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Computational procedures for stochastic multi-echelon production systems

G.J. van Houtum
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Abstract

This paper is concerned with the numerical analysis of multi-echelon production systems. In these systems, materials and components are ordered from outside suppliers and next proceed through a number of manufacturing and/or assembly stages. Each stage requires a fixed predetermined leadtime; furthermore, we assume a stochastic, stationary end-item demand process.

In a previous paper, we have presented an exact analysis of such multi-echelon systems under an average cost criterion. All three basic structures, i.e. serial, assembly and distribution systems, have been considered. In particular, it has been shown that, by transforming penalty and holding costs into appropriate echelon cost functions, an exact decomposition of these systems can be obtained, thus reducing complex multi-dimensional problems to a series of more simple one-dimensional problems.

The current paper is based on this analytical theory but discusses numerical aspects, in particular for serial and assembly systems. The one-dimensional problems arising after the (exact) decomposition of a multi-echelon system involve incomplete convolutions of distribution functions, which are only recursively defined. We develop numerical procedures for analyzing these incomplete convolutions; these procedures are based on approximations of distribution functions by mixtures of Erlang distributions. The combination of the analytically obtained (exact) decomposition results with these numerical procedures enables us to analyze fairly complex systems in only a few seconds on a microcomputer.

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1. Introduction

In this paper, we are concerned with the planning and control of the materials flow in a production chain. At the beginning of the chain, components and raw materials are ordered from outside suppliers. Upon arrival, these materials and components may be used immediately in a subsequent manufacturing stage if needed, or else may be stored in a component store. In particular when several components have to be assembled in one subassembly or final product, it may happen that components have to be temporarily stored since one other critical component is missing. After proceeding through a number of manufacturing stages, final products are stored in a factory warehouse from which they are sold to the market.

In controlling such a chain of purchasing and manufacturing/assembly activities, one generally tries to balance a desired customer service level against reasonable production and inventory holding costs. Often, such chains are controlled by MRP-type systems (e.g. Orlicky[1975], Wight[1981], Vollmann, Berry and Whybark[1984]). Basically, MRP systems are materials coordination systems, driven by a Master Production Schedule (MPS) which operates on end-item level in a make-to-stock environment (as we discuss here). Safety stocks coping with demand uncertainty are usually allowed only at this MPS level, all upstream production and procurement actions are derived in a purely deterministic way (using the product structure or Bill of Materials) from this MPS. This contradicts the Base Stock Control system in which order-up-to levels are determined at each stage, based on end-item demand forecasts and the complete echelon stock associated with that stage (i.e. all stock present and downstream in the chain). In particular, uncertainty on end-item demand is explicitly taken into account when determining echelon order-up-to levels at each separate stage.

The concept of echelon stock has been introduced by Clark[1958] and was exploited in a famous paper by Clark and Scarf[1960] to establish the optimality of echelon stock based policies in a serial inventory system (in a discounted cost framework). Under certain assumptions, they have established an exact decomposition of the multi-echelon serial system into a series of single stock systems (each associated with the echelon stock of a certain stage). Schmidt and Nahmias[1985] extended their approach to a two stage assembly system in which only two components are assembled in one final
product, again under discounted cost assumptions. The analysis in these papers unfortunately leads to somewhat cumbersome formulas while the work of Schmidt and Nahmias seems not to be generalizable to systems with more than two components. In an average cost framework, Eppen and Schrage[1981] have presented a thorough analysis of the stockless depot distribution system (one depot, which only acts as a pure distribution centre, and a number of end-item warehouses). A large number of authors have further contributed to the analysis of distribution systems (e.g. Federgruen and Zipkin[1984], Zipkin[1984]).

Langenhoff and Zijm[1989] have unified and extended the above contributions in one general theory, which establishes exact decomposition results for arbitrary logistics chains in which assembly systems (with an arbitrary number of components), serial and distribution systems may all be present. They study multi-echelon production/distribution systems under an average cost criterion. The linear penalty and holding costs are transformed into appropriate echelon cost functions which form the basis of the (exact) decomposition results. As Clark and Scarf[1960], they come up with artificial penalty cost functions at each echelon, which reflect the inability of a certain installation to satisfy downstream demand; due to the average cost structure these functions however are much less complicated than in the discounted cost analysis. Moreover, it can be shown that assembly systems can be transformed into almost purely serial systems, allowing for an almost direct application of the results for these serial systems to assembly systems as well (a similar result was recently established by Rosling[1989]). For distribution systems, the results parallel those of Eppen and Schrage[1981], although Langenhoff and Zijm[1989] also consider the case in which the central depot may hold stock.

The current paper is based on the analytical theory presented in Langenhoff and Zijm[1989] but discusses numerical aspects. Although the exact decomposition results reviewed above reduces the analysis of a complex multi-dimensional problem to the analysis of a series of one-dimensional problems, the latter are not necessarily easy to handle. The one-dimensional functions which have to be minimized often involve incomplete convolutions of demand distribution functions; moreover, these functions are only recursively defined. We approximate these functions by matching mixtures of Erlang distributions to their first two moments (see e.g. Tijms[1986]). These matchings have been shown to perform extremely well, even when approximating convolutions of distributions in which one of the distributions has a certain probability mass
at one point (De Kok and Seidel [1990]). In between we note that matching
distributions to the first two moments makes sense anyhow; often one does not
have any further knowledge on the shape of the original distributions, in
particular when forecasting market demand.

Based on these approximations, we develop numerical procedures for minimizing
the single echelon cost functions arising after decomposition of the
original multi-echelon system. We restrict ourselves to production systems
with serial and assembly structures. The decomposition of distribution systems
leads to one-dimensional functions with additional complexities and will be
discussed elsewhere. For production systems, the combination of the analytical
theory discussed in Langenhoff and Zijm [1989] with the numerical procedures
developed here enables us to analyze fairly complex systems in only a few
seconds on an IBM-compatible PC/AT.

