Weighted subcoercive operators on Lie groups

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Weighted subcoercive operators on Lie groups

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Abstract

Let $U$ be a continuous representation of a Lie group $G$ on a Banach space $\mathcal{X}$ and $a_1, \ldots, a_d$ an algebraic basis of the Lie algebra $g$ of $G$, i.e., the $a_1, \ldots, a_d$ together with their multi-commutators span $g$. Let $A_i = dU(a_i)$ denote the infinitesimal generator of the continuous one-parameter group $t \mapsto U(\exp(-t a_i))$ and set $A^\alpha = A_{i_1} \cdots A_{i_n}$ where $\alpha = (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, d\}$. We analyze properties of $m$-th order differential operators

$$dU(C) = \sum_{\alpha: |\alpha| \leq m} c_\alpha A^\alpha$$

with coefficients $c_\alpha \in \mathbb{C}$.

If $L$ denotes the left regular representation of $G$ in $L_2(G)$ then $dL(C)$ satisfies a Gårding inequality on $L_2(G)$ if and only if the closure of each $dU(C)$ generates a holomorphic semigroup $S$ on $\mathcal{X}$ which is quasi-contractive, i.e., $\|S_t\| \leq e^{\omega t}$, in an open representation independent subsector of the sector of holomorphy and the action of $S_t$ is determined by a smooth, representation independent, kernel $K_t$ which, together with its derivatives $A^\alpha K_t$, satisfies $m$-th order Gaussian bounds.

Alternatively, $dL(C)$ satisfies a Gårding inequality on $L_2(G)$ if, and only if, the closure of $dL(C)$ generates a holomorphic, quasi-contractive, semigroup satisfying bounds $\|A_i S_t\|_{2 \rightarrow 2} \leq c t^{-1/m} e^{\omega t}$ for all $t > 0$ and $i \in \{1, \ldots, d\}$.

These results extend to operators for which the directions $a_1, \ldots, a_d$ are given different weights. The unweighted Gårding inequality is a stability condition on the the principal part, i.e., the highest order part, of $dU(C)$ but in the weighted case the condition is on the part of $dU(C)$ with the highest weighted order.

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1 Introduction

The theory of partial differential operators extends naturally from the Euclidean space $\mathbb{R}^d$ to a general $d$-dimensional Lie group. The operators are defined in any continuous Banach space representation $U$ of $G$ as polynomials in the associated representatives of the Lie algebra $g$ of $G$. The first interesting case occurs for polynomials formed from the representatives of a vector space basis of $g$. An operator of this type is defined to be strongly elliptic if the coefficients of the highest order part of the polynomial satisfy the usual strong ellipticity condition of the $\mathbb{R}^d$-theory. Langlands, in an unpublished thesis [Lan1] (see also [Lan2]), proved that the closure of each strongly elliptic operator $H$ generates a holomorphic semigroup $S$ with a smooth, fast decreasing, representation-independent, integral kernel $K$. Subsequently, Robinson [Rob1] established that $K$, together with its derivatives, satisfies Gaussian bounds of the appropriate order. Moreover, Bratteli, Goodman, Jørgensen and Robinson [BGJR] proved that each unitary representation $S$ is quasi-contractive and $H$ satisfies a Gårding inequality, i.e., a coercivity condition. (All these results are described at length in [Rob2].) Finally a limiting argument of Folland [Fol2] shows that the Gårding inequality for $H$ in the left regular representation of $G$ on $L^2(G)$ implies strong ellipticity. Thus strong ellipticity, or $\mathbb{R}^d$-coercivity, is equivalent to $G$-coercivity for a partial differential operator $H$ expressed in terms of a vector space basis of $g$ and these conditions imply that $H$ is the pregenerator of a semigroup with good boundedness and analyticity properties and a universal ‘Gaussian’ kernel. One of our results is a converse of the last conclusion: if $H$ is the pregenerator of a quasi-contractive semigroup on $L^2(G)$ with a good ‘Gaussian’ kernel then $H$ must be a $G$-coercive operator. Hence one concludes that there is an equivalence between $\mathbb{R}^d$-coercivity, $G$-coercivity and good semigroup properties. Our main result establishes a similar equivalence for weighted subelliptic operators.

We consider operators $H$ which are polynomials in the representatives of a (Lie-)algebraic basis of $g$ with different weights assigned to each of the directions in the basis. The order of $H$ is defined as the weighted order of the polynomial and the weighting is taken into account in the definition of distance etc.. Since in general there is no obvious direct definition of coercivity in terms of the coefficients of the polynomial we introduce a notion of (weighted) $G$-subcoercivity in terms of an appropriately weighted Gårding inequality. We then establish that $H$ is $G$-subcoercive if and only if it generates a holomorphic, quasi-contractive, semigroup on $L^2(G)$ with a universal ‘Gaussian’ kernel. This equivalence encompasses all earlier known results and gives a straightforward characterization of the ‘heat’ semigroups on the Lie group $G$. The proof of the statements relies on a combination of all previous arguments together with two new techniques.

First, we introduce the notion of a reduced weighted basis. The reduced basis is an algebraic subbasis of the original basis in which certain overweight directions have been eliminated. (The definition of overweight is related to the Lie-algebraic structure.) Our strategy is to establish the main structural features for operators defined with a reduced basis and then, by auxiliary arguments, to lift the results to operators expressed in terms of the original unreduced basis. If all weights are equal to one, or if the weights satisfy the compatibility conditions imposed in our earlier analysis [EIR5] of weighted strongly elliptic operators, then the reduction process has no effect. It is, however, interesting to
note that a weighted vector space basis of $g$ usually yields a weighted algebraic basis after reduction. Therefore subelliptic techniques automatically enter the analysis of weighted strongly elliptic operators.

Secondly, we associate with each weighted algebraic basis of $g$ a homogeneous (nilpotent) group $G_0$ which serves as a local approximation of $G$. The group $G_0$ is uniquely determined by $G$ and the weighted algebraic basis by a canonical contraction of $g$. Langlands original analysis of (unweighted) strongly elliptic operators was based on the local approximation of $G$ by $\mathbb{R}^d$. But in our approach $G_0 = \mathbb{R}^d$ if and only if the basis used in the construction is a full vector space basis with weights satisfying the compatibility conditions of [EIR5]. Therefore the analysis of general weighted operators, or even unweighted subelliptic operators, compels the use of non-commutative approximants $G_0$. Since the work of Rothschild and Stein [RoS] it has become a standard procedure to construct local nilpotent approximants to a Lie group. Given an algebraic basis of $g$ with $d'$ elements and rank $r$ the Rothschild—Stein approximant $\tilde{G}$ of $G$ is the nilpotent group with $d'$ generators which is free of step $r$. This group was used in our earlier work on subelliptic operators [EIR3] [EIR7]. The disadvantage of the Rothschild—Stein approach is that $\tilde{G}$ is usually of larger dimension that $G$ and the extra dimensions introduce superfluous conditions and complications. But the approximant $G_0$ used in the current analysis has the same dimension as $G$ and this is advantageous in the application of the parametrix arguments which are used to lift results from $G_0$ to $G$.

In order to give a more precise description of our results it is necessary to introduce more detailed notation and definitions.

Generally we adopt the notation of [Rob2] with the modifications used in [EIR3] and [AER]. Let $G$ be a $d$-dimensional connected Lie group with Lie algebra $g$ and $(\mathcal{X}, G, U)$ a strongly, or weakly, continuous representation of $G$ on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. If $a_i \in g$ then $A_i = dU(a_i)$ will denote the generator of the one-parameter subgroup $t \mapsto U(\exp(-ta_i))$ of the representation. Let $a_1, \ldots, a_{d'}$ be an algebraic basis of $g$, i.e., a set of linearly independent elements which together with their multi-commutators span $g$, and $w_1, \ldots, w_{d'} \in [1, \infty)$ a $d'$-tuple of numbers which we call weights. The group $G$ can be equipped with a distance $| \cdot |$ which is naturally determined by the algebraic basis $a_1, \ldots, a_{d'}$ and the weights $w_1, \ldots, w_{d'}$. The detailed definition of this distance will be given in Section 6. The distance then determines a local ‘dimension’ $D' > 0$ of the group such that $c^{-1} \delta^{D'} \leq |B'_\delta| \leq c \delta^{D'}$

for some $c > 0$ and all $\delta \in (0, 1]$ where $|B'_\delta|$ denotes the volume of the ball $B'_\delta = \{ g \in G : |g|' < \delta \}$ with respect to left invariant Haar measure $dg$.

Next we introduce a multi-index notation suited to the definition of products. If $n \in \mathbb{N}_0$ we set

$$J_n(d') = \bigoplus_{k=0}^n \{1, \ldots, d'\}^k, \quad J^+_n(d') = \bigoplus_{k=1}^n \{1, \ldots, d'\}^k$$

and

$$J(d') = \bigcup_{n=0}^{\infty} J_n(d'), \quad J^+(d') = \bigcup_{n=1}^{\infty} J_n^+(d').$$

Then $A^\alpha = A_{i_1} \cdots A_{i_n}$ for $\alpha = (i_1, \ldots, i_n)$ etc.. Alternatively, we set $a^\alpha = a_{i_1} \cdots a_{i_n}$ in the universal enveloping algebra and write $A^\alpha = dU(a^\alpha)$. 2
The weighted length $\|\alpha\|$ of $\alpha = (i_1, \ldots, i_n) \in J(d')$ is defined by

$$\|\alpha\| = \sum_{k=1}^{n} w_{i_k}$$

and the Euclidean length $n$ is denoted by $|\alpha|$.

If the algebraic basis is extended to a full vector space basis $a_1, \ldots, a_d$ and $n \in \mathbb{N}$ we define $\mathcal{X}_n = \mathcal{X}_n(U) = \cap_{\alpha \in J(d')} D(A^\alpha)$ and introduce norms and seminorms by

$$\|x\|_n = \|x\|_{U,n} = \max_{\alpha \in J(d')} \|A^\alpha x\|, \quad N_n(x) = N_{U,n}(x) = \max_{\alpha \in J(d')} \|A^\alpha x\|.$$

These spaces are independent of the choice of the full basis up to equivalence of norms. Similarly, for $n \in \mathbb{R}$ with $n \geq 0$, we define the weighted spaces

$$\mathcal{X}_n' = \mathcal{X}_n'(U) = \bigcap_{\|\alpha\| \leq n} D(A^\alpha)$$

corresponding to the weighted algebraic basis. Now, however, it can happen for a given $n$ that there are no multi-indices $\alpha$ such that $\|\alpha\| = n$. Therefore the corresponding norms and seminorms are given by

$$\|x\|'_n = \|x\|_{U,n}' = \begin{cases} 
\max_{\alpha \in J(d')} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d') \text{ with } \|\alpha\| = n, \\
0 & \text{otherwise},
\end{cases}$$

$$N_n'(x) = N_{U,n}'(x) = \begin{cases} 
\max_{\alpha \in J(d')} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d') \text{ with } \|\alpha\| = n, \\
0 & \text{otherwise}.
\end{cases}$$

The definition of $\|x\|'_n = 0$ in case $n \not\in \{\|\alpha\| : \alpha \in J(d')\}$ is to avoid complications in the proofs of some statements. In Section 11 we remove this part of the definition. In any case, these values of $n$ are not interesting.

Let $\mathcal{X}_\infty = \mathcal{X}_\infty(U) = \cap_{n=1}^{\infty} \mathcal{X}_n$. Since $a_1, \ldots, a_d$ is an algebraic basis one also has $\mathcal{X}_\infty = \cap_{n=1}^{\infty} \mathcal{X}_n'$. It then follows by the proof of Lemma 2.4 of [EIR1] that the space $\mathcal{X}_\infty$ is dense in $\mathcal{X}_n'$ for all $n \geq 0$. The density is with respect to the weak, or weak*, topology. If $U$ is the left regular representation on $L_p(G) = L_p(G; dg)$ we denote the corresponding spaces by $L_{p,n}$, $L'_{p,n}$, $L_{p,\infty}$ and the norms and seminorms by $\|\cdot\|_{p,n}$ etc. Further we let $L = L_G$ denote the left regular representation of $G$ in $L_2(G; dg)$.

A function $C : J(d') \to \mathbb{C}$ such that $C(\alpha) = 0$ if $\|\alpha\| > m$ but there exists at least one $\alpha \in J(d')$ with $\|\alpha\| = m$ and $C(\alpha) \neq 0$, where $m \in [1, \infty)$, is defined to be an $m$-th order form $C$. Here, and in the sequel, the order $m$ is understood to be the weighted order. We write $c_\alpha = C(\alpha)$.

The principal part $P$ of the $m$-th order form $C$ is the $m$-th order form given by

$$P(\alpha) = \begin{cases} 
C(\alpha) & \text{if } \|\alpha\| = m, \\
0 & \text{if } \|\alpha\| < m,
\end{cases}$$
and \( C \) is called homogeneous if \( C = P \).

The formal adjoint \( C^\dagger \) of \( C \) is the function \( C^\dagger : J_m(d') \to C \) defined by

\[
C^\dagger(\alpha) = (-1)^{|\alpha|} \overline{C(\alpha^*)},
\]

where \( \alpha^* = (i_n, \ldots, i_1) \) if \( \alpha = (i_1, \ldots, i_n) \). The real and imaginary parts of \( C \) are

\[
\Re C = 2^{-1}(C + C^\dagger) \quad \text{and} \quad \Im C = (2i)^{-1}(C - C^\dagger).
\]

We will consider the \( m \)-th order operators

\[
dU(C) = \sum_{\alpha \in J(d')} c_\alpha A^\alpha
\]

with domain \( D(dU(C)) = \mathcal{X}'_m \) associated with the form. If \( (\mathcal{F}, G, U_\ast) \) is the dual representation of \( (\mathcal{X}, G, U) \) then \( dU_\ast(C^\dagger) \) with the domain \( D(dU_\ast(C^\dagger)) = \mathcal{F}'_m \) is called the dual operator.

The \( m \)-th order form \( C \) is defined to be a \( G \)-weighted subcoercive form if \( m/w_i \in 2\mathbb{N} \) for each \( i \in \{1, \ldots, d'\} \) and the corresponding operator \( dL_G(C) \) satisfies a local Gårding inequality. Specifically we demand that

\[
\Re(\varphi, dL_G(C)\varphi) \geq \mu (N_{m/2}^i(\varphi))^2 - \nu \|\varphi\|^2_2
\]

for some \( \mu > 0 \) and \( \nu \in \mathbb{R} \), uniformly for all \( \varphi \in C_c^\infty(V) \) where \( V \) is some open neighbourhood of the identity \( e \in G \). For example, let \( c_{\alpha, \beta} \in \mathbb{C} \), with \( \alpha, \beta \in J(d') \) and \( \|\alpha\| = m/2 = \|\beta\| \), satisfy

\[
\Re \sum_{\alpha, \beta} c_{\alpha, \beta} \xi_\alpha \xi_\beta > 0
\]

for all non-zero complex \( (\xi_\alpha) \). Then the operator

\[
H = \sum_{\alpha, \beta} (-1)^{|\alpha|} c_{\alpha, \beta} A^{\alpha^*} A^{\beta}
\]

satisfies the Gårding inequality. This follows because

\[
\Re(\varphi, H\varphi) = \Re \sum_{\alpha, \beta} c_{\alpha, \beta} (A^{\alpha^*} \varphi, A^{\beta} \varphi) \geq \mu \sum_{\|\alpha\| = m/2} \|A^{\alpha^*} \varphi\|^2_2 \geq \mu (N_{m/2}^i(\varphi))^2
\]

where \( \mu \) is the strictly positive lowest eigenvalue of the real part of the matrix \( (c_{\alpha, \beta}) \).

Our main result establishes that subcoercivity gives an infinitesimal characterization of generators of semigroups with kernels satisfying Gaussian bounds.

**Theorem 1.1** Let \( C \) be an \( m \)-th order form and assume that the weights \( w_i \) satisfy \( m/w_i \in 2\mathbb{N} \). Then the following conditions are equivalent.

I. The form \( C \) is \( G \)-weighted subcoercive.

II. There are \( c, \mu > 0 \) and an open neighbourhood \( V \) of the identity of \( G \) such that

\[
\mu \varepsilon^{2w_i} \|A_i \varphi\|^2_2 \leq \varepsilon^m \Re(\varphi, dL_G(C)\varphi) + c \|\varphi\|^2_2
\]

for all \( \varphi \in C_c^\infty(V) \), all \( \varepsilon \in (0, 1] \) and all \( i \in \{1, \ldots, d'\} \).
III. The closure of \( dL(G)(C) \) generates a holomorphic semigroup \( S \) on \( L^2(G) \) with the following properties.

i. The semigroup \( S \) is quasi-contractive in an open subsector of the sector of holomorphy, i.e., there exist \( \varphi \in (0, \pi/2] \) and \( \omega \geq 0 \) such that \( \|S_z\| \leq e^{\omega|z|} \) for all \( z \in \Lambda(\varphi) = \{ z \in C \setminus \{ 0 \} : \arg z < \varphi \} \).

ii. \( S_t \subset L^2(G) \subseteq \bigcap_{i=1}^{d'} D(A_i) \) and there exist \( c > 0 \) and \( \omega \geq 0 \) such that

\[
\|A_i S_t\|_{2 \to 2} \leq c t^{-\omega/m} e^{\omega t}
\]

for all \( t > 0 \) and \( i \in \{1, \ldots, d'\} \).

