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Note on Urbanik's Class $L_n$

B.G. Hansen
NOTE ON URBANIK'S CLASS $L_n$

by

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ABSTRACT

In this note we consider the analog of multiple self-decomposability for characteristic functions $\phi$ satisfying $\phi(t) = \phi^c(\phi t) \phi_c(t)$. We obtain results analogous to those known for multiple self-decomposable characteristic functions.

1. INTRODUCTION AND PRELIMINARIES

A characteristic function $\phi$ of a random variable on $\mathbb{R} := (-\infty, \infty)$ is said to be self-decomposable if for every $c \in (0, 1)$ there exists a characteristic function $\phi_c$ such that

$$\phi(t) = \phi(ct) \phi_c(t),$$

(cf. Lukacs (1970)). The set of all self-decomposable characteristic functions is denoted by $L$. Urbanik (1973) considered self-decomposable characteristic functions $\phi$, where $\phi_c$ also has a decomposability property. He defined, inductively, the sets $L_n$, $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$, of $n$-times self-decomposable characteristic functions by $L_0 := L$ and $L_n := \{ \phi \mid \phi \in L_{n-1} \text{ and } \phi_c \in L_{n-1} \text{ for all } c \in (0, 1) \}$. O'Connor (1979) and (1981), Jurek (1988) and (1989) and Hansen (1988) generalized the notion of self-decomposability. An infinitely divisible characteristic function $\phi$ is said to belong to the set $U_\alpha$ if $\phi$ satisfies some differentiability conditions and

$$\phi(t) = \phi^c(\phi t) \phi_c(t),$$

for all $c \in (0, 1)$. In this note we introduce the sets $U^\alpha_n$ of multiple self-decomposable characteristic functions in the sense of (1.2). It is shown that a random variable has its characteristic function in $U^\alpha_n$ if and only if it is the weak limit of a sequence of partial sums of a uan triangular array of $n$-times $\alpha$-unimodal (cf. Section 2) random variables.
We also establish a canonical form for characteristic functions in $U_0^\alpha$ and show that the Lévy spectral function of $\phi$ is $n$-times $\alpha$-unimodal. We also characterize the set $\cup_{\alpha \in \mathbb{R}} U_0^\alpha$. We conclude this section with three theorems. For the proof of the first two theorems we refer to Loève (1977) and of the third to Lukacs (1970).

**Theorem 1.1.** A function $\phi$ is an infinitely divisible characteristic function if and only if it can be written in the form
\[
\ln \phi(t) = i t a_\phi - \frac{1}{2} \sigma_\phi^2 t^2 + \int_{\mathbb{R} \setminus \{0\}} k(t, x) dM(x),
\]
where $a_\phi \in \mathbb{R}$, $\sigma_\phi^2 \in \mathbb{R}_+$, $k(t, x) = e^{i t x} - 1 - it(x^2-1)$, and such that the function $M$ (called the Lévy spectral function) satisfies

(i) $M(x)$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$;

(ii) $M(-\infty) = M(\infty) = 0$;

(iii) The integrals $\int_{-1}^{0} x^2 \, dM(x)$ and $\int_{1}^{0} x^2 \, dM(x)$ are finite.

The representation is unique.

An infinitely divisible characteristic function $\phi$ is uniquely determined by the triple $[a_\phi, \sigma_\phi^2, M]$ in Theorem 1.1. We therefore write $\phi = [a_\phi, \sigma_\phi^2, M]$.

**Theorem 1.2.** Let $X$ be a random variable with characteristic function $\phi$ and let $(X_{k,l})$, $k = 1, 2, \ldots, l, \in \mathbb{N}_+$ be a uan triangular array of random variables with distribution functions $(F_{k,l})$, $k = 1, 2, \ldots, l, \in \mathbb{N}_+$. There exists a sequence $(a_l)$ such that
\[
\sum_{k=1}^{l} X_{k,l} + a_l \xrightarrow{w} X \text{ as } l \to \infty,
\]
if and only if

(i) there exists a function $M$ satisfying (i) and (iii) of Theorem 1.1 such that
\[
M_l := \sum_{k=1}^{l} F_{k,l} \xrightarrow{w} M \text{ as } l \to \infty
\]
outside every neighbourhood of the origin.

(ii) $\lim_{l \to \infty} \lim_{\varepsilon \to 0} \frac{1}{l} \sum_{k=1}^{l} \int_{|x| \leq \varepsilon} x^2 \, dF_{k,l}(x) - \int_{|x| \leq \varepsilon} x \, dF_{k,l}(x)^2 = \sigma_\phi^2$. 