We conclude this section with an outline of the contents of the paper. In
section 2, we review the results of Langenhoff and Zijm [1989] on the decom-
position of general multi-echelon production/distribution systems. In section
3, we discuss preliminaries on the approximation of incomplete convolutions by
mixtures of Erlang distributions. Section 4 shows how this approximation
technique can be used to numerically analyze the most simple multi-echelon
system: the two-stage serial system. In section 5 we present the general
algorithm for both the serial and the assembly system. Section 6 is a preview
on future work on distribution and finite capacity production systems.

2. Fundamental decomposition results for multi-echelon systems

In this section we review the exact decomposition results derived by
Langenhoff and Zijm [1989]. Throughout this paper, we assume that demand
originates at the lowest installations only (i.e. the installations at the
downstream side of the logistic chain). We assume that all excess demand is
backlogged which permits us to neglect any variable production or distribution
costs. In addition, we do not consider fixed production and distribution
costs. For a discussion on these assumptions as well as remarks on relaxation
of some of them, see Langenhoff and Zijm [1989].

The echelon stock of a given installation includes all stock at that
installation plus in transit to or on hand at any installation downstream
minus the backlogs at the most downstream installations (which do not have a successor). The chain under consideration is called the echelon. An echelon stock may be negative, indicating that the backlogs are larger than the total inventory in that echelon. Note that the echelon stock, associated with a component store, includes components on hand in that store, plus all components on hand at and in transit to any downstream installation, no matter whether these components have been used already in assembled products, minus possible backlogs. The echelon inventory position finally denotes the echelon stock plus the materials already ordered but not yet available at the highest (most upstream) installations. When an echelon consists of one installation, this definition coincides with the one given in Silver and Peterson[1985]. Echelons are numbered according to the highest installation in that echelon.

Consider first a serial system which consists of N installations. Installation 1 is the most downstream installation from which products are sold to the market. Products present at installation n or in transfer between echelon n and echelon n-1 are charged at a rate \( \sum_{k=n}^{N} (h_k) \) per unit of product per period. A penalty cost p per unit of product and per period is incurred if the lowest echelon 1 is unable to meet market demand. Both penalty and inventory costs are charged at the end of a period.

Let \( F_{\ell} \) denote the distribution of the \( \ell \)-period cumulative demand, for all \( \ell \). If \( \ell = 1 \), we suppress the index. Furthermore, \( l_n \) denotes the delivery leadtime for goods ordered by installation n from the preceding installation if these goods are available \( (n-1, 2, \ldots, N-1) \) or from the (infinite capacity) initial supplier \( (n=N) \). Now, transform the above defined costs into echelon holding and penalty cost functions, as follows:

\[
L_1(x_1) = \sum_{n=1}^{N} (h_n) \int_{0}^{x_1} (x_1-u) dF(u) + p \int_{x_1}^{\infty} (u-x_1) dF(u) - \sum_{n=2}^{N} h_n x_1 \quad \text{if } x_1 \geq 0,
\]

\[
L_1(x_1) = p \int_{0}^{x_1} (u-x_1) dF(u) - \sum_{n=2}^{N} h_n x_1 \quad \text{if } x_1 < 0,
\]

\[
L_n(x_n) = h_n x_n \quad \text{for all } x_n \ (n = 2, \ldots, N).
\]

This transformation of traditional costs into echelon cost functions proves to be fundamental in decomposing the system. We have (Langenhoff and Zijm[1989]):
Theorem 2.1. Consider a policy which, at the beginning of every period, increases the echelon inventory position of echelon \( n \) to \( y_n \) \((n = 1, 2, \ldots, N)\). Let \( D^{(N)}(y_1, y_2, \ldots, y_N) \) be the associated average costs (which is defined only on \( \{(y_1, y_2, \ldots, y_N) \mid y_1 \leq y_2 \leq \cdots \leq y_N \} \)). We have

\[
D^{(N)}(y_1, y_2, \ldots, y_N) = C_1(y_1) + \ldots + C_N(y_1, \ldots, y_N)
\]

where

\[
C_1(y_1) = -\int_0^\infty L_1(y_1 - u_1) dF_1(u_1)
\]

\[
C_n(y_1, \ldots, y_n) = -\int_0^\infty L_n(y_n - u_n) dF_n(u_n) + \\
\int_{y_n - y_{n-1}}^\infty \left[ C_{n-1}(y_1, \ldots, y_{n-2}, y_n - u_n) - C_{n-1}(y_1, \ldots, y_{n-2}) \right] dF_n(u_n)
\]

for \( n = 2, \ldots, N \).

The second term in the definition of \( C_n \) is in fact the extra penalty incurred by installation \( n \) due to its inability to completely satisfy the demand of installation \( n-1 \). The restriction to the area \( \{(y_1, y_2, \ldots, y_N) \mid y_1 \leq y_2 \leq \cdots \leq y_N \} \) reflects the fact that the echelon inventory position of installation \( n \) can never be smaller than the echelon inventory position of installation \( n-1 \). Now define for convenience

\[
D^{(n)}(y_1, \ldots, y_n) = C_1(y_1) + \ldots + C_n(y_1, \ldots, y_n) \quad n = 1, 2, \ldots, N.
\]

Then the following procedure yields the global minimum of \( D^{(N)}(y_1, \ldots, y_N) \) in the area \( \{(y_1, y_2, \ldots, y_N) \mid y_1 \leq y_2 \leq \cdots \leq y_N \} \).

**Step 1.** (Initialization). \( n := 1 \). Minimize \( D^{(1)}(y_1) \). Let \( S_1 \) denote the value that minimizes \( D^{(1)}(y_1) \).

**Step 2.** \( n := n+1 \). If \( n > N \) stop.

Let \( (S_1, \ldots, S_n) \) minimize \( D^{(n-1)}(y_1, \ldots, y_{n-1}) \). Minimize next \( D^{(n)}(S_1, \ldots, S_n, y_n) \) and let \( S_n \) denote the corresponding minimizing value. If \( S_n \geq S_{n-1} \) then goto 2.
Step 3. Let $k$ be the smallest index such that $S_k > S_n$. Set $S_m := S_n$ for $m = k, \ldots, n$. Goto 2.

