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IMPLICATION
A survey of the different logical analyses
of "if...,then ..."

by

H.C.M. de Swart R.P. Nederpelt

91/02

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Implication
A survey of the different logical analyses of
"if ..., then ..."

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February 6, 1991

Abstract
This paper discusses different forms of implication as they are used in logic, mathematics, linguistics and artificial intelligence. The most commonly used form is that of material implication. It has turned out, however — as the paper makes clear — that there are aspects that are not covered by this material implication. This has led to strict implication (in order to provide for necessity) and relevant implication (for relevance).
There are still other forms of implication. The paper discusses counterfactuals and intuitionistic implication.
Finally, a survey is given of decision procedures for different kinds of implication.

1 Introduction
In what follows we shall study the main connective of logic: the implication "if ..., then ...". More precisely, we shall study the question what one can or should mean with the validity of an implication "if A, then B", where A and B may be composite formulas, built from atomic formulas P, Q, .... The question whether the atomic expressions themselves are true or false (with respect to a given interpretation) is not to the purpose. The domain of logic is reasoning, which should be applicable to all fields of interest, like mathematics, physics, philosophy and so on. Therefore in logic one is only interested in the validity of composite formulas in relation to their components. One does not regard the truth or falsehood of atomic propositions, which is studied in the science in question.
A special class of logical formulas consists of the valid formula's, which are true independently of the truth values of their components. For example, it will be clear that "if P, then P" is valid, i.e. it is true no matter what kind of proposition P expresses and no matter whether that proposition is false or true. The same holds for "if P, then P or Q".
In Section 2 different forms of implication are presented. We start with a discussion of the (commonly used) material implication. This form of implication, although adequate for mathematics, is deficient in more than one respect.
We explain how Grice uses pragmatic principles of conversation to account for these deficiencies. According to this theory, the shortcomings of material implication are caused by the fact that one restricts oneself to a purely semantic analysis of "if ..., then ...", in terms of

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truth-conditions. Thereby one does not take into account the \textit{pragmatic} principles governing discourse.

The use of material implication as an analysis of “if ..., then ...” ignores considerations of \textit{necessity}; one may use \textit{strict implication} to take necessity into account. Both material implication and strict implication do not take into account considerations of \textit{relevance}; modifications designed to accommodate this important feature of the intuitive logical “if ..., then ...” yield \textit{relevant implication}. Combining necessity and relevance leads to a form of implication called \textit{entailment}.

Subsequently we will study \textit{counterfactuals}, which are sentences of the form “if it were the case that ..., then it would be the case that ...”.

Finally, \textit{intuitionistic implication} will be discussed. This form of implication is quite different from the foregoing types, because not the notions of truth and falsehood are taken into consideration, but rather the notion of “proof”. The implication “if $P$, then $Q$” is interpreted as the fact that one has a construction at hand which permits to transform any proof of $P$ into a proof of $Q$.

In Section 3 we show how for each of the implications considered, we have a \textit{decision procedure}. We describe methods which, for each implicational formula $A$, either yield a formal proof of $A$ or construct a countermodel for $A$.

2 Different forms of implication

Conditionals can be divided in two classes: those in the \textit{indicative} mood and those in the \textit{subjunctive} mood. In the examples below\footnote{These examples are from [Adams, 70]}, the conditional in (1) is in the indicative mood, while the conditional in (2) is a subjunctive one.

(1) If Oswald did not kill Kennedy, someone else did.
(2) If Oswald had not killed Kennedy, someone else would have.

The first of these sentences is certainly true (someone killed Kennedy), but the second is probably false. Hence, these sentences cannot fit in the same logical frame.

In Subsection 2.4 we shall discuss subjunctive conditionals, in particular those called \textit{counterfactuals}. In the other subsections of the present section we will restrict our attention to indicative conditionals.

2.1 Material implication

W. & M. Kneale mention that the first logicians to debate the nature of conditional statements were Diodorus Cronus and his pupil Philo ($\pm$ 300 B.C.). Sextus Empiricus ($\pm$ 200 A.D.) lists four views which had been held about the nature of conditionals, of which we mention here the first two.

1. Philo says that a sound conditional is one that does not begin with a truth and end with a falsehood, e.g. when it is day and I am conversing, the statement “If it is day, I am conversing” is sound.

2. But Diodorus says it is one that neither \textit{could nor can} begin with a truth and end with a falsehood. According to him the conditional statement just quoted seems to be false,
since when it is day and I have become silent it will begin with a truth and end with a falsehood. (See [Kneale and Kneale, 62, pp. 128-129]).

Later on, the symbol “⊨”, called material implication\(^2\), has been used to indicate the conditional Philo had in mind. It will be the main subject of the present subsection. In the next subsection we will discuss strict implication, which comes close to the conditional Diodorus had in mind.

Indicating truth with the symbol “1” and falsehood with the symbol “0”, Philo’s conditional can be characterized by the following matrix, called a truth-table.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( \text{P} \supset \text{Q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The conditional “\( \supset \)” has the property that it is a truthfunctional connective, i.e. the truth-value of \( \text{A} \supset \text{B} \) is completely determined by the truth-values of \( \text{A} \) and \( \text{B} \).

It is interesting to note that in mathematics the “if ..., then ...” is used precisely as described in the matrix above, as can be seen from the following. The statement “For all integers \( a \) and \( b \), if \( a = b \), then \( a^2 = b^2 \)” is generally accepted as being true, because it does not happen that \( a = b \) is true and \( a^2 = b^2 \) is false.

<table>
<thead>
<tr>
<th>( a = b )</th>
<th>( a^2 = b^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = 2, \ b = 2 )</td>
<td>1 1 1</td>
</tr>
<tr>
<td>( a = 2, \ b = -2 )</td>
<td>0 1 1</td>
</tr>
<tr>
<td>( a = 2, \ b = 3 )</td>
<td>0 0 1</td>
</tr>
</tbody>
</table>

No disagreement exists that “if \( P \), then \( Q \)” is false if \( P \) is true and \( Q \) is false. Problems arise with Philo’s claim that “if \( P \), then \( Q \)” is false ONLY if \( P \) is true and \( Q \) is false, and is true in all other cases.

In what follows, strict implication (Subsection 2.2) and relevant (Subsection 2.3) implication will give a more subtle analysis of “if \( P \), then \( Q \)” than Philo’s one in the case that both \( P \) and \( Q \) are true (the first line in our truth-table for \( P \supset \text{Q} \)). Furthermore, counterfactuals (Subsection 2.4) will give a more subtle analysis of “if \( P \), then \( Q \)” in the case that \( P \) is false (the last two lines in our truth-table for \( P \supset \text{Q} \)).

Some material implications, like \( P \supset P \) and \( P \supset (Q \supset P) \), turn out to be valid, i.e. have the value 1, no matter what the values of their components \( P \) and \( Q \) are.

---

\(^2\)The term consequentiae materiales already appears in “In Universam Logicam Questiones”, formerly attributed to John Duns the Scot (1266-1308). See [Kneale and Kneale, 62, pp. 278-280].

Whitehead and Russell used the phrase “\( P \) materially implies \( Q \)” as a rendering for \( P \supset Q \) in their informal comments on their own system of logic ([Kneale and Kneale, 62, p. 549]). The truth of \( P \supset Q \) for given propositions \( P \) and \( Q \) will ordinarily depend on circumstances outside of logic, e.g., on matters of empirical fact; hence the name “material implication” for “\( \supset \)”
Validity of a formula $A$ is indicated by writing $\models A$.

This semantic notion of validity in the logic with material implication has an interesting syntactic counterpart in the notion of (formal) provability. It turns out — and this is not a trivial result — that $A$ is valid if and only if (iff) $A$ is (formally) provable in a sense that we will now describe.

We consider three so-called axiom schemata and one rule, called Modus Ponens. The axiom schemata are the following:

axiom schema 1: $A \implies (B \implies A)$
axiom schema 2: $((A \implies B) \implies A) \implies A$ (Peirce's law)
axiom schema 3: $(A \implies B) \implies ((B \implies C) \implies (A \implies C))$

Each of these axiom schemata is supposed to contain infinitely many axioms, one for each substitution of "well-formed" formulas in the logic with material implication, for the metavariables $A$, $B$ and $C$.

The Modus Ponens rule has the following format:

$$
\text{MP: } \frac{B}{\therefore C}.
$$

This rule should be read as follows: when formulas $B$ and $B \implies C$ have been obtained, then the formula $C$ may be added.

Now a formula is formally provable if and only if it can be obtained by finitely many applications of the Modus Ponens rule to instances of the above axiom schemata. The result mentioned above states that precisely all valid formulas in the logic with material implication are formally provable. See [Church, 56, §26].