Necessarily \( \phi \) is infinitely divisible with \( \phi = [a_\phi , \sigma^2_\phi , M] \) for some \( a_\phi \in \mathbb{R} \).

**Theorem 1.3.** A function \( \phi \) is the characteristic function of a stable distribution if and only if \( \phi \) is either normal or \( \phi \) can be written in the form

\[
\ln \phi(t) = ita_\phi - c \|t\|^\delta (1 + i \beta \text{sgn}(t) w(\|t\|, \delta))
\]

where \( c \geq 0, \|\beta\| \leq 1, \delta \in (0,2) \) and \( a_\phi \in \mathbb{R} \). The function \( w(\|t\|, \delta) \) is given by

\[
w(\|t\|, \delta) = \begin{cases} \\
\tan(\pi \delta/2) & \text{if } \delta \neq 1 \\
-(2/\pi)\ln \|t\| & \text{if } \delta = 1
\end{cases}
\]

Equivalently, \( \phi \) is the characteristic function of a stable distribution if and only if \( \phi \) is infinitely divisible and either (cf. Theorem 1.1) \( \sigma^2_\phi > 0 \) and \( M(x) = 0 \) or \( \sigma^2_\phi = 0 \) and \( M(x) = C_- \|x\|^{-\delta} \) for \( x < 0 \) and \( M(x) = C_+ x^{-\delta} \) for \( x > 0 \). The parameters satisfy \( \delta \in (0,2), C_- \geq 0, C_+ \geq 0 \) and \( C_- + C_+ \geq 0 \). The parameter \( \delta \) is called the exponent of stability of \( \phi \) and we write \( \phi \text{STABLE}(\delta) \).

2. \( \alpha \)-Unimodality

A random variable \( X \) with distribution function \( F \) and density \( f \) is said to be unimodal, with mode at \( x_0 \) (not necessarily unique), if \( f(x) \) is non-decreasing for \( x < x_0 \) and non-increasing for \( x > x_0 \). We will throughout this paper assume that \( x_0 = 0 \), i.e., if a function is said to be unimodal (or \( n \)-times \( \alpha \)-unimodal) it is understood that its mode is at the origin. Khintchine (1938) showed that \( X \) is unimodal (at zero) if \( X = UY \), with \( U \) and \( Y \) independent and \( U \) uniform on \( (0,1) \). Olshen and Savage (1970) generalized this concept; a random variable is said to be \( \alpha \)-unimodal (at zero) if it is of the form \( U^{1/\alpha} Y \), with \( U \) and \( Y \) independent, \( U \) uniformly distributed on \( (0,1) \) and \( \alpha > 0 \). If \( Y \) has distribution function \( G \), then

\[
f(x) = \alpha x^{\alpha-1} \int_x^\infty v^{-\alpha} dG(v), \; x \in \mathbb{R}_+ ,
\]

\[
f(x) = \alpha \|x\|^{\alpha-1} \int_{-\infty}^x |v|^{-\alpha} dG(v), \; x \in \mathbb{R}_- := (-\infty,0] ,
\]

or equivalently

\[
F(x) = \alpha \int_0^1 G(\nu^{-1}x) \nu^{\alpha-1} d\nu .
\]

This result corresponds to Corollary 2, p. 28 in Olshen and Savage (1970). Hence, \( f \) is \( \alpha \)-unimodal if and only if \( \|x\|^{1-\alpha} f(x) \) is non-decreasing on \((-\infty,0)\) and non-increasing on \((0,\infty)\). By iteration we see that a random variable \( X \) is of the form
$X = \cup_{i=1}^{n+1} \cdots \cup_{i=1}^{n+1} Y$, with $U_i$, $i = 1, 2, \ldots, n+1$, mutually independent, independent of $Y$ and all uniformly distributed on $(0, 1)$ if and only if

$$f(x) = x^{\alpha+1} \frac{(n+1)!}{x} \int_{-\infty}^{\infty} (\ln(x/v))^n v^\alpha dG(v), \quad x \in \mathbb{R}_+,$$

or equivalently

$$F(x) = x^{\alpha+1} \frac{(n+1)!}{x} \int_{0}^{1} G(v^{-1}x) (\ln v^{-1})^n v^{\alpha-1} dv.$$

We will use the notion of $\alpha$-unimodality in connection with Lévy spectral functions, so a more general definition is needed.