Hence, the global minimum of $D^{(N)}(y_1, \ldots, y_N)$ can be found by subsequently minimizing a number of one-dimensional functions. The proof of this statement is based on the convexity of the functions $D^{(n)}(S_1, \ldots, S_{n-1}, y_n)$ $(n = 1, \ldots, N)$. Moreover, it can be shown that the average cost optimal policy is contained in the class of policies mentioned in theorem 2.1, and hence that this average cost optimal policy corresponds with the minimum of $D^{(N)}(y_1, \ldots, y_N)$. This is summarized in the following theorem (Langenhoff and Zijm[1989]):

**Theorem 2.2.** The procedure outlined above yields the global minimum of $D^{(N)}(y_1, \ldots, y_N)$ in a finite number of steps. The associated policy (which in every period increases the echelon inventory position of echelon $n$ to $S_n$) is average cost optimal for the infinite horizon problem.

This concludes our review of basic results for serial systems. Next, we consider a production system in which several components are assembled into a single end item (cf. fig 1). Without loss of generality we may assume that only one component of each type is needed to assemble one product. Components are delivered by (infinite capacity) outside suppliers where a supply leadtime $l_i$ is needed for a component of type $i$. Let $l_0$ denote the assembly leadtime. The fact that final products can be assembled only if components of all types are available in sufficient quantities clearly demands for coordinated ordering of the different component types.

![Diagram](image)

*Fig. 1. An assembly system with different supply leadtimes.*
As before, only final products are subject to outside demand. Components in the system (in stock at the component store or as part of work-in-process in the assembly phase) are subject to a holding cost $h_i$ (for component type $i$), final products are stored at a holding cost $h_0 + \sum_{n=1}^{N} h_n$, while a penalty $p$ is incurred if demand cannot be met immediately and has to be backlogged. All costs are calculated at the end of a period.

Recall that the echelon stock of a component includes components already assembled in end items stored in the final product warehouse. Let $x_i$ denote the echelon stock of echelon $i$, for $i = 0, 1, \ldots, N$ (where a negative value denotes a backlog again). Define

$$L_0(x_0) = (h_0 + \sum_{n=1}^{N} h_n) \int_{0}^{x_0} (x_0 - u)dF(u) + \sum_{n=1}^{N} h_n x_0 \text{ if } x_0 \geq 0,$$

$$L_0(x_0) = \int_{0}^{x_0} (u-x_0)dF(u) - \sum_{n=1}^{N} h_n x_0 \text{ if } x_0 < 0,$$

$$L_n(x_n) = h_n x_n \text{ for all } x_n \ (n = 1, \ldots, N).$$

As in the case of the serial system, there is also for assembly systems a class of policies which deserves special attention. The structure of these policies however is less obvious than in the serial case and therefore will be discussed in some detail.

Without loss of generality we may assume $\ell_1 < \ell_2 < \ldots < \ell_N$ (since all leadtimes are deterministic we may treat components with equal order leadtime as one "aggregate" component). We will consider policies that are characterized by so-called potential order-up-to levels $y_n$ for each component type $n$ ($n = 1, \ldots, N$). Now suppose that, at the beginning of period $t$, the inventory position of components of type $N$ is increased to $y_N$. Let $u_{N-1}^N$ denote the outside demand (translated in terms of components of type $N$) between $t$ and $t+\ell_N^N - \ell_{N-1}^N$. Then it does not make sense to increase the inventory position of components of type $N-1$ to a level higher than $y_N - u_{N-1}^N$, at time $t+\ell_N^N - \ell_{N-1}^N$, since any larger order would result in temporarily useless stock of components of type $N-1$ at time $t+\ell_N^N$ (note that the echelon stocks of both component types are subject to the same demand pattern between time $t+\ell_N^N - \ell_{N-1}^N$ and time $t+\ell_N^N$).

On the other hand, we may consider an independent order-up-to level $y_{N-1}$ (also called a potential order-up-to level) which serves as an upper bound for the
inventory position of component type N-1. Therefore, we increase the inventory position of component N-1 to \( \min(y_{N-1}, y_{N-1} - u_{N-2} - u_{N-1}) \) at time \( t + l_{N-1} \).

More general, at time \( t + l_{N-k} \), we increase the echelon inventory position of component \( k \) to \( \min(y_k, y_{k+1} - u_{k+1} - l_k, \ldots, y_N - u_{N} - l_N) \). Here \( u_{N-k} - l_k \) denotes the outside demand (in terms of components of type \( m \)) between \( t + l_{N-k} \) and \( t + l_{N-k} \), for \( m = k+1, \ldots, N \). At time \( t + l_{N} \), finally, we decide to increase the echelon inventory position of final products to a level \( y_0 \), say. However, if the echelon stock of any component is smaller than \( y_0 \), i.e. if

\[
\min(y_N - u_N, y_{N-1} - u_{N-1}, \ldots, y_1 - u_1) < y_0
\] (4.1)

we simply assemble as much as possible while furthermore a backlog occurs at some of the component stores.

Let \( D^{(N)}(y_0, y_1, \ldots, y_N) \) denote the resulting costs of these \( N+1 \) decisions (at time points \( t, t + l_{N-1}, \ldots, t + l_1 \) and \( t + l_N \)) which arise at the end of period \( t + l_N \). It is not hard to see that we must have \( y_0 \leq y_1 \leq \cdots \leq y_N \). The following result now corresponds to theorem 2.1 (Langenhoff and Zijm[1989]).