IV. In each continuous representation \((X, G, U)\) the closure of \( dU(C) \) generates a continuous semigroup \( S \) with the following properties.

i. The semigroup \( S \) is holomorphic in a sector which contains an open representation independent subsector \( \Lambda(\theta_C) \).

ii. If \( U \) is unitary then the semigroup \( S \) is quasi-contractive in each subsector of \( \Lambda(\theta_C) \), i.e., for each \( \varphi \in (0, \theta_C) \) there is an \( \omega \geq 0 \) such that \( \|S_z\| \leq e^{\omega|z|} \) for all \( z \in \Lambda(\varphi) \).

iii. The semigroup \( S \) has a representation independent, fast decreasing, kernel \( K \in L^{1,\infty}(G) \cap C_{0,\infty}(G) \) such that

\[
A^\alpha S_z x = \int_G dg (A^\alpha K_z)(g) U(g)x
\]

for all \( \alpha \in J(d') \), \( z \in \Lambda(\theta_C) \) and \( x \in X \).

iv. For each \( \varphi \in (0, \theta_C) \) and all \( \alpha \in J(d') \) there exist \( b, c > 0 \) and \( \omega \geq 0 \) such that

\[
|(A^\alpha K_z)(g)| \leq c |z|^{-(D'+||\alpha||)/m} e^{\omega|z|} e^{-b(|\alpha'|)^{m-1}} |z|^{1/(m-1)}
\]

for all \( g \in G \) and \( z \in \Lambda(\varphi) \).

A crucial element in the proof of the theorem is the local approximation of \( G \) by the homogeneous (nilpotent) group \( G_0 \) alluded to above. The group \( G_0 \), which has the same dimension as \( G \), is constructed, following an idea of Kashiwara and Vergne [KaV], through exponentiation of a contraction of the Lie algebra \( g \) of \( G \). In the unweighted vector space case the contraction process is as follows. For \( t > 0 \) define the commutation relation \( [\cdot, \cdot]_t \) on \( g \) by \( [a, b]_t = t[a, b] \). Then the limit \( t \to 0 \) of \( [\cdot, \cdot]_t \) exists and \( (g, [\cdot, \cdot]_0) \) is the Lie algebra of the commutative group \( G_0 = \mathbb{R}^d \). In general the contraction is determined by the algebraic basis \( a_1, \ldots, a_d \) and the weights \( w_1, \ldots, w_d \) and the homogeneous group \( G_0 \) is not commutative. (Hebisch [Heb2] recently used this contraction procedure to prove kernel bounds similar to ours for higher-order Hörmander operators of type 2, but by quite different arguments.) The contraction mechanism provides a family of groups \( G_t, t \in [0, 1] \), which interpolate between \( G = G_1 \) and its homogeneous contraction \( G_0 \). One can use this interpolation to establish that each \( G \)-weighted subcoercive form is automatically a \( G_0 \)-weighted subcoercive form. Then the implication \( I \Rightarrow IV \) in Theorem 1.1 is proved by first applying the results of [AER] to \( C \) on the homogeneous group \( G_0 \) to obtain the corresponding implication for \( G_0 \) and subsequently lifting the result to \( G \) by parametrix.
arguments. The latter reasoning makes essential use of the results of Helffer and Nourrigat [HeN] for homogeneous groups.

The proof of $IV \Rightarrow III$ is straightforward. The $L_2$-bound on $A_i S_t$ follows from the corresponding kernel bound by a quadrature argument; one deduces that $\|A_i K_t\|_1 \leq c' t^{-\omega'/m} e^{\omega't}$ for some $c' > 0$ and $\omega' \geq 0$.

The circle of arguments used to prove $I \Rightarrow IV$ allows one to establish the equivalence of $G_0$-weighted subcoercivity and $G$-weighted subcoercivity. This equivalence is one of the most important structural features of the theory. It provides the starting point for the proof of $III \Rightarrow I$ and $II \Rightarrow I$ in the theorem since it then suffices to prove that $C$ is $G_0$-subcoercive. The latter property follows by exploitation of the contraction mechanism and the homogeneity of $G_0$.

The proof that $I \Rightarrow II$ is straightforward. Since the $A_i$ are group generators one has the inequalities $e^{2w_i} \|A_i \varphi\|_2^2 \leq e^{m} \|A_i^{m/(2w_i)} \varphi\|_2^2 + c \|\varphi\|_2^2$ for all $\varepsilon \in (0, 1]$.

In general the homogeneous contraction $G_0$ of the $d$-dimensional group $G$ is non-abelian. But $G_0 = R^d$ precisely in the setting of [EIR5] if, and only if, $d' = d$, i.e., the $a_1, \ldots, a_{d'}$ form a vector space basis of $g$, and $w_i + w_j - w_k > 0$ whenever the corresponding structure constant $c_{ij}^k \neq 0$ (see Proposition 3.2). The property of $R^d$-weighted subcoercivity can, however, be expressed directly in terms of the coefficients of the form $C$; it is equivalent to the strong ellipticity condition

$$\sum_{|\alpha| \leq m} \Re c_\alpha (i\xi)^\alpha > 0$$

for all $\xi \in R^d \setminus \{0\}$ (see Example 4.2). In particular, if one considers operators formed from a full vector space basis with all weights equal to one the corresponding operators $dU(C)$ generate semigroups with smooth Gaussian kernels if and only if the principal coefficients satisfy the strong ellipticity property.

Finally a simple illustration of our results is given by the group $SO(3)$ of rotations in $R^3$. If $a_1, a_2, a_3$ is a basis of $so(3)$ satisfying $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$ and $[a_3, a_1] = a_2$ then $a_1, a_2$ is an algebraic basis. If one specifies weights $w_1 = 3$, $w_2 = 2$ the operator

$$H = A_1^4 - A_2^6 - A_3^2 A_2^3$$

has (weighted) order 12 and satisfies the Gårding inequality because a straightforward calculation gives

$$\Re(\varphi, H\varphi) \geq 2^{-1}(\|A_1^2 \varphi\|_2^2 + \|A_2^3 \varphi\|_2^2) \geq 2^{-1}(N_{\rho,0}(\varphi))^2$$

Hence $H$ generates a holomorphic semigroup with a smooth kernel satisfying Gaussian bounds in each continuous representation of the group.

The paper is organized as follows. First, in Section 2 we introduce reduced algebraic bases. All the proofs in Sections 3–10 are carried out for reduced algebraic bases but in Section 11 we remove this restriction on the bases. In Section 3 we introduce the contraction mechanism and prove several uniform properties for the right invariant vector fields on each of the interpolating groups $G_t$. In Section 4 we define $G$-weighted subcoercive forms and establish several structural properties. The most important is that any $G$-weighted subcoercive form is a $G_0$-weighted subcoercive form. In Section 5 we prove the
implication $I \Rightarrow IV$ in Theorem 1.1 for $G_0$ and $U$ the left regular representation of $G_0$ in $L_2(G_0)$. In Section 6 we define a distance on $G$ and $G_t$ associated with the weighted algebraic basis and in Section 7 the kernel on $G_0$ of Section 5 is lifted to a ‘kernel’ on $G$ by a parametric argument which uses $G_0$ as a local approximation of $G$. In Section 8 we prove the implication $I \Rightarrow IV$ in Theorem 1.1 for reduced bases, but under the (weaker) assumption that $C$ is merely a $G_0$-weighted subcoercive form. Under the same conditions we prove in Section 9 regularity results. In Section 10 we prove that a form is a $G$-weighted subcoercive form if, and only if, it is a $G_0$-weighted subcoercive form. Moreover, we prove Theorem 1.1 for reduced bases and derive other equivalent characterizations for $G$-weighted subcoercive forms. In the last section we extend the results for reduced bases to general bases.

2 Reduced bases

Let $g$ be a $d$-dimensional Lie algebra with Lie product $[\cdot,\cdot]$ . We adopt the multi-index notation introduced in Section 1. If $\alpha = (i_1,\ldots,i_n)$ with $i_j \in \{1,\ldots,d'\}$ is a multi-index of length $|\alpha| = n \neq 0$ and $a_1,\ldots,a_{d'} \in g$ are elements of $g$ we set the multi-commutator $a_{[\alpha]}$ of order $n$ by

$$a_{[\alpha]} = [a_{i_1},\ldots,[a_{i_{n-1}},a_{i_n}]] \in g$$

where $a_{[i]} = a_i$. Our principal interest is in algebraic bases $a_1,\ldots,a_{d'}$ of $g$. The smallest integer $r$ for which the $a_1,\ldots,a_{d'}$ together with all their multi-commutators of order less than or equal to $r$ span $g$ is called the rank of the algebraic basis.

We also consider algebraic bases with weights $w_1,\ldots,w_{d'} \in [1,\infty)$ assigned to each of the $d'$ directions. We call $a_1,\ldots,a_{d'}$ a weighted algebraic basis. The unweighted algebraic basis $a_1,\ldots,a_{d'}$ can be considered as a weighted algebraic basis with all weights $w_i = 1$.

Next we introduce a special class of weighted algebraic bases for which the weights are minimal. One only has directions for which the weight is not too large compared with the other directions and their weights. These bases will then be used throughout the subsequent analysis of subcoercive operators and the associated semigroups. Nevertheless, the key results are not dependent on the special properties of the bases.

A filtration for $g$ is a family of vector subspaces $(g_\lambda)_{\lambda \geq 0}$ of $g$ with the following four properties. First $g_\lambda \subseteq g_\mu$ if $\lambda \leq \mu$, secondly $g_\lambda = \{0\}$ if $\lambda < 1$, thirdly $[g_\lambda,g_\mu] \subseteq g_{\lambda+\mu}$ for all $\lambda,\mu \in [0,\infty)$ and fourthly $g_\lambda = g$ for large $\lambda$. If $a_1,\ldots,a_{d'}$ is a weighted algebraic basis, $\lambda \geq 0$ and we set

$$g_\lambda = \text{span} \bigcup_{n=1}^{\lambda} \{[a_{i_1},\ldots,[a_{i_{n-1}},a_{i_n}]\ldots] : i_1,\ldots,i_n \in \{1,\ldots,d'\}, w_{i_1} + \ldots + w_{i_n} \leq \lambda \}$$

$$= \text{span} \{a_{[\alpha]} : \alpha \in J(d'), 0 < ||\alpha|| \leq \lambda \}$$

then $(g_\lambda)_{\lambda \geq 0}$ is a filtration, which we call the filtration corresponding to the weighted algebraic basis. Note that it is possible that $a_i \in g_\lambda$ for a $\lambda \geq 0$ with $\lambda < w_i$.

Next for a general filtration define

$$g_\lambda = \bigcup_{\lambda' < \lambda} g_{\lambda'}$$

7
for each \( \lambda > 0 \). Further let \( 1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_k \) be such that \( g_{\lambda_k} = g \) and

\[
\{ \lambda_j : j \in \{1, \ldots, k\} \} = \{ \lambda \geq 0 : g_{\lambda} \neq g_\lambda \}.
\]

We will call \( \lambda_1, \ldots, \lambda_k \) the weights of the filtration \( (g_{\lambda})_{\lambda \geq 0} \).

Note that in the sequel we sometimes set \( g_{\lambda+\mu} = g_{\lambda+\mu} \) for clarity of notation.

We are interested in algebraic bases \( a_1, \ldots, a_{d'} \) with weights \( w_1, \ldots, w_{d'} \) such that \( a_i \notin g_{w_i} \) for all \( i \in \{1, \ldots, d'\} \), where \( (g_{\lambda})_{\lambda \geq 0} \) is the filtration corresponding to the weighted algebraic basis. A basis with this latter property will be called a reduced weighted algebraic basis. The definition of the reduced basis is such that the corresponding weights are minimal.

One can pass from a general weighted algebraic basis to a reduced weighted algebraic basis by eliminating the 'overweight' directions.

**Proposition 2.1** Let \( a_1, \ldots, a_{d'} \) be an algebraic basis with weights \( w_1, \ldots, w_{d'} \) and corresponding filtration \( (g_{\lambda})_{\lambda \geq 0} \).

Then there exists a reduced weighted algebraic basis \( b_1, \ldots, b_{d''} \) with weights \( v_1, \ldots, v_{d''} \) such that \( \{b_1, \ldots, b_{d''}\} \subseteq \{a_1, \ldots, a_{d'}\} \) and \( v_i = w_j \) if \( b_i = a_j \). Moreover, \( (g_{\lambda})_{\lambda \geq 0} \) is the filtration corresponding to \( b_1, \ldots, b_{d''} \).

**Proof** After reordering one may assume \( w_1 \leq w_2 \leq \ldots \leq w_{d'} \). Then \( g_{w_1} \) is spanned by \( a_1, \ldots, a_l \) together with the multi-commutators \( a_{[\alpha]} \) with \( \alpha \in J(l) \) and \( 0 < \|\alpha\| \leq w_l \), where \( l \in \{1, \ldots, d'\} \) is such that \( w_i = w_i < w_{i+1} \). Now suppose that \( a_j \in g_{w_j} \). Let \( i \in \{1, \ldots, d' - 1\} \) be such that \( w_i < w_j = w_{i+1} \). Then there exist \( c_{j\alpha} \in \mathbb{R} \) such that

\[
a_j = \sum_{\alpha \in J^+(l)} c_{j\alpha} a_{[\alpha]}. \tag{2}
\]

Hence the subbasis obtained from \( a_1, \ldots, a_{d'} \) by removal of \( a_j \) remains a weighted algebraic basis with the same filtration \( (g_{\lambda})_{\lambda \geq 0} \) as the original weighted algebraic basis. By repeating this process a finite number of times one obtains the desired reduced weighted algebraic basis.

It is important to note that the \( a_i \) in the directions eliminated in this construction of a reduced basis can be expressed in terms of the remaining directions by (2). In Section 11 this will be used to lift results established for the reduced algebraic basis back to the general algebraic basis.

In the sequel we also need to extend a reduced weighted algebraic basis to a full vector space basis in an appropriate manner.

**Remark 2.2** Each reduced weighted algebraic basis \( a_1, \ldots, a_{d'} \) has an extension to a (vector space) basis \( b_1, \ldots, b_{d_1}, \ldots, b_{d_2}, \ldots, b_{d_k} \) for \( g \) with the following properties. First \( b_{11}, \ldots, b_{1d_1}, \ldots, b_{k1}, \ldots, b_{kd_k} \) is a basis for \( g_{\lambda_1} \) for all \( i \in \{1, \ldots, k\} \), where \( \lambda_1 < \ldots < \lambda_k \) are the weights for the filtration \( (g_{\lambda})_{\lambda \geq 0} \). Secondly

\[
\{a_1, \ldots, a_{d'}\} \subseteq \{b_{ij} : i \in \{1, \ldots, k\}, j \in \{1, \ldots, d_i\}\},
\]
with \( w_i = \lambda_i \) if \( a_i = b_{ij} \). Thirdly the other \( b_{ij} \) equal some commutator

\[
[a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}], \ldots]
\]

with \( w_i + \ldots + w_n = \lambda_i \). If \( b_{ij} \) is given the weight \( w_{ij} = \lambda_i \) one obtains an extension of the weighted algebraic basis to a weighted vector space basis \( a_1, \ldots, a_d, \ldots, a_d \) such that \( a_i \) has the weight \( w_i = \lambda_i \) if \( a_i = b_{ij} \).

**Example 2.3** Let \( a_1, \ldots, a_d \) be a basis for \( g \) and set all weights equal to one. Then \( a_1, \ldots, a_d \) is a reduced (weighted algebraic) basis for \( g \). The operators which we will construct with respect to such a basis are strongly elliptic operators and are studied in detail in [Rob2].

**Example 2.4** Let \( a_1, \ldots, a_d' \) be an algebraic basis for \( g \) and set all weights equal to one. Then \( a_1, \ldots, a_d' \) is a reduced (weighted) algebraic basis for \( g \). The subcoercive and subelliptic operators studied in [ElR3] and [ElR7] are examples of the operators which we will construct with respect to such a basis.

**Example 2.5** Let \( g \) be the usual three-dimensional Heisenberg algebra \( h_1 \) and \( a_1, a_2, a_3 \) a (vector space) basis with \([a_1, a_2] = a_3\). Assign weights \( w_1 = w_2 = 1 \) and \( w_3 = 3 \). Then the corresponding filtration is given by \( g_1 = \text{span}\{a_1, a_2\}, g_2 = h_1 = g_3 \) and the weights of the filtration are \( \lambda_1 = 1 \), and \( \lambda_2 = 2 \). Therefore \( a_1, a_2, a_3 \) is not a reduced weighted algebraic basis since \( a_3 \in g_3 \). If one deletes the direction \( a_3 \) then \( a_1, a_2 \) is a reduced weighted algebraic basis with the same filtration as the weighted algebraic basis \( a_1, a_2, a_3 \).

**Example 2.6** Let \( g \) be the four-dimensional Lie algebra \( k_3 \) with basis \( a_1, \ldots, a_4 \) and commutation relations \([a_4, a_3] = a_2 \) and \([a_4, a_2] = a_1\). Then \( a_1, a_3, a_4 \) is an algebraic basis. Assign weights \( w_1 = 8, w_3 = 3 \) and \( w_4 = 2 \). Then the corresponding filtration is given by \( g_2 = \text{span}\ a_4, g_3 = \text{span}\{a_3, a_4\}, g_5 = \text{span}\{a_2, a_3, a_4\} \) and \( g_7 = g \). Therefore \( a_1, a_3, a_4 \) is not a reduced weighted algebraic basis since \( a_1 \in g_7 = g_8 \). If one deletes the direction \( a_1 \) then \( a_3, a_4 \) is a reduced weighted algebraic basis with the same filtration as the weighted algebraic basis \( a_1, a_3, a_4 \).