**Definition 2.1.** A function $f$ (not necessarily non-negative) is said to be $n$-times $\alpha$-unimodal and belong to the set $UN^\alpha_n$ for some $\alpha \in \mathbb{R}$ and some $n \in \mathbb{N}_0$, if there exists constants $C_+, C_- \in \mathbb{R}_+$ and a right continuous function $N$, non-decreasing on $(-\infty, 0)$ and $(0, \infty)$ such that

$$f(x) =
\begin{cases}
  x^{\alpha-1} \left( \int_{-\infty}^{\infty} (\ln(x/v))^n v^\alpha dN(v) + C_+ \right) & x > 0 \\
  |x|^{\alpha-1} \left( \int_{-\infty}^{\infty} (\ln(x/v))^n v^\alpha dN(v) + C_- \right) & x < 0
\end{cases},$$

and such that the integrals converge for every $x \in \mathbb{R}$.

It is clear that $UN^\alpha_n$ form an increasing sequence of sets in $\alpha$ and a decreasing sequence of sets in $n$. Let $F$ be absolutely continuous with Radon-Nikodym derivative $F' = f$ and let $xy > 0$. It can be shown that (2.2) is equivalent to

$$F(x) - \lambda = \int_{0}^{1} N(v^{-1}x) (\ln v^{-1})^n v^{\alpha-1} dv + \alpha-1 \text{sign}(x) C_{\text{sign}(x)} |x|^\alpha,$$

for some $\lambda \in \mathbb{R}$. We will henceforth assume, without loss of generality, that $\lambda = 0$. In the rest of this section we study the functions in $UN^\alpha_n$ closer, in particular the distribution functions in $UN^\alpha_n$ and we also give some properties of characteristic functions having Lévy spectral functions $M$ with Radon-Nikodym derivative $M' \in UN^\alpha_n$. The following theorem is an immediate consequence of Definition 2.1.
THEOREM 2.2. The set $\mathcal{U}N_\alpha^n$ is closed under limits.

It can be verified that $N$ in (2.2) is, up to additive constants, unique. We first prove a lemma which shows that a function is in $\mathcal{U}N_\alpha^n$ if and only if it satisfies some inequalities. Let $T_c$ be a linear operator, acting on set functions, and defined by $T_c F(B) = F(\{ cb \mid b \in B \})$, for any Borel set $B$. Also, denote the Raydon-Nikodym derivative of a function $F$ (or $M$) by $F'$ (or $M'$).

LEMMA 2.3. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}(-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$ and let $c(k) := (c_1, c_2, \ldots, c_k), \ k \in \mathbb{N}_+$. Define, inductively, the functions $F_{c(k)}$ by $F_{c(1)} := F$ and

$$F_{c(k)}(x) := F_{c(k-1)}(x) - c_k^\alpha F_{c(k-1)}(c_k^{-1} x), \ k \in \mathbb{N}_+, \ (2.4)$$

for all $c_k \in (0, 1), 1 \leq i \leq k$. $F$ is absolutely continuous on $\mathbb{R}\{0\}$ and $F' \in \mathcal{U}N_\alpha^n$ if and only if for all $c \in (0, 1)$ and all $c_k \in (0, 1), 1 \leq k \leq n$ $F_{c(n)}(B) \geq c^\alpha T_c F_{c(n)}(B)$ and $F_{c(n)}(B) \geq 0$ for all $B \in \beta$.

PROOF. Let $\alpha > 0$ and let $F_{c(n)}$ satisfy (2.5). Fix $\varepsilon > 0$ and let $B = (a, b), 0 < \varepsilon \leq a < b$. Then (2.5) is equivalent to

$$F_{c(n)}(b) - F_{c(n)}(a) \geq c^\alpha (F_{c(n)}(b/c) - F_{c(n)}(a/c)).$$

Let $w(x) := F_{c(n)}(x)^{1/\alpha}$. Let $x = (b/c)^{-\alpha}$, $y = (a/c)^{-\alpha}$, $x' = b^{-\alpha}$ and $y' = a^{-\alpha}$. It follows that

$$
\frac{w(y') - w(x')}{y' - x'} \geq \frac{w(y) - w(x)}{y - x}.
$$

Hence $w$ is convex. By Theorem A, p. 4, Roberts and Varberg (1973), $w$ and hence also $F_{c(n)}$ is absolutely continuous on $(\varepsilon, \infty)$. Observe that

$$\int_B F'_{c(n)}(x) dx = F_{c(n)}(B) \geq c^\alpha T_c F_{c(n)}(B) = c^\alpha \int_B F'_{c(n)}(c^{-1} x) dx, \ (2.6)$$