**Theorem 2.3.** Let \( \ell_1 < \ell_2 < \ldots < \ell_N \). Then

\[
D^{(N)}(y_0, y_1, \ldots, y_n) = C_0(y_0) + C_1(y_0, y_1) + \ldots + C_N(y_0, y_1, \ldots, y_N)
\]

\[
C_0(y_0) = \int_0^\infty L_0(y_0 - u_0) dF_{\ell_0}(u_0),
\]

\[
C_1(y_0, y_1) = \int_0^\infty L_1(y_1 - u_1) dF_{\ell_1}(u_1) + \int_0^\infty [C_0(y_1 - u_1) - C_0(y_0)] dF_{\ell_1}(u_1)
+ \int_0^\infty [C_0(y_0 - u_1) - C_0(y_0) - y_1 - y_0] dF_{\ell_1}(u_1)
\]

\[
C_n(y_0, y_1, \ldots, y_n) = \int_0^\infty L_n(y_n - u_n) dF_{\ell_n}(u_n) + \int_0^\infty [C_{n-1}(y_0, y_1, \ldots, y_{n-2}, y_{n-1} - u_{n-1} - l_{n-1}) - C_{n-1}(y_0, y_1, \ldots, y_{n-2}, y_{n-1})] dF_{\ell_n}(u_n - l_{n-1}) (n=2, \ldots, N).
\]
A comparison of theorem 2.3 and theorem 2.1 suggests a strong similarity of the parallel system with a serial system with leadtimes $\ell_n - \ell_{n-1}$ ($n = 2, \ldots, N$), $\ell_1$ and $\ell_0$. However, the reader may note that the first term in the definition of $C_n(y_0, y_1, \ldots, y_n)$ is not equal to

$$\int_0^\infty L_n(y_n - u_{\ell_n - \ell_{n-1}}) dF_{\ell_n - \ell_{n-1}}(u_{\ell_n - \ell_{n-1}})$$

For an explanation of the similarities as well as the differences between serial and parallel systems we refer to Langenhoff and Zijm [1989]. However, the structure of the function $D^{(N)}(y_0, y_1, \ldots, y_N)$ in theorem 2.3 suggests a minimization procedure similar to the one given for the serial system. Moreover, it can be shown that the order-up-to levels $y_N$, $y_{N-1}$, $\ldots$, $y_1$, $y_0$, together with the decision structure outlined above, also characterize the structure of an optimal policy for the infinite horizon problem under an average cost criterion. The last theorem of this section summarizes these results.

**Theorem 2.4.** The function $D^{(N)}(y_0, y_1, \ldots, y_N)$, which is defined on the area $(y_0, \ldots, y_N | y_0 \leq \ldots \leq y_N)$, can be minimized by subsequently minimizing a series of one-dimensional convex functions, by using the procedure described above for serial systems. Let the global minimum of $D^{(N)}(y_0, y_1, \ldots, y_N)$ be reached in $(S_0, S_1, \ldots, S_N)$. Then an average cost optimal policy for the infinite horizon problem can be formulated as follows:

- at each decision moment, increase the inventory position
- of echelon 0 to $S_0$,
- of echelon $N$ to $S_N$,
- of echelon $k$ to $\min(S_k, S_{k+1} - \ell_{k+1} - \ell_k, \ldots, S_N - \ell_N - \ell_k)$, for $k = 1, \ldots, N-1$.

Here $u_{\ell_m - \ell_k}$ denotes the outside demand (translated in terms of components of type m) in the last $\ell_m - \ell_k$ periods, for $k+1 \leq m \leq N$, $k = 2, \ldots, N$.

Concluding, for convergent assembly systems it is again possible to decompose the system into a series of single echelon problems, even though several component types run parallel through the system. This concludes our summary of basic results on assembly systems.
Both for serial and assembly systems, the above reviewed decomposition results leave us with the problem to minimize one-dimensional convex functions. Unfortunately, these functions are only recursively defined and involve incomplete convolutions of distribution functions (as will be explained in detail in the forthcoming sections). Therefore, the minimization of these functions is far from trivial. The rest of this paper is devoted to the development of (fast) procedures for solving these minimization problems. Combined with the above outlined decompositions, these procedures enable us to determine optimal policies in fairly complex networks in just a few seconds on a microcomputer.

For the sake of completeness we note that decomposition results can also be established for distribution systems (under a so-called balance assumption which states that, if a central depot is unable to satisfy all local warehouse demand, it is still possible to supply these warehouses such that they face equal stockout probabilities, see also Eppen and Schrage[1981]). An analysis along the same lines as given above can be found in Langenhoff and Zijm[1989]. However, the average cost functions arising in these systems are even more complex than those arising in serial and assembly systems. Therefore, they will be treated separately. For the rest of this paper, we restrict ourselves to production systems with a serial or an assembly structure.

3. Approximations of distributions and incomplete convolutions

In this section it is shown how to fit mixtures of Erlang distributions to the first two moments of any general distribution. Note that mixtures of Erlang distributions with the same scale parameter can be used to approximate any distribution function arbitrarily close (Schassberger[1973]).

Let $x$ be a random variable with distribution function $F(x)$, where we assume $F(x) = 0$ for $x \leq 0$. Let the first two moments of $x$ be given by $\mu_{1,x}$ and $\mu_{2,x}$ respectively. Define the coefficient of variation $c_x$ by

$$c_x^2 = (\mu_{2,x} - \mu_{1,x}^2)/\mu_{1,x}^2.$$
If $c_x^2 \leq 1$ then we approximate the distribution function of $x$ by a mixture of an Erlang-$(k-1)$ and an Erlang-$k$ distribution (which is unimodal), which has probability density

$$f(x) = p\lambda^{k-1} \frac{x^{k-2}}{(k-2)!} e^{-\lambda x} + (1-p)\lambda^k \frac{x^{k-1}}{(k-1)!} e^{-\lambda x}$$

with $k$ chosen such that $1/k \leq c_x^2 \leq 1/(k-1)$, and $p$ and $\lambda$ defined by

$$p = \frac{kc_x^2 - (k(1+c_x^2)-k^2 c_x^2)^{1/2}}{(1+c_x^2)}$$

$$\lambda = \frac{(k-p)/\mu_1}{c_x}.$$ 

If $c_x^2 > 1$ then we approximate the distribution function of $x$ with a hyperexponential distribution with the following density function:

$$f(x) = p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}$$

with $\lambda_1$, $\lambda_2$ and $p$ defined by

$$\lambda_1 = \frac{2}{\mu_1 x} \{1 + [(c_x^2-0.5)/(c_x^2+1)]^{1/2}\},$$

$$\lambda_2 = \frac{4}{\mu_1 x} - \lambda_1,$$

$$p = \lambda_1(\lambda_2^{-1} - 1)/(\lambda_2^{-1} - \lambda_1).$$

If the original distribution is a Gamma distribution then the hyperexponential fit has even the same first three moments as this original distribution. The above results are easily verified (cf. Tijms[1986]).