For instance, $P \implies P$ can be obtained as follows:

1. $P \implies ((P \implies P) \implies P)$ by axiom schema 1, taking $P$ for $A$ and $P \implies P$ for $B$.
2. $(P \implies ((P \implies P) \implies P)) \implies (((P \implies P) \implies P) \implies (P \implies P))$ by axiom schema 3, taking $P$ for $A$, $(P \implies P) \implies P$ for $B$ and $P$ for $C$.
3. $(((P \implies P) \implies P) \implies (P \implies P))$ from 1 and 2 by Modus Ponens, taking $P \implies ((P \implies P) \implies P)$ for $B$ and $(((P \implies P) \implies P) \implies (P \implies P))$ for $C$.
4. $((P \implies P) \implies P)$ by axiom schema 2, taking $P$ for both $A$ and $B$.
5. $P \implies P$ from 3 and 4 by Modus Ponens.

The truth-functional reading of "if $A$, then $B$" (viz. as $A \implies B$) is equivalent to the one of $\neg A \lor B$ (not-$A$ or $B$). This seems to conflict with judgments we ordinarily make. We give two examples.

Consider the formulas in the so-called paradoxes of material implication: $\models \neg P \implies (P \implies Q)$ and $\models Q \implies (P \implies Q)$. These valid formulas lead to the following two logical inferences (at left), whereas the English instantiations (at right) seem incorrect.
There is no oil in my coffee

If there is oil in my coffee, then I like it

I’ll ski tomorrow

If I break my leg today, then I’ll ski tomorrow

(The first inference reads as follows: if \( \neg P \) has the value 1, then \( P \rightarrow Q \) has the value 1 as well. This is in accordance with our truth-table. The English instantiation is read as: “When I have established that there is no oil in my coffee, I may conclude that if there is oil in my coffee, then I like it”. This inference is questionable, at the least.

Analogous things can be said about the second inference and its instantiation.)

Grice uses principles of conversation to explain facts about the use of conditionals that seem to conflict with the truth-functional analysis of the ordinary indicative conditional. In the 1967 William James Lectures (published in [Grice, 89]), Grice works out a pragmatic theory, which he calls the theory of conversational implicature. Generally speaking, in conversation we usually obey or try to obey rules something like the following:

(QUALITY) Tell the truth

(QUANTITY) Be informative

(RELATION) Be relevant

(MODE) Avoid obscurity, prolixity, etc.

One says that \( A \) conversationally implicates \( B \) if the following is the case: \( B \) is logically entailed by the fact that \( A \) has been said, plus the assumption that the speaker is observing the above rules, plus other reasonable assumptions about the speaker’s purposes and intentions in the context.

It is possible for \( A \) to conversationally implicate many things which are in no way part of the meaning of \( A \). Speaking the truth is only one of the conversation rules one is expected to obey in daily discourse; one is also expected to be as informative and relevant as possible³.

For example, if \( X \) says “Your hat is either upstairs in the back bedroom or down in the hall closet”, this remark conversationally implicates “I don’t know which”, since if \( X \) did know which, this remark would not be the most informative one he could provide.

Now, if one has at one’s disposal the information \( \neg A \) (or \( B \), respectively) and at the same time provides the information \( A \lor B \), i.e. \( \neg A \lor B \), then one is speaking the truth, but a truth calculated to mislead, since the premiss \( \neg A \) (or \( B \), respectively) is so much simpler and more informative than the conclusion \( A \lor B \). If one knows the premiss \( \neg A \) (or \( B \), respectively), the conversation rules force us to assert this premiss instead of \( A \lor B \). Quoting [Jeffrey, 81, pp. 77-78]:

“Thus defenders of the truth-functional reading of everyday conditionals point out that the disjunction “\( \neg A \lor B \)” shares with the conditional “if \( A \) then \( B \)” the feature that normally it is not to be asserted by someone who is in a position to deny “\( A \)” or to assert “\( B \)”...

³In this paper we shall not discuss the fourth rule, concerning the “mode”
Normally, then, conditionals will be asserted only by speakers who think the antecedent false or the consequent true, but do not know which. Such speakers will think they know of some connection between the components, by virtue of which they are sure (enough for the purposes at hand) that the first is false or the second is true.

An example concerning the third requirement, relevance, is the following. If X says: “I'm out of gas” and Y says: “There's a gas station around the corner”, Y's remark conversationally implicates that the station in question is open. (Since the information that the station is there would be irrelevant to X's predicament otherwise.)

Summarizing in a slogan:
indicative conditional = material implication + conversation rules.

As a final remark for this subsection we note that not everyone agrees on this point of view. See, for instance, [Skyrms, 80]. Also, the following inferences (at left) do hold for material implication, whereas the English versions (at right) seem incorrect.

\[ A \supset C \]  If there is sugar in the coffee, then it will taste good.

\[(A \land B) \supset C \]  If there is sugar in the coffee and diesel-oil as well, then it will taste good.

\[ A \supset B \]  If Smith dies before the election, Jones will win it.
\[ B \supset C \]  If Jones wins the election, Smith will retire.

\[ A \supset C \]  If Smith dies before the election, he will retire.

The “paradoxes” in the English versions (at right) can be explained by the aid of Grice's theory of conversational implicature, as we can see as follows:

(1) It is a conversational implicature of the first conclusion that the coffee may well contain diesel-oil. But we are not prepared to add this implicature as a second premiss to the original argument.

(2) The conclusion of the second argument conversationally implicates that Smith may die before the election. Once we reckon with this implicature of the conclusion, the second premiss of the original argument turns out to be unacceptable.

See [Veltman, 86, pp. 164-165] for an explanation of the paradoxes in terms of Grice's theory, and [Nieuwint, 90].

2.2 Strict implication

C. I. Lewis' starting point is to avoid the paradoxes of material implication. In [Lewis, 18] he therefore introduces strict implication: one proposition implies another in the strict sense of the word if, and only if, it is impossible that the first should be true and the second false; notation: \( P \rightarrow Q \). (See [Lemmon, 77, pp. 5-6] and [Kneale and Kneale, 62, p. 549]).

Lewis' strict implication can quite easily be defined in terms of material implication with the help of the modal operator \( \square \). This operator stands for "it is necessary that". It is one of a pair of modal operators, the other one being \( \Diamond \), meaning "it is possible that".
Now one can define strict implication in terms of material implication plus the modal operator $\Box$:

$$P \rightarrow Q := \Box(P \supset Q).$$

This can be justified as follows: $P \rightarrow Q$ can be considered to mean "it is impossible that $P$ and not $Q$" (written in a formula: $\neg\Box(\neg(P \land \neg Q))$), which is equivalent to "it is necessary that not $(P \land \neg Q)$" (or $\Box\neg(P \land \neg Q)$), which is again equivalent to "it is necessary that if $P$, then $Q$" (or $\Box(P \supset Q)$).

Note that, as we have seen at the beginning of Subsection 2.1, already Diodorus seems to have suggested that "if ..., then ..." should be analysed as something like a necessary material implication.

Also many medieval writers, e.g. Abelard (1079-1142), insist that any true conditional must be necessarily true: to affirm "if $A$, then $B$" is to affirm that it is not possible for $A$ to be the case without $B$ being the case. See [Lemmon, 77, pp. 4-5].

The interpretation of the modal notion $\Box$ (necessity) is not without problems. (See [de Swart, 9].) However, the following semantic analysis is generally accepted.

We imagine that there are many possible worlds $w$ and that our actual world $w_0$ is one of them. World $w$ can be different from our actual world $w_0$ by having different physical laws, by having a present king of France or by having different ethical norms, and so on.

A number of worlds $w$ will be accessible from our actual world $w_0$, denoted by $w_0Rw$, but some will not; for example a world with different mathematical laws will be supposed not to be accessible from our actual world $w_0$.

The notion of truth of $B$ relative to a world $w$ will be denoted by $w \models B$. Now we may agree that $\Box A$ ("$A$ is necessary") is true in world $w$, denoted $w \models \Box A$, if and only if $A$ holds in every world $w'$ which is accessible from world $w$:

$$w \models \Box A \iff \text{for all worlds } w', \text{if } wRw', \text{then } w' \models A.$$ 

By varying the accessibility relation $R$, different notions of necessity result:

1. For $wRw'$ := $w$ and $w'$ have the same physical laws:

$$w \models \Box A \iff A \text{ is necessitated by the physical laws of } w, \text{ and hence holds in all worlds accessible from } w.$$ 

2. For $wRw'$ := $w$ and $w'$ have the same ethical norms and $w \neq w'$:

$$w \models \Box A \iff A \text{ is obligated by the ethical norms of } w.$$ 

As we learn from Boethius, Diodorus Cronus defines the necessary as that which, being true, will not be false (quod cum verum sit, non erit falsum) ([Kneale and Kneale, 62, p. 117]). This suggests a connection between the modalities and temporal notions.