**Example 2.7** Let \( a_1, \ldots, a_d \) be a basis for \( g \) with weights \( w_1, \ldots, w_d \in \mathbb{N} \) and suppose that the structure constants \( c_{ij}^k \), defined by \([a_i, a_j] = \sum_{k=1}^d c_{ij}^k a_k\), are such that \( c_{ij}^k \neq 0 \) implies \( w_i + w_j - 1 \geq w_k \), i.e., one has

\[
[a_i, a_j] = \sum_{k \in \{1, \ldots, d\} \atop w_k \leq w_i + w_j - 1} c_{ij}^k a_k
\]

Let \( \lambda_1 < \ldots < \lambda_k \) be such that

\[
\{w_i : i \in \{1, \ldots, d\}\} = \{\lambda_j : j \in \{1, \ldots, k\}\}
\]

and let \((g_\lambda)_{\lambda \geq 0}\) be the filtration corresponding to the weighted basis \( a_1, \ldots, a_d \). Then \( g_\lambda = \{0\} \) if \( \lambda < \lambda_1 \) and \( g_{\lambda_1} = \text{span}\{a_i : i \in \{1, \ldots, d\}, w_i = \lambda_1\} \). Suppose that \( j \in \{1, \ldots, k-1\} \) and

\[
g_{\lambda_j} = \text{span}\{a_i : i \in \{1, \ldots, d\}, w_i \leq \lambda_j\}
\]
Further suppose $g_{\lambda} \neq g_{\lambda_j}$ for some $\lambda \in (\lambda_j, \lambda_{j+1})$. Let $\lambda = \min\{\mu \in (\lambda_j, \lambda_{j+1}) : g_{\mu} \neq g_{\lambda_j}\}$. The minimum exists since $g$ is finite-dimensional. Then there are $n \in \mathbb{N}$, $n \geq 2$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$ such that $w_{i_1} + \ldots + w_{i_n} = \lambda$ and

$$[a_{i_1}, \ldots, [a_{i_{n-1}}, a_{i_n}]] \in g_{\lambda \setminus g_{\lambda_j}}.$$ 

But by assumption

$$[a_{i_1}, \ldots, [a_{i_{n-1}}, a_{i_n}]] \in \text{span}\{a_i : i \in \{1, \ldots, d\}, w_i \leq \lambda - (n - 1)\} \subseteq g_{\lambda - (n-1)} \subseteq g_{\lambda_j}$$

since $g_{\mu} = g_{\lambda_j}$ for all $\mu \in (\lambda_j, \lambda)$. So $g_{\lambda} = g_{\lambda_j}$ for all $\lambda \in (\lambda_j, \lambda_{j+1})$.

Therefore $g_{\lambda_{j+1}} = \text{span}\{a_i : i \in \{1, \ldots, d\}, w_i \leq \lambda_{j+1}\}$. It follows from the above argument that $a_1, \ldots, a_d$ is a reduced weighted algebraic basis for $g$. The operators which we will construct with respect to such a basis are the weighted strongly elliptic operators studied in detail in [EIR5].

**Example 2.8** Let $g$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and $a_1, \ldots, a_d$ an algebraic basis for $g$ such that $\gamma_t(a_i) = t w_i a_i$ for some $w_i \in [1, \infty)$ and all $t > 0$. Then $a_1, \ldots, a_d$ is a weighted algebraic basis with weights $w_1, \ldots, w_d$. We describe the corresponding filtration and show that $a_1, \ldots, a_d$ is a reduced weighted algebraic basis.

Extend the algebraic basis to a vector space basis $a_1, \ldots, a_{d'}$, $\ldots$, $a_d$ such that for each $i \in \{d' + 1, \ldots, d\}$ there exists a $w_i \in [1, \infty)$ such that $\gamma_t(a_i) = t w_i a_i$ for all $t > 0$. For $\lambda > 0$ set

$$g^{(\lambda)} = \{a \in g : \gamma_t(a) = t^\lambda a \text{ for all } t > 0\}.$$ 

Then $g^{(\lambda)} = \text{span}\{a_i : i \in \{1, \ldots, d\}, w_i = \lambda\}$ and

$$g = \bigoplus_{\lambda > 0} g^{(\lambda)}.$$ 

Next by definition of $g_\lambda$ one obtains the inclusions

$$g_\lambda \subseteq \text{span}\{a \in g : \exists \mu \in [0, \lambda] \forall t > 0 [\gamma_t(a) = t^\mu a]\} \subseteq \bigoplus_{\mu \leq \lambda} g^{(\mu)}.$$ 

Conversely, let $i \in \{1, \ldots, d\}$. For all $\alpha \in J_r(d')$, with $r$ the rank of the algebraic basis, there exist $c_{\alpha} \in \mathbb{R}$ such that $a_i = \sum_{\alpha \in J_r(d')} c_{\alpha} a_{[\alpha]}$. Then by scaling

$$a_i = \sum_{\alpha \in J_r(d')} c_{\alpha} a_{[\alpha]} \in g_{w_i}.$$ 

Therefore $g^{(w_i)} \subseteq g_{w_i}$ and hence

$$g_\lambda = \bigoplus_{\mu \leq \lambda} g^{(\mu)}$$

for all $\lambda > 0$. So $g_\lambda \neq g_\lambda$ if, and only if, $g^{(\lambda)} \neq \{0\}$.

Now suppose $a_i \in g_{w_i}$ for some $i \in \{1, \ldots, d'\}$. Then

$$a_i \in \bigcup_{\lambda < w_i} g_\lambda \subseteq \bigoplus_{\mu < w_i} g^{(\mu)}$$

which is a contradiction. Therefore $a_1, \ldots, a_{d'}$ is a reduced weighted algebraic basis of $g$.  

10
Example 2.9 Let \( g \) be the nilpotent Lie algebra with generators \( a_1, \ldots, a_d' \) which is free of step \( r \). Let \( w_1, \ldots, w_d' \geq 1 \) be weights and let \((g_\lambda)_{\lambda \geq 0}\) be the corresponding filtration. Before presenting another description of \( g_\lambda \) we recall several definitions.

Let \( V = \text{span}\{a_1, \ldots, a_d'\} \) and for \( t > 0 \) define the linear map \( \gamma_t : V \to V \) such that

\[
\gamma_t(a_i) = t^{w_i}a_i.
\]

Let \( \mathcal{I} = \bigoplus_{n=0}^{\infty} V^\otimes n \) be the associative tensor algebra over \( V \). We identify \( V \) with the subspace of tensors of degree one of \( \mathcal{I} \). There exists a unique algebra homomorphism \( \gamma_t \) on \( \mathcal{I} \) such that \( \gamma_t(a) = \gamma_t(a) \) for all \( a \in V \). We will not distinguish between \( \gamma_t \) and \( \gamma_t \) and write \( \gamma_t \). The associative tensor algebra \( \mathcal{I} \) is a Lie algebra with the usual commutation relations. Let \( \mathcal{G} \) be the Lie subalgebra of \( \mathcal{I} \) generated by \( a_1, \ldots, a_d' \). Then \( \mathcal{G} \) is the free Lie algebra generated by \( a_1, \ldots, a_d' \). The restriction, again denoted by \( \gamma_t \), of \( \gamma_t \) to \( \mathcal{G} \) is a Lie algebra homomorphism. Let \( I \) be the ideal in \( \mathcal{G} \) spanned by all commutators of order larger than or equal to \( r + 1 \). Then the nilpotent Lie algebra \( g \) with generators \( a_1, \ldots, a_d' \) which is free of step \( r \) is equal to \( \mathcal{G} / I \). Since \( \gamma_t(I) \subseteq I \), there exists a unique Lie algebra homomorphism \( \gamma_t : g \to g \) such that \( \gamma_t(a + I) = \gamma_t(a) + I \) for all \( a \in \mathcal{G} \). Again we write \( \gamma_t \) for \( \gamma_t \). One easily verifies that \( \gamma_{st} = \gamma_s \circ \gamma_t \) for all \( s, t > 0 \), so \( g \) equipped with the dilations \( \gamma_t, t > 0 \), becomes a homogeneous Lie algebra. Now it follows from Example 2.8 that \( a_1, \ldots, a_d' \) is a reduced weighted algebraic basis for \( g \).

We call \( g \) the weighted nilpotent Lie algebra with generators \( a_1, \ldots, a_d' \) and weights \( w_1, \ldots, w_d' \) which is free of step \( r \). The corresponding connected simply connected Lie group \( G \) with Lie algebra \( g \) is called the weighted nilpotent Lie group with generators \( a_1, \ldots, a_d' \) and weights \( w_1, \ldots, w_d' \) which is free of step \( r \). We denote \( g \) and \( G \) by \( g(d', r, w_1, \ldots, w_d') \) and \( G(d', r, w_1, \ldots, w_d') \).

3 Homogenization by contraction

Let \( a_1, \ldots, a_d' \) be a reduced weighted algebraic basis of the Lie algebra \( g \). We next use the basis and its weighting to construct a family of Lie products \([\cdot, \cdot], t > 0\), on \( g \). Then we examine the contraction of the Lie algebras \((g, [\cdot, \cdot], t)\) obtained in the limit \( t \to 0 \). We establish that this contraction yields a homogeneous Lie algebra \((g, [\cdot, \cdot], 0)\). The simply connected homogeneous Lie group \( G_0 \) corresponding to this Lie algebra will subsequently play a fundamental role in the analysis of elliptic operators on the connected Lie group \( G \) corresponding to the original Lie algebra.

First let \( b_{i1}, \ldots, b_{id_1}, \ldots, b_{ik_1}, \ldots, b_{kd_k} \) be a basis for the filtration \((g_\lambda)_{\lambda \geq 0}\) corresponding to an extension of the reduced weighted algebraic basis with the properties described in Remark 2.2. So

\[
\{a_1, \ldots, a_d'\} \subseteq \{b_{ij} : i \in \{1, \ldots, k\}, j \in \{1, \ldots, d_i\}\}
\]

and \( w_i = w_{ij} = \lambda_i \) if \( a_{ij} = b_{ij} \), where \( \lambda_1 < \ldots < \lambda_k \) are the weights for the filtration. Moreover, for all \( i \) and \( j \) there exists a multi-index \( \alpha_{ij} \) such that \( b_{ij} = a_{\alpha_{ij}} \).

Secondly, we follow the ideas of Kashiwara and Vergne [KaV], or Hebisch [Heb2] Lemma 4.1. For \( t > 0 \) define the linear bijection \( \gamma_t : g \to g \) by

\[
\gamma_t(b_{ij}) = t^{w_i}b_{ij} = t^{\lambda_i}b_{ij}.
\]
Moreover, define \([ \cdot, \cdot ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) by

\[
[a, b]_t = \gamma_t^{-1}([\gamma_t(a), \gamma_t(b))]
\]

Then \((\mathfrak{g}, [\cdot, \cdot])\) is a Lie algebra, which equals \((\mathfrak{g}, [\cdot, \cdot])\) if \(t = 1\). Now we have a whole scale of Lie algebras and we examine the limit \([\cdot, \cdot]_0\) of the Lie brackets \([\cdot, \cdot]_t\) as \(t \to 0\). The limit defines an algebraic structure on \(\mathfrak{g}\) and the Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_0)\) is the contraction of \((\mathfrak{g}, [\cdot, \cdot]_t)\) in the sense of Saletan [Sal]. This construction is uniquely determined by the reduced weighted algebraic basis \(a_1, \ldots, a_{d'}\).

**Proposition 3.1**

I. The limit

\[
[a, b]_0 = \lim_{t \to 0} [a, b]_t
\]

exists for all \(a, b \in \mathfrak{g}\).

II. The Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_0)\) is uniquely determined, up to isomorphism, by the filtration corresponding to the reduced weighted algebraic basis.

III. \((\mathfrak{g}, [\cdot, \cdot]_0)\) is a homogeneous Lie algebra with dilations \((\gamma_t)_{t > 0}\).

IV. The \(a_1, \ldots, a_{d'}\) form an algebraic basis for the Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_t)\) for all \(t \in [0, \infty)\).

V. The reduced weighted algebraic basis \(a_1, \ldots, a_{d'}\) is a reduced weighted algebraic basis for the Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_t)\) for all \(t \in [0, \infty)\). Moreover, the filtrations with respect to the Lie algebras \((\mathfrak{g}, [\cdot, \cdot]_t)\) are equal to the filtration \((\mathfrak{g}_\lambda)_{\lambda \geq 0}\) as vector spaces.

VI. For all \(i_1, j_1, i_2, j_2\) one has

\[
[b_{i_1 j_1}, b_{i_2 j_2}]_t - [b_{i_1 j_1}, b_{i_2 j_2}]_0 \in \mathfrak{g}(\lambda_{i_1} + \lambda_{i_2})
\]

for all \(t > 0\).

**Proof** For \(\lambda \geq 0\) set

\[
\mathfrak{g}(\lambda) = \begin{cases} 
\text{span}\{b_{ij} : j \in \{1, \ldots, d_i\}\} & \text{if } \lambda = \lambda_i \text{ for some } i \in \{1, \ldots, k\}, \\
\{0\} & \text{if } \lambda \notin \{\lambda_1, \ldots, \lambda_k\}.
\end{cases}
\]

Then \(\mathfrak{g} = \bigoplus_{\lambda \geq 0} \mathfrak{g}(\lambda)\). Let \(\pi_\lambda\) be the projection of \(\mathfrak{g}\) onto \(\mathfrak{g}(\lambda)\). For all \(i_1, j_1, i_2, j_2\) one has

\[
[\gamma_t(b_{i_1 j_1}), \gamma_t(b_{i_2 j_2})] = t^{\lambda_{i_1} + \lambda_{i_2}} [b_{i_1 j_1}, b_{i_2 j_2}] = t^{\lambda_{i_1} + \lambda_{i_2}} \sum_{\lambda \leq \lambda_{i_1} + \lambda_{i_2}} \pi_\lambda([b_{i_1 j_1}, b_{i_2 j_2}]).
\]

(Note that the sum is finite.) So

\[
[b_{i_1 j_1}, b_{i_2 j_2}]_t = \pi_{\lambda_{i_1} + \lambda_{i_2}} ([b_{i_1 j_1}, b_{i_2 j_2}]) + \sum_{\lambda < \lambda_{i_1} + \lambda_{i_2}} t^{\lambda_{i_1} + \lambda_{i_2} - \lambda} \pi_\lambda([b_{i_1 j_1}, b_{i_2 j_2}]).
\]

Therefore \(\lim_{t \to 0} [b_{i_1 j_1}, b_{i_2 j_2}]_t = \pi_{\lambda_{i_1} + \lambda_{i_2}} ([b_{i_1 j_1}, b_{i_2 j_2}])\) exists and by linearity it follows that \([\cdot, \cdot]_0\) can be defined as in Statement I. Statements III and VI are easy. Moreover, it follows for all \(t \geq 0\) that

\[
[b_{i_1 j_1}, b_{i_2 j_2}]_t = [b_{i_1 j_1}, b_{i_2 j_2}] \mod \mathfrak{g}(\lambda_{i_1} + \lambda_{i_2}).
\]
for all $\mu_1, \mu_2 \geq 1$, $a \in g_{\mu_1}$ and $b \in g_{\mu_2}$.

Let $t \geq 0$. For $\lambda \geq 0$ set

$$g^t_\lambda = \text{span} \bigcup_{n=1}^\lambda \{[a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}] \ldots]_t : i_1, \ldots, i_n \in \{1, \ldots, d'\}, w_{i_1} + \ldots + w_{i_n} \leq \lambda\}.$$ 

We shall prove that for all $l \in \{1, \ldots, k\}$ one has $g^t_\lambda = g_\lambda$ for all $\lambda \leq \lambda_l$ and

$$b_{l_1}, \ldots, b_{l_{d_l}}, \ldots, b_{l_1}, \ldots, b_{l_{d_l}}$$

is a basis for $g^t_{\lambda_l}$. If we have proved this, then $g^t_{\lambda_k} = g_{\lambda_k} = g$ so $a_1, \ldots, a_{d'}$ is an algebraic basis for $(g, [\cdot, \cdot])$.

The case $l = 1$ is trivial. Let $l \in \{2, \ldots, k\}$ and suppose the case $l-1$ is valid. Let $\lambda \in (\lambda_{l-1}, \lambda_l)$, $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, d'\}$. Suppose that $w_{i_1} + \ldots + w_{i_n} \leq \lambda$. Consider the multi-commutator $[a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}] \ldots]_t$. It follows from (3), even in the case $n = 1$, that

$$[a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}] \ldots]_t = [a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}] \ldots]_t \mod g_{\lambda_{l-1}}$$

by the induction hypothesis. Consequently

$$[a_{i_1}, \ldots [a_{i_{n-1}}, a_{i_n}] \ldots]_t \mod g^t_{\lambda_{l-1}}.$$ 

So by (4) and (5) one has $g^t_{\lambda_k} = g_\lambda$.

Similarly, if $\mu + \lambda = \lambda_l$ then

$$[a^t_{\mu}, g^t_{\lambda_k}] = [a^t_{\mu}, g_\lambda] = [a^t_{\mu}, g_{\lambda_k}] \mod g_{\lambda_k}.$$ 

Hence $g^t_{\lambda_k} = g_{\lambda_k}$ and $b_{l_1}, \ldots, b_{l_{d_l}}, \ldots, b_{l_1}, \ldots, b_{l_{d_l}}$ is a basis for $g^t_{\lambda_k}$. This establishes Statements IV and V.