Now define for $a \geq 0$ the distribution $F^a(y)$ by

$$F^a(x) = F(x+a) \text{ if } x \geq 0,$$

$$F^a(x) = 0 \text{ if } x < 0.$$ 

In the next section we will have to evaluate distributions of the form $F^a \ast G$, where $G$ is a continuous distribution function on $[0,\infty)$ with $G(0) = 0$, and $\ast$ denotes the usual convolution operator of the distribution functions of
two independent random variables. The function \((F^a \ast G)(x)\) is also called an incomplete convolution because of the alternative representation

\[
(F^a \ast G)(x) = \int_0^x F(x+a-u)dG(u), \tag{3.1}
\]

which is easily shown by partial integration and by exploiting the definition of \(F^a(x)\). In order to evaluate these incomplete convolutions we now proceed as follows.

Consider first the case in which \(c^2_x \leq 1\) and let \(F(x)\) be approximated by an appropriately chosen mixture of an Erlang-(k-1) and an Erlang-k distribution. Then it is not hard to verify that the first two moments of the stochastic variable with distribution function \(F^a, X^a\) say, can be approximated by

\[
\mu_1, x^a = pA(k-1, \lambda, a) + (1-p)A(k, \lambda, a),
\]

\[
\mu_2, x^a = pB(k-1, \lambda, a) + (1-p)B(k, \lambda, a),
\]

where \(A(k, \lambda, a)\) and \(B(k, \lambda, a)\) are given by

\[
A(k, \lambda, a) = -a(1-G_{k-1}(a)) + (k/\lambda)(1-G_k(a)),
\]

\[
B(k, \lambda, a) = a^2(1-G_{k-2}(a)) - (2ak/\lambda)(1-G_{k-1}(a)) + (k(k+1)/\lambda^2)(1-G_k(a))
\]

and \(G_k(a)\) the Erlang-k distribution function, i.e.

\[
G_k(a) = 1 - \sum_{j=0}^{k-1} \frac{\lambda^j a^j}{j!} e^{-\lambda a}, \quad \text{for } a \geq 0.
\]

If on the other hand \(c^2_x > 1\) and \(F(x)\) is approximated by a hyperexponential distribution, the first two moments of \(X^a\) are approximated by

\[
\mu_1, x^a = \left( p/\lambda_1 \right) e^{-\lambda_1 a} + \left( (1-p)/\lambda_2 \right) e^{-\lambda_2 a},
\]

\[
\mu_2, x^a = \left( 2p/\lambda_1^2 \right) e^{-\lambda_1 a} + \left( 2(1-p)/\lambda_2^2 \right) e^{-\lambda_2 a}.
\]

From these two moments and the first two moments of the random variable with distribution function \(G, Y\) say, the first two moments of the convolution
of $x^a$ and $y$ is derived immediately. Note that this latter random variable is again continuous, with distribution function $F^a \ast G$ while $(F^a \ast G)(0) = 0$. We now again fit a mixture of two Erlang distributions with the same scale parameter or a hyperexponential distribution to the first two moments of $F^a \ast G$ (which have just been calculated) and use this approximation instead of the true distribution $F^a \ast G$.

The procedure just described has been tested extensively by De Kok and Seidel[1990], with very satisfactory results. We will come back to this issue in the next section. The above procedure can be used to define an approximation scheme for distribution functions of the type

\[(F_{\ell_1}^a \ast F_{\ell_0}^a) \ast F_{\ell_0}^{a-1} \ast F_{\ell_0}^{a-2} \ast \ldots \ast F_{\ell_2}^a \ast F_{\ell_1}^{a+1}(x)\]

which appear to arise naturally in multi-echelon serial systems (where the $a_i$'s are positive real numbers and the $\ell_i$'s are leadtimes again). The approximation scheme works as follows.

From the first two moments of $F_{\ell_1}^a$ and the approximation by means of either a mixture of Erlang distributions with the same scale parameter or a hyper-exponential distribution we may calculate the first two moments of $F_{\ell_1}^a$ and hence of $F_{\ell_1}^a \ast F_{\ell_0}^a$. After approximating this latter distribution function again we may calculate the first two moments of $(F_{\ell_1}^a \ast F_{\ell_0}^a)^{a-1}$, etc. This approximation scheme will be used extensively in the next two sections, in which we return to multi-echelon production systems.

4. The two-echelon serial system

In order to demonstrate the calculation of the optimal control strategies in a multi-echelon system we first analyze the serial system, consisting of two installations. Recall that the optimal order-up-to levels $S_1$ and $S_2$ have to be found by minimizing

\[D(y_1, y_2) = C_1(y_1) + C(y_1, y_2).\]
and that these values $S_1$ and $S_2$ are also the minimizing values of $C_1(y_1)$ and $C_2(S_1,y_2)$ respectively. It is not hard to verify (Langenhoff and Zijm[1989]) that (note that $F_{\xi_1+1}(y) = 0$ and $F_{\xi_2}(y) = 0$ for $y < 0$)

\[ C_1'(y_1) = \frac{\partial}{\partial y_1} C_1(y_1) = -(h_2 + p) + (h_1 + h_2 + p) F_{\xi_1+1}(y_1), \quad (4.1) \]

\[ \frac{\partial}{\partial y_2} C_2(S_1,y_2) = h_2 + \int_{y_2}^{\infty} C_1'(y_2 - u_2) dF_{\xi_2}(u_2). \]