with $B = (a, b)$. Differentiating (2.6) with respect to $a$ and multiplying both sides by $a^{1-\alpha}$, we see that $a^{1-\alpha} F'_{c(n)}(a)$ is non-increasing on $\mathbb{R}_+$. Similarly for $b < a \leq \varepsilon < 0$ we obtain that $a^{1-\alpha} F'_{c(n)}(a)$ is non-decreasing on $\mathbb{R}_-$. Hence $F'_{c(n)} \in \mathcal{U}N_0^n$. Suppose that $n > 0$. Since $F_{c(n)}(B) \geq 0$, we see from (2.4) that $F_{c(n-1)}$ satisfies (2.5) and so $F_{c(n-1)}$ exists and $F'_{c(n-1)} \in \mathcal{U}N_0^n$. For $x > 0$

$$N_n(x) := \lim_{c_n \downarrow 1} (1-c_n)^{-1} F_{c(n)}(x) = \frac{\partial}{\partial c_n} c_n^\alpha F_{c(n-1)}(c_n^{-1} x) \big|_{c_n = 1}
$$

$$= \alpha F_{c(n-1)}(x) - x F_{c(n-1)}(x) = x^{1+\alpha} \frac{\partial}{\partial x} x^{-\alpha} F_{c(n-1)}(x),$$

From (2.3) it follows that $x^{-\alpha} F_{c(n-1)}(x) \to -\alpha^{-1} C_+$ as $x \to \infty$, and so
\[ F_{c(n-1)}(x) = x^\alpha \left[ \int_{x}^{\infty} N_n(v) v^{-\alpha-1} dv + C_+ \right] = \int_{0}^{1} N_n(v^{-1}x) v^{-\alpha-1} dv + C_+ x^\alpha. \]

Similarly for \( x < 0 \). By Theorem 2.2, \( N'_n \in UN_0^0 \) and so \( F'_{c(n-1)} \in UN_0^1 \). Proceeding similarly we see that \( F'_{c(0)} := F' \in UN_0^0 \). The proof for \( \alpha \leq 0 \) is almost identical and therefore omitted.

Conversely, suppose \( F \) is absolutely continuous with \( F' \in UN_0^0 \). Then (cf. (2.3))

\[ F(x) = \int_{0}^{1} \cdots \int_{0}^{1} N(v_1^{-1} \cdots v_{n+1}^{-1} x) (v_1 \cdots v_{n+1})^{\alpha-1} dv_{n+1} \cdots dv_1 + \alpha^{-1} \text{sign}(x) C_{\text{sign}(x)} |x|^{\alpha}. \]

Hence

\[ F_{c(1)}(x) = \int_{0}^{1} \cdots \int_{c_1}^{1} N(v_1^{-1} \cdots v_{n+1}^{-1} x) (v_1 \cdots v_{n+1})^{\alpha-1} dv_{n+1} \cdots dv_1 + \alpha^{-1} \text{sign}(x) C_{\text{sign}(x)} |x|^{\alpha}, \]

where

\[ N_{c(1)}(x) := \int_{c_1}^{1} N(v_{n+1}^{-1} x) v_{n+1}^{-\alpha-1} dv_{n+1}. \]

Proceeding similarly \( n-1 \) times we have that

\[ F_{c(n)}(x) = \int_{0}^{1} N_{c(n)}(v_{n+1}^{-1} x) v_{n+1}^{-\alpha-1} dv_{n+1} + \alpha^{-1} \text{sign}(x) C_{\text{sign}(x)} |x|^{\alpha}. \]

Obviously \( F_{c(n)}(B) \geq 0 \) and

\[ F_{c(n)}(B) = \int_{0}^{1} T_{v_1} N_{c(n)}(B) v_{n+1}^{\alpha-1} dv_1 + \int_{B} C_{\text{sign}(x)} |x|^{\alpha-1} dx \]

\[ = c^\alpha \int_{0}^{c} T_{u} N_{c(n)}(B) u^{\alpha-1} du + c^\alpha \int_{T_c^{-1}B} C_{\text{sign}(x)} |x|^{\alpha-1} dx \]

\[ \geq c^\alpha \int_{0}^{1} T_c T_u N_{c(n)}(B) u^{\alpha-1} du + c^\alpha \int_{T_c^{-1}B} C_{\text{sign}(x)} |x|^{\alpha-1} dx \]

\[ = c^\alpha T_c F_{c(n)}(B). \]

\[ \square \]

REMARK 2.4. It can be shown (cf. Lemma 2.3) that a distribution function has a density in \( UN_0^0 \) if and only if \( \alpha > 0 \) and for any \( c \in (0,1) \) there exists a distribution function \( F_c \) such that
Let $X$ and $X'$ have distribution function $F$, $X_c$ have distribution function $F_c$ and define the random variable $Y_c$ by

$$P(Y_c = 1) = e^\alpha = 1 - P(Y_c = 0).$$

Then (2.7) is equivalent to

$$X = Y_c c X' + (1 - Y_c) X_c,$$

with $X$, $X'$, $X_c$ and $Y_c$ all independent. Let $DF$ denote the set of all distribution functions. Define the sets $F^n_\alpha$ by $F^n_\alpha := DF$ and

$$F^n_\alpha := \{ F \in F^{n-1}_\alpha \mid F \text{ satisfies (2.7) with } F_c \in F^{n-1}_\alpha \}, \ n \in \mathbb{N}_0.$$

Proceeding as in the proof of Lemma 2.3 and using the above (see also the proof of Theorem 3.4) it can be verified that the following statements are equivalent.