Finally we prove the uniqueness statement, Statement II. Let $a_1^{(1)}, \ldots, a_{d'}^{(1)}$ be a reduced weighted algebraic basis with weights $w_1, \ldots, w_{d'}$ and $a_1^{(2)}, \ldots, a_{d'}^{(2)}$ a second reduced weighted algebraic basis with weights $v_1, \ldots, v_{d'}$ such that the filtrations with respect to the two weighted algebraic bases coincide. Let $a_1^{(1)}, \ldots, a_{d'}^{(1)}$, $a_1^{(1)}, \ldots, a_{d'}^{(2)}$, $a_1^{(2)}, \ldots, a_{d'}^{(2)}$, $a_1^{(2)}, \ldots, a_{d'}^{(2)}$ be two extensions of the reduced weighted algebraic basis $a_1^{(1)}, \ldots, a_{d'}^{(1)}$ and $a_1^{(2)}, \ldots, a_{d'}^{(2)}$ to a vector space basis as in Remark 2.2, respectively. The weights $w_i$ and $v_i$ of $a_i^{(1)}$ and $a_i^{(2)}$ can, after a possible reordering, be chosen to be identical, since they only depend on the dim$g_{\lambda_k}$. (This could possibly mix the original algebraic basis elements with the extended directions.) Let $\pi^{(1)}_\lambda$ and $\pi^{(2)}_\lambda$ be the projections and $[\cdot, \cdot]^{(1)}_0$ and $[\cdot, \cdot]^{(2)}_0$ the Lie brackets with respect to the bases $a_1^{(1)}, \ldots, a_{d'}^{(1)}$ and $a_1^{(2)}, \ldots, a_{d'}^{(2)}$, respectively. Define the linear map $\Phi: g \rightarrow g$ such that

$$\Phi(a_i^{(1)}) = \pi^{(2)}_w(a_i^{(1)}).$$

13
for all $i \in \{1, \ldots, d\}$. Then $\gamma_i^2(\Phi(a_i^{(1)})) = t^{w_i} \Phi(a_i^{(1)})$. So $\Phi \circ \gamma_i^1 = \gamma_i^2 \circ \Phi$. Next, $\Phi(a_i^{(1)}) = a_i^{(1)} \mod g_{\mu_i}$. Therefore $\Phi$ is a bijection. Moreover, $\Phi(a) = a \mod g_\mu$ for all $a \in g_\mu$ and $\mu > 0$. Therefore, if $a \in g_{\mu_1}$ and $b \in g_{\mu_2}$ then $[\Phi(a), \Phi(b)] = [a, b] \mod g_{(\mu_1+\mu_2)}$. And

$$[\Phi(a), \Phi(b)]_{(2)} = [\Phi(a), \Phi(b)] \mod g_{(\mu_1+\mu_2)},$$

$$= [a, b] \mod g_{(\mu_1+\mu_2)},$$

$$= [a, b]_{(1)} \mod g_{(\mu_1+\mu_2)} = \Phi([a, b]_{(1)}) \mod g_{(\mu_1+\mu_2)}.$$

Let $i, j \in \{1, \ldots, d\}$. Let $c \in g_{(w_i+w_j)}$ be such that $[\Phi(a_i), \Phi(a_j)]_{(2)} = \Phi([a_i, a_j]^{(1)}) + c$. Then for all $t > 0$ one has

$$t^{w_i+w_j} [\Phi(a_i^{(1)}), \Phi(a_j^{(1)})]_{(2)} = [\Phi(\gamma_i^1(a_i^{(1)})), \Phi(\gamma_j^1(a_j^{(1)}))]_{(2)}$$

$$= \gamma_i^2([\Phi(a_i^{(1)}), \Phi(a_j^{(1)})]_{(2)})$$

$$= \gamma_i^2([\Phi(a_i^{(1)}), \Phi(a_j^{(1)})]_{(2)})) + c$$

$$= \ldots = t^{w_i+w_j} \Phi([a_i^{(1)}, a_j^{(1)}]^{(1)}) + \gamma_i^2(c).$$

So $[\Phi(a_i^{(1)}), \Phi(a_j^{(1)})]_{(2)} = \Phi([a_i^{(1)}, a_j^{(1)}]^{(1)})$ and $\Phi$ is a Lie algebra isomorphism.

For all $t \in [0, 1)$ let $G_t$ be the connected, simply connected, Lie group with the Lie algebra $(g_i, [\cdot, \cdot])$ constructed by the above procedure. The group $G_0$ is unique and is called the homogeneous contraction of $G$. The homogeneous contraction $G_0$ will be used as a 'local approximation' of $G$ in much of the subsequent analysis. This is a generalization of the standard local approximation of $G$ by $\mathbb{R}^d$ which is only appropriate for a full vector space basis. Note that the connection between the weights and the structure constants in the next proposition is precisely the condition in [ElR5] in case all weights are integers.

**Proposition 3.2** The following conditions are equivalent.

I. $G_0 = \mathbb{R}^d$ ,

II. $d' = d$ and $w_i + w_j - w_k > 0$ whenever the structure constant $c_{ij}^k$ of the basis $a_1, \ldots, a_{d'}$ is non-zero.

**Proof** I$\Rightarrow$II. Assume that $d' < d$ then there exists $l \in \{2, \ldots, k\}$, to be chosen minimal, such that $g_{w_l} \neq \text{span}\{a_i : i \in \{1, \ldots, d'\}, w_i \leq w_l\}$. Then there are $i, j \in \{1, \ldots, d'\}$ such that $w_i + w_j = w_l$ and $[a_i, a_j] \in g_{w_i} \setminus g_{w_l}$. Since $[a_i, a_j]_0 = [a_i, a_j] \mod g_{w_l}$ this implies that $[a_i, a_j]_0 \neq 0$. Therefore $d' = d$. But then

$$[a_i, a_j]_t = \sum_{k \in \{1, \ldots, d\}} t^{w_i+w_j-w_k} c_{ij}^k a_k$$

and $[a_i, a_j]_0 = 0$ if and only if $w_i + w_j - w_k > 0$ for those $k$ such that $c_{ij}^k \neq 0$. 

14
II⇒I. Since
\[ [a_i, a_j] = \sum_{k \in \{1, \ldots, d\}} c_{ij}^k a_k \]

it follows that \([a_i, a_j]_0 = 0.\]

In the unweighted case, i.e., if the \(w_i = 1\), the proposition demonstrates that \(G_0 = \mathbb{R}^d\) if and only if one is dealing with a full vector space basis. Therefore the analysis of strict algebraic bases automatically enforces the introduction of non-commutative approximations \(G_0\) of \(G\). The previous analysis of sub coercive operators \([\text{EIR}3]\ [\text{EIR}7]\) was based on approximation with a nilpotent group \(\bar{G}\) with \(d'\) generators which was free of step \(r\), the rank of the algebraic basis, and hence \(\bar{G}\) is usually of larger dimension than the group \(G\). One clear advantage in using the group \(G_0\) as an approximant is that it has the same dimension as \(G\).

In order to analyze \(G\) and \(G_0\) more fully we need to collect some information about the intermediate groups \(G_t\), \(t \in (0, 1)\). Occasionally we write \(G_t = G\). We will identify quantities associated with \(G_t\) by indices and suffices \(t\) but in the case \(t = 1\) we often omit these indices or suffices.

Let \(\exp_t : (g, [\cdot, \cdot]) \to G_t, t \in [0, 1]\), denote the exponential map. We use the map \(\exp_0\) to lift the dilations \(\gamma_t\) on the Lie algebra \((g, [\cdot, \cdot])\) to dilations on \(G_0\), which we also denote by \(\gamma_t\). Complete the weighted algebraic basis \(a_1, \ldots, a_{d'}\) to a full vector space basis \(a_1, \ldots, a_d\) as in Remark 2.2. Let \((c_{ij}^k)\) be the structure constants of \((g, [\cdot, \cdot])\) with respect to the basis \(a_1, \ldots, a_d\). We may assume that \(c_{ij}^k \leq d^{-3}\) for all \(i, j, k \in \{1, \ldots, d\}\) and we let \(\|\cdot\|\) be the Euclidean norm with respect to the basis \(a_1, \ldots, a_d\). Then \(\|[a, b]\| \leq \|a\| \|b\|\) for all \(a, b \in g\).

**Lemma 3.3**

I. There exists a \(u_1 \in (0, 1)\) such that \(\exp_t\) is a diffeomorphism from \(\{a \in g : \|a\| < u_1\}\) onto an open neighbourhood of the identity in \(G_t\), uniformly for all \(t \in [0, 1]\).

II. There exists a \(u_2 \in (0, u_1]\) such that the Campbell–Baker–Hausdorff formula with respect to \((g, [\cdot, \cdot])\) is absolutely convergent on \(\{a \in g : \|a\| < u_2\}^2\) uniformly for \(t \in [0, 1]\).

III. There exists a \(u_3 \in (0, u_2]\) such that
\[ \exp_t a \exp_t b \in \exp_t \{c \in g : \|c\| < u_1\} \]
uniformly for all \(a, b \in \{c \in g : \|c\| < u_3\}\) and \(t \in [0, 1]\).

IV. Setting
\[ a \ast_t b = \log_t(\exp_t a \exp_t b), \]
where \(\log_t\) denotes the local inverse of \(\exp_t\) onto \(\{a \in g : \|a\| < u\}\) one has
\[ a \ast_t b = \gamma_t^{-1}(\gamma_t(a) \ast_1 \gamma_t(b)) = \gamma_t^{-1}(\gamma_t(a) \ast \gamma_t(b)) \]
for all \(t \in (0, 1]\) and \(a, b \in \{c \in g : \|c\| < u_3\}\).
V. There exists a \( u_4 \in (0, u_3) \) such that

\[
\exp_t a \exp_t b \in \exp_t \{ c \in \mathfrak{g} : \|c\| < u_4 \}
\]

uniformly for all \( a, b \in W = \{ c \in \mathfrak{g} : \|c\| < u_4 \} \) and \( t \in [0, 1] \).

**Proof** The diffeomorphic property in Statement I is well known for each \( \exp_t \) and the problem is to show uniformity of \( u_1 \) in \( t \). For \( t \in (0, 1] \) let

\[ V_t = \{ a \in \mathfrak{g} : \text{Im} \lambda < \pi \text{ for each eigenvalue } \lambda \text{ of } \text{ad}_t a \} \]

where \( \text{ad}_t \) denotes the adjoint action with respect to the Lie product \( [\cdot, \cdot]_t \). Since

\[
\text{ad}_t a = \gamma_t^{-1}(\text{ad}\gamma_t(a))\gamma_t
\]

(6)

it follows that \( \lambda \) is an eigenvalue of \( \text{ad}_t a \) if, and only if, it is an eigenvalue of \( \text{ad}\gamma_t(a) \). Therefore \( V_t = \gamma_t^{-1}(V_1) \supseteq V_1 \) where the inclusion follows because \( t \leq 1 \). But by [Var], Lemma 2.14.5, the subset \( V_1 \) is an open neighbourhood of \( 0 \) in \( \mathfrak{g} \). Since \( G_t \) is connected and simply connected, for all \( t \in (0, 1) \), by construction, it follows by [Var], Theorem 2.14.6, that \( \exp_t \) is a diffeomorphism from \( V_1 \) onto its image in \( G_t \). Statement I follows immediately.

For the proof of the second statement we need the following version of a standard result.

**Proposition 3.4 (Campbell–Baker–Hausdorff)** Let \( G \) be a Lie group with Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) and let \( \| \cdot \| \) be a Euclidean norm on \( \mathfrak{g} \) such that \( \| [a, b] \| \leq \|a\| \|b\| \) for all \( a, b \in \mathfrak{g} \). Then there exist \( M > 0 \) and \( \delta \in (0, (2s-1)^{-1}) \) and for each \( \alpha \in J(2) \) with \( |\alpha| \neq 0 \) there is a \( c_\alpha \in \mathbb{R} \) with \( |c_\alpha| \leq Ms^{|\alpha|} \), all independent of \( G, \mathfrak{g} \) and \( \| \cdot \| \), such that

\[
\exp b_1 \exp b_2 = \exp c(b_1, b_2)
\]

for all \( b_1, b_2 \in \mathfrak{g} \) with \( \|b_1\|, \|b_2\| < \delta \), where

\[
c(b_1, b_2) = \sum_{\substack{\alpha \in J(2) \\ |\alpha| \neq 0}} c_\alpha b_{[\alpha]}.
\]

In particular this series converges absolutely.

**Proof** This follows from the discussion in [Hoc] pages 111–112.

We continue with the proof of Lemma 3.3. The structure constants of the Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) with respect to the basis \( a_1, \ldots, a_d \) are equal to \( t^{w_i+w_j-w_k}c_{ij}^k \), where \( (c_{ij}^k) \) are the structure constants of \( (\mathfrak{g}, [\cdot, \cdot]) \) with respect to the basis \( a_1, \ldots, a_d \). Since \( w_k \leq w_i + w_j \) if \( c_{ij}^k \neq 0 \), they are also bounded by \( d^{-3} \) if \( t \in [0, 1] \). So \( \|[a, b]_t\| \leq \|a\| \|b\| \) for all \( a, b \in \mathfrak{g} \). Now Statement II follows from Proposition 3.4.

If \( M, s, \delta, c_\alpha \) are as in Proposition 3.4 then

\[
\|c(b_1, b_2)\| \leq \sum_{\substack{\alpha \in J(2) \\ |\alpha| \neq 0}} Ms^{|\alpha|}(2^{-1}\delta)^{|\alpha|} \leq \delta Ms(1 - \delta s)^{-1}
\]

16
for all \( b_1, b_2 \in \mathfrak{g} \) with \( \|b_1\|, \|b_2\| < 2^{-1} \delta \). Therefore \( \|c(b_1, b_2)\| < u_1 \) if \( \delta \) is taken small enough. Finally, if \( b_1, b_2 \in \mathfrak{g} \) with \( \|b_1\|, \|b_2\| < \delta \) and \( t \in (0, 1] \) then

\[
b_1 * b_2 = \sum_{\alpha \in J(2)} c_\alpha b_{[\alpha]} = \sum_{\alpha \in J(2)} c_\alpha \gamma_\alpha^{-1} b'_{[\alpha]} = \gamma_\alpha^{-1} (b'_1 * b'_2) = \gamma_\alpha^{-1} (\gamma_1(b_1) * \gamma_2(b_2)) ,
\]

where \( b'_1 = \gamma_1(b_1) \) and \( b'_2 = \gamma_2(b_2) \). This completes the proof of Lemma 3.3. \( \square \)

Next we need some information concerning the vector fields in the directions \( a_1, \ldots, a_d \) with respect to the left regular representation of \( G_t \) on \( C^\infty(G_t) \). Let \( t \in [0, 1] \). For \( i \in \{1, \ldots, d\} \) and \( \varphi \in C^\infty(G_t) \) define \( Y_i(t) \varphi : G_t \to \mathbb{C} \) by

\[
(Y_i(t) \varphi)(g) = \frac{d}{ds} \varphi(\exp_t(-sa_i) g) \big|_{s=0} .
\]

Moreover, for \( g \in G \) define \( R(t)(g) : G_t \to G_t \) by

\[
R(t)(g) = hg .
\]

Further introduce \( \pi : \mathfrak{g} \to \mathbb{R}^d \) by

\[
\pi \left( \sum_{j=1}^d \xi_j a_j \right) = (\xi_1, \ldots, \xi_d) .
\]

Then

\[
(Y_i(t) \varphi)(g) = -R(t)(g)_e \exp_{t*} \left( \frac{\partial}{\partial \pi_i |_0} \right)(\varphi) ,
\]

where \( R(t)(g)_e \exp_{t*} \left( \frac{\partial}{\partial \pi_i |_0} \right)(\varphi) \) denotes the differential of \( R(t)(g) \) at the identity, etc.. Next, for all \( \psi \in C_c^\infty(W) \), where \( W \) is the uniform open neighbourhood introduced in Lemma 3.3.V, define \( X_i(t) \psi \in C_c^\infty(W) \) by

\[
(X_i(t) \psi)(a) = (Y_i(t)(\psi \circ \log_t))(\exp_t a) .
\]

Then \( \exp_{t*}(X_i(t)_a) = Y_i(t) \big|_{\exp_t a} \) for all \( a \in W \).

Fix \( a \in W \) and set \( g = \exp_t a \). Let \( \gamma \) be a \( C^\infty \)-path from an open neighbourhood of \( 0 \in \mathbb{R} \) to \( \mathfrak{g} \) such that \( \gamma(0) = a \). Since \( X_1(t)_a, \ldots, X_d(t)_a \) span the tangent space at \( a \) there exist \( c_1, \ldots, c_d \in \mathbb{R} \) such that

\[
\gamma(0) = \sum_{i=1}^d c_i X_i(t)_a |_{a} .
\]

We calculate the constants \( c_1, \ldots, c_d \). Since

\[
\exp_{t*} |_{a} \gamma(0) = \sum_{i=1}^d c_i Y_i(t) |_{g}
\]

one obtains

\[
-\log_t \big|_{g} R(t)(g^{-1}) \exp_{t*} |_{a} \gamma(0) = \sum_{i=1}^d c_i \frac{\partial}{\partial \pi_i |_0} .
\]

17
Hence
\[
c_i = \left( - \log_t \left| R^{(t)}(g^{-1}) \right| \exp_t \left| \gamma(0) \right| (\pi^i) \right) \]
\[
= - \frac{d}{ds} \pi^i(\log_t(\exp_t(\gamma(s)) \exp_t(-a)))\bigg|_{s=0} = - \frac{d}{ds} \pi^i(\gamma(s) * (-a))\bigg|_{s=0} .
\]

In particular,
\[
\gamma(0) = \sum_{i=1}^d \frac{d}{ds} \pi^i(\gamma(s) * (-a))\bigg|_{s=0} X_i(0)\bigg|_{a} .
\]

Now let \( t \in [0,1] \), \( \psi \in C_c^\infty(W) \) and \( a \in W \). Then
\[
(X_i^{(t)} \psi)(a) = \left( Y_i^{(t)}(\psi \circ \log_t)\right)(\exp_t a)
\]
\[
= \frac{d}{ds} \psi\left( \log_t(\exp_t(-sa_i) \exp_t a) \right)\bigg|_{s=0}
\]
\[
= \frac{d}{ds} \psi(-sa_i * t a)\bigg|_{s=0} = -\gamma(0)\psi ,
\]
where \( \gamma(s) = sa_i * t a \). So
\[
(X_i^{(t)} \psi)(a) = \sum_{j=1}^d \frac{d}{ds} \pi^j(\gamma(s) * (-a))\bigg|_{s=0} X_j(0)^{i}\bigg|_{a} (\psi)
\]
\[
= \sum_{j=1}^d \frac{d}{ds} \pi^j((sa_i * t a) * (-a))\bigg|_{s=0} X_j(0)^{i}\bigg|_{a} (\psi)
\]
and hence
\[
X_i^{(t)}\bigg|_a = \sum_{j=1}^d \frac{d}{ds} \pi^j((sa_i * t a) * (-a))\bigg|_{s=0} X_j(0)^{i}\bigg|_{a} .
\]

Since the Campbell–Baker–Hausdorff formula converges absolutely on the set \( \{ c \in g : \| c \| < u_2 \} \) in Lemma 3.3.II it follows that there exist \( M, \delta > 0 \) and for all \( n \in \mathbb{N} \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \{0,1\} \) there exist \( c_{\varepsilon_1,\ldots,\varepsilon_n} \in \mathbb{R} \) such that \( |c_{\varepsilon_1,\ldots,\varepsilon_n}| \leq M\delta^n \) and
\[
\frac{d}{ds}(sa_i * t a) * (-a)\bigg|_{s=0} = a_i + \sum_{n=1}^\infty \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}} c_{\varepsilon_1,\ldots,\varepsilon_n}(ad_{\varepsilon_1 a}) \ldots (ad_{\varepsilon_n a})(a_i) (9)
\]
for all \( a \in W \) and \( t \in [0,1] \) and such that this series converges absolutely, uniformly for all \( a \in W \) and \( t \in [0,1] \).