A further investigation of these expressions leads to

**Lemma 4.1.** Let $S_1$ be the minimizing value of $C_1(y_1)$. Then

\[ \frac{\partial}{\partial y_2} C_2(S_1,y_2) = -p + (h_1 + h_2 + p)(F_{\xi_2}^y)^{-1}(S_1) \quad \text{for} \quad y_2 \geq S_1, \quad (4.2.a) \]

\[ \frac{\partial}{\partial y_2} C_2(S_1,y_2) = -p + (h_1 + h_2 + p)(F_{\xi_2}^y)^{-1}(y_2) \quad \text{for} \quad y_2 < S_1. \quad (4.2.b) \]

**Proof.** For $y_2 < 0$ we have

\[ \frac{\partial}{\partial y_2} C_2(S_1,y_2) = h_2 + \int_{0}^{\infty} C_1'(y_2 - u_2) dF_{\xi_2}(u_2) = h_2 + \int_{0}^{\infty} [-(h_2 + p)] dF_{\xi_2}(u_2) - h_2 - (h_2 + p) = -p. \]

For $0 \leq y_2 < S_1$ we may write

\[ \frac{\partial}{\partial y_2} C_2(S_1,y_2) = h_2 + \int_{0}^{y_2} C_1'(y_2 - u_2) dF_{\xi_2}(u_2) + \int_{y_2}^{\infty} C_1'(y_2 - u_2) dF_{\xi_2}(u_2) = h_2 + \int_{0}^{y_2} [-(h_2 + p) + (h_1 + h_2 + p) F_{\xi_1+1}^y](y_2 - u_2) dF_{\xi_2}(u_2) \]

\[ + \int_{y_2}^{\infty} [-(h_2 + p)] dF_{\xi_2}(u_2) = h_2 - (h_2 + p) + (h_1 + h_2 + p) \int_{0}^{y_2} F_{\xi_1+1}^y(y_2 - u_2) dF_{\xi_2}(u_2). \]
\[-p + (h_1 + h_2 + p)(F_{k_2} S_{k_1,1} + 1)(y_2) .\]

Finally, for \( y_2 \geq S_1 \) we may write
\[
\frac{\partial}{\partial y_2} C_2(S_1, y_2) = h_2 + \int_{y_2 - S_1}^{\infty} C_1'(y_2 - u_{k_2}) dF_{k_2}(u_{k_2}) -
\]
(\text{use } u_{k_2} = u_{k_2} - (y_2 - S_1))

\[
= h_2 + \int_0^{\infty} C_1'(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2} + (y_2 - S_1)) -
\]

\[
= h_2 + \int_0^{\infty} C_1'(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2}) -
\]

(\text{use } C_1'(S_1) = 0)

\[
= h_2 + \int_0^{\infty} C_1'(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2}) -
\]

\[
= h_2 + \int_0^{S_1} C_1'(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2}) + \int_{S_1}^{\infty} C_1'(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2}) -
\]

\[
= h_2 + \int_0^{S_1} \left[-(h_2 + p) + (h_1 + h_2 + p)F_{k_1,1} + 1(S_1 - \hat{u}_{k_2}) \right] dF_{k_2}(\hat{u}_{k_2}) +
\]

\[
\int_{S_1}^{\infty} \left[-(h_2 + p) \right] dF_{k_2}(\hat{u}_{k_2}) -
\]

\[
= h_2 - (h_2 + p) + (h_1 + h_2 + p) \int_0^{S_1} F_{k_1,1}(S_1 - \hat{u}_{k_2}) dF_{k_2}(\hat{u}_{k_2}) -
\]

\[
= -p + (h_1 + h_2 + p)(F_{k_2} S_{k_1,1} + 1)(S_1) .
\]

\[\Box\]

It is easily seen from lemma 4.1 that \( \frac{\partial}{\partial y_2} C_2(S_1, y_2) \) is indeed a continuous, monotone non-decreasing function of \( y_2 \). For completeness, we also express \( D^{(2)}(S_1, S_2) \) in terms of \( F_{k_2} S_{k_1,1} \) and \( F_{k_1,1} + 1 \). We have
Lemma 4.2. The average cost function $D^{(2)}(S_1, S_2)$ satisfies

$$D^{(2)}(S_1, S_2) = h_2 \left[ \int_{0}^{\infty} (S_2-u) dF_{\hat{\xi}_2}(u) - \int_{0}^{\infty} (S_1-u) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1})(u) \right] +$$

$$+ (h_1 + h_2) \int_{0}^{S_1} (S_1-u) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1+1})(u) + p \int_{S_1}^{\infty} (u-S_1) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1+1})(u)$$

Proof. By applying the same transformation as in the last part of the proof of lemma 4.1 it is easy to show that

$$D^{(2)}(S_1, S_2) = h_2 \left[ \int_{0}^{\infty} (S_2-u \hat{\xi}_2) dF_{\hat{\xi}_2}(u \hat{\xi}_2) \right] +$$

$$+ \int_{0}^{\infty} \left( C_1(S_1-u \hat{\xi}_2) dF_{\hat{\xi}_2}^{S_2-S_1}(u \hat{\xi}_2) \right)$$

For the second expression at the right hand side of this equation we may write

$$\int_{0}^{\infty} C_1(S_1-u \hat{\xi}_2) dF_{\hat{\xi}_2}^{S_2-S_1}(u \hat{\xi}_2) =$$

$$= (h_1 + h_2) \int_{0}^{S_1} (S_1-u) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1+1})(u) + p \int_{S_1}^{\infty} (u-S_1) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1+1})(u)$$

$$- \int_{0}^{\infty} \left[ \int_{0}^{\infty} h_2 (S_1-u \hat{\xi}_2-\hat{\xi}_1) dF_{\hat{\xi}_1}(u \hat{\xi}_1) \right] dF_{\hat{\xi}_2}^{S_2-S_1}(u \hat{\xi}_2)$$

Since the last term of the latter expression equals

$$- h_2 \int_{0}^{\infty} (S_1-u) d(F_{\hat{\xi}_2}^{S_2-S_1} \cdot F_{\hat{\xi}_1})(u)$$

the result now follows immediately.