(i) $F \in F^n_\alpha$;

(ii) There exists a unique distribution function $G$ such that

$$F(x) = \alpha^{n+1} (n!)^{-1} \int_0^1 G(v^{-1} x) (\ln v^{-1})^n v^{\alpha-1} \, dv;$$

(iii) $F$ is absolutely continuous and $F' \in UN^n_\alpha$. \hfill \Box

It is well known that the sum of two unimodal random variables need not be unimodal (cf. for example Olshen and Savage (1970)). With the help of Theorem 1.2 we characterize in the following theorem the random variables which can be obtained as the weak limit of a sequence of partial sums of $n$-times $\alpha$-unimodal random variables. Note that if $X$ has a density in $UN^n_\alpha$, then necessarily $\alpha > 0$.

THEOREM 2.5. Let $\alpha > 0$. There exists a triangular array $(X_{k,l})$, $k = 1, 2, \ldots, l, l \in \mathbb{N}_+$, with $X_{k,l}$ $n$-times $\alpha$-unimodal (i.e., its distribution function $F_{k,l}$ has a density $f_{k,l}$ of the form (2.1) for some distribution function $G_{k,l}$) and a sequence $(a_l)$ such that

$$\sum_{k=1}^l X_{k,l} + a_l \rightarrow X \text{ as } l \rightarrow \infty.$$

if and only if $X$ is a random variable with characteristic function $\phi = [a_\phi, \sigma^2_\phi, M]$ where $M$ is absolutely continuous and $M' \in UN^n_\alpha$. Necessarily $\sum_{k=1}^l G_{k,l} \rightarrow N$ as $l \rightarrow \infty$ outside every neighbourhood of the origin with $M'$ and $N$ related by (2.2) with $f$ replaced by $M'$. 

PROOF. The necessity follows from Theorem 1.2 and Theorem 2.2. Suppose \( \phi = [a_\phi, \sigma_\phi^2, M] \), with \( M \in UN^n_\alpha \). We consider three cases.

CASE I. Let \( M(0^+) = -\infty \) and \( M(0^-) = \infty \). Let \( (b_l) \) and \( (c_l) \) be two sequences with \( b_l \downarrow 0, c_l \uparrow 0 \) as \( l \to \infty \) and such that

\[
-M(b_l) = M(c_l) \quad \text{and} \quad M'(c_l)(b_l - c_l) + M(c_l) - M(b_l) = 1.
\]

Also let \( F'_1(x) = M'(x) l^{-1} \) if \( x \geq b_l \) or if \( x \leq c_l \) and \( F'_1(x) = M'(c_l) l^{-1} \) if \( x \in (c_l, b_l) \). Hence \( F_1 \) is a distribution function with an \( n \)-times \( \alpha \)-unimodal density, i.e., it has a density almost everywhere of the form (2.1) for some distribution function \( G_1 \).

Note that for any Borel set \( B \), bounded away from the origin, there exists an \( l_0 \) such that for all \( l > l_0 \), \( l F_1(B) = M(B) \) and that for all \( l \in \mathbb{N}_+ \), \( l F_1(B) \leq M(B) \). Let \( X_{k,l} \) have distribution function \( F_1 \) for each \( 1 \leq k \leq l \). Necessarily \( (X_{k,l})_1, 2, \ldots, l, l \in \mathbb{N}_+ \) is uan and

\[
0 \leq l \left( \int_{-\infty}^{\infty} x^2 dF_1(x) \right)^2 \leq l \int_{-\infty}^{\infty} x^2 dF_1(x) \leq \int_{x \in S} x^2 dM(x).
\]

By Theorem 1.2, \( \sum_{k=1}^l X_{k,l} \to X \) as \( l \to \infty \), with \( \sigma_\phi = 0 \). Since \( l F_1(B) = M(B) \) for sufficiently large \( l \) we have that \( l F_1(x) = M'(x) \) for sufficiently large \( l \) and hence by the uniqueness of the measure \( N \) in (2.2), \( l G_1(B) = N(B) \) for sufficiently large \( l \). Thus \( N \) is the weak limit of \( l G_1 \) as \( l \to \infty \) outside every neighbourhood of the origin.