These observations immediately imply the following continuity property of the vector fields.

**Lemma 3.5** For each \( m \in \mathbb{N} \), \( i_1, \ldots, i_m \in \{1, \ldots, d\} \) and \( \psi \in C_c^\infty(W) \) one has
\[
\lim_{t \to 0} X_{i_1}^{(t)} \ldots X_{i_m}^{(t)} \psi = X_{i_1}^{(0)} \ldots X_{i_m}^{(0)} \psi
\]
uniformly on \( W \).
Next for all \( \psi \in C_c^\infty(g) \) and \( t > 0 \) define \( \psi_t \in C_c^\infty(g) \) by

\[
\psi_t(a) = t^{D'/2} \psi(\gamma_t(a))
\]

where \( D' = \sum_{i=1}^d w_i \).

**Lemma 3.6** If \( t \in (0, 1] \), \( \psi \in C_c^\infty(\gamma_t(W)) \) and \( i \in \{1, \ldots, d\} \) then

\[
X_i^{(t)} \psi_t = t^{w_i}(X_i \psi)_t
\]

where \( X_i = X_i^{(1)} \).

**Proof** Let \( a \in g \). Then by Lemma 3.3.IV

\[
(X_i^{(t)} \psi_t)(a) = \frac{d}{ds} \psi_t(-sa; \ast t a) \bigg|_{s=0}
\]

\[
= t^{D'/2} \frac{d}{ds} \psi(-s\gamma_t(a); \ast \gamma_t(a)) \bigg|_{s=0}
\]

\[
= t^{D'/2} \frac{d}{ds} \psi(-st \gamma_t(a); \ast \gamma_t(a)) \bigg|_{s=0}
\]

\[
= t^{D'/2} w_i \frac{d}{ds} \psi(-sa; \ast \gamma_t(a)) \bigg|_{s=0} = t^{w_i}(X_i \psi)_t(a)
\]

where the first and last steps use (8).

**Corollary 3.7** For each \( m \in \mathbb{N} \), \( i_1, \ldots, i_m \in \{1, \ldots, d\} \) and \( \psi \in C_c^\infty(g) \) one has

\[
(X_{i_1}^{(0)} \ldots X_{i_m}^{(0)} \psi)(a) = \lim_{t \to 0} t^{D'/2} t^{w_1 + \ldots + w_m} (X_{i_1} \ldots X_{i_m} \psi_{t^{-1}})(\gamma_t(a))
\]

uniformly for \( a \in g \).

**Proof** If \( \text{supp} \psi \subseteq W \) and \( i \in \{1, \ldots, d\} \) then

\[
X_i^{(t)} \psi = t^{w_i}(X_i \psi_{t^{-1}})_t
\]

by Lemma 3.6 and hence the statement follows from Lemma 3.5. If, however, \( \text{supp} \psi \not\subseteq W \) one can fix an \( r \geq 1 \) such that \( \text{supp} \psi_r \subseteq W \) and then applying the previous result to \( \psi_r \) gives

\[
(X_{i_1}^{(0)} \ldots X_{i_m}^{(0)} \psi)(a) = r^{-D'/2} r^{-w_1 - \ldots - w_m} (X_{i_1}^{(0)} \ldots X_{i_m}^{(0)} \psi_r)(\gamma_{r^{-1}}(a))
\]

\[
= \lim_{t \to 0} (r^{-1} t)^{D'/2} (r^{-1} t)^{w_1 + \ldots + w_m} (X_{i_1} \ldots X_{i_m} \psi_r(t^{-1}))(\gamma_{r^{-1}}(a))
\]

\[
= \lim_{t \to 0} t^{D'/2} t^{w_1 + \ldots + w_m} (X_{i_1} \ldots X_{i_m} \psi_{t^{-1}})(\gamma_t(a))
\]

uniformly for all \( a \in \text{supp} \psi \).

Finally it will be necessary to examine the left regular representation of the groups \( G_t \) on the \( L_2 \)-spaces, \( L_2(G_t) \), with respect to a suitably normalized Haar measure. It follows
from [Var], Theorem 2.14.3 and Exercise 2.26(d), that there exists a unique Haar measure \( \rho_t \) on \( G_t \) such that
\[
\int_{G_t} d\rho_t(g) \psi(\log_t g) = \int_W da \sigma_t(a) \psi(a) = \int_g da \sigma_t(a) \psi(a)
\]
for all \( \psi \in C_c(W) \), where
\[
\sigma_t(a) = \left| \det \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \left(\text{ad}_t a\right)^n \right|
\]
for all \( a \in W \) and \( t \in [0,1] \). In particular this fixes a Haar measure on \( G = G_1 \). Then \( \sigma_0(a) = 1 \) since \((g,[\cdot,\cdot])_0\) is nilpotent.

We have the following transformation property under scaling.

**Lemma 3.8** If \( t \in (0,1] \) and \( \varphi, \psi \in C^\infty_c(\gamma_t(W)) \) then
\[
\int_g da \sigma_t(a) \overline{\varphi_t(a)} \psi_t(a) = \int_g da \sigma(a) \overline{\varphi(a)} \psi(a)
\]
where \( \sigma = \sigma_1 \).

**Proof** It follows from (6) that
\[
\sigma_t(a) = \left| \det \left( \gamma_t^{-1} \circ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \left(\text{ad}_t a\right)^n \right) \right| = \sigma(\gamma_t(a))
\]
for all \( a \in W \). Therefore
\[
\int_g da \sigma_t(a) \overline{\varphi_t(a)} \psi_t(a) = \int_g da \sigma(\gamma_t(a)) t^{D'} \overline{\varphi(\gamma_t(a))} \psi(\gamma_t(a)) = \int_g da \sigma(a) \overline{\varphi(a)} \psi(a)
\]
by a change of variables. \( \square \)

**Example 3.9** Let \( g \) be a homogeneous Lie algebra with respect to a family of dilations \((\gamma_t)_{t>0}\) and \( a_1, \ldots, a_d \) an algebraic basis such that \( \gamma_t(a_i) = t^{w_i}a_i \) for some \( w_i \in [1,\infty) \). Then \([a,b]_t = [a,b] \) for all \( t > 0 \) and hence also for \( t = 0 \). Thus all the Lie algebras coincide.

**Example 3.10** Let \( g \) be the two-dimensional Lie algebra of the \((ax+b)-\)group and \( a_1, a_2 \) a basis with the relations \([a_1,a_2] = a_1 \). Assign weights \( w_1 \) and \( w_2 \) to \( a_1 \) and \( a_2 \). Then \([a_1,a_2]_0 = 0 \) and \((g,[\cdot,\cdot])_0\) is commutative. Thus \( G_0 = \mathbb{R}^2 \).

**Example 3.11** Let \( g = \text{so}(3) \) be the three-dimensional Lie algebra of rotations in \( \mathbb{R}^3 \) with a basis \( a_1, a_2, a_3 \) satisfying \([a_1,a_2] = a_3, [a_2,a_3] = a_1 \) and \([a_3,a_1] = a_2 \). Thus \( a_1, a_2 \) is an algebraic basis of rank 2. Assign weights \( w_1 \) and \( w_2 \) to \( a_1 \) and \( a_2 \). Then \( a_1, a_2 \) is a reduced weighted algebraic basis for all choices of \( w_1 \) and \( w_2 \). Now \([a_1,a_2]_t = a_3 \) but \([a_2,a_3]_t = t^{w_2}a_1 \) and \([a_3,a_1] = t^{2w_1}a_2 \). Therefore \((g,[\cdot,\cdot])_0\) is the usual three-dimensional Heisenberg algebra \( h_1 \) and \( G_0 \) the simply connected Heisenberg group.
Example 3.12 Let \( \mathfrak{g} \) be the five-dimensional Heisenberg algebra \( \mathfrak{h}_2 \). Thus one has a basis \( a_1, \ldots, a_5 \) with \( [a_1, a_2] = [a_3, a_4] = a_5 \). Consider the algebraic basis \( a_1, \ldots, a_4 \). If \( w_1 = \ldots = w_4 = 1 \) then \( \mathfrak{g} \) is a homogeneous Lie algebra with dilations \( \gamma_t(a_i) = t^{w_i}a_i \) when \( w_5 = 2 \). With this weighting \( (\mathfrak{g}, [\cdot, \cdot]_0) = (\mathfrak{g}, [\cdot, \cdot]) = \mathfrak{h}_2 \) by Example 3.9. Alternatively, if \( w_1 = w_2 = w_3 = 1 \) and \( w_4 = 2 \) then \( [a_1, a_2]_t = a_5 \) but \( [a_3, a_4]_t = ta_5 \). Therefore \( (\mathfrak{g}, [\cdot, \cdot]_0) = \mathfrak{h}_1 \times \mathbb{R}^2 \). More generally, if \( w_1 + w_2 = w_3 + w_4 \) then the contraction gives \( \mathfrak{h}_2 \) but if \( w_1 + w_2 \neq w_3 + w_4 \) one obtains \( \mathfrak{h}_1 \times \mathbb{R}^2 \).

Lemma 3.13 Let \( a_1, \ldots, a_{d'} \) be an algebraic basis of rank \( r \) of a Lie algebra \( \mathfrak{g} \) and set all weights equal to one as in Example 2.4. Then \( (\mathfrak{g}, [\cdot, \cdot]_0) \) is a homogeneous nilpotent Lie algebra of rank \( r \).

Proof We use the notation of the proof of Proposition 3.1. Since all weights are equal to one, it follows that \( \mathfrak{g}_1 \) is the span of all commutators of \( a_1, \ldots, a_{d'} \) of (unweighted) order less than or equal to \( i \). Because the rank of the algebraic basis \( a_1, \ldots, a_{d'} \) in \( (\mathfrak{g}, [\cdot, \cdot]) \) equals \( r \) there is a multi-index \( \alpha \) with \( ||\alpha|| = |\alpha| = r \) such that \( a_\alpha \notin \mathfrak{g}_{i-1} = \mathfrak{g}_i \). Then \( a_\alpha \) is a \( \mathfrak{g}_r \) mod \( \mathfrak{g}_1 \neq 0 \) and hence \( a_\alpha \notin 0 \). So the rank of the nilpotent Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]_0) \) is at least \( r \).

Conversely, in \( (\mathfrak{g}, [\cdot, \cdot]_0) \) one has \( \mathfrak{g}_r = \mathfrak{g}_r = \mathfrak{g}_r \) and therefore the \( [\cdot, \cdot]_0 \)-commutators of \( a_1, \ldots, a_{d'} \) of order less than or equal to \( r \) span \( \mathfrak{g} \). So the rank of the algebraic basis \( a_1, \ldots, a_{d'} \) in \( (\mathfrak{g}, [\cdot, \cdot]_0) \) is at most \( r \).

Next suppose that the rank of the Lie algebra is larger than \( r \). Then there is an \( n > r \) together with \( b_1, \ldots, b_n \in \mathfrak{g} \) such that \( [b_1, \ldots, b_{i-1}, b_i]_0 \) \( \neq 0 \). But expressing the \( b_i \) as linear combinations of a vector space basis \( a_1, \ldots, a_d \) of \( \mathfrak{g} \) one deduces that there are \( i_1, \ldots, i_n \in \{1, \ldots, d\} \) such that \( [a_{i_1}, \ldots, [a_{i_{n-1}}, a_{i_n}]]_0 \) \( \neq 0 \). But we have chosen the additional elements \( a_{d'+1}, \ldots, a_d \) such that the basis, up to reordering, is the basis of Remark 2.2. This then implies, by the Jacobi identity, that there are \( n > r \) and \( i_1, \ldots, i_n \in \{1, \ldots, d'\} \) such that \( [a_{i_1}, \ldots, [a_{i_{n-1}}, a_{i_n}]]_0 \) \( \neq 0 \). Since the rank of \( a_1, \ldots, a_{d'} \) is at most \( r \) one then has

\[
[a_1, \ldots, [a_{i_{n-1}}, a_{i_n}]]_0 = \sum_{\beta} c_{\beta} a_\beta \|_{\cdot} \]

where the sum is over all multi-indices \( \beta \) in \( \{1, \ldots, d'\} \) with \( |\beta| \leq r \) and with \( c_{\beta} \in \mathbb{R} \). But by scaling it follows that the left hand side equals zero and hence \( n \leq r \).

Remark In general the rank of the nilpotent Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]_0) \) is larger than the rank of the algebraic basis \( a_1, \ldots, a_{d'} \). A simple example is the Heisenberg algebra \( \mathfrak{h}_1 \) with algebraic basis \( a_1, a_2, a_3 \) and weights \( w_1 = w_2 = 1 \) and \( w_3 = 2 \). Then the rank of the algebraic basis equals one, but the rank of \( \mathfrak{h}_1 \) is two. Even if one assumes additionally that \( a_1 \notin \text{span} \{a_{i_1} + \ldots + a_{i_n} : [a_1, [a_{i_1}, \ldots, a_{i_n}]]_0 = 0 \} \) where \( (\mathfrak{g}_i)_{i \geq 0} \) is the filtration corresponding to the weighted algebraic basis then the rank of the nilpotent Lie algebra can be larger than the rank of the algebraic basis. An example is the five-dimensional nilpotent (homogeneous) Lie algebra \( \mathfrak{g} \) with commutation relations

\[
[a_1, a_2] = a_4, \quad [a_1, a_4] = a_5, \quad [a_1, a_3] = a_5,
\]

and algebraic basis \( a_1, a_2, a_3 \) with weights \( w_1 = w_2 = 1 \) and \( w_3 = 2 \). Then the rank of the algebraic basis equals two, whilst the rank of \( \mathfrak{g} \) equals three.
Moreover, in general the rank of the algebraic basis \( a_1, \ldots, a_{d'} \) in \((g, [\cdot, \cdot]_0)\) is larger than the rank of the algebraic basis in \((g, [\cdot, \cdot])\). An example is in the five-dimensional nilpotent Lie algebra \( g \) with basis \( a_1, \ldots, a_5 \) and commutation relations \([a_2, a_3] = a_4, [a_2, a_4] = a_5 \) and \([a_1, a_2] = a_3 \). If we consider the weighted algebraic basis \( a_1, a_2, a_3 \) with weights \( w_1 = 3 \) and \( w_2 = w_3 = 1 \) then the rank of the algebraic basis is 2 in \( g \), but 3 in \((g, [\cdot, \cdot]_0)\).

4 Weighted subcoercive forms. Part I

Let \( G \) be a connected Lie group with Lie algebra \( g \) and \( a_1, \ldots, a_{d'} \) a weighted algebraic basis with weights \( w_1, \ldots, w_{d'} \in [1, \infty) \). We now assume that \( w_1, \ldots, w_{d'} \) have a common multiple, i.e., \( \cap_{i=1}^{d'} w_i N \neq \emptyset \), and set

\[
w = \min \{ x \in [1, \infty) : x \in w_i N \text{ for all } i \in \{1, \ldots, d'\} \} .
\]

Further we adopt the definitions and notation introduced in Section 1 for continuous representations \((X, G, U)\) of \( G \), the \( C^n \)-subspaces \( X_0, X_1, \text{etc.} \). In addition we refer to Section 1 for the definition of \( m \)-th order forms \( C \) and the corresponding operators \( dU(C) \).

Now we examine the definition of weighted subcoercivity given in the introduction. Let \( C: J(d') \to C \) be an \( m \)-th order form with \( m \in 2wN \). We call \( C \) a \( G \)-weighted subcoercive form, with respect to the weighted algebraic basis \( a_1, \ldots, a_{d'} \) of \( g \), if \( dL_G(C) \) satisfies the local Garding inequality (1) of Section 1, i.e., if

\[
\text{Re}(\varphi, dL_G(C)\varphi) \geq \mu (N_{2m/2}(\varphi))^2 - \nu \|\varphi\|^2_2
\]

for some \( \mu > 0 \) and \( \nu \in \mathbb{R} \), uniformly for all \( \varphi \in C^\infty_c(V) \) where \( V \) is some open neighbourhood of the identity \( e \in G \). (Note that the condition \( m \in 2wN \) implies that there exist several \( \alpha \in J(d') \) with \( \|\alpha\| = m/2 \).) The least upper bound \( \mu_{C,G} \) of the \( \mu \) for which (12) is satisfied is called the ellipticity constant.

Now for a general representation \((X, G, U)\) of the Lie group \( G \) and a weighted algebraic basis \( a_1, \ldots, a_{d'} \) of the corresponding Lie algebra \( g \) we study the \( m \)-th order operators \( dU(C) \) associated with the \( G \)-weighted subcoercive forms \( C \). We call \( dU(C) \) a \( G \)-weighted subcoercive operator.

In addition we want to assign an angle to each subcoercive form. This is analogous to the association of an angle with a sectorial form and the angle will subsequently be used in a similar way as an estimate of a lower bound for the holomorphy sector. It is, however, convenient to define the angle in a different manner to the usual sectorial angle. Set

\[
\theta_{C,G} = \theta_C = \sup \{ \theta \in [0, \pi/2] : \forall \psi \in [\theta, \theta] [e^{i\theta} C \text{ is a } G \text{-weighted subcoercive form} \} .
\]

Then \( \theta_{C,G} \in [0, \pi/2] \). We shall prove in Section 5 and Theorem 10.1 that in fact \( \theta_{C,G} > 0 \).

The foregoing notation explicitly identifies the relevant group \( G \). But if this is clear from the context we will omit the \( G \), e.g., we use the phrase weighted subcoercive instead of \( G \)-weighted subcoercive.