Next we show how to calculate the order-up-to levels $S_1$ and $S_2$ from (4.1) and (4.2). We use the procedure given in section 2. Note that, by definition, $D^{(1)}(y_1) = C_1(y_1)$ while $\frac{\partial}{\partial y_2} D^{(2)}(S_1, y_2) = \frac{\partial}{\partial y_2} C_2(S_1, y_2)$. If $F_{\hat{\xi}_1+1}(y_1)$ is given we determine $S_1$ directly from $C_1'(y_1) = 0$ (using (4.1)), otherwise we match either a mixture of Erlang distributions with the same scale parameter or a
hyperexponential distribution to the first two moments of the \((l_1+1)\)-period demand (compare section 3). Call this approximative distribution \(G_{l_1+1}(y_1)\) and determine next \(S_1\) from \(C_1(y_1) = 0\) with \(F_{l_1+1}(y_1)\) replaced by \(G_{l_1+1}(y_1)\).

To calculate \(S_2\) we first determine the sign of \(\frac{\partial}{\partial y_2} C_2(S_1,S_1)\) (use (4.2.b)). If \(\frac{\partial}{\partial y_2} C_2(S_1,S_1) \geq 0\), we have \(y_2 \leq S_1\) and we may solve the equation

\[-p + (h_1+h_2+p)(F_{l_2} \ast F_{l_1+1})(y_2) = 0\]

to obtain \(S_2\) (where we again may use an approximative distribution). If however \(\frac{\partial}{\partial y_2} C_2(S_1,S_1) < 0\) then we have to determine \(S_2\) from

\[-p + (h_1+h_2+p)(F_{l_2} \ast F_{l_1+1})(S_1) = 0\]

(with \(y_2\) the unknown parameter) by using e.g. a bisection method (note that \(F_{l_2} \ast F_{l_1+1})(S_1)\) is monotone in \(y_2\)). In each bisection step (i.e. for every choice of \(y_2\)), we fit a distribution to the first two moments of \(F_{l_2} \ast F_{l_1+1}\) by using the approximation described in section 3. This bisection method finally yields the solution \(y_2 = S_2\). The (approximately) optimal policy is now the policy characterized by the parameters \((S_1, S_2)\).

In all cases tested the resulting value of \(S_2\) deviated less than 0.5 % from the exact value found by a discretization procedure. Hence, the approximation procedure described in section 3 yields results accurate enough for our purposes. However, the results can even be improved by applying three moment fits. Details are left to the reader.

5. Analysis of general multi-echelon production systems

In this section we show how to calculate optimal order-up-to levels in general multi-echelon serial or assembly systems. Consider first a \(N\)-echelon serial system as described in section 2, with leadtimes \(l_N, l_{N-1}, \ldots, l_1\). Define

\[H_1(y) = F_{l_1+1}(y),\]
\[ H_n^*(y) = (F_{n-1} \ast \ldots \ast F_2 \ast F_{n+1})(y) \]

and for \(2 \leq k \leq n\) (with again \(n \geq 2\))

\[ H_k^n(y) = ((\ldots ((F_{n-1} \ast \ldots \ast F_{k+1})^y S_{k-1} \ast F_{k-1})^y S_{k-2} \ast \ldots \ast F_2)^y S_1 \ast F_{n+1})(y) \]

where \(\ast\) denotes the convolution operator again. Finally, let

\[ J_k^n(y) = -(p + \sum_{j=n+1}^N h_j) + (p + \sum_{j=1}^N h_j)(H_k^n(y)) \]

The following generalization of lemma 4.1 is stated without proof:

**Theorem 5.1.** Let, for each \(n\) with \(1 \leq n \leq N\), \(D^{(n-1)}(y_1, \ldots, y_{n-1})\) take its absolute minimum value in \((5, 1, 5_2, \ldots, S_{n-1})\). Then the partial derivatives of \(D^{(n)}(S_1, \ldots, S_{n-1}, y_n)\) to \(y_n\) satisfy

\[ \frac{\partial}{\partial y_n} D^{(n)}(S_1, \ldots, S_{n-1}, y_n) = J_n^{(n)}(y_n) \quad \text{for } y_n > S_{n-1}, \]

\[ \frac{\partial}{\partial y_n} D^{(n)}(S_1, \ldots, S_{n-1}, y_n) = J_k^{(n)}(y_n) \quad \text{for } S_{k-1} < y_n \leq S_k, \quad k = 2, \ldots, n-1, \]

\[ \frac{\partial}{\partial y_n} D^{(n)}(S_1, \ldots, S_{n-1}, y_n) = J_1^{(n)}(y_n) \quad \text{for } y_n \leq S_1. \]

\[\square\]

Theorem 5.1 is easily proved by induction. Hence, the procedure described in section 2 can be applied to determine the values \(S_1, S_2, \ldots, S_N\) if we are able to evaluate the functions \(J_k^{(n)}(y_n)\) or \(H_k^n(y_n)\). However, for each value of \(y_n\) we may approximate \(H_k^n(y_n)\) by successive fits of mixtures of Erlang distributions with the same scale parameter (or hyperexponential distributions) as described at the end of section 3. Below, we describe how to select successive trial values of \(y_n\).

Suppose we wish to determine \(y_n = S_n\) as the solution of

\[ \frac{\partial}{\partial y_n} D^{(n)}(S_1, \ldots, S_{n-1}, y_n) = 0, \]

where we now suppose \(n \geq 2\) (the case \(n = 1\) is easy, we simply have to solve \(J_1^1(y) = 0\)). The following procedure yields \(S_n\).
Step 1. Set \( k = n \).
Step 2. Determine \( H_k^n(S_{k-1}) \) and subsequently \( J_k^n(S_{k-1}) \). If \( J_k^n(S_{k-1}) < 0 \) then go to Step 3, otherwise set \( k := k - 1 \). If \( k = 1 \) then goto Step 3, otherwise repeat Step 2.
Step 3. Apply a bisection method to solve \( J_k^n(y_n) = 0 \), yielding \( y_n = S_n \). Stop.