CASE II. Let \( M(0^+) > -\infty \) and \( M(0^-) < \infty \). Assume without loss of generality that \( M(\mathbb{R}) = 1 \). Let \( F_1(B) := l^{-1} M(B) + (1 - l^{-1}) l_0(B) \), where \( l_0(B) = 0 \) if \( 0 \notin B \) and \( l_0(B) = 1 \) if \( 0 \in B \). The rest of the proof is as in case I.

CASE III. If \( M(0^+) > -\infty \) and \( M(0^-) = \infty \) or \( M(0^+) = -\infty \) and \( M(0^-) < \infty \), then by combining the approaches in cases I and II we can prove the theorem. \( \square \)

From Theorem 2.5 we see that \( \sum_{k=1}^l G_{k,l} \to N \) as \( l \to \infty \), with \( M' \) and \( N \) related by (2.2) where \( f \) is replaced by \( M' \). In view of Theorem 1.2 we can expect that \( N \) is a Lévy spectral function. We will use the following lemma, which is easily proved by applying induction to Lemma 5.4.3 in Hansen (1988).

**Lemma 2.6.** Let \( \alpha > -2 \) and let \( \mathcal{M} \) be such that \( \mathcal{M} \in UN^n_\alpha \), i.e.,

\[
M'(x) = \begin{cases} 
\varepsilon^\alpha - 1 \int_x^\infty (\ln(x/v))^\alpha v^\alpha dN(v) + C_+ & x > 0 \\
\varepsilon^\alpha - 1 \int_x^\infty (\ln(x/v))^\alpha v^\alpha dN(v) + C_- & x < 0
\end{cases}
\]

(2.8)
with the integrals converging for every $x \neq 0$. Then $M$ is a Lévy spectral function if and only if $N$ is a Lévy spectral function with $C_+ = C_- = 0$ for $\alpha \geq 2$.

In Hansen (1988) it is shown that if $M$ is a non-zero Lévy spectral function with $M' \in UN^\alpha$, then $\alpha > -2$. If $\phi$ is infinitely divisible, then $\phi$ has no real zeros (cf. Lukacs (1970)) and hence $\ln \phi(t)$ is a well defined function for $t \in \mathbb{R}$. We now state a lemma which is essentially proved in Hansen (1988), Lemma 5.4.6.

**Lemma 2.7.** Let $\alpha > -2$. If $\phi$ is an infinitely divisible characteristic function with an absolutely continuous Lévy spectral function $M$ where $M' \in UN^\alpha$, then there exists an infinitely divisible characteristic function $\gamma$ and a stable characteristic function $\Phi_{\text{STABLE}(\alpha)}(t)$, possibly degenerate, such that

$$
\ln \phi(t) = \int_0^1 \ln \gamma(vt) \left( \ln v^{-1} \right)^\alpha v^{\alpha-1} dv + \ln \Phi_{\text{STABLE}(\alpha)}(t),
$$

with $\Phi_{\text{STABLE}(\alpha)}(t) \equiv 1$ if $\alpha \geq 0$.

Theorem 2.5 shows that the set of infinitely divisible characteristic functions with $M' \in UN^\alpha$ is the solution of a central limit problem. This central limit problem provided the motivation for this work. In the next section we study this set closer. Our starting point is however different than that of Theorem 2.5, but similar to that used by for example Lukacs (1970) to define the set of self-decomposable characteristic functions.

### 3. SELF-DECOMPOSABILITY

Let $ID$ denote the set of all infinitely divisible characteristic functions. We begin with two definitions.

**Definition 3.1.** The characteristic function $\phi$ is said to belong to $U_\alpha$, for some $\alpha \in \mathbb{R}$, if $\phi$ is infinitely divisible, $\phi'(t)$ exists for $t \neq 0$, $\phi'(t) \to 0$ as $t \to 0$ and for every $c \in (0,1)$ there exists a characteristic function $\phi_c$ such that

$$
\phi(t) = \phi^c(c t) \phi_c(t), \quad t \in \mathbb{R}. \tag{3.1}
$$

**Definition 3.2.** Define the sets $U^n_\alpha$, $n \in \mathbb{N}_0$, (or $U^n_\alpha$ for short), inductively by $U^0_\alpha := U_\alpha$ and
It is evident from Definition 3.2 that \( U^n_{\alpha}, \alpha \in \mathbb{N}_0, \) form a decreasing sequence of sets for each fixed \( \alpha, \) i.e.,

\[
U_{\alpha} := U_{\alpha}^0 \supseteq U_{\alpha}^1 \supseteq \cdots \supseteq U_{\alpha}^n = \bigcap_{n \in \mathbb{N}_0} U_{\alpha}^n.
\]

We will need the following lemma, which gives a canonical representation of \( \phi_c \) in Definition 3.1. For a proof we refer to Theorems 5.4.7 and 5.4.8 in Hansen (1988).