Example 4.1 If \( G = \mathbb{R}^d \) with the usual basis \( a_1, \ldots, a_d \) and with \( w_1, \ldots, w_d = 1 \) then an \( m \)-th order form \( C \) is a \( (w \text{-weighted}) \) subcoercive form if and only if

\[
\sum_{\alpha, |\alpha| = m} \text{Re} c_{\alpha} (i \xi)^\alpha > 0
\]
for all $\xi \in \mathbb{R}^d \backslash \{0\}$. This last condition is equivalent to the Gårding inequality as a simple consequence of Fourier theory, the differential operator is a multiplication operator on the Fourier transform.

**Example 4.2** Again let $G = \mathbb{R}^d$ with the usual basis $a_1, \ldots, a_d$ but with general weights. Then an $m$-th order form $C$ is a weighted subcoercive form if, and only if,

$$ \sum_{\alpha: ||\alpha|| = m} \text{Re} \ c_\alpha (i\xi)^\alpha > 0 $$

for all $\xi \in \mathbb{R}^d \backslash \{0\}$. This condition is again equivalent to the Gårding inequality by Fourier theory.

**Example 4.3** Let $g$ be a general Lie algebra, $m \in 2\omega \mathbb{N}$ and for all $\alpha, \beta \in J(d')$ with $||\alpha|| = m/2 = ||\beta||$ let $c_{\alpha, \beta} \in \mathbb{C}$ satisfy $\text{Re} \sum_{\alpha, \beta} c_{\alpha, \beta} \bar{\xi}_\alpha \xi_\beta > 0$ for all non-zero complex $(\xi_\alpha)$. Then the argument given in Section 1 establishes that the operator $H = \sum_{\alpha, \beta} (-1)^{||\alpha||} c_{\alpha, \beta} A^{\alpha*} A^\beta$ is a weighted subcoercive operator with respect to any representation.

**Example 4.4** Let all weights be equal to one, $d', s \in \mathbb{N}$ and consider the free group $G = G(d', s, 1, \ldots, 1)$. Then a form $C: J(d') \rightarrow \mathbb{C}$ of order $m \in \mathbb{N}$ is a $G$-weighted subcoercive form of order $m$ if, and only if, $C$ is a subcoercive form of order $m$ and step $s$ (see [EIR3]).

**Example 4.5** Let $g$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t > 0}$ and fix an algebraic basis $a_1, \ldots, a_{d'}$ such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty)$. Let $G$ be the corresponding simply connected Lie group with Lie algebra $g$. Then $G$ is a homogeneous group with the dilations $(\gamma_t)_{t > 0}$. A form $P$ is called a Rockland form if $P$ is homogeneous and the operator $dU(P)$ is injective on the space $X_\infty(U)$ for every non-trivial irreducible unitary representation $U$ of $G$. The Helffer–Nourrigat theorem, [HeN], states that a homogeneous form $P$ is a positive Rockland form if, and only if, the operator $dL(P)|_{C^{\infty}(G)}$ is hypoelliptic. A Rockland form $P$ is called a positive Rockland form if $dL(P)$ is symmetric and positive (see [AER]). In that case the operator $dL(P)$ is referred to as a positive Rockland operator.

Let $P$ be a positive Rockland form of order $m$. By [EIR6], Lemma 2.2, there exist a basis $b_1, \ldots, b_d$ of $g$, $d'' \in \{1, \ldots, d\}$ and $v_1, \ldots, v_d \in [1, \infty)$ such that $[g, g] \subseteq \text{span}\{b_{d'+1}, \ldots, b_d\}$ and $\gamma_t(b_i) = t^{v_i} b_i$ for all $i \in \{1, \ldots, d\}$ and $t > 0$. Moreover, $b_1, \ldots, b_{d''}$ is an algebraic basis of $g$. We give $b_1, \ldots, b_{d''}$ the weights $v_1, \ldots, v_{d''}$. It follows from Example 2.8 that the filtration corresponding to the algebraic basis $a_1, \ldots, a_d'$ equals the filtration corresponding to the weighted algebraic basis $b_1, \ldots, b_{d''}$. It then follows from Lemma 2.4 in [EIR6] that $m \in 2v\mathbb{N}$ for all $i \in \{1, \ldots, d''\}$. Set

$$ v = \min\{x \in [1, \infty) : x \in v_i \mathbb{N} \text{ for all } i \in \{1, \ldots, d''\}\} $$

Then by definition of $v$ one deduces that $m \in 2v\mathbb{N}$. (We do not assume that the $v_i$ are integers, in which case $v = \text{lcm}(v_1, \ldots, v_d)$, and in which case it is well known that $m \in 2v\mathbb{N}$. In the present situation one writes $m = 2qv + x$, with $q \in \mathbb{N}$ and $x \in [0, 2v)$ and easily establishes that $x = 0$.) Moreover, it follows from [EIR6], Theorem 2.5, that $dL(P)$ satisfies a Gårding inequality. So every positive Rockland operator is a weighted
subcoercive operator associated with a weighted subcoercive form with respect to a suitable weighted algebraic basis of \( g \).

On the other hand, if \( m \in 2wN \) then it follows from [EIR6], Theorem 2.5, that \( dL(P) \) satisfies a Gårding inequality and hence \( P \) is a \( G \)-weighted subcoercive form.

If \( m \in 2wN \) then there are many positive Rockland operators of order \( m \). For example, if \( P \) is the form such that

\[
dU(P) = \sum_{i=1}^{d'} (-1)^{m/(2w)} A_i^{m/2w_i}
\]

for any representation \((\mathcal{A}, G, U)\) (see [FoS] (4.20)) then \( dL(P) \) is a positive Rockland operator. These operators have been studied in [FoS] [Heb1] [DzH] [Dzi] [DHZ] [AER] and [EIR6].

The definition of \( G \)-subcoercivity is local insofar the Gårding inequality (12) is only required for \( \varphi \) supported in some arbitrarily small neighbourhood \( V \) of the identity. We will, however, show that this is equivalent to a global condition, i.e., we will conclude that the local Gårding inequality implies that (12) is valid for all \( \varphi \in L_{2;\infty}(G) \). This equivalence is not hard to understand if the group is homogeneous.

Let \( G \) be a homogeneous group with dilations \((\gamma_t)_{t>0}\). The action of \( \gamma_t \) on \( G \) lifts to an isometric action \( \varphi \mapsto \varphi_t \) on \( L^2(G) \) by the definition

\[
\varphi_t(g) = t^{d'/2} \varphi(\gamma_t(g))
\]

Moreover,

\[
N'_{2;k}(\varphi_t) = t^k N'_{2;k}(\varphi)
\]

for all \( k \geq 0 \). But if \( V \) is a fixed neighbourhood of the identity and \( \varphi \in L_{2;\infty}(G) \) has compact support then \( \varphi_t \in C_\infty(V) \) for all sufficiently large \( t \). Then

\[
t^m \left( \text{Re}(\varphi, dL_G(P)\varphi) - \mu \left( N'_{2;m/2}(\varphi) \right)^2 \right) = \text{Re}(\varphi_t, dL_G(P)\varphi_t) - \mu \left( N'_{2;m/2}(\varphi_t) \right)^2 \\
\geq \text{Re}(\varphi_t, dL_G(P - C)\varphi_t) - \nu \|\varphi_t\|_2^2
\]

where \( P \) is the principal part of \( C \), which scales as \( t^m \). Since \( C - P \) is of lower order than \( P \) one deduces that

\[
\lim_{t \to \infty} t^{-m} \left( \text{Re}(\varphi_t, dL_G(C - P)\varphi_t) - \nu \|\varphi_t\|_2^2 \right) = 0
\]

Therefore one concludes that

\[
\text{Re}(\varphi, dL_G(P)\varphi) \geq \mu \left( N'_{2;m/2}(\varphi) \right)^2
\]

uniformly for all \( \varphi \in L_{2;\infty}(G) \) with compact support, and then by continuity for all \( \varphi \in L_{2;\infty}(G) \), and for all \( \mu < \mu_{C,G} \). Hence, taking the supremum over \( \mu \),

\[
\text{Re}(\varphi, dL_G(P)\varphi) \geq \mu_{C,G} \left( N'_{2;m/2}(\varphi) \right)^2
\]

for all \( \varphi \in L_{2;\infty}(G) \). Thus the local Gårding inequality (12) for \( C \) implies the global homogeneous Gårding inequality (14) for the principal part \( P \). The converse statement is part of the next proposition.
In the sequel we establish for general $G$ that the local Gårding inequality for $C$ is in fact equivalent to a global inequality for the principal part $P$. But the proof of this assertion is very indirect and will result from detailed analysis of the operators $dL_G(C)$. The key to the analysis is the passage to a reduced weighted algebraic basis and the introduction of the corresponding homogeneous contraction $G_0$ of $G$. It is a remarkable fact that the Gårding inequalities on $G$ and $G_0$ are equivalent. The next proposition compares various versions of the Gårding inequalities for $G$ and $G_0$. It should be emphasized that in the following proposition all the conditions are equivalent and, in addition, all the ellipticity constants are equal. But at this stage we are only able to establish some of these connections. (We prove the equivalence of the remaining implications in Section 10.)

**Proposition 4.6** Let $G$ be a connected Lie group, $a_1, \ldots, a_n$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of $G$, $G_0$ the corresponding homogeneous contraction of $G$ and $V, V_0$ open neighbourhoods of the identity in $G$ and $G_0$, respectively. Further let $m \in 2\mathbb{W}\mathbb{N}$ and $C$ be an $m$-th order form with principal part $P$. Consider the following conditions.

1 (1') There is a $\mu > 0$ and $\nu \in \mathbb{R}$ such that
\[
\Re(\varphi, dL_G(C)\varphi) \geq \mu (N_{2;m/2}(\varphi))^2 - \nu \|\varphi\|_2^2
\]
for all $\varphi \in L_{2;\infty}(G)$ (for all $\varphi \in C_c^\infty(V)$).

2 (2') There is a $\mu > 0$ and $\nu \in \mathbb{R}$ such that
\[
\Re(\varphi, dL_G(P)\varphi) \geq \mu (N_{2;m/2}(\varphi))^2 - \nu \|\varphi\|_2^2
\]
for all $\varphi \in L_{2;\infty}(G)$ (for all $\varphi \in C_c^\infty(V)$).

3 (3') There is a $\mu > 0$ and $\nu \in \mathbb{R}$ such that
\[
\Re(\varphi, dL_{G_0}(C)\varphi) \geq \mu (N_{2;m/2}(\varphi))^2 - \nu \|\varphi\|_2^2
\]
for all $\varphi \in L_{2;\infty}(G_0)$ (for all $\varphi \in C_c^\infty(V_0)$).

4 (4') There is a $\mu > 0$ and $\nu \in \mathbb{R}$ such that
\[
\Re(\varphi, dL_{G_0}(P)\varphi) \geq \mu (N_{2;m/2}(\varphi))^2 - \nu \|\varphi\|_2^2
\]
for all $\varphi \in L_{2;\infty}(G_0)$ (for all $\varphi \in C_c^\infty(V_0)$).

5 (5') There is a $\mu > 0$ such that
\[
\Re(\varphi, dL_{G_0}(P)\varphi) \geq \mu (N_{2;m/2}(\varphi))^2
\]
for all $\varphi \in L_{2;\infty}(G_0)$ (for all $\varphi \in C_c^\infty(V_0)$).

Then $1 \Rightarrow 1' \Rightarrow 3 \Leftrightarrow 3' \Leftrightarrow 4 \Leftrightarrow 4' \Leftrightarrow 5 \Leftrightarrow 5' \Leftrightarrow 2' \Leftrightarrow 2$. Moreover, if 1' is valid then $\mu_{C,G} \leq \mu_{P,G_0} = \mu_{C,G_0}$ and if 2' is valid $\mu_{P,G} \leq \mu_{P,G_0} = \mu_{C,G_0}$.

**Proof** Clearly each of the five unprimed conditions implies its primed version and $5 \Rightarrow 4$ and $5' \Rightarrow 4'$. But we have already argued that $3' \Rightarrow 5$ and hence $4' \Rightarrow 5$. Therefore we
have $3' \Rightarrow 5 \Leftrightarrow 4 \Leftrightarrow 4'$. Hence it suffices to prove that $1' \Rightarrow 4'$ and $5 \Rightarrow 3$, because $2' \Rightarrow 4'$ follows from $1' \Rightarrow 4'$.

First we prove $5 \Rightarrow 3$. Let $P_0$ be the form defined in (13). It was established in [AER], Proposition 2.1, that $dL(P_0)$ is a positive self-adjoint operator and there exists a $c > 0$ such that

$$\|\varphi\|_{2,m} \leq c (\|dL(P_0)\varphi\|_2 + \|\varphi\|_2)$$

for all $\varphi \in D(dL(P_0)) = L'_{2,m}$. Now let $C_1$ be a form of order less than or equal to $m$. Then the operator $dL(C_1 + C_1^*)$ is symmetric and $dL(P_0)$-bounded, by the foregoing bounds. Hence by Theorem VI.1.38 of [Kat] one deduces that there exist $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$|(\varphi, dL(C_1 + C_1^*)\varphi)| \leq c_1 (\varphi, dL(P_0)\varphi) + c_2 \|\varphi\|_2^2$$

for all $\varphi \in L'_{2,m}$. The same argument applies to the operator $i(dL(C_1 - C_1^*))$ and by linear combination it follows that there exist $c_1' > 0$ and $c_2' \in \mathbb{R}$ such that

$$|(\varphi, dL(C_1)\varphi)| \leq c_1' (\varphi, dL(P_0)\varphi) + c_2' \|\varphi\|_2^2 \cdot$$

Next there exists a $c_3 > 0$ such that

$$(\varphi, dL(P_0)\varphi) = \|dL(P_0)^{1/2}\varphi\|_2^2 \leq c_3 (N'_{2,m/2}(\varphi))^2$$

for all $\varphi \in L'_{2,m}$ (see [ElR6] Corollary 2.6.II). Combining these estimates it follows that for all $\alpha \in J(d')$ with $\|\alpha\| \leq m$ there exist $c_1, c_2 > 0$ such that

$$|(\varphi, A^\alpha \varphi)| \leq c_1 (N'_{2,m/2}(\varphi))^2 + c_2 \|\varphi\|_2^2$$

for all $\varphi \in L'_{2,m}$. Therefore, by a scaling argument, one concludes that

$$|(\varphi, A^\alpha \varphi)| \leq c_1 e^{m-\|\alpha\|} (N'_{2,m/2}(\varphi))^2 + c_2 e^{-\|\alpha\|} \|\varphi\|_2^2$$

(15)

for all $\varepsilon > 0$ if $\|\alpha\| < m$ and

$$|(\varphi, A^\alpha \varphi)| \leq c_1 (N'_{2,m/2}(\varphi))^2$$

(16)

if $\|\alpha\| = m$. Now the implication $5 \Rightarrow 3$ follows from (15). Moreover, it follows from the proof that $\mu_{C,G_0} = \mu_{P,G_0}$.

Now we prove the implication $1' \Rightarrow 4'$. We temporarily denote by $(\cdot, \cdot)_0$ the inner product on $L_2(G_0)$ and $(\cdot, \cdot)$ the inner product on $L_2(G)$. Let $W$ be the open neighbourhood of 0 in $g$ as introduced in Lemma 3.3.V. We may assume that $\exp(W) \subseteq V$ and that $\exp_0(W) = V_0$. We will use the notation of Section 3. Let $\varphi \in C_\infty^0(V_0)$ and set $\psi = \varphi \circ \exp_0 \in C_\infty^0(W)$. For $t \in [0,1]$ set

$$P_{X(t)} = \sum_{\alpha \in J(d') \atop \|\alpha\|=m} c_\alpha X(t)^\alpha,$$

where we have used multi-index notation. Then

$$\text{Re}(\varphi, dL_{G_0}(P)\varphi)_0 = \text{Re} \int da \overline{\psi(a)} (P_{X(t)})\psi(a)$$

26
and we approximate $\text{Re} \int da \overline{\psi(a)} (P_{X(t)} \psi)(a)$ by $\text{Re} \int da \sigma_t(a) \overline{\psi(a)} (P_{X(t)} \psi)(a)$.