If step 3 is entered with \( k = n \) then we know that \( S_n > S_{n-1} \). If \( 2 \leq k < n \) then \( S_{k-1} < S_n \leq S_k \) and if \( k = 1 \) then \( S_n \leq S_1 \). In all cases the right function \( J_k^n(y_n) \) is evaluated.

The above procedure should be used to execute the second step in the procedure of section 2, for every \( n \). This solves the serial system.

For completeness, we also state the analogon of lemma 4.2. Define recursively

\[
B_k(u) = F_k(u) \quad \text{for } u \geq 0,
\]

\[
B_{k-1}(u) = (B_k^S - S_{k-1} \ast F_{k-1})(u) \quad \text{for } 2 \leq k \leq N, u \geq 0,
\]

\[
\hat{B}_1(u) = (B_1 \ast F)(u) \quad \text{for } u \geq 0.
\]

Finally, let

\[
\beta_k = \int_{0^-}^{\infty} u dB_k(u).
\]

Then the following result is easily shown by induction.

**Theorem 5.2.** The average cost function \( D^{(N)}(S_1, S_2, \ldots, S_N) \) satisfies

\[
D^{(N)}(S_1, S_2, \ldots, S_N) = \sum_{n=2}^{N} \left( \sum_{j=n}^{n-1} h_j \right) \left[ (S_n - \beta_n) - (S_{n-1} - \beta_{n-1}) \right] + \sum_{j=1}^{N} h_j \int_{S_1}^{\infty} (S_1 - u) d\hat{B}_1(u) + p \int_{S_1}^{\infty} (u - S_1) d\hat{B}_1(u).
\]
The analogon of theorem 5.1 and the associated calculation scheme for the order-up-to levels to assembly systems with one final product and N components is rather straightforward. We have indicated already the similarity between an assembly system and a serial system with leadtimes \( l_n - l_{n-1} \) \( n = 2, \ldots, N \), \( l_1 \) and \( l_0 \). And although the first term of \( C_n(y_0, y_1, \ldots, y_n) \) in an assembly system differs from the first term of \( C_n(y_0, y_1, \ldots, y_n) \) in a serial system with leadtimes \( l_n - l_{n-1} \) for \( n \geq 2 \) (compare theorem 2.1 and theorem 2.3) this difference completely disappears when taking derivatives. In other words: the functions

\[
\frac{\partial}{\partial y_n} C_n(y_0, y_1, \ldots, y_{n-1}, y_n)
\]

\( n = 0, 1, \ldots, N \),

in an assembly system are completely equal to those in a serial system with leadtimes \( l_n - l_{n-1} \) \( n = 2, \ldots, N \), \( l_1 \) and \( l_0 \). This follows immediately from the definitions of these functions (cf. theorem 2.1 and theorem 2.3). The same then obviously holds also for the functions

\[
\frac{\partial}{\partial y_n} D^{(n)}(y_0, y_1, \ldots, y_{n-1}, y_n)
\]

\( n = 0, 1, \ldots, N \).

As a result, the order-up-to levels in an assembly system can be found by applying theorem 5.1 and the approximation scheme outlined above to the serial system with leadtimes \( l_n - l_{n-1} \) \( n = 2, \ldots, N \), \( l_1 \) and \( l_0 \). This concludes our discussion of assembly systems.

6. Conclusions and suggestions for future research

In this paper, we have reviewed our theoretical analysis of multi-echelon production systems, in particular the fundamental decomposition results for serial and assembly systems. These (exact) decompositions leave us with the problem to determine numerically order-up-to levels in complex one-dimensional inventory systems. An approximation scheme has been defined to solve this latter problem. For systems which are not too large (4 to 5 echelons), numerical tests have shown a performance which is surprisingly accurate. In particular, the order-up-to levels proved to be sufficiently close to their true values to justify application of the approximation.
Our first goal will be to extend the approximation scheme to distribution systems as well. Although these systems satisfy similar decomposition properties as serial and assembly systems (but under an additional so-called balance assumption, see the end of section 2 or Langenhoff and Zijm[1989]), numerical difficulties arise if the central depot is unable to satisfy all demand from the local warehouses. The distribution function appears as the solution of a parametric nonlinear convex minimization problem and is not easily characterized in general, not even under a balance assumption (except for some very special demand distributions).

A second extension includes the assumption of finite capacities in some phases, both in distribution systems (in the first production phase) as well as in assembly systems (in the latter assembly phase and in some component fabrication phases). Here we exploit a result of Federgruen and Zipkin[1986] on the optimality of (S,S)-policies for single stage inventory systems without fixed costs but with finite capacity (under stationary stochastic demand). However, the calculation of the order-up-to levels in finite capacity systems is far from trivial again. It can be shown that the analysis resembles the optimization of certain parameters in a D/G/1 queue with a cost structure (where G may be a mixture of Erlang distributions with the same scale parameter again). This will be the topic of a forthcoming paper.

Further work will include multi-product environments as well as the discussion of nonstationary demand environments. In particular, we will focus on the combination of ideas from a more capacity oriented approach such as Hierarchical Production Planning (see e.g. Bitran, Haas and Hax[1981]) and the (stochastic) material coordination approach reviewed here.

References

Clark, A.J.[1958], A dynamic, single-item, multi-echelon inventory model, RM-2297, The Rand Corporation, Santa Monica,
Clark, A.J. and H. Scarf[1960], Optimal policies for a multi-echelon inventory problem, Management Science 6, pp. 475-490,
De Kok, A.G. and H. Seidel[1990], unpublished manuscript,


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