As can be seen in [van Leeuwen, 90, vol. B], there is recently a renewed interest in the modalities from the field of computer science and artificial intelligence, which has greatly stimulated further research. We mention temporal logic, reading $\Box A$ as "$A$ is and will be

4Because propositions which are obligated by the ethical norms of $w$ do not always hold in $w$ itself, one needs the clause "$w \neq w'$" in the definition of $R$.

5Boethius (480–526) translated and commented some of Aristotle's (logical) works and also wrote himself about logic, in particular about syllogisms.
the case” (see [Goldblatt, 87]); epistemic logic, reading \( \Box A \) as “the agent (processor) knows that \( A \)” (see [Meyer, 87]); deontic logic, reading \( \Box A \) as “it ought to be the case that \( A \)”, or, equivalently, as “\( A \) is obligatory” (see [van Eck, 81]); doxastic logic, reading \( \Box A \) as “I believe that \( A \)” (see [Hintikka, 62]); and dynamic logic or the logic of programs\(^6\) (see [Harel, 79]).

Because \( B \rightarrow \Box C \) iff \( \Box (B \rightarrow \Box C) \) iff for all \( w' \) holds that if \( wRw' \) and \( w' \models B \), then \( w' \models C \). Otherwise said: \( B \rightarrow \Box C \) holds in world \( w \) iff \( C \) holds in each world \( w' \) which is accessible from \( w \) and in which \( B \) holds.

For formulas possibly containing \( \Box \), we also have a notion of validity: \( A \) is valid, denoted again by \( \models A \), iff \( A \) holds in all possible worlds.

More precisely, the semantics of modal logic can be defined as follows.

We say that \( M = (K, R, \models) \) is a (Kripke-)model iff

1. \( K \) is a non-empty set (regarded as playing the role of the set of possible worlds),
2. \( R \) is a binary relation on \( K \) (regarded as the accessibility relation), such that \( R \) is reflexive (i.e., for all \( w \) in \( K \), \( wRw \)) and such that \( R \) is transitive (i.e. if \( wRw' \) and \( w'Rw'' \), then \( wRw'' \) for all \( w, w', w'' \) in \( K \))^7,
3. \( \models \) is a relation between elements of \( K \) and atomic expressions \( P \). (“\( w \models P \)” is to be read as “\( P \) is true in world \( w \)”).

Next we define, for given \( M = (K, R, \models) \) and \( w \in K \), and for arbitrary formulas \( A \), the notion \( M \models_w A \) (to be read as: \( A \) is true in world \( w \) of the model \( M \)):

\[
\begin{align*}
M \models_w P & \iff w \models P \quad (P \text{ atomic}), \\
M \models_w B \rightarrow C & \iff \text{not } M \models_w B \text{ or } M \models_w C, \\
M \models_w \Box B & \iff \text{for all } w' \in K, \text{ not } wRw' \text{ or } M \models_{w'} B.
\end{align*}
\]

Now a given model \( M \) is a model for \( A \), denoted by \( M \models A \), iff for all \( w \) in the \( K \) belonging to \( M \) it holds that \( M \models_w A \). And \( A \) is valid, denoted again by \( \models A \), iff every model \( M \) is a model for \( A \).

Now it turns out — and again this is not a trivial result — that precisely all valid formulas in the logic based on strict implication are formally provable, i.e., can be obtained by Modus Ponens, \( \frac{B \quad B \rightarrow C}{C} \), from the following axiom schemata:

1. \( A \rightarrow \exists A \)
2. \( (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \),
3. \( (A \rightarrow B) \rightarrow (C \rightarrow (A \rightarrow B)) \),

which is essentially the implicational fragment of Lewis’ system \( S_4 \) (supposing that the accessibility relation \( R \) is reflexive and transitive).

\(^6\)Here one has a necessity operator \([\alpha]\) for each program \( \alpha \), and \( \omega \models [\alpha]P \) is read as: \( P \) is true in all final states \( \omega' \) accessible from state \( \omega \) via the program \( \alpha \).

\(^7\)Other conditions on \( R \) will give somewhat different notions of necessity and hence of strict implication; the conditions just mentioned are typical for the modal logic called \( S_4 \).
We are now almost ready to show that the strict implication versions of the original paradoxes of material implication ($\not\vdash \neg\phi \rightarrow \neg\psi$ and $\vdash \phi \rightarrow \neg\psi$) do not hold. The only thing we still need is the definition of $M \vDash_\omega \neg \psi$, which obviously should be as follows: $M \vDash_\omega \neg \psi$ iff not $M \vDash_\omega \psi$.

1. not $\vdash \neg\phi \rightarrow (\neg\psi \rightarrow \neg\phi)$.
   In order to show this, we provide a “countermodel” $M$.
   Let $M$ be the model which contains two worlds $w$ and $w'$ such that $\phi$ holds in $w'$ but not in $w$, and such that $\psi$ holds in neither of the worlds. Let $wRw$, $w'Rw'$ and $wRw'$. This model can be depicted as follows:

   $\begin{array}{c}
   w \\uparrow \\
   Q \quad w'
   \end{array}$

   Then $M \vDash_\omega \neg \phi$, since not $M \vDash_\omega \phi$. Moreover, not $M \vDash_\omega \neg \phi \rightarrow \neg \psi$, since if this were the case, then the strict implication would entail that not only $\phi$, but also $\psi$ would hold in $w'$. Hence, by the definition of strict implication and the reflexivity of $R$, not $M \vDash_\omega \neg \phi \rightarrow (\neg \psi \rightarrow \neg \phi)$. The conclusion is that not $M \vDash_\omega \neg \phi \rightarrow (\neg \psi \rightarrow \neg \phi)$.

2. not $\vdash \psi \rightarrow (\phi \rightarrow \psi)$.
   Again we give a countermodel.
   Let $M$ be the model containing two worlds $w$ and $w'$ such that $\psi$ holds in $w$ but not in $w'$ and such that $\phi$ holds in $w'$ but not in $w$. Let $wRw$, $w'Rw'$ and $wRw'$.

   $\begin{array}{c}
   \psi \quad w' \\
   \phi \quad w
   \end{array}$

   Then $M \vDash_\omega \psi$, but not $M \vDash_\omega \phi \rightarrow \psi$ (again by the properties of strict implication). Therefore, analogously to the previous case, we obtain: not $M \vDash \psi \rightarrow (\phi \rightarrow \psi)$.

Another manner to obtain the system $S_4$ from classical logic, i.e. the logic of material implication, results from adding the following axioms and rule to it:

$(\Box(A \Box B)) \supset ((\Box A) \supset (\Box B))$

$(\Box A) \supset A$

$(\Box A) \supset (\Box \Box A)$

If $A$ is provable, then $\Box A$ is provable.

The semantical counterpart of the formulas $(\Box A) \supset A$ and $(\Box A) \supset (\Box \Box A)$ is the reflexivity and the transitivity, respectively, of the accessibility relation $R$.

We already noted that $\Box$ may have many different, sometimes rather vague, meanings. One particular reading, which is well-defined and interesting, is the following: $\Box A$ means that $A^*$ is formally provable in Peano Arithmetic, where $A^*$ results from $A$ by replacing each atomic formula $P$ in $A$ by a sentence of Peano Arithmetic. See [Boolos and Jeffrey, 82, Ch. 27].
2.3 Relevant implication and entailment

As was already noticed by Abélard (1079–1142), the paradoxes of material implication (viz. \( \models \neg P \supset (P \supset Q) \) and \( \models Q \supset (P \supset Q) \)) reappear for strict implication in the following form.

1. If \( P \) is impossible, then \( P \supset Q \) is true for any proposition \( Q \): \( \models (\neg \Diamond P) \supset (P \supset Q) \), or equivalently, \( \models (\Box \neg P) \supset (P \supset Q) \). Using his example: since Socrates being a stone is impossible \( \neg \Diamond P \), it is impossible for him to be a stone plus something else, e.g. to be no ass \( \neg \Box (P \wedge \neg Q) \), which is equivalent to \( P \supset \neg Q \).

2. Later, Kilwardby (± 1270) realized that if \( Q \) is necessary, then it is not possible to have \( P \) without \( Q \), so that again \( P \supset Q \). Therefore \( \models (\Box Q) \supset (P \supset \exists Q) \) ([Lemmon, 77, p. 5]).

Also C.I. Lewis saw that his definition of strict implication led to the validity of these strict implications, which can be questioned. We can phrase these so-called paradoxes of strict implication\(^8\) as follows:

1. \( \models (\Box \neg P) \supset (P \supset Q) \): “An impossible proposition \( P \) must imply every proposition \( Q \)”,
   and
2. \( \models \Box Q \supset (P \supset \exists Q) \): “A necessary proposition \( Q \) is implied by every proposition \( P \)”.