Let $t \in (0, 1]$. Now Lemma 3.6, applied to $\psi_{t-1}$ gives

$$P_{X(t)} \psi = P_{X(t)}(\psi_{t-1})_t = t^n (P_X \psi_{t-1})_t,$$

where $P_X = P_{X(t)}$. Since $\psi_{t-1}, P_{X(t)} \psi_{t-1} \in C_c^\infty(\gamma_t(W))$ one can then use Lemma 3.8 to deduce that

$$\text{Re} \int da \sigma_t(a) \overline{\psi(a)} (P_{X(t)} \psi)(a)$$

$$= t^n \text{Re} \int da \sigma_t(a) \overline{\psi_{t-1}(a)} (P_X \psi_{t-1})(a)$$

$$= t^n \text{Re} \int da \sigma(a) \overline{\psi_{t-1}(a)} (P_X \psi_{t-1})(a)$$

$$= t^n \text{Re} \left( (\psi_{t-1} \circ \log), dL_G(P)(\psi_{t-1} \circ \log) \right)$$

$$= t^n \text{Re} \left( (\psi_{t-1} \circ \log), dL_G(C)(\psi_{t-1} \circ \log) \right)$$

$$- \sum_{\|\alpha\| < m} t^n \text{Re} \left( (\psi_{t-1} \circ \log), c_\alpha dL_G(A^\alpha)(\psi_{t-1} \circ \log) \right).$$

Since $dL_G(C)$ satisfies the Gårding inequality on $C_c^\infty(V)$ one then obtains the estimate

$$\text{Re} \int da \sigma_t(a) \overline{\psi(a)} (P_{X(t)} \psi)(a)$$

$$\geq \mu t^n (N'_t;2;m/2(\psi_{t-1} \circ \log))^2 - \nu t^n \|\psi_{t-1} \circ \log\|^2_{(1),2}$$

$$- \sum_{\|\alpha\| < m} t^n \text{Re} \left( (\psi_{t-1} \circ \log), c_\alpha dL_G(A^\alpha)(\psi_{t-1} \circ \log) \right)$$

$$= \mu t^n \max_{\|\alpha\| = m/2} \int da \sigma(a) |(X^{(1)}^\alpha \psi_{t-1})(a)|^2 - \nu t^n \int da \sigma(a) |\psi_{t-1}(a)|^2$$

$$- \sum_{\|\alpha\| < m} t^n \text{Re} c_\alpha \int da \sigma(a) \overline{\psi_{t-1}(a)} (X^{(1)}^\alpha \psi_{t-1})(a),$$

where $N'_{(t);2;m/2}(\varphi)$ and $\|\varphi\|_{(1),2}$ denote the seminorm and norm of $\varphi$ on the group $G_t$. Since $\psi_{t-1}, X^{(1)}^\alpha \psi_{t-1} \in C_c^\infty(\gamma_t(W))$ one can again use Lemmas 3.8 and 3.6 to deduce that

$$\text{Re} \int da \sigma_t(a) \overline{\psi(a)} (P_{X(t)} \psi)(a)$$

$$\geq \mu t^n \max_{\|\alpha\| = m/2} \int da \sigma_t(a) |(X^{(1)}^\alpha \psi_{t-1})(a)|^2 - \nu t^n \int da \sigma_t(a) |(\psi_{t-1})(a)|^2$$

$$- \sum_{\|\alpha\| < m} t^n \text{Re} c_\alpha \int da \sigma_t(a) \overline{(\psi_{t-1})(a)} (X^{(1)}^\alpha \psi_{t-1})(a)$$

$$= \mu \max_{\|\alpha\| = m/2} \int da \sigma_t(a) |(X^{(1)}^\alpha \psi)(a)|^2 - \nu t^n \int da \sigma_t(a) |\psi(a)|^2$$

$$- \sum_{\|\alpha\| < m} t^n - \|\alpha\| \text{Re} c_\alpha \int da \sigma_t(a) \overline{\psi(a)} (X^{(1)}^\alpha \psi)(a).$$

27
Now it follows from (11) that $\lim_{t \to 0} \sigma_t(a) = \lim_{t \to 0} \sigma_t(0) = 1 = \sigma_0(a)$ uniformly for all $a \in W$. Therefore, as an application of Lemma 3.5, one estimates

$$\text{Re}(\varphi, dL_G(P)\varphi)_0 = \text{Re} \int da \overline{\psi(a)} (P_{X(0)} \psi)(a)$$

$$= \lim_{t \to 0} \text{Re} \int da \sigma_t(a) \overline{\psi(a)} (P_{X(0)} \psi)(a)$$

$$\geq \lim_{t \to 0} \left( \mu \max_{||\alpha||=m/2} \int da \sigma_t(a) |(X^{(t)\alpha} \psi)(a)|^2 - \nu t^m \int da \sigma_t(a) |\psi(a)|^2ight.$$

$$- \sum_{||\alpha||<m} t^{m-||\alpha||} \text{Re} c_{\alpha} \int da \sigma_t(a) \overline{\psi(a)} (X^{(t)\alpha} \psi)(a) \left. \right)$$

$$= \mu \max_{||\alpha||=m/2} \int da \sigma_0(a) |(X^{(0)\alpha} \psi)(a)|^2 = \mu (N_{(0)}; 2m/2(\varphi))^2.$$ 

This completes the proof of the proposition. \qed

## 5 Homogeneous groups

Our approach to the analysis of subcoercive operators on the Lie group $G$ is to study the comparable problem on the homogeneous contraction $G_0$ of $G$ and then to extend the results to $G$ by a parametrix argument. Therefore we must first analyze subcoercive operators on homogeneous groups.

Let $g$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and $a_1, \ldots, a_d$ an algebraic basis such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty)$. Then $a_1, \ldots, a_d$ is a reduced weighted algebraic basis by Example 2.8. Further let $G$ be the corresponding connected, simply connected, homogeneous Lie group and $C: J(d') \to C$ a $G$-weighted subcoercive form of order $m$ where $m \in 2wN$. In this section we also assume the form $C$ is homogeneous, so $C = P$, the principal part of $C$. Let $H = dL(C)$ be the corresponding weighted subcoercive operator on $L^2(G)$. We frequently use the fact that $H$ is a homogeneous operator.

First we prove that $H|_{C^\infty}$ is hypoelliptic and then, using a variety of standard $L^2$-techniques, we deduce that $H$ generates a holomorphic semigroup $S$ on the sector $\Lambda(\theta_C)$ with a smooth kernel $K$ satisfying Gaussian type bounds. The homogeneous structure of the Lie algebra $g$ is exploited through scaling arguments. The arguments are an amalgamation of methods used in [HeN] [ElR3] and [AER].

As a preliminary we observe that $\theta_C > 0$. Set

$$\tau = \sup \{ |\text{Im}(dL_G(P)\varphi, \varphi)|/(N_{2m/2}^2(\varphi))^2 : \varphi \in L_2(\infty)(G), \varphi \neq 0 \}.$$ 

Then $\tau < \infty$ by (16) and $\theta_C \geq \arctan(\mu^{-1}) > 0$.

**Lemma 5.1** The operator $H|_{C^\infty} = dL(C)|_{C^\infty}$ is hypoelliptic.

**Proof** We prove that $dL(C)$ is a Rockland operator, i.e., the operator $dU(C)$ is injective on $\mathcal{H}_\infty(U)$ for each non-trivial irreducible representation $(\mathcal{H}, G, U)$. But this is equivalent to hypoellipticity by the Helffer–Nourrigat theorem [HeN].

28
It follows by the definition of subcoercivity and the homogeneity of $G$ that

$$\text{Re}(\varphi, H\varphi) \geq \mu (N'_{2m/2}(\varphi))^2$$

for all $\varphi \in L_{2;\infty}(G)$, where $\mu = \mu_{C,G}$ is the ellipticity constant (see Proposition 4.6). So

$$\text{Re}(\varphi, dL(C)\varphi) \geq \mu \max_{\alpha \in \mathcal{I}_G} \text{Re}(-1)^{|\alpha|} (\varphi, A^{(\alpha, \alpha)}\varphi)$$

(17)

for all $\varphi \in L_{2;\infty}(G)$. Now we argue as in Section 2 and the proof of Proposition 4.6.2 in Helffer–Nourrigat [HeN] that

$$\text{Re}(x, dU(C)x) \geq \mu \max_{\alpha \in \mathcal{I}_G} \text{Re}(-1)^{|\alpha|} (x, A^{(\alpha, \alpha)}x)$$

(18)

for every irreducible unitary representation $(\mathcal{H}, G, U)$ and every $x \in \mathcal{H}_{\infty}(U)$.

At this point we merely sketch the proof. In the proof of Proposition 10.2 we need a refinement of this result and we then give full details. First we note that Proposition 2.1 of [HeN], which gives a norm comparison for two operators, has a version expressed in terms of the forms of the operators. This is evident from the proof of the proposition which relies on Fourier transformation. Replacing the Fourier transforms of the operator norms by similar transforms of the operator forms does not affect the argument.

Secondly, one deduces (18) from (17) by the same argument that establishes Proposition 4.6.2 as a consequence of Proposition 4.6.1 in [HeN]. This relies upon the form version of Proposition 2.1.

One immediately deduces from (18) that

$$\text{Re}(x, dU(C)x) \geq \mu (N'_{2m/2}(x))^2$$

(19)

for every irreducible unitary representation $(\mathcal{H}, G, U)$ and every $x \in \mathcal{H}_{\infty}(U)$.

Now let $(\mathcal{H}, G, U)$ be a non-trivial irreducible unitary representation and for $x \in \mathcal{H}_{\infty}(U)$ suppose $dU(C)x = 0$. Then $(x, dU(C)x) = 0$ and one deduces from (19) that $N'_{2m/2}(x) = 0$. Then $dU(a_i)x = 0$ and, by spectral theory, one also has $dU(a)x = 0$ for all $i \in \{1, \ldots, d'\}$. Since $a_1, \ldots, a_{d'}$ is an algebraic basis for $\mathfrak{g}$ it follows that $dU(a)x = 0$ for all $a \in \mathfrak{g}$ and since $U$ is non-trivial this implies that $x = 0$. Thus $dL(C)$ is a Rockland operator and therefore hypoelliptic.

The Helffer–Nourrigat theorem even states that the operator $H + \lambda I$ is hypoelliptic for all $\lambda \in \mathbb{C}$. Next we establish a variant of hypoellipticity involving an extension of $H$.

**Lemma 5.2** Let $\varphi \in D((H^t)^*)$ where $H^t = dL(C^t)$. If $((H^t)^* + \lambda I)\varphi \in L_{2;\infty}$ for some $\lambda \in \mathbb{R}$ then $\varphi \in L_{2;\infty}$.

**Proof** For all $\psi \in L_{2;\infty}$ the function $g \mapsto (L(g)\psi, \varphi)$ is infinitely differentiable and

$$\int_G dg \, ((H^t + \lambda I)\tau)(g) (\psi, L(g^{-1})\varphi) = \int_G d\tau(g) ((H^t + \lambda I)L(g)\psi, \varphi)$$

$$= \int_G d\tau(g) (\psi, L(g^{-1})((H^t)^* + \lambda I)\varphi)$$

29
for all \( \tau \in C^\infty_c(G) \). Since \( L_{2;00} \) is norm dense in \( L_2 \) it follows from the Lebesgue dominated convergence theorem that

\[
\int_G dg \left( (H^* + \lambda I) \psi, L(g^{-1}) \phi \right) = \int_G \bar{\psi} (g) \left( (H^* + \lambda I) \phi \right)
\]

for all \( \tau \in C^\infty_c(G) \) and \( \psi \in L_2 \).

Now let \( \psi \in L_2 \) and define the functions \( u, v : G \to \mathbb{C} \) by \( u(g) = (\psi, L(g^{-1}) \phi) \) and \( v(g) = (\psi, L(g^{-1}) (H^* + \lambda I) \phi) \). Then \( (H + \lambda I) u = v \) as distributions. Since \( H + \lambda I \) is hypoelliptic, by Lemma 5.1, and \( v \in C^\infty_c(G) \), it follows that \( u \in C^\infty(G) \). This is valid for all \( \psi \in L_2 \), so \( \phi \in L_{2;00} \).

The hypoellipticity of \( H \) has many consequences, in particular for regularity properties.

**Corollary 5.3**

I. The operator \( H \) is closed in \( L_2 \).

II. For all \( n \in \mathbb{N} \) one has \( D(H^n) = L_{2;n,m}^2 \), with equivalent norms. There exists a \( c > 0 \) such that

\[
c N_{2;n,m}^\prime(\varphi) \leq \|H^n \varphi\|_2
\]

for all \( \varphi \in D(H^n) \).

III. The spaces \( L_{2;00} \) and \( C^\infty_c(G) \) are cores for \( H^n \), for all \( n \in \mathbb{N} \).

IV. If \( n \in \mathbb{N} \) and \( k \in (0, nm) \) then there exists a \( c > 0 \) such that

\[
N_{2;ik}^\prime(\varphi) \leq c \varepsilon^{nm-k} N_{2;n,m}^\prime(\varphi) + c \varepsilon^{-k} \|\varphi\|_2
\]

for all \( \varepsilon > 0 \) and \( \varphi \in L_{2;n,m}^2 \).

V. If \( n \in \mathbb{N} \) and \( k \in (0, nm) \) then there exists a \( c > 0 \) such that

\[
\|\varphi\|_{2;ik}^2 \leq c \varepsilon^{nm-k} \|\varphi\|_{2;n,m}^2 + c \varepsilon^{-k} \|\varphi\|_2^2
\]

for all \( \varepsilon > 0 \) and \( \varphi \in L_{2;n,m}^2 \).

**Proof** This follows from the Helffer-Nourrigat theory and scaling. See also [AER] Proposition 2.1.

**Proposition 5.4** The operator \( H = dL(C) \) generates a holomorphic semigroup \( S \) on \( L_2(G) \), with a holomorphy sector containing the sector \( \Lambda(\theta_C) \), which satisfies \( \|S_z\|_{2 \to 2} \leq 1 \) for all \( z \in \Lambda(\theta_C) \).

**Proof** The Gårding inequality establishes that

\[
\|(H + \lambda I) \varphi\|_2 \geq \Re(\varphi, (H + \lambda I) \varphi) \geq \lambda \|\varphi\|_2^2
\]

for all \( \lambda > 0 \) and \( \varphi \in L_{2;00} \). Hence

\[
\|(H + \lambda I) \varphi\|_2 \geq \lambda \|\varphi\|_2
\]

(20)
for all \( \varphi \in D(H) \). So \( H + \lambda I \) is injective for all \( \lambda > 0 \). If we show that the range \( R(H + \lambda I) \) of \( H + \lambda I \) is equal to \( L_2 \) for some \( \lambda > 0 \) then it follows from the Hille–Yosida theorem that \( H \) generates a contraction semigroup. It suffices to prove that \( R(H + \lambda I) \) is dense since the space \( R(H + \lambda I) \) is closed by (20).

The formal adjoint \( C^\dagger \) is also a \( G \)-weighted subcoercive form so we can apply the above reasoning to \( H^\dagger = dL(C^\dagger) \) and deduce that

\[
\| (H^\dagger + \lambda I) \varphi \|_2 \geq \lambda \| \varphi \|_2 \tag{21}
\]

for all \( \varphi \in D(H^\dagger) = L_{2;m}^2 \) and \( \lambda > 0 \). Fix \( \lambda > 0 \). Let \( P \) be the projection of \( L_2 \) onto \( R(H^\dagger + \lambda I) \). By (21) the map \( T \colon (H^\dagger + \lambda I) \varphi \mapsto \varphi \) from \( R(H^\dagger + \lambda I) \) into \( L_2 \) is continuous. Let \( E = TP \). Then \( E \) is continuous and \( E(H^\dagger + \lambda I) \varphi = \varphi \) for all \( \varphi \in L_{2;m}^2 \). So for all \( \psi \in L_2 \) one has

\[
(E^* \psi, (H^\dagger + \lambda I) \varphi) = (\psi, E(H^\dagger + \lambda I) \varphi) = (\psi, \varphi)
\]

for all \( \varphi \in L_{2;m}^2 \). Therefore \( E^* \psi \in D((H^\dagger + \lambda I)^*) \) and

\[
(H^\dagger + \lambda I)^* E^* \psi = \psi .
\]

Now it follows from Lemma 5.2 that \( E^* \psi \in L_{2;\infty}^2 \subseteq D(H + \lambda I) \) for all \( \psi \in L_{2;\infty}^2 \). Hence

\[
(H + \lambda I) E^* \psi = \psi
\]

for all \( \psi \in L_{2;\infty}^2 \). Therefore \( L_{2;\infty}^2 \subseteq R(H + \lambda I) \) and \( R(H + \lambda I) \) is dense. Thus \( H \) generates a contraction semigroup.

Finally, for any \( \theta \in (-\theta_C, \theta_C) \) the form \( e^{it\theta} \mathcal{C} \) is also a \( G \)-weighted subcoercive form of order \( m \), so the operator \( e^{it\theta} \mathcal{H} = dU(e^{it\theta} \mathcal{C}) \) is the generator of a contraction semigroup. Then by [Kat], Theorem IX.1.23, it follows that \( H \) is the generator of a holomorphic semigroup which is holomorphic in a sector with angle at least \( \theta_C \). This completes the proof of the proposition.

Let \( \cdot \) be a homogeneous modulus on \( G \) (see [HeS]). Extend the algebraic basis to a vector space basis \( a_1, \ldots, a_d, \ldots, a_d \) such that for each \( i \in \{d' + 1, \ldots, d\} \) there exists a \( w_i \in [1, \infty) \) such that \( \gamma_i(ta_i) = t^{w_i}a_i \) for all \( t > 0 \). Set \( D' = \sum_{i=1}^d w_i \). Note that \( D' = \sum_{\lambda>0} \lambda \dim (g_\lambda/g_{\lambda-1}) \), where \( (g_\lambda)_{\lambda>0} \) is the filtration corresponding to the weighted algebraic basis.

**Proposition 5.5** The holomorphic semigroup \( S \) generated by \( H \) has a smooth kernel \( K \in L_{1;\infty}(G) \cap C_{0;\infty}(G) \) such that

\[
(A^\alpha S_t \varphi)(g) = \int_G dh \ (A^\alpha K_z)(h) \varphi(h^{-1} g)
\]

for all \( \alpha \in J(d''), z \in \Lambda(\theta_C), \varphi \in L_2(G) \) and \( g \in G \). Moreover, the function \( z \mapsto K_z(g) \) is analytic on \( \Lambda(\theta_C) \), uniformly for \( g \in G \), and for each \( \alpha \in J(d'') \) and \( \varepsilon \in (0, \theta_C) \) there exist \( b, c > 0 \) such that

\[
| (A^\alpha K_z)(g) | \leq c |z|^{-(D'+||\alpha||)/m} e^{-b(|z|^m|z|^{-1})^{(m-1)}}
\]

for all \( z \in \Lambda(\theta_C - \epsilon) \) and all \( g \in G \).
Proof  The proof is almost the same as the proofs of Proposition 2.2 and Corollary 2.3 in [AER], so we only indicate the one significant difference. This occurs at the beginning of the proof of Proposition 2.2, where the spectral theorem was used to establish for all $n \in \mathbb{N}$ the existence of a $C_1 > 0$ such that

\[
||| (\lambda I + H)^{-n} |||_{L^2 \rightarrow L^2_{2,nn}} \geq c_1^{-1}
\]  

uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ with $|\lambda| = 1$ for the self-adjoint operator $H$ in [AER]. Since the present operator $H$ is not necessarily symmetric we have to give a new proof of (22), uniformly for $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ with $|\lambda| = 1$.

