A Note on "The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling"

I.J.B.F. Adan
J. Wessels
W.H.M. Zijm

Eindhoven, May 1992
The Netherlands
Eindhoven University of Technology
Department of Mathematics and Computing Science
Probability theory, statistics, operations research and systems theory
P.O. Box 513
5600 MB Eindhoven - The Netherlands

Secretariate: Dommelbuilding 0.03
Telephone: 040-47 3130

ISSN 0926 4493
A Note on “The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling”

I.J.B.F. Adan, J. Wessels and W.H.M. Zijm

Abstract
In the paper entitled “The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling”, Kojima et al. derive an expression in the form of a product of powers for the state probabilities of a threshold-type shortest queue problem. In this note it is demonstrated that this expression is essentially more complicated and has the form of an infinite sum of products of powers. In fact, Kojima et al. find the first term in this infinite sum only.

1 Introduction
In [5] Kojima et al. consider a system consisting of two parallel servers with service rates \( \mu_1 \) and \( \mu_2 \) respectively, where \( \mu_1 \geq \mu_2 \). Jobs arrive according to a Poisson stream with rate \( \lambda \), where, to guarantee stability, \( \rho \equiv \lambda/(\mu_1 + \mu_2) < 1 \). If an arriving job finds \( n_1 \) jobs in queue 1 and \( n_2 \) jobs in queue 2, then the job joins queue 1 if \( n_1 < n_2 + m \) and queue 2 if \( n_1 > n_2 + m \). In case \( n_1 = n_2 + m \), then the job joins either queue with probability \( p_1 \) and \( p_2 = 1 - p_1 \) respectively. The integer \( m \) is called the threshold of the system. Jobs require exponentially distributed service times with unit mean. The service times are supposed to be independent. This system can be represented by a continuous-time Markov process, whose natural state space consists of the pairs \( (n_1, n_2) \) where \( n_1 \) and \( n_2 \) are the lengths of the two queues. Kojima et al. show that the state probabilities can be expressed as a product of powers. Unfortunately, this is incorrect, which can easily be verified for the case \( \mu_1 = \mu_2 \), \( p_1 = p_2 \) and \( m = 0 \) by considering the classical result of Kingman [4]. In this note it is demonstrated that, in fact, the expression of Kojima et al. is the first term of the solution, which has the form of an infinite sum of products of powers. The case \( m = 0 \) has been worked out in detail in [2]. The case \( m > 0 \) can be treated analogously. This will be sketched briefly in the next section.


1International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria.

2University of Twente, Department of Mechanical Engineering, p.o. Box 217, 7500 AE - Enschede, The Netherlands.
2 Analysis

Instead of \( n_1 \) and \( n_2 \) we shall use the variables \( m_1 = \min(n_1, n_2 + m) \) and \( m_2 = n_2 + m - n_1 \). So, \( m_1 \geq 0 \) and \( m_2 \geq m - m_1 \) if \( m_1 < m \) and \( m_2 \) may attain any integer value if \( m_1 \geq m \). Let \( \{ p(m_1, m_2) \} \) be the equilibrium distribution. The equilibrium equations can be obtained by equating in each state the rate into and the rate out of that state. In doing so, we distinguish between the following regions:

\[
\begin{align*}
& m_1 > 0, m_2 > 1, m_1 + m_2 > m; \\
& m_1 > m, m_2 < -1; \\
& m_1 \geq m, m_2 = 1; \\
& m_1 \geq m, m_2 = 0; \\
& m_1 \geq m, m_2 = -1; \\
& m_1 = 0, m_2 > m; \\
& m_1 = m, m_2 < -1; \\
& m_1 \geq 0, m_2 > 0, m_1 + m_2 = m.
\end{align*}
\]

In region (1) the following type of equation holds:

\[
p(m_1, m_2)(\lambda + \mu_1 + \mu_2) = p(m_1 - 1, m_2 + 1)\lambda + p(m_1, m_2 + 1)\mu_2 + p(m_1 + 1, m_2 - 1)\mu_1. \tag{9}
\]

The equations in other regions can be found similarly. The ones in region (8) and in the states \((m, 0)\) and \((m, -1)\) may be left out because of their minor importance to the analysis. The probabilities \( p(m_1, m_2) \) can be found by using a compensation approach. In [2] this approach has been worked out for the case \( m = 0 \). Since the case \( m > 0 \) can be treated analogously, we sketch this approach only and formulate the final results. For details and rigorous proofs the reader is referred to [1], [2] and [3].