Other so called paradoxes of strict implication are\(^9\):

3. \( \models (\neg Q \wedge Q) \supset P \), and
4. \( \models Q \supset (P \supset Q) \)

Note that the formula \( (\neg Q \wedge Q) \supset P \) is equivalent with \( \neg Q \supset (Q \supset P) \), but \( \neg Q \supset (Q \supset P) \) is valid and \( \neg Q \supset (Q \supset P) \) is invalid (see Subsection 2.2). Also, the formula \( Q \supset (P \supset P) \) is equivalent with \( P \supset (Q \supset P) \), but \( Q \supset (P \supset P) \) is valid and \( P \supset (Q \supset P) \) is invalid (again, see Subsection 2.2).

The problem with the paradoxes of strict implication is that for the validity of an inference from \( A \) to \( B \), one might reasonably require that \( A \) should be relevant to \( B \). And in both 1: \( (\Box \neg P) \supset (P \supset Q) \), and 2: \( (\Box Q) \supset (P \supset Q) \), we can easily see that \( P \) is not relevant to \( Q \). In 3: \( (\neg Q \wedge Q) \supset P \), the antecedens \( \neg Q \wedge Q \) is not relevant to the consequens \( P \), and in 4: \( Q \supset (P \supset P) \), the antecedens \( Q \) is not relevant to the consequens \( P \supset P \).

S. Moh (see [Moh, 50]) and A. Church (see [Church, 51]) were the first to succeed in excluding these so-called paradoxes without also excluding at the same time arguments which every one regards as valid.

In his paper Church presents essentially the following axiom schemata for what he calls weak implication, but what one now prefers to call relevant implication, which we denote by the symbol \( \Rightarrow \):

1. \( A \Rightarrow A \),
2. \( (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \),

\(^8\) [Lemmon, 77, p. 6].

\(^9\) [Anderson and Belnap, 75, p. 18].
3. \((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))\),

4. \((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))\),

together with the rule Modus Ponens, \[
\frac{B}{\therefore C}
\]

The conditional \(\Rightarrow\) satisfies principles of relevance in the following mathematically definite sense ("Deduction theorem for relevant implication"):

\[A_1, \ldots, A_{n-1}, A_n \vdash \ A \Rightarrow B \text{ iff } A_1, \ldots, A_{n-1}, A_n \vdash^* B.\]

Here one may read \(A_1, \ldots, A_{n-1}, A_n \vdash^* B\) as: \(B\) is deducible from \(A_1, \ldots, A_{n-1}, A_n\). It means that the formula \(B\) can be obtained by a finite number of applications of Modus Ponens to \(A_1, \ldots, A_{n-1}, A_n\) and to instances of the axiom schemata 1, 2, 3, 4, such that all of \(A_1, \ldots, A_{n-1}, A_n\) are actually used in the deduction of \(B\). More precisely: \(B\) should get the relevance-index \(\{1, \ldots, n-1, n\}\) if we assign to each \(A_i\) (\(1 \leq i \leq n\)) the index \(\{i\}\), to each axiom the index \(\emptyset\) and to each consequence of an application of Modus Ponens the union of the indices of its premises.

For instance, \(A \Rightarrow B, B \Rightarrow C \vdash^* A \Rightarrow C\), for the following diagram is a deduction (in the new sense) of \(A \Rightarrow C\) from \(A \Rightarrow B\) and \(B \Rightarrow C\):

\[
\frac{(A \Rightarrow B)_{\{1\}} \quad (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))_{\emptyset}}{(B \Rightarrow C)_{\{2\}} \quad (B \Rightarrow C) \Rightarrow (A \Rightarrow C)_{\{1\}}} \quad (A \Rightarrow C)_{\{1, 2\}}
\]

However, it is not the case that \(Q \vdash^* (P \Rightarrow P)^{10}\). (We note that \(Q \vdash (P \supset P)\) does hold; i.e., under the assumption \(Q\), we can give a proof of \(P \supset P\), since \(P \supset P\) is formally provable. See Subsection 2.1. The irrelevance of the assumption \(Q\) for the proof of \(P \supset P\) does not play a role there.).

In order to develop semantics for Church’s relevant implication, we consider pieces of information. Given any two pieces of information, \(a\) and \(b\), we may put them together to form a new piece of information, \(a \cup b\), containing all the information in \(a\) together with all the information in \(b\). We must also allow for the empty piece of information, \(\emptyset\), for which we have the equation \(a \cup \emptyset = a\) for all \(a\).

A piece of information \(a\) may determine a basic formula \(P\) in the sense that it may be concluded that \(P\) is true on the basis of the information in \(a\). Let us write \(a \models P\) if \(a\) determines \(P\). To say that \(a\) determines \(B \Rightarrow C\) is to say that whenever we can conclude \(B\) on the basis of a piece of information \(b\), we can conclude \(C\) on the basis of \(a\) and \(b\) jointly, that is, on the basis of \(a \cup b\). Hence, the valuation rule for \(\Rightarrow\) is: \(a \models B \Rightarrow C\) if and only if for all \(b\), either not \(b \models B\) or \(a \cup b \models C\).

A formula \(A\) is valid \((\models A)\) when it is always determined by the empty piece of information, that is, when \(\emptyset \models A\).

It is to be noted that we do not in general require that if \(a \models A\), then \(a \cup b \models A\). This is because we interpret the phrase “on the basis of” in a rather strong sense, one which requires

\[^{10}\text{[Anderson and Belnap, 75, §3]}\]
that for $a \models A$ to be true, the information in $a$ must be relevant to $A$. For example, we might have $\{2+2 = 4\} \models 2+2 = 4$, but not $\{2+2 = 4\}, \text{grass is green} \models 2+2 = 4$; and $\emptyset \models P \Rightarrow P$, but not $\{2+2 = 4\} \models P \Rightarrow P$ ([Urquhart, 72]). (If we do require the relation $\models$ to satisfy the additional requirement "if $a \models A$, then $a \cup b \models A$", then the schema $A \Rightarrow (B \Rightarrow A)$ would be valid. The set of formulas valid with this added requirement coincides with the pure implicational fragment of intuitionistic logic (see Subsection 2.5).)

More precisely, the semantics of relevance logic can be described as follows.

We define $\langle S, \models \rangle$ to be a model if and only if

1. $S$ is a collection of sets, closed under the union $\cup$ (the elements of $S$ regarded as playing the role of the pieces of information),
2. $S$ contains the empty set $\emptyset$ (regarded as the empty piece of information),
3. $\models$ is a relation between elements of $S$ and atomic formulas $P$ ("$a \models P$" is to be read as "$P$ is true on the basis of the information in $a$").

Next, for given $\langle S, \models \rangle$ and $a \in S$ we define $\models_a A$ (A is true on the basis of the information $a$ of the model $\langle S, \models \rangle$) for an arbitrary formula $A$ as follows.

$\models_a P$ iff $a \models P$ (P atomic).

$\models_a B \iff C$ iff for all $b$ in $S$, not $\models_b B$ or $\models_{a\cup b} C$.

$M$ is a model for $A$ (in symbols: $M \models A$) iff $M \models_a A$. And $A$ is valid ($\models A$) iff for all models $M$, $M \models A$.

It turns out — and again this is not a trivial result — that the valid relevant implications coincide exactly with the provable relevant implications ([Urquhart, 72]).

One can easily show now that the relevant implication versions of the original paradoxes of strict implication do not hold. For example:

3. Not $\models (\neg Q \land Q) \Rightarrow P$, where $\neg Q := Q \Rightarrow \bot$ and $\bot$ is a special symbol denoting falsity.

Let $S = \{\emptyset, \{1\}\}$ and let $\models$ be such that precisely $\{1\} \models Q$ and $\{1\} \models \bot$. Then $M \models_{\{1\}} \neg Q$, hence — using $\land$ in the usual manner — $M \models_{\{1\}} \neg Q \land Q$, and not $M \models_{\{1\}} P$. Hence not $M \models_{\emptyset} (\neg Q \land Q) \Rightarrow P$.

4. Not $\models Q \Rightarrow (P \Rightarrow P)$.

Let $S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and let $\models$ be such that precisely $\{1\} \models Q$ and $\{2\} \models P$.

Then $M \models_{\{1\}} Q$, but not $M \models_{\{1\}} P \Rightarrow P$, since $M \models_{\{2\}} P$ and not $M \models_{\{1, 2\}} P$. Hence not $M \models_{\emptyset} Q \Rightarrow (P \Rightarrow P)$.

