\pi} (t - \frac{\pi}{4})^2 \left(\frac{\sin t}{t}\right)^{2p} \, dt - \int_0^\infty g(t) \left(\frac{\sin t}{t}\right)^{2p} \, dt \right\} =$$

$$= \frac{2\sqrt{p}}{\pi S(p,p)} \left\{ \int_0^\infty (t - \frac{\pi}{4})^2 \left(\frac{\sin t}{t}\right)^{2p} \, dt - \sum_{j=0}^\infty \int_0^{\frac{\pi}{2}} (t - \frac{\pi}{4})^2 \left(\frac{\sin (t + j\pi)}{t + \frac{j\pi}{2}}\right)^{2p} \, dt \right\}.$$

In view of lemma 2.1 the first integral in the right-hand side of (5.4) is easy to evaluate. The speed of convergence of the infinite series in (5.4) is, even for $p = 2$, quite satisfactory, and the range of integration is now finite. Formula (5.4) has been used to compute $b_p$. Table 5.1 contains the numerical values of $a_p$, $b_p$ and $c_p$ for $p = 2(1)10; 20(10)50$. 
Table 5.1

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<th>a_p</th>
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<th>c_p</th>
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5.2. Case n ≥ 2, p → ∞. We now investigate the limiting behaviour of c_{n,p} for n ≥ 2 fixed and p → ∞. We begin by considering c_{n,p}/T_{n,p}, where c will be given a convenient value. Leaving out the details of the computation, one has

\[ g_{c_{n,p}}(t) = \frac{1}{c_{n,p}} k_{n,p} \left( \frac{t}{c_{n,p}} \right) + g(n)(t) \quad (p \rightarrow \infty), \]

where

\[ g(n)(t) = \left( \int_{-\infty}^{\infty} \exp\left( -\frac{t^2(n-1)}{12n^2c^2} \right) dt \right)^{-1} \exp\left( -\frac{t^2(n-1)}{12n^2c^2} \right). \]

If we take \((n^2-1)/(6n^2c^2) = 1\) it follows that \(\sqrt{\frac{1}{6}np(n^2-1)}T_{n,p}\) is asymptotically standard normal. By dominated convergence, putting

\[ a_{n,p} = \sqrt{\frac{1}{6}np(n^2-1)}, \]

this means that
We have now proved the following theorem.

Theorem 5.2. For $n \geq 2$

$$\lim_{p \to \infty} \lim_{n \to \infty} \int_{-\pi}^{\pi} \tilde{f}_n(t) k_n,p(t) dt = \sqrt{\frac{3}{\pi}} \frac{n}{\sqrt{n^2 - 1}}.$$  

We note that if $n = 2$ we have $\lim_{p \to \infty} c_{2,p} = \frac{2}{\sqrt{\pi}}$ (cf. (4.17)). Furthermore, it follows that $\left\{ \lim_{p \to \infty} c_{n,p} \right\}_{n=2}^{\infty}$ is a decreasing sequence.

5.3. Case $n \to \infty$, $p \to \infty$. From theorems 5.1 and 5.2 we obtain (cf. table 5.1)

$$\lim_{n \to \infty} \lim_{p \to \infty} \int_{-\pi}^{\pi} \tilde{f}_n(t) k_n,p(t) dt = \sqrt{\frac{3}{\pi}} = 0.97720502,$$

where the limits may be taken in both orders.

Proof. It is an immediate consequence of theorem 5.2 that

$$\lim_{n \to \infty} \lim_{p \to \infty} c_{n,p} = \sqrt{\frac{3}{\pi}}.$$  

On the other hand we obtain from theorem 5.1, using dominated convergence for integrals and sum,

$$\lim_{p \to \infty} \lim_{n \to \infty} c =$$

$$= \lim_{p \to \infty} \left\{ A^{-1} \left[ \int_0^\infty u \left( \frac{\sin \frac{u}{\sqrt{p}}}{\sqrt{p}} \right) \left( \frac{\sin \frac{\pi u}{\sqrt{p}}}{\sqrt{p}} \right)^{2p} du + 2 \sum_{j=1}^{\infty} \int_{\frac{\pi u}{2}}^{\pi u} \left( u - \pi j \frac{\sqrt{v}}{2} \right) \left( \frac{\sin \frac{u}{\sqrt{p}}}{\sqrt{p}} \right)^{2p} \left( \frac{\sin \frac{\pi u}{\sqrt{p}}}{\sqrt{p}} \right) \left( \frac{\sin \frac{\pi v}{\sqrt{p}}}{\sqrt{p}} \right) \left( \frac{\sin \frac{\pi u}{\sqrt{p}}}{\sqrt{p}} \right) \left( \frac{\sin \frac{\pi v}{\sqrt{p}}}{\sqrt{p}} \right) du \right] \right\} =$$

$$= \left( \int_0^\infty \exp\left(-\frac{u^2}{3}\right) du \right)^{-1} \int_0^\infty \exp\left(-\frac{u^2}{3}\right) du = \sqrt{\frac{3}{\pi}},$$

where the limits may be taken in both orders.
where we have written \( \int_0^\infty \left( \frac{\sin \frac{u}{\sqrt{p}}}{\sqrt{p}} \right)^{2p} du = A_p \).