**Lemma 3.3.** Let \( \alpha > -2 \) and let \( \phi = [a_\phi, \sigma^2_\phi, M]. \) If \( \phi \in U_{\alpha} \) then there exists a unique infinitely divisible characteristic function \( \gamma \) such that

\[
\ln \phi_c(t) = \int_0^1 \ln \gamma(vt) v^{\alpha-1} dv,
\]

and so \( \phi_c \in ID \) for every \( c \in (0, 1). \)

We are now ready to prove the main theorem of this section (compare with Remark 2.4).

**Theorem 3.4.** Let \( \alpha > 0, \) let \( n \in \mathbb{N}_0 \) and let \( \phi = [a_\phi, \sigma^2_\phi, M]. \) The following statements are equivalent.

1. \( \phi \in U^n_{\alpha} \)
2. There exists a unique infinitely divisible characteristic function \( \gamma \) and a stable, possibly degenerate, characteristic function \( \phi_{STABLE(-\alpha)} \) such that

\[
\ln \phi(t) = \int_0^1 \ln \gamma(vt) (\ln v)^n v^{\alpha-1} dv + \ln \phi_{STABLE(-\alpha)}(t),
\]

with \( \phi_{STABLE(-\alpha)}(t) \equiv 1 \) if \( \alpha \geq 0. \)
3. \( \sigma_\phi = 0 \) if \( \alpha < 2 \) and \( M \) is absolutely continuous with \( M' \in UN^n_{\alpha}. \)

**Proof.** Lemma 2.7 proves (iii) \( \Rightarrow \) (ii). That (ii) \( \Rightarrow \) (i) is proved as in the first part of the converse part of the proof of Lemma 2.3. Suppose (i) holds. By Lemma 3.3, \( \phi_c \in ID. \) Let \( c(k) := (c_1, \ldots, c_k), \) \( k \in \mathbb{N}_+, \) and define the characteristic functions \( \phi_{c(k)} \) inductively, by \( \phi_{c(0)} := \phi \) and

\[
\ln \phi_{c(k)}(t) := \ln \phi_{c(k-1)}(t) - c_k \ln \phi_{c(k-1)}(c_k t).
\]

If \( M_{c(k)} \) is the Lévy spectral function of \( \phi_{c(k)} \), then \( M_{c(k)} \) is defined by (2.4) with \( F \)
replaced by $M$. By Definition 3.2, $\phi_c(n) \in U^n_\alpha$, i.e.,

$$\ln \phi_c(t) := \ln \phi_c(n)(t) - c^\alpha \ln \phi_c(n)(ct), \ c \in (0, 1),$$

for some $\phi_c \in ID$. Let $M_c$ be the Lévy spectral function of $\phi_c$. Then

$$M_c(x) = M_c(n)(x) - c^\alpha M_c(n)(c^{-1}x)$$  \quad (3.5)

(cf. Lukacs (1970), p. 163), and so $M_c(n)$ satisfies (2.5). By Lemma 2.3, $M' \in U^n_\alpha$. \[\square\]

**COROLLARY 1.** $U^n_\alpha$ is a multiplication semigroup, closed under limits.

In the next theorem we prove a relationship between $\phi$ and $\gamma$ in (3.4) when $n = 0$.

**THEOREM 3.5.** Let $\phi$ and $\gamma$ be related by

$$\ln \phi(t) = \int_0^1 \ln \gamma(vt) v^{\alpha-1} dv + \ln \phi_{\text{STABLE}(-\alpha)}(t),$$

and let $n \in \mathbb{N}_+$. $\phi \in U^n_\alpha$ if and only if $\gamma \in U^{n-1}_\alpha$ and $\gamma$ has no stable component with exponent $\alpha$.

**PROOF.** Let $\phi \in U^n_\alpha$, i.e., let $\phi$ satisfy (3.1) with $\phi_c \in U^{n-1}_\alpha$ for every $c \in (0, 1)$. From the proofs of Lemma 3.3 and Theorem 3.4 we have that (cf. (3.2))

$$\ln \gamma(t) = \lim_{c \uparrow 1} (1 - c)^{-1} \ln \phi_c(t).$$

Also from Lemma 3.3, $\phi_c$ has no stable component with exponent $\alpha$ and so neither does $\gamma$. Since $\phi_c \in U^{n-1}_\alpha \subset ID$, we have that $\phi^{1/(1-c)}_c \in U^{n-1}_\alpha$. By Corollary 1 to Theorem 3.4, $U^{n-1}_\alpha$ is closed under limits and hence $\gamma \in U^{n-1}_\alpha$.