The combination of the possible world semantics for necessity, $\Box$, with the information-pieces semantics for relevant implication, $\Rightarrow$, makes clear in an analogous way that also both $(\Box \neg P) \Rightarrow (P \Rightarrow Q)$ and $\Box Q \Rightarrow (P \Rightarrow Q)$ are invalid.

Yet, also relevant implication does not fully satisfy. By axiom schema 1 of relevant implication we have $(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$ and hence, by axiom schema 4 and Modus Ponens, $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$. This says that if $A$ is true, then it follows from $A \Rightarrow A$. On the other hand, we expect that any logical consequence of $A \Rightarrow A$ should be necessarily true ([Anderson and Belnap, 75, p. 23]).

In order to repair this, one can consider $\rightarrow$, defined by $P \rightarrow Q := \Box(P \Rightarrow Q)$, sometimes called entailment, which was considered for the first time in [Ackermann, 56]. In this paper, Ackermann presents essentially the following axiomatic system for $\rightarrow$:
1. \( A \rightarrow A \),
2. \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\),
3. \((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\),
4. \((A \rightarrow B) \rightarrow (((A \rightarrow B) \rightarrow C) \rightarrow C)\),

together with the rule Modus Ponens: \[
\begin{array}{c}
P \\
P \rightarrow Q \\
\hline
Q
\end{array}
\] \( ([\text{Anderson and Belnap}, 75, \text{p. 23-24}]) \).

Entailment satisfies both principles of relevance and principles of necessity in certain mathematically definite senses: all valid entailments are necessarily valid and in all valid entailments the antecedents is relevant to the consequents.

Does entailment really capture the implication "if . . . , then . . . " of natural language? We do not think so. If one tries to combine entailment with the other connectives like conjunction, disjunction and negation, one may arrive at different systems and it is not clear at all which is the right one (see [Anderson and Belnap, 75, p. 340-341]).

For more information we refer the interested reader to Studia Logica 1979, vol. XXXVIII, no. 2, on Problems in Relevance Logic.

2.4 Counterfactuals

As stated in the beginning of this section, conditionals can occur in the indicative or in the subjunctive mood. This subsection considers the latter case.

We first note that the "if . . . then . . . " in subjunctive sentences is clearly not truthfunctional, as shown by the following two examples.

1. If I had jumped out of the window of the 5th floor, I would have been wounded.
2. If I had jumped out of the window of the 5th floor, I would have changed into a bird.

Although in both examples the truth-values of the two sentences of which the whole sentence is composed are the same (in sentence (i) 0-0 and in sentence (ii) 0-0), the truth-value of example (i) is 1 and that of example (ii) is 0.

In the rest of this subsection we will confine ourselves to a special kind of subjunctive sentences called counterfactuals. A counterfactual is an expression of the form "if \( A \) were the case, then \( B \) would be the case", where \( A \) is supposed to be false.

A counterfactual is a subjunctive conditional, but not all subjunctive conditionals are counterfactuals. Consider the argument: "The murderer used an ice pick. But if the butler had done it, he wouldn't have used an ice pick. So the murderer must have been someone else". If the subjunctive conditional (the "if . . . , then . . ."-clause) were a counterfactual, then the speaker would assume that the butler was not guilty. Hence, the speaker would be presupposing that the conclusion of his argument is true, which makes the whole argument useless \(^{11}\).

We shall denote the counterfactual by \( \Box \rightarrow \).

The counterfactual has other properties than the material, strict and relevant implication:

\(^{11}\)This example is from [Stalnaker, 81].
1. the counterfactual is not transitive, i.e. not \[ A \rightarrow B \quad B \rightarrow C \implies A \rightarrow C \];

2. it does not have the property of contraposition, i.e. not \[ A \rightarrow B \implies \neg B \rightarrow \neg A \];

3. and it does not have the property of strengthening, i.e. not \[ A \rightarrow B \implies (A \land C) \rightarrow B \].

In order to show this, consider the following counterexamples (from [Lewis, 73b, p. 10 and pp. 33–35]).

1. If J. Edgar Hoover had been born a Russian, then he would have been a communist.

   If he had been a communist, he would have been a traitor.

   \[ \therefore \] If he had been born a Russian, he would have been a traitor.

   (One might agree with the first two sentences without accepting the last one as true.)

2. If Boris had gone to the party, Olga would still have gone.

   \[ \therefore \] If Olga had not gone, Boris would still not have gone.

   (Suppose that Boris wanted to go, but stayed away solely in order to avoid Olga, so the conclusion if false; but Olga would have gone all the more willingly if Boris had been there, so the premiss is true.)

3. If I walked on the lawn, no harm at all would come of it.

   \[ \therefore \] If I and my elephant walked on the lawn, no harm at all would come of it.

   (One may easily accept the first sentence and reject the second one.)

   We say that \( A \rightarrow B \) is true in world \( w \) (\( w \models A \rightarrow B \)) if either \( A \) is impossible in \( w \) or there is an accessible \( A \land B \)-world \( w' \), which is "closer" to \( w \) than every \( A \land \neg B \)-world is, where we mean by a \( C \)-world simply a world in which \( C \) is true. Here the notion "\( w_1 \) is closer to \( w \) than \( w_2 \)" is a primitive relation. (These ideas are from R. Stalnaker and D. Lewis, ±1970.)

   We now describe how the counterfactual is related to other conditionals. First we connect the counterfactual with the material implication.

   Counterfactuals with true antecedents reduce to material conditionals. In other words, the following two inference-patterns are valid:

   \[ (a) \quad \frac{A \land \neg B}{\neg (A \rightarrow B)} \quad \text{and} \quad (b) \quad \frac{A \land B}{A \rightarrow B}; \]

   that is, our truth conditions guarantee that whenever the premiss is true in a world, so is the conclusion.

   The validity of the first inference-pattern, (a), guarantees also the validity of the inference from a counterfactual to a material conditional and the validity of modus ponens for a counterfactual conditional:

   \[ \frac{A \rightarrow B}{A \supset B} \quad \text{and} \quad \frac{A \quad A \rightarrow B}{B} \] (see [Lewis, 73b, p. 26–27]).
We also have the following inference: \[ \square(A \supset B) \]
\[ \frac{A}{B} \square \rightarrow B. \]

There is also a relation between the counterfactual and necessity. It is interesting to note that necessity can be defined in terms of the counterfactual and falsity: \( \square A \leftrightarrow (\neg A \square \rightarrow \bot) \) ("A is necessary").

We finally note that there exists a connection between the counterfactual and the so-called comparative possibility <, where \( A < B \) is true in a world \( w (w \models A < B) \) if and only if some accessible \( A \)-world is closer to \( w \) than any \( B \)-world. One can express comparative possibility in terms of the counterfactual and the modal operator \( \diamond \) (for possibility) as follows: \( A < B \equiv (\diamond A \land ((A \lor B) \square \rightarrow (A \land \neg B))) \) ("it is less remote from actuality that \( A \) than that \( B \)"").

Conversely, we can take comparative possibility as primitive and define the counterfactual in terms of it as follows:
\[ \diamond A \leftrightarrow (A < \bot) \) ("A is possible") and \((A \square \rightarrow B) \leftrightarrow (\neg \diamond A) \lor ((A \land B) < (A \land \neg B)).\]

By specifying mathematically what we mean by "\( w_1 \) is closer to \( w \) than \( w_2 \)", we can develop adequate possible-world semantics (see Subsection 2.2) for the counterfactual, a notion of validity (\( \models \)) and a notion of provability (\( \vdash \)), such that a counterfactual formula \( A \) is valid if and only if \( A \) is provable. See [Lewis, 73a, pp. 441-443] or [Lewis, 73b].

Unfortunately, the analysis of counterfactuals given above is not without philosophical problems.

Consider Simplification, i.e. the inference pattern \[ \frac{(A \lor B) \square \rightarrow C}{A \square \rightarrow C} \text{ or } \frac{(A \lor B) \square \rightarrow C}{B \square \rightarrow C}.\]

Now Simplification and Interchange of logical equivalents combine to give clearly unacceptable inferences. For example, suppose \( A \square \rightarrow B \); then by Interchange \( (A \land C) \lor (A \land \neg C) \square \rightarrow B \) and hence by Simplification \( (A \land C) \square \rightarrow B \). So, we have the property of strengthening. See the Journal of Philosophical Logic 6 (1977), pp. 355-363, for this objection of Ellis, Jackson and Pargetter (and for other references) and D. Lewis' reply.

We also refer the interested reader to the Special Issue on Counterfactuals and Laws of Nous, Volume XIII, Number 4, November 1979.

2.5 Intuitionistic implication

In this subsection we sketch the ideas and the possibilities of intuitionistic implication. We do not go as much into details as we did in the previous subsection. For more explicit information, we refer to the cited works mentioned below.