5.4. Case \( n \to \infty, p = 1 \). In this section we consider the limiting behaviour of the constants \( c_{n,1} \) as defined in (1.5). We note (cf. [3], formulae (22), (23)) that

\[
\text{var } T_{n,1} = E^2 T_{n,1} = 2 \int_0^\pi t^2 k_{n,1}(t) dt = \frac{4 \log 2}{n} + O\left(\frac{1}{n^2}\right),
\]

(n \to \infty)

(5.5) \( E \left| T_{n,1} \right| = 2 \int_0^\pi t k_{n,1}(t) dt = \frac{2 \log n}{\pi n} + O\left(\frac{1}{n}\right) \).

As by (3.10) and (3.11) we have

\[
\tilde{f}'_{n}(t) = \begin{cases} \frac{nt}{\pi} + O(1) & (0 < t < \frac{\pi}{2}) \\ n(\pi-t) & (\pi/2 < t < \pi), \end{cases}
\]

(n \to \infty)

it follows that

\[
\tilde{f}_{n}(t) = \begin{cases} \frac{nt^2}{2\pi} + t O(1) & (0 < t < \frac{\pi}{2}) \\ \frac{n\pi}{4} - \frac{n(\pi-t)^2}{2\pi} + t O(1) & (\pi/2 < t < \pi) \end{cases}
\]

(n \to \infty)

(5.6)

In view of (5.5) and (5.6), and taking into account that \( A_{n,1} = 2\pi n \) (cf. Lemma 2.4), one has

\[
\lim_{n \to \infty} c_{n,1} = \lim_{n \to \infty} \int_{-\pi}^{\pi} \tilde{f}_{n}(t) k_{n,1}(t) dt = \lim_{n \to \infty} \int_0^{\pi} \left( \frac{n^2}{2} - (\pi-t)^2 \right) \frac{\sin \frac{nt}{\pi}}{\sin \frac{1}{2}t} dt = \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{\pi} t^2 \sin \frac{nt}{\pi} (\sin \frac{nt}{\pi})^2 dt =
\]
\[
\frac{1}{4\pi^2} \int_0^{\frac{\pi}{2}} \frac{t^2}{\sin^2 \frac{t}{2}} \, dt + \frac{1}{4\pi} \int_{\frac{\pi}{2}}^{\pi} \left( \frac{\pi^2}{2} - (\pi - t)^2 \right) \frac{1}{\sin^2 \frac{t}{2}} \, dt ,
\]

by an application of the Riemann-Lebesgue lemma. It follows that
(cf. [6], p. 123)

\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} \tilde{f}_n(t) k_{n,1}(t) \, dt =
\]

\[
= \frac{2}{\pi} \int_0^{\frac{\pi}{4}} \frac{t^2}{\sin^2 t} \, dt + \frac{1}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 t} \, dt - \frac{2}{\pi} \int_0^{\frac{\pi}{4}} \frac{t^2}{\cos^2 t} \, dt =
\]

\[
= \frac{2}{\pi} \left( 2\theta - \frac{\pi^2}{8} \right) + \frac{1}{4} = \frac{4\theta}{\pi} ,
\]

where \( \theta \) denotes Catalan's constant: \( \theta := \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596559 \).

We have thus proved the following theorem (compare table 4.2).

**Theorem 5.4.**

\[
\lim_{n \to \infty} c_{n,1} = \lim_{n \to \infty} \int_{-\pi}^{\pi} \tilde{f}_n(t) k_{n,1}(t) \, dt = \frac{4\theta}{\pi} = 0.37122687 .
\]

**Corollary.** Let \( f \in C_{2\pi}^1 \) and let \( \omega_1(f;\delta) := \omega(f';\delta) \) be the modulus of continuity of \( f' \), then for \( n \in \mathbb{N} \) one has the following inequality for the Fejér operators

\[
(5.7) \quad \max_x \left| L_{n,1}(f;x) - f(x) \right| \leq c_{n,1} \omega_1(f;\pi) ,
\]

where

\[
\lim_{n \to \infty} c_{n,1} = \frac{4\theta}{\pi^2} = 0.37122687 .
\]
Remark. Inequality (5.7) can also be stated in the form

$$\max_x \left| \frac{1}{n} \sum_{j=0}^{n-1} S_j(f;x) - f(x) \right| \leq c_n \omega_1(f;1/n)$$

where $S_j(f;x)$ is the $j$-th Fourier approximant to $f$.

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