Conversely, let $\gamma \in U^{n-1}_\alpha$ (with no stable component with exponent $\alpha$). Observe that

$$\ln \phi_c(t) = \ln \phi(t) - c^\alpha \ln \phi(ct)$$

$$= \int_0^1 [\ln \gamma(vr) - c^\alpha \ln \gamma(vct)] v^{\alpha-1} dv$$

$$= \int_0^1 \ln \gamma_c(vr) v^{\alpha-1} dv.$$ 

Obviously, if $\gamma$ can be decomposed $n$ times in this fashion, then so can $\phi$. Hence $\phi \in U^n_\alpha$. \[\square\]

We can also prove that (i)$\Rightarrow$(ii) in Theorem 3.4, by applying induction to Theorem 3.5. In the next remark we give necessary and sufficient conditions for the
convergence of the integral in (3.4).

REMARK 3.6. Let $M$ and $N$ be the Lévy spectral functions of $\phi$ and $\gamma$, respectively, in (3.4). From the proof of Theorem 3.4, $M$ and $N$ are related by (2.8). Hence,

(i) if $\alpha > 0$, then the integral in (3.4) converges for all infinitely divisible characteristic functions $\gamma$;

(ii) if $\alpha = 0$, then the integral in (3.4) converges if and only if

$$
\int_{-\infty}^{\infty} (\ln x^{-1})^{n+1} dN(x) < \infty \quad \text{and} \quad \int_{-\infty}^{-1} (\ln |x|^{-1})^{n+1} dN(x) < \infty ;
$$

(iii) if $\alpha \in (-1,0)$, then the integral in (3.4) converges if and only if

$$
\int_{1}^{\infty} x^\alpha (\ln x^{-1})^n dN(x) < \infty \quad \text{and} \quad \int_{-\infty}^{1} |x|^\alpha (\ln |x|^{-1})^n dN(x) < \infty ;
$$

(iv) if $\alpha \in (-2,-1]$, then the integral in (3.4) converges if and only if $\gamma$ has degenerate component and

$$
\int_{1}^{\infty} x^\alpha (\ln x^{-1})^n dN(x) < \infty \quad \text{and} \quad \int_{-\infty}^{1} |x|^\alpha (\ln |x|^{-1})^n dN(x) < \infty ;
$$

If $\alpha(1) > \alpha(2)$ and $\phi \in U_{\alpha(2)}^n$, then (cf. (3.1))

$$
\phi(t) = \phi^{e^{\alpha(1)}}(ct) \phi^{e^{\alpha(2)} - e^{\alpha(1)}}(ct) \phi_c(t).
$$

Since $\phi$ is infinitely divisible, $\phi^{e^{\alpha(2)} - e^{\alpha(1)}}$ is a characteristic function and hence $\phi \in U_{\alpha(1)}^n$. Let $\Gamma\alpha(u)$ denote the distribution function of a gamma $(\alpha, n+1)$ distributed random variable. Theorem 3.4 (ii) implies for any $\phi \in ID$,

$$
\phi_{\alpha}(t) := \exp \left( \int_{0}^{1} \ln \phi(vt) \alpha^{n+1} (\ln v^{-1})^n (n!)^{-1} v^{\alpha-1} dv \right)
$$

$$
= \exp \left( \int_{0}^{\infty} \ln \phi(e^{-u}t) d\Gamma\alpha(u) \right).
$$

is in $U_{\alpha}^n$ for all $\alpha > 0$, with the integral converging by Remark 3.6. Obviously $\Gamma\alpha(u)$ tends to a distribution function with total mass at zero, and so by Helly's second theorem $\phi_{\alpha} \to \phi$ as $\alpha \to \infty$. Hence

$$
\bigcup_{\alpha \in \mathbb{R}} U_{\alpha}^n = ID.
$$

We collect these results in the following theorem.
THEOREM 3.7. The sets $U^n_\alpha$ are multiplication semigroups, closed under limits and provide a classification of $\Lambda D$, i.e.,

(i) If $\alpha(2) < \alpha(1)$ then $U^n_{\alpha(2)} \subset U^n_{\alpha(1)}$;

(ii) $\bigcup_{\alpha \in \mathbb{R}} U^n_\alpha = \Lambda D$;
References