In the analysis of the conditionals discussed so far, we have spoken in terms of truth and falsehood of propositions and we did not worry about the question whether I have or do not have a proof of a given proposition.

In intuitionistic logic one speaks rather in terms of "I have a proof of \( A \)" , "I have a proof of \( \neg \neg A \)" or "I neither have a proof of \( A \) nor a proof of \( \neg A \)." (See [Troelstra and van Dalen 88].)

The analysis of \textit{intuitionistic implication}, denoted by "\( \dashv \)", is the following:
\( B \vdash C \) holds ("I have a proof of \( B \vdash C \)"") if I have a construction which transforms each proof of \( B \) into a proof of \( C \).
An implicate formula is by definition *intuitionistically provable* iff it can be obtained by a finite number of applications of Modus Ponens, \( \frac{B \rightarrow B \rightarrow A}{C} \), to instances of the following two axiom schemata:

1. \( A \rightarrow (B \rightarrow A) \)
2. \( (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \).

We note the following. The intuitionistic version of axiom 2 for material implication, viz. \( (A \rightarrow B) \rightarrow A \), does not hold. However, \( (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \), the intuitionistic version of axiom 3 for material implication, is intuitionistically provable.

If we assign to \( P \) the value 1 if we have a proof of \( P \) and the value 0 if we do not (yet) have a proof of \( P \), then we might make the following table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>?</td>
</tr>
</tbody>
</table>

Now one can associate to each proposition \( P \) an infinite sequence \( a_1 a_2 a_3 \ldots \) of zero's and one's. In such a sequence we interpret \( a_n = 1 \) as: "I have a proof of \( P \) at time \( n \)", and \( a_n = 0 \) as: "at time \( n \) I do not have a proof of \( P \)". Furthermore, we require that such a sequence \( a_1 a_2 a_3 \ldots \) has the property that, for all natural numbers \( n \), if \( a_n = 1 \), then \( a_{n+1} = 1 \) (i.e. if I have a proof of \( P \) at time \( n \), then I also have a proof of \( P \) at any later time). Note that it is possible to have no proof of \( P \) at time \( n \) (so \( a_n = 0 \)), but to have a proof of \( P \) at a later time (\( a_m = 1 \) for some \( m > n \)).

Along these lines one can develop an adequate notion of validity for intuitionistic implication (see [Beth, 59], [Kripke, 65], [de Swart, 77]) as follows.

\( M = (K, R, \models) \) is a *Kripke model (for intuitionistic logic)* if and only if

1. \( K \) is a non-empty set (of possible proof-situations, rather than of possible worlds),
2. \( R \) is a binary relation on \( K \), such that \( R \) is reflexive and transitive (\( w Rw' \) is to be read as: \( w' \) is a later proof-situation than \( w \)),
3. \( \models \) is a relation between elements of \( K \) and atomic formulas \( P \), such that if \( w \models P \) and \( w Rw' \), then \( w' \models P \) (if \( P \) has been proved in the situation \( w \) and \( w' \) is a later proof-situation than \( w \), then \( P \) has also been proved in the situation \( w' \)).

Now \( M \models_w A \) ("\( A \) has been proved in the situation \( w \) of the model \( M \)") is defined as follows:

\( M \models_w A \) iff \( w \models P \) (\( P \) atomic),
\( M \models_w B \rightarrow C \) iff for all \( w' \in K \), if \( w Rw' \) and \( M \models_{w'} B \), then \( M \models_{w'} C \).

(Condition 3 above can be generalized to: if \( M \models_w A \) and \( w Rw' \), then \( M \models_{w'} A \), for all (atomic or non-atomic) formulas \( A \).)
$M$ is a model for $A$ iff for all $w \in K$ it holds that $M \models_w A$. And $A$ is (intuitionistically) valid, once more denoted by $\models A$, iff every model $M$ is a model for $A$. And again one can show that the (intuitionistically) valid (implicational) formulas are precisely the intuitionistically provable formulas.

In Subsection 3.5 we shall construct a Kripke countermodel for $((A \rightarrow B) \rightarrow A) \rightarrow A$.

One can also introduce other intuitionistic connectives. In particular, intuitionistic negation is defined by $\neg A := A \rightarrow \bot$, where $\bot$ stands for a contradiction ("falsity"). Consequently, $M \models_w \neg B$ iff for all $w' \in K$, if $wRw'$, then not $M \models_w B$ (supposing there can be no proof-situation $w'$ in which $\bot$ is true).

Finally we note that the notion of intuitionistic implication described above can be mimicked constructively in typed lambda-calculus. In so-called type theories one may encode implications as product types, and their "proofs" as terms. For more details, see [Nederpelt, 90].

3 Decision procedures

For each type of implication discussed so far, there is an effective method which for each formula $A$ in a finite number of steps either yields a countermodel for $A$ (showing that $A$ is not valid), or a (formal) proof of $A$ (showing that $A$ is valid).

The methods described are uniform and efficient, and "work" in all cases that we consider. The general approach is often called the method of semantic tableaux. It was invented by Gentzen in 1934 and rediscovered by Beth, Hintikka and others in the mid-fifties. See [Hodges, 83, pp. 21-23] or [Sundholm, 83, pp. 180-186] for details and references. Our presentation is very close to [Smullyan, 68] and [Fitting, 83].

The methods amount to the following: we try to construct a countermodel for $A$ according to certain rules. If such a countermodel exists, then we actually find one. If it turns out to be impossible to do so, we have in fact constructed a (formal) proof of $A$. So, we might also say that we try to construct in a systematic way a formal proof of $A$.

For the different types of implication we have to develop different rules, according to which we try to construct countermodels.

It should be noted here that we can easily adjust our decision methods to the case that we extend any one of the implicational logics considered, with conjunction, disjunction and/or negation.

But if we add quantifiers we lose in general our decidability. The intuitive reason for this is that we may continue to apply some of the quantifier rules again and again, each time introducing new variables.

3.1 Material implication

Let us start with material implication.

Let $F(A)$ mean that $A$ is false and $T(A)$ that $A$ is true. In order to see whether it is possible to construct a countermodel for $A$, we suppose $F(A)$, i.e., that $A$ is false and we analyze under what conditions this is possible. If it turns out to be possible to satisfy these conditions, we can construct such a countermodel; if not, then we can show that $A$ is provable.

The conditions for material implication are expressed by the following two rules, the $T\lor$- and the $F\land$-rule, respectively:
where $S$ is a sequent, i.e., any set of expressions of the form $T(E)$ or $F(E)$, with $E$ being a formula. In words: if $B \supset C$ is true, then either $B$ is false or $C$ is true (the vertical bar (|) differentiates between these two cases) and if $B \supset C$ is false, then $B$ is true and $C$ is false.

To see how the method works, let us apply the rules $T \mathbin{\supset}$ and $F \mathbin{\supset}$ to $(P \mathbin{\supset} Q) \mathbin{\supset} Q$. We obtain the following diagram:

\[
\begin{align*}
F((P \mathbin{\supset} Q) \mathbin{\supset} Q) & : \text{suppose that } (P \mathbin{\supset} Q) \mathbin{\supset} Q \text{ is false;} \\
T(P \mathbin{\supset} Q), F(Q) & : \text{then, by the rule } F \mathbin{\supset}, \text{ } P \mathbin{\supset} Q \text{ is true and } Q \text{ is false;} \\
F(P), F(Q) & : \text{then both } P \text{ and } Q \text{ are false,} \\
T(Q), F(Q) & : \text{or both } Q \text{ is true and } Q \text{ is false.}
\end{align*}
\]

Now this latter case $(T(Q), F(Q))$ is impossible, but the former case $(F(P), F(Q))$ is quite possible. So let us give to both $P$ and $Q$ the value 0, then $P \mathbin{\supset} Q$ gets the value 1, corresponding with the occurrence of $T(P \mathbin{\supset} Q)$ at the second line, and $(P \mathbin{\supset} Q) \mathbin{\supset} Q$ gets the value 0, corresponding with the occurrence of $F((P \mathbin{\supset} Q) \mathbin{\supset} Q)$ at the first line.

So we actually have constructed a countermodel for $(P \mathbin{\supset} Q) \mathbin{\supset} Q$.

If we apply the rules $T \mathbin{\supset}$ and $F \mathbin{\supset}$ to $P \mathbin{\supset} (Q \mathbin{\supset} P)$, we find the following diagram:

\[
\begin{align*}
F(P \mathbin{\supset} (Q \mathbin{\supset} P)) & : \text{suppose that } P \mathbin{\supset} (Q \mathbin{\supset} P) \text{ is false;} \\
T(P), F(Q \mathbin{\supset} P) & : \text{then } P \text{ is true and } Q \mathbin{\supset} P \text{ is false;} \\
T(P), T(Q), F(Q) & : \text{then both } P \text{ and } Q \text{ are true and } Q \text{ is false.}
\end{align*}
\]

Since this is impossible, we find that it is impossible for $P \mathbin{\supset} (Q \mathbin{\supset} P)$ to be false and one can conceive the diagram as a formal proof of $P \mathbin{\supset} (Q \mathbin{\supset} P)$.