The essence of the compensation approach is to characterize the products satisfying the conditions in the inner regions (1)-(2) and then to use these products to construct a linear combination which also satisfies the conditions on the boundaries (3)-(7). This construction is of a compensation-type. After introducing the first term, the approach consists of adding on terms so as to compensate alternately for errors on one of the boundaries and in doing so, builds up an infinite sum of terms. Products satisfying the conditions in region (1) are easily characterized. Substituting \( \alpha^{m_1} \beta^{m_2} \) into (9) and then dividing by the common factors, it readily follows that this product satisfies (9) iff

\[
\alpha \beta (\lambda + \mu_1 + \mu_2) = \beta^2 \lambda + \alpha \beta^2 \mu_2 + \alpha^2 \mu_1. \tag{10}
\]

A symmetrical characterization holds for products \( \alpha^{-m_1} \beta^{-m_2} \) satisfying the conditions in region (2). For fixed \( \alpha \) the quadratic equation (10) in \( \beta \) is solved by

\[
X(\alpha) = \alpha \frac{\lambda + \mu_1 + \mu_2 \pm \sqrt{(\lambda + \mu_1 + \mu_2)^2 - 4(\lambda + \alpha \mu_2)\mu_1}}{2(\lambda + \alpha \mu_2)}.
\]

The roots of (10) for fixed \( \beta \) are denoted by \( Y(\beta) \). Similarly \( x(\alpha) \) and \( y(\beta) \) are the roots of the quadratic equation in the lower region for fixed \( \alpha \) and \( \beta \) respectively. The products
satisfying the conditions in the inner regions are used to build up an infinite sum which also satisfies the conditions on the boundaries. Using the same arguments as in [2] we start with \(d_1 \alpha_0^m \beta_1^m\) in the upper region, \(d_2 \alpha_0^m \beta_2^m\) in the lower region and \(e_0 \alpha_0^m\) on the \(m_1\)-axis, where \(\alpha_0 = \rho^2\), \(\beta_1 = X_-(\alpha_0)\) and \(\beta_2 = x_-(\alpha_0)\). The terms satisfy the conditions in the inner regions (1)-(2) and \(d_1\), \(d_2\) and \(e_0\) are chosen such that the conditions on the boundary (3)-(5) are also satisfied. In fact, these initial terms are equivalent to solution (22) in [5] and describe the behaviour of \(p(m_1, m_2)\) for large \(m_1\). However, \(d_1 \alpha_0^m \beta_1^m\) violates the conditions on the vertical boundary (6). To compensate for this error we try to find \(c_1\), \(\alpha\), \(\beta\) with \(\alpha\), \(\beta\) satisfying (10) such that \(d_1 \alpha_1^m \beta_1^m + d_1 c_1 \alpha_1^m \beta_2^m\) satisfies the conditions on (6). Then we are forced to take \(\beta = \beta_1\) and thus \(\alpha = \alpha_1\), where \(\alpha_1\) is the second root of (10) for \(\beta = \beta_1\), i.e. \(\alpha_1 = Y_-(\beta_1)\). The coefficient \(c_1\) can now be determined such that the conditions on (6) are satisfied. Compensation of \(d_2 \alpha_0^m \beta_2^m\) on the boundary (7) is symmetrical and results in the addition of \(d_2 c_2 \alpha_2^m \beta_2^m\) with \(\alpha_2 = y_-(\beta_2)\) and some suitably chosen \(c_2\). The two new terms violate the conditions on the boundary (3)-(5). Compensation of these terms requires the addition of terms with the same \(\alpha\)-factor as the error terms. Since the two error terms have different \(\alpha\)-factors, we have to compensate for each of them separately. To compensate for \(d_1 c_1 \alpha_1^m \beta_1^m\) we determine \(d_3\), \(d_4\), \(e_1\) such that the terms \(d_1 c_1 \alpha_1^m \beta_1^m + d_3 c_3 \alpha_1^m \beta_2^m\) in the upper region, \(d_4 c_4 \alpha_4^m \beta_4^m\) in the lower region and \(e_1 c_1^m\) on the \(m_1\)-axis satisfy the conditions on (3)-(5), where \(\beta_3 = X_-(\alpha_1)\) and \(\beta_4 = x_-(\alpha_1)\). The compensation of \(d_2 c_2 \alpha_2^m \beta_2^m\) is symmetrical. The compensation for the errors on the boundary (3)-(5) introduces new errors on the boundaries (6)-(7), but it is clear how to continue. The procedure consists of adding on terms so as to alternately compensate for errors on the vertical boundaries (6)-(7) and the horizontal boundary (3)-(5). This results in an infinite sum of products, which has a binary tree structure due to the compensation on the horizontal boundary. A detailed definition of the resulting infinite sum \(x(m_1, m_2)\) is given below.

The parameters \(\alpha_i\) and \(\beta_i\) in the sum \(x(m_1, m_2)\) can be represented in the tree in figure 1. For specifying the recursion relations to generate \(\alpha_i\) and \(\beta_i\) we use the following notations.

\[\beta_{l(i)} = \text{left descendant of } \alpha_i; \quad \beta_{r(i)} = \text{right descendant of } \alpha_i; \quad \alpha_{p(i)} = \text{parent of } \beta_i.\]

Further, \(L\) and \(R\) are defined as the set of indices \(i\) of \(\beta_i\)'s that are left, respectively right descendants. Then for all \(i = 0, 1, \ldots\)

\[\beta_{l(i)} = X_-(\alpha_i); \quad \beta_{r(i)} = x_-(\alpha_i); \quad \alpha_{l(i)} = Y_-(\beta_{l(i)}); \quad \alpha_{r(i)} = y_-(\beta_{r(i)}),\]

with starting value \(\alpha_0 = \rho^2\).