More precisely, from such a diagram one can construct a formal proof of the formula in question by reading the rules upwards instead of downwards, interpreting the sequents $T(B_1), \ldots, T(B_m), F(C_1), \ldots, F(C_n)$ as $B_1 \supset \ldots (B_m \supset C_1 \vee \ldots \vee C_n) \ldots$. The lowest sequents in such a diagram are of the form $T(P), F(P), \ldots$ and correspond to a formula $\ldots \supset (P \mathbin{\supset} P \vee \ldots)$, which is formally provable. The other sequents in the diagram correspond to a formula which is formally deducible from the formula(s) corresponding to the sequents immediately below it. Since $F(A)$ is at the top of the diagram, it follows that $A$ is formally provable.

So, the decision-method for material implication is as follows: for any formula $A$, start with $F(A)$ and apply the rules $T \mathbin{\supset}$ and $F \mathbin{\supset}$ as many times as possible.

If all cases, mentioned in the last line, turn out to be impossible, then one can show along the lines sketched above that the diagram constructed actually yields a (formal) proof of $A$, and hence $A$ is then valid.

If at least one case in the last line is possible, then assign 1 to $P$ if $T(P)$ occurs in this case and assign 0 to $P$ if $F(P)$ occurs in this case. Then one easily shows that for any formula $B$,

1. if $T(B)$ occurs in the diagram constructed, $B$ gets the value 1 under the assignment given above; 
2. if $F(B)$ occurs in the diagram constructed, $B$ gets the value 0 under the assignment given above.

Since the diagram constructed starts with $F(A)$, the formula $A$ will get the value 0 under the given assignment and we have constructed a countermodel for $A$, showing that $A$ is not valid.
3.2 Strict implication

Strict implication, $\rightarrow$, is defined by $B \rightarrow C := \Box(B \supset C)$. In order to adapt our decision-method to strict implication, we therefore only have to add rules for necessity ($\Box$), called $T\Box$ and $F\Box$, respectively:

$$
\begin{align*}
T\Box &: S, T(\Box B) \quad F\Box &: S, F(\Box B) \\
\to &: S, T(B) &\quad \to &: S, F(B)
\end{align*}
$$

where $S^{\Box}$ consists of those expressions occurring in $S$ that have the form $T(\Box E)$. Since in the transition from $S$ to $S^{\Box}$ formulas may get lost, we have drawn a thick line in rule $F\Box$ to emphasize the possible "loss of information".

Intuitively, the rules $T\Box$ and $F\Box$ say the following.

$T\Box$: if $\Box B$ is true in a world $w$, then $B$ is true in this $w$ (supposing that each world $w$ is accessible from itself).

$F\Box$: if $\Box B$ is false in a world $w$, then there is some world $w'$ accessible from $w$ in which $B$ is false. Only propositions which are necessary in $w$ will also be true in $w'$ (supposing that the accessibility-relation is transitive), hence we made the transition from $S$ to $S^{\Box}$.

Applying, for instance, our rules to $P \supset (Q \supset P)$, i.e. $\Box(P \supset (Q \supset P))$, we get the following.

$$
\begin{align*}
F(\Box(P \supset (Q \supset P))) &: \text{suppose that } P \supset (Q \supset P) \text{ is false in our actual world } w_0; \\
F(P \supset (Q \supset P)) &: \text{then there is an accessible world } w \\
\text{in which } P \supset (Q \supset P) \text{ is false;} \\
T(P), F(\Box (Q \supset P)) &: \text{and hence } P \text{ is true in } w \text{ and } \Box (Q \supset P) \text{ is false in } w; \\
F(Q \supset P) &: \text{then there is an accessible world } w' \text{ in which } Q \supset P \text{ is false;} \\
T(Q), F(P) &: \text{and hence } Q \text{ is true in } w' \text{ and } P \text{ is false in } w'.
\end{align*}
$$

And from the above we can immediately construct a countermodel for $P \supset (Q \supset P)$, which happens to be the same as the one considered in Subsection 2.2:

$$
\begin{align*}
F(\Box(P \supset (Q \supset P))) \quad F(P \supset (Q \supset P)) \\
T(P), F(\Box (Q \supset P)) \\
F(Q \supset P) \\
T(Q), F(P)
\end{align*}
$$

However, applying our decision-method to $Q \supset (P \supset P)$, i.e. $\Box(Q \supset (P \supset P))$, we find the following:
Supposing that \( Q \rightarrow (P \rightarrow P) \) is false, we find that in some world, \( P \) is both true and false, which is impossible. Hence there is no countermodel for \( Q \rightarrow (P \rightarrow P) \) and one can prove that the diagram constructed actually yields a (formal) proof of \( Q \rightarrow (P \rightarrow P) \).

In order to see this, note that, going from bottom to top, the formulas corresponding to the sequents in the diagram above are respectively

\[ P \supset P, \]
\[ P \supset P, \]
\[ Q \supset \Box (P \supset P), \]
\[ Q \supset \Box (P \supset P), \]
\[ \Box (Q \supset \Box (P \supset P)). \]

The formula \( P \supset P \) is (formally) provable; hence \( \Box (P \supset P) \) is provable; therefore \( Q \supset \Box (P \supset P) \) is provable; and consequently \( \Box (Q \supset \Box (P \supset P)) \) is provable.

Of course, this decision method does not only work for formulas with strict implication, but also, more generally, for formulas containing material implication and possibility (\( \Box \)) as well. For example, we apply our decision method to \( (Q \rightarrow P) \supset \Box Q \supset (P \rightarrow Q) \). After elimination of \( \rightarrow \), this formula becomes \( (\Box (Q \supset P) \supset \Box Q) \supset (P \supset Q) \). The method gives:

\[
\begin{align*}
\frac{\varphi((\Box (Q \supset P) \supset \Box Q) \supset (P \supset Q))}{\top (\Box (Q \supset P) \supset \Box Q), \varphi((P \supset Q))} & \quad \mid \quad \varphi((\Box (Q \supset P) \supset \Box Q)), \varphi((P \supset Q)) \\
\vdots & \quad \mid \quad \top (\Box Q), \varphi((P \supset Q)) \\
\top (\Box Q), \top (P), \varphi(Q) & \quad \mid \quad \top (Q), \top (P), \varphi(Q)
\end{align*}
\]

Note that in the right-hand part another continuation would have been possible; we have chosen the one leading to closure (\( Q \) is both true and false).

For the left-part there are two ways to go on: either apply rule \( \varphi \) to \( \varphi((Q \supset P)) \), in which case \( \varphi((P \supset Q)) \) is lost, or apply rule \( \varphi \) to \( \varphi((P \supset Q)) \), in which case \( \varphi((Q \supset P)) \) is lost:

\[
\begin{align*}
\varphi((Q \supset P)), \varphi((P \supset Q)) & \quad \mid \quad \varphi((Q \supset P)), \varphi((P \supset Q)) \\
\top (Q), \varphi(P) & \quad \mid \quad \varphi((P \supset Q)) \\
\varphi((Q \supset P)) & \quad \mid \quad \varphi((Q \supset P)) \\
\top (Q), \top (P), \varphi(Q) & \quad \mid \quad \top (Q), \top (P), \varphi(Q)
\end{align*}
\]

From this we can immediately construct a countermodel for \( (\Box (Q \supset P) \supset \Box Q) \supset (P \supset Q) \), as shown to the right above.
The rules $T\square$ and $F\square$ given above are actually the rules for the system $S_4$ of modal logic. Rules $T\square$ and $F\square$ for many other systems of modal logic and effective decision methods for these logics are described in [de Swart, 80].

3.3 Relevant implication

The decision-method for relevant implication is afforded by the following two rules.

$$
T \Rightarrow \begin{array}{c}
S, T(B \Rightarrow C)_{a} \\
S, \ F(B)_{b} \mid S, T(C)_{a \cup b} \\
\text{for any } b
\end{array}
F \Rightarrow \begin{array}{c}
S, \ F(B \Rightarrow C)_{a} \\
S, T(B)_{\{k\}}, \ F(C)_{a \cup \{k\}} \\
\text{where } k \text{ does not occur in } a.
\end{array}
$$

Here $a$ and $b$ are relevance-indices, which in particular are finite sets of natural numbers, corresponding with the information-pieces of Subsection 2.3.

Intuitively, these rules say the following.

$T \Rightarrow$: If $B \Rightarrow C$ is true on the basis of the information $a$, then for any piece of information $b$, either it is not the case that $B$ is true on the basis of $b$, or $C$ is true on the basis of $a \cup b$.

$F \Rightarrow$: If it is not the case that $B \Rightarrow C$ is true on the basis of $a$, then for some piece of information $\{k\}$ it holds that $B$ is true on the basis of $\{k\}$, and it is not the case that $C$ is true on the basis of $a \cup \{k\}$. Since this piece of information $\{k\}$ needs not contain any information already available, we have to demand that $k$ is not in $a$.

To decide whether a relevant-implication formula $A$ is provable (valid) or not, we start with $F(A)_\emptyset$, where $\emptyset$ is the empty set, and next apply our two rules for $\Rightarrow$ as many times as possible. We give again two examples.

In the first example we examine the formula $Q \Rightarrow (P \Rightarrow P)$:

$$
\begin{align*}
&\ F((Q \Rightarrow (P \Rightarrow P))_{\emptyset} \\
&\ F(Q)_{\{1\}}, T(F(P \Rightarrow P))_{\{1\}} \\
&\ T(Q)_{\{1\}}, T(P)_{\{2\}}, F(P)_{\{1,2\}}
\end{align*}
$$

So we do not find a contradiction; we only find that $Q$ is true on the basis of $\{1\}$, that $P$ is true on the basis of $\{2\}$ and that it is not the case that $P$ is true on the basis of $\{1,2\}$. Hence we have actually constructed a countermodel $M$ for $Q \Rightarrow (P \Rightarrow P)$, the same as the one considered in Subsection 2.3. Given that $\{1\} \models Q$, $\{2\} \models P$ and not $\{1,2\} \models P$, we can compute that not $M \models \{1\} P \Rightarrow P$, corresponding with the occurrence of $F(P \Rightarrow P)_{\{1\}}$ at the second line, and that not $M \models \emptyset Q \Rightarrow (P \Rightarrow P)$, corresponding with the occurrence of $F(Q \Rightarrow (P \Rightarrow P))_{\emptyset}$ at the first line.

The second example is the formula $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

$$
\begin{align*}
&\ F((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))_{\emptyset} \\
&\ T(A \Rightarrow B)_{\{1\}}, T((B \Rightarrow C) \Rightarrow (A \Rightarrow C))_{\{1\}} \\
&\ T(A \Rightarrow B)_{\{1\}}, T(A)_{\{2\}}, T(B \Rightarrow C)_{\{2\}}, F(A \Rightarrow C)_{\{1,2\}} \\
&\ T(A \Rightarrow B)_{\{1\}}, T(A)_{\{3\}}, T(B \Rightarrow C)_{\{2\}}, F(C)_{\{1,2,3\}}
\end{align*}
$$

$F(A)_{\{3\}}, T(B \Rightarrow C)_{\{2\}}, T(A)_{\{3\}}, F(C)_{\{1,2,3\}} | T(B)_{\{1,3\}}, T(B \Rightarrow C)_{\{2\}}, T(A)_{\{3\}}, F(C)_{\{1,2,3\}}$

$S_1 | S_2 | S_3 | S_4$
where $S_1 = \mathcal{F}(A)_{\{3\}}, \mathcal{F}(B)_{\{1,3\}}, T(A)_{\{3\}}, \mathcal{F}(C)_{\{1,2,3\}}$

$S_2 = \mathcal{F}(A)_{\{3\}}, T(C)_{\{1,2,3\}}, T(A)_{\{3\}}, \mathcal{F}(C)_{\{1,2,3\}}$

$S_3 = T(B)_{\{1,3\}}, \mathcal{F}(B)_{\{1,3\}}, T(A)_{\{3\}}, \mathcal{F}(C)_{\{1,2,3\}}$

$S_4 = T(B)_{\{1,3\}}, T(C)_{\{1,2,3\}}, T(A)_{\{3\}}, \mathcal{F}(C)_{\{1,2,3\}}$

Since all of $S_1$ through $S_4$ contain a contradiction, we find that there is no countermodel for $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$. Hence it is a valid formula in relevant-implication logic.

### 3.4 Counterfactuals

In a similar, but more complicated, way one can develop effective decision methods for counterfactuals. See [de Swart, 83] and [Gent, 9-].

### 3.5 Intuitionistic implication

The rules for intuitionistic implication are:

\[
\text{T} \quad \frac{S, T(B \Leftarrow C)}{S, T(B \Leftarrow C), \mathcal{F}(B)} \quad \text{\mathcal{F}} \quad \frac{S, \mathcal{F}(B \Leftarrow C)}{S, T(B \Leftarrow C), T(C)}
\]

where $S_T$ only contains all expressions of the form $T(E)$ that occur in $S$. Since in the transition from $S$ to $S_T$ formulas may get lost, we have drawn a thick line in rule $\mathcal{F} \leftarrow$.

In these rules, $T(A)$ has to be read as: "I have a proof of $A$" and $\mathcal{F}(A)$ as: "I do not have a proof of $A$" (which does not necessarily mean that $A$ is false!). Then the readings of $\text{T} \Leftarrow$ and $\mathcal{F} \Leftarrow$ are as follows:

- $\text{T} \Leftarrow$: If I have a proof of $B \Leftarrow C$ in proof-situation $w$, then in all proof-situations $w'$ which are accessible from $w$, either I do not have a proof of $B$, or I have a proof of $C$. In other words: for all such $w'$, if I have a proof of $B$ in $w'$, then I also have a proof of $C$ in $w'$.

- $\mathcal{F} \Leftarrow$: If I do not have a proof of $B \Leftarrow C$, then I can imagine a situation in which I do have a proof of $B$ without having a proof of $C$. In this new situation formulas may have been proved that were not proved before. Hence the transition from $S$ to $S_T$.

Applying our rules to, for instance, $((P \Leftarrow Q) \Leftarrow P) \Leftarrow P$, we find:

\[
\text{T} \quad \frac{\mathcal{F}((P \Leftarrow Q) \Leftarrow P) \Leftarrow P)}{\mathcal{F}(P \Leftarrow Q), \mathcal{F}(P)} \quad \text{\mathcal{F}} \quad \frac{\mathcal{F}(P \Leftarrow Q), \mathcal{F}(P)}{T(P), \mathcal{F}(P)}
\]

This shows that the formula in question is not intuitionistically provable, since the left part does not contain a contradiction (i.e. both $T(A)$ and $\mathcal{F}(A)$ for some formula $A$). Actually, from the left part above one can construct the following Kripke-model $M$ such that not $M \models ((P \Leftarrow Q) \Leftarrow P) \Leftarrow P$:
4 Conclusions

We showed in this short overview that one can have different views on the notion of implication. In mathematics, material implication is the most commonly used version. It satisfies the purposes in that field of science in a large measure.

In natural language, however, this view on implication is problematic. Conversation rules have to be taken into account and one may require more of an implication, for example necessity and relevance. The discussion about these matters started already in the time of the Greek philosophers, about three hundred years B.C.

In the present time, the nature of the implication is once more of great importance. Renewed interest in this subject arises in computer science and in linguistics.

In computer science, aspects of time can be expressed in terms of the modal operator for necessity. This has a direct influence on the type of implication that is suitable. Temporal matters play an important role in e.g. message passing and in real-time applications. In artificial intelligence, one uses the same modal operator for expressing facts about, for example, knowledge and belief.

We gave an overview of the most important aspects connected with “implication”. Having given the main features of the truthfunctional material implication, we discussed strict implication (for capturing “necessity”) and relevant implication (for “relevance”). Thereupon we summarized the main facts about a special class of subjunctive implication, viz. counterfactuals. The last type of implication of which we recapitulated the characteristics, was the intuitionistic implication, where “truth” is considered as “having a proof at hand”.

Each of the implications under consideration has been treated under two different angles: the semantic angle, describing the meaning of the formulas in a more or less formal way, and the syntactic one, where derivation of formulas is the first purpose. We mentioned the fact that (semantic) validity and (syntactic) derivability coincide, referring to the literature for the respective proofs.

Finally, we accentuated the constructive character of the implicational logics by giving decision methods for each of the implications mentioned. We presented these methods, which are described in the literature, in a uniform manner. The decision methods are effective, in the following sense: for any given implicational formula one obtains in a finite number of steps either a formal proof of the formula in question (showing that the formula is valid) or a counterexample, showing that it is not valid (and hence not provable).

The last word about the time-honoured notion of implication has not yet been said. However, the present situation, as summarized in this paper, covers many important aspects of this notion. The purpose of the authors was to inform the interested reader about the main topics in this area, which is anew of current interest.
5 Acknowledgements

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