![Figure 1: The sequences \(\{\alpha_i\}\) and \(\{\beta_i\}\).](image)
The sum $x(m_1, m_2)$ is given by

$$x(m_1, m_2) = \begin{cases} 
\sum_{i \in L} d_i (c_{p(i)} a_{\alpha_i}^{m_1} + c_{i} a_{\alpha_i}^{m_2}) \beta_i^{m_2} & (m_1 \geq 0, m_2 > 0, m_1 + m_2 > m); \\
\sum_{i \in R} d_i (c_{p(i)} a_{\alpha_i}^{m_1} + c_{i} a_{\alpha_i}^{m_2}) \beta_i^{m_1} & (m_1 \geq m, m_2 < 0); \\
\sum_{i=0}^{\infty} c_{i} a_{\alpha_i}^{m_1} & (m_1 \geq m). 
\end{cases}$$

The sums in the upper and lower regions can be expressed in the tree in figure 2. The left descendants add up to the sum in the upper region and the right ones to the sum in the lower region.

The recursion relations for the coefficients are formulated below. The coefficients $c_i$ can be obtained from $c_{p(i)}$ by

$$c_i = \frac{Y_- (\beta_i) - \beta_i}{Y_+ (\beta_i) - \beta_i} c_{p(i)} \quad (i \in L); \quad c_i = \frac{y_- (\beta_i) y_- (\beta_i) - \beta_i}{y_- (\beta_i) y_+ (\beta_i) - \beta_i} c_{p(i)} \quad (i \in R),$$

with initially $c_0 = 1$. The coefficients $d_{l(i)}$, $d_{r(i)}$, and $e_i$ can be obtained from $d_i$ by

$$d_{l(i)} = \frac{\alpha_1 (\alpha_1 + p_1 \lambda)}{X_+ (\alpha_1) - \lambda - \mu_1 - \mu_2} - \frac{\alpha_2 (\alpha_2 + p_1 \lambda)}{X_+ (\alpha_2) - \lambda - \mu_1 - \mu_2} d_i \quad (i \in L);$$

$$d_{r(i)} = \frac{\mu_1 (\alpha_1 + p_1 \lambda)}{\mu_2 (\alpha_1 + p_1 \lambda)} \left( \frac{1}{X_+ (\alpha_1)} - \frac{1}{X_+ (\alpha_1)} \right) d_i \quad (i \in L);$$

$$e_i = \frac{\alpha_1 (\alpha_1 + p_1 \lambda)}{X_+ (\alpha_1) - \lambda - \mu_1 - \mu_2} - \frac{\alpha_2 (\alpha_2 + p_1 \lambda)}{X_+ (\alpha_2) - \lambda - \mu_1 - \mu_2} d_i \quad (i \in L),$$

and symmetrical relations hold for all $i \in R$ (by interchanging $p_1$ and $p_2$, $\mu_1$ and $\mu_2$, $X$ and $x$). The starting values are given by

$$d_1 = \frac{\alpha_0 \mu_1 + p_2 \lambda}{\alpha_0 \mu_1}; \quad d_2 = \frac{\alpha_0 \mu_2 + p_1 \lambda}{\alpha_0 \mu_2}; \quad e_0 = 1.$$

This concludes the definition of $x(m_1, m_2)$. It may happen that the series $x(m_1, m_2)$ diverges for small $m_1$ and $m_2$, but it can be shown that there exists an integer $N \geq m$, depending on the values of $\lambda$, $\mu_1$, $\mu_2$ and $p_1$, such that $x(m_1, m_2)$ absolutely converges for all states $(m_1, m_2)$ with $m_1 + |m_2| > N$ and for $(m_1, m_2) = (N, 0)$. The next theorem states that on this set $x(m_1, m_2)$ equals $p(m_1, m_2)$ up to some normalizing constant.
Theorem: For all states \((m_1, m_2)\) with \(m_1 + |m_2| > N\) and for \((m_1, m_2) = (N, 0)\),
\[ p(m_1, m_2) = Cx(m_1, m_2) \]
where \(C\) is the normalizing constant.

The series-representation for \(p(m_1, m_2)\) easily leads to similar expressions for the moments of the waiting time or other quantities of interest. These results offer efficient numerical algorithms with tight error bounds on each partial sum. Furthermore, an efficient and numerically stable algorithm can be derived for solving the equilibrium equations for states \((m_1, m_2)\) with \(m_1 + |m_2| \leq N\).

3 Conclusion

In this note it has been demonstrated that, by using a compensation approach, an expression for the state probabilities can be derived which has the form of an infinite sum of products of powers. In [5] Kojima et al. find the first term of this infinite sum only. Although the resulting expression looks rather complicated, it is well suited for computation of the equilibrium probabilities as well as the relevant performance criteria.

Acknowledgement

The authors are grateful to professor Paul J. Schweitzer from the University of Rochester for drawing their attention to the paper of Kojima et al. and the relation of that paper to the work of the present authors.

References