Let $n \in \mathbb{N}$. By Corollary 5.3 there exist $c_1, c_2 > 0$ such that $N_{2,nn}'(\varphi) \leq c_1 ||H^n\varphi||_2$ and

\[
||H^{n-j}\varphi||_2 \leq \delta^j ||H^n\varphi||_2 + c_2 \delta^{-n+j}||\varphi||_2
\]

for all $\varphi \in L^2_{2,nn}$, $j \in \{1, \ldots, n - 1\}$ and $\delta > 0$. Let $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$. Then

\[
||H^n\varphi||_2 \leq ||(\lambda I + H)^n\varphi||_2 + \sum_{j=1}^{n} \binom{n}{j} |\lambda|^j ||H^{n-j}\varphi||_2
\]

\[
\leq ||(\lambda I + H)^n\varphi||_2 + \sum_{j=1}^{n} \binom{n}{j} |\lambda|^j (\delta^j ||H^n\varphi||_2 + c_2 \delta^{-n+j}||\varphi||_2)
\]

\[
= ||(\lambda I + H)^n\varphi||_2 + (1 + |\lambda|\delta)^n - 1 ||H^n\varphi||_2 + \delta^{-n}c_2((1 + |\lambda|\delta)^n - 1) ||\varphi||_2
\]

for all $\varphi \in L^2_{2,nn}$.

Let $\theta \in (-\theta_C + \varepsilon/2, \theta_C - \varepsilon/2)$ be such that $\rho = e^{i\theta} \lambda \in \Lambda(\pi/2 - \varepsilon/2)$. Then the Gårding inequality, applied to the form $e^{i\theta}C$, gives

\[
||(\lambda I + H)\varphi||_2 ||\varphi||_2 = ||(\rho I + e^{i\theta}H)\varphi||_2 ||\varphi||_2 \geq \text{Re}(\varphi, (\rho I + e^{i\theta}H)\varphi)
\]

\[
\geq (\text{Re}\rho)||\varphi||_2^2 \geq |\rho| \sin(\varepsilon/2) ||\varphi||_2^2 = |\lambda| \sin(\varepsilon/2) ||\varphi||_2^2
\]

for all $\varphi \in L^2_{2,m}$. Hence by induction

\[
||\varphi||_2 \leq |\lambda|^{-n}(\sin(\varepsilon/2))^{-n} ||(\lambda I + H^n)\varphi||_2
\]

uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ and $\varphi \in L^2_{2,nn}$.

Taking $\delta > 0$ such that $(1 + |\lambda|\delta)^n - 1 = 1/2$ one establishes that

\[
N_{2,nn}'(\varphi) \leq c_1 ||H^n\varphi||_2 \leq c_3 ||(\lambda I + H^n)\varphi||_2
\]

uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ and $\varphi \in L^2_{2,nn}$, where the value of $c_3$ is given by $c_3 = 2c_1 + 2c_1c_2((3/2)^{1/n} - 1)\sin(\varepsilon/2))^{-n}$. Next, using Proposition 5.3.IV one deduces that there exists a $c_4 > 0$ such that

\[
||\varphi||_2 \leq 2^{-1}c_4(1 + |\lambda|^{-n})||(\lambda I + H^n)\varphi||_2
\]

So the operators $(\lambda I + H)^{-n}$ map the Hilbert space $L^2_{2,nn}$ continuously into the Banach space $L^2_{2,nn}$ and

\[
||| (\lambda I + H)^{-n} |||_{L^2 \rightarrow L^2_{2,nn}} \geq c_4^{-1}
\]
uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \epsilon)$ with $|\lambda| = 1$.

Now one can proceed as in [AER]. We omit the details. \hfill \Box

Define $K_t = 0$ for $t \leq 0$. Then $(t, g) \mapsto K_t(g)$ is a $C^\infty$-function on $(\mathbb{R} \times G) \setminus \{(0, e)\}$ and the Gaussian bounds imply that this function is a distribution on $\mathbb{R} \times G$.

**Proposition 5.6** One has $((\partial_t + H)K_t)(g) = \delta(t) \delta(g)$ as distributions.

**Proof** It follows from [AER] that $((\partial_t + H)K_t)(g) = 0$ pointwise if $(t, g) \neq (0, e)$. The fact that $(t, g) \mapsto K_t(g)$ is a fundamental solution of the operator $\partial_t + H$ then follows as in Folland [Fol1] Proposition 3.3. \hfill \Box

Next we introduce another class of differential operators which turns out to be very useful in Sections 7 and 8 in the proof for general Lie groups that weighted subcoercive operators generate holomorphic semigroups. It is evident that one has bounds

$$|g|^n |(A^\alpha K_t)(g)| \leq c |t|^{-(D'+|\alpha| - n)/m} e^{-b(|\alpha|^m |g|)^{1/(m-1)}}$$

for all $\alpha \in J(d')$ and $n \in [0, \infty)$. Thus differentiating introduces an additional singularity $t^{-|\alpha|/m}$ but multiplication with $|g|^n$ introduces a factor $t^{n/m}$, which effectively removes the singularity. This motivates the following definition. Let $M_f$ denote the operator of multiplication with the function $f$. An $n$-th order differential operator with variable $C^\infty$-coefficients $f_\alpha$,

$$L = \sum_{\alpha \in J(d')} \frac{M_{f_\alpha} A^\alpha}{|\alpha|!^n}$$

on an open set $V$ containing the identity element $e$, is defined to be an operator of actual order $N$ if there exists a $c > 0$ and an open neighbourhood $B$ of the identity $e$ such that

$$|f_\alpha(g)| \leq c |g|^{|n(\alpha)|}$$

for all $\alpha \in J(d')$ with $|\alpha| \leq n$ and $g \in B \cap V$ where $n(\alpha) = (|\alpha| - N) \vee 0$.

One can restate the foregoing inequalities.

**Corollary 5.7** Let $L$ be a differential operator on $V$ of actual order $N$ with $N \in [0, \infty)$. Then for each compact subset $B$ of $V$ there exist $b, c > 0$ and $\omega \geq 0$ such that

$$|(LK_t)(g)| \leq c |t|^{-(D'+N)/m} e^{\omega t} e^{-b(|\alpha|^m |g|)^{1/(m-1)}}$$

uniformly for all $g \in B$ and $t > 0$.

The next lemma gives another description of the actual order of a differential operator.

**Lemma 5.8** Let $V$ be an open neighbourhood of the identity element $e$ in $G$ and $\varphi: V \to \mathbb{C}$ a $C^\infty$-function. Then for each $n \in [1, \infty)$ the following are equivalent.

I. $(A^\alpha \varphi)(e) = 0$ for all $\alpha \in J(d')$ with $|\alpha| < n$.

II. For every compact neighbourhood $K$ of $e$ such that $K \subset V$ there exists $c > 0$ such that $|\varphi(g)| \leq c |g|^n$ for all $g \in K$. 

33
III. There exist a compact neighbourhood $K$ of $e$ such that $K \subseteq V$ and a $c > 0$ such that $|\varphi(g)| \leq c|g|^n$ for all $g \in K$.

Proof Extend the algebraic basis $a_1, \ldots, a_d'$ to a vector space basis $a_1, \ldots, a_d$ such that for all $i > d'$ there exists a $w_i \in [1, \infty)$ with $\gamma_i(a_i) = t^{w_i}a_i$ for all $t > 0$. Define a modulus $|\cdot|$ on $g$ by

$$\sum_{i=1}^{d} |\xi_i a_i|^{2w_i} = \sum_{i=1}^{d} |\xi_i|^{2w_i/w_i},$$

where $w = \min\{x \in [1, \infty) : x \in w_iN \text{ for all } i \in \{1, \ldots, d\}\}$. Then by scaling there exists a $c \geq 1$ such that

$$c^{-1}|a| \leq |\exp a| \leq c|a|$$

for all $a \in g$. Moreover, $\|a\| \leq d^{1/2}|a|$ if $\|a\| \leq 1$. Next, let $X_i = X_i^{(1)}$ be the vector fields defined by (7), but now for the full basis of the Lie algebra $g$. Let $\psi = \varphi \circ \exp$ and $N \in \mathbb{N}$ with $N \geq n$. Then for all $a = \sum_{i=1}^{d} \xi_i a_i \in g$ one has by the usual Taylor formula

$$\varphi(\exp a) = \varphi(a) = \sum_{\alpha \in J(d)} \alpha!^{-1}(X^\alpha \psi)(0) \xi^\alpha + O(||a||^N) = \sum_{\alpha \in J(d)} \alpha!^{-1}(A^\alpha \varphi)(e) \xi^\alpha + O(||a||^N)$$

for $a \to 0$. Here $\alpha! = k_1! \cdots k_d!$ if $\alpha = (i_1, \ldots, i_m)$ and $k_l = \#\{j \in \{1, \ldots, m\} : i_j = l\}$. But if $\alpha \in J(d)$ with $|\alpha| \leq N$ and $||\alpha|| \geq n$ then

$$(A^\alpha \varphi)(e) \xi^\alpha = O(||a||^{||\alpha||}) = O(|a|^n), \quad (23)$$

where we have used the inequality $|\xi_i| \leq |a|^{w_i}$. Hence

$$\varphi(\exp a) = \sum_{\substack{\alpha \in J(d) \\ ||\alpha|| \leq n}} \alpha!^{-1}(A^\alpha \varphi)(e) \xi^\alpha + O(|a|^n). \quad (24)$$

Now the implication $\text{I} \Rightarrow \text{II}$ is obvious.

The implication $\text{II} \Rightarrow \text{III}$ is trivial.

III $\Rightarrow$ I. Suppose that $|\varphi(g)| \leq c|g|^n$ for all $g \in K$. Then $\varphi(e) = 0$. We shall prove by induction on $k$ that $(A^\alpha \varphi)(e) = 0$ for all $\alpha \in J(d)$ with $||\alpha|| = k$. Let $k \in [0, n)$ and suppose that $(A^\alpha \varphi)(e) = 0$ for all $\alpha$ with $||\alpha|| < k$. Then for all $a = \sum_{i=1}^{d} \xi_i a_i \in g$ one has

$$\sum_{\substack{\alpha \in J(d) \\ ||\alpha|| = k}} \alpha!^{-1}(A^\alpha \varphi)(e) \xi^\alpha = \varphi(\exp a) \sum_{\substack{\alpha \in J(d) \\ n > ||\alpha|| > k}} \alpha!^{-1}(A^\alpha \varphi)(e) \xi^\alpha + O(|a|^n) = o(|a|^k)$$

by (23) and (24). Next fix $a = \sum_{i=1}^{d} \xi_i a_i \in g$. Then the scaling $\eta_i = u^{w_i} \xi_i$ gives

$$u^k \sum_{\substack{\alpha \in J(d) \\ ||\alpha|| = k}} (A^\alpha \varphi)(e) \eta^\alpha = \sum_{\substack{\alpha \in J(d) \\ ||\alpha|| = k}} (A^\alpha \varphi)(e) \eta^\alpha = o(|\gamma_u(a)|^k) = o(u^k)$$

34
for all small $u > 0$. Therefore
\[ \sum_{\alpha \in J(d) \atop ||\alpha|| = k} \alpha^{-1} (A^\alpha \varphi)(e) \xi^\alpha = 0 \quad . \] (25)

We next prove that $(A^\alpha \varphi)(e) = 0$ for all $M \in \mathbb{N}$ and all $\alpha \in J(d)$ with $||\alpha|| = k$ and $|\alpha| = M$. The proof is by induction on $M$. If $M = 1$ then $\alpha = (i)$ for some $i \in \{1, \ldots, d\}$ and one substitutes $\xi = (0, \ldots, 0, 1, \ldots, 0)$ with the non-zero entry in the $i$-th place. Therefore $(A^\alpha \varphi)(e) = 0$ if $|\alpha| = 1$. Next let $M \in \mathbb{N}$, $M \geq 2$ and suppose that $(A^\alpha \varphi)(e) = 0$ for all $\alpha \in J(d)$ with $||\alpha|| = k$ and $|\alpha| < M$. Let $\alpha = (i_1, \ldots, i_M) \in J(d)$, $j \in \{1, \ldots, M - 1\}$ with $||\alpha|| = k$. If $\beta = (i_1, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_M)$ then
\[ (A^\alpha \varphi)(e) = (A^\beta \varphi)(e) + \sum_{l=1}^{d} c_{ij} (A^\delta \varphi)(e) \]
where the $c_{ij}$ are the structure constants of $g$ with respect to the basis $a_1, \ldots, a_d$ and $\delta_j = (i_1, \ldots, i_{j-1}, l, i_{j+1}, \ldots, i_M)$. Since $g$ is homogeneous one has $c_{ij} = 0$ if $w_l \neq w_{i_j} + w_{i_{j+1}}$. Therefore $||\delta|| = k$ for all $l$ such that the term in the sum does not vanish. But $|\delta| = M - 1$. Hence by the induction hypothesis $(A^\delta \varphi)(e) = 0$. It follows that $(A^\alpha \varphi)(e) = (A^\beta \varphi)(e)$. Consequently, $(A^\alpha \varphi)(e) = (A^\alpha \varphi)(e)$ for all $\alpha = (i_1, \ldots, i_M) \in J(d)$ with $||\alpha|| = k$ and all $\sigma \in S_M$, the permutation group, where $\alpha_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(M)})$. Now it follows from (25) by differentiation that $(A^\alpha \varphi)(e) = 0$. \( \square \)

One important implication of the Lemma 5.8 is that if $L_1$ and $L_2$ are differential operators with variable coefficients and actual orders $N_1$ and $N_2$, respectively, then $L_1 \circ L_2$ is a similar operator but with actual order $N_1 + N_2$.

Finally we need an estimate for the kernel of the resolvent of the closed operator $H = dL(C)$. The resolvent is defined by the Laplace transform
\[ (\lambda I + H)^{-1} = \int_0^\infty dt e^{-\lambda t} S_t \]
for all $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$. But it then follows from the estimates of Proposition 5.5 that $(\lambda I + H)^{-1}$ has a kernel $R_\lambda$ given by
\[ R_\lambda(g) = \int_0^\infty dt e^{-\lambda t} K_t(g) \]
and $R_\lambda \in L_1(G)$ with
\[ \|R_\lambda\|_1 \leq c(\Re \lambda)^{-1} \]
for a suitable $c > 0$ and all $\lambda \in \mathbb{C}$ such that $\Re \lambda > 0$. Since $K_t \in C^\infty(G)$ for $t > 0$ it follows that $R_\lambda \in C^\infty(G \setminus \{e\})$. Moreover, $A^\alpha R_\lambda \in L_1(G)$ for all $\alpha \in J(d)$ with $||\alpha|| < m$ and
\[ \|A^\alpha R_\lambda\|_1 \leq c'(\Re \lambda)^{-(m-||\alpha||)/m} \]
for $\Re \lambda > 0$. (For a related detailed discussion of strongly elliptic operators see [Rob2] Section III.6b and Appendix A of [EIR4].) Higher derivatives of $R_\lambda$ are, however, not in $L_1(G)$ because of singularities at the identity $e$. Nevertheless we now note that differential operators of order larger or equal to $m$ but with actual order less than $m$ do map $R_\lambda$ into $L_1(G)$. 35
Lemma 5.9 Let $L$ be a differential operator on $V$ of actual order $N$ with $N \in [0, m)$ and $B$ a compact subset of $V$. If $1_B$ denotes the characteristic function of $B$ then $1_B L R_\lambda \in L_1$ and there are $c > 0$ and $\rho \geq 0$ such that

$$\|1_B L R_\lambda\|_1 \leq c(\text{Re } \lambda)^{- (m - N)/m}$$

for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > \rho$.

**Proof** It follows directly from Corollary 5.7 that

$$\|1_B L R_\lambda\|_1 \leq c' \int_B dg \int_0^{\infty} dt e^{- (\lambda - \omega)t - (D' + N)/m} e^{- k(|s|^m|t|^{1/(m - 1)})}$$

$$\leq c' \int_0^{\infty} dt \int_B dh e^{- (\lambda - \omega)t - N/m} e^{- k(|h|^{m/(m - 1)})} \leq c''(\lambda - \omega)^{-1 + N/m}.$$ 

Therefore one can choose $\rho = 2\omega$ and $c = 2c''$.

6 Distances

In this section we introduce a distance on the (general) connected Lie group $G$, together with a new distance on the homogeneous contraction $G_0$. Let $a_1, \ldots, a_d$ be a reduced weighted algebraic basis for the (general) Lie algebra $\mathfrak{g}$ and extend this algebraic basis to a full basis $a_1, \ldots, a_d, \ldots, a_d$, with weights $w_1, \ldots, w_d$ as in Remark 2.2. So each $a_i$, for $i > d'$, is a commutator of elements of the algebraic basis. Set

$$D' = \sum_{i=1}^d w_i = \sum_{\lambda > 0} \lambda \dim(\mathfrak{g}_\lambda/\mathfrak{g}_\lambda).$$

Then $D'$ is independent of the extension of the algebraic basis to a full basis and we refer to it as the **local dimension** of $G$ with respect to the (reduced) weighted algebraic basis. This name is justified by the estimates of Proposition 6.1.11 given below.

Let $G_t$ and $\exp_t$, for $t \in [0, 1]$, be as in Section 3. We introduce a distance $| \cdot |_t$ on $G_t$. Although we are mainly interested in the cases $t = 0$ and $t = 1$ the construction is identical for all $t \in [0, 1]$. Let $B^{(t)}_i, i \in \{1, \ldots, d\}$, be the left invariant vector fields on $G_t$ corresponding to $a_i$, i.e.,

$$(B^{(t)}_i \psi)(g) = \frac{d}{ds} \psi(g \exp_t(sa_i))|_{s=0}$$

for $\psi \in C^\infty(G_t)$. Then for $\delta > 0$ let $C_t(\delta)$ be the set of all absolutely continuous functions $\varphi : [0, 1] \to G_t$ which satisfy the differential equation

$$\varphi(s) = \sum_{i=1}^{d'} \varphi_i(s) B^{(t)}_i|_{\varphi(s)}$$

almost everywhere with

$$|\varphi_i(s)| < \delta^{w_i}$$

for all $i \in \{1, \ldots, d'\}$ and $s \in [0, 1]$. Now define the distance $d'_t(g; h)$ between two elements $g, h \in G_t$ by

$$d'_t(g; h) = \inf\{\delta > 0 : \exists \varphi \in C_t(\delta)[\varphi(0) = g \text{ and } \varphi(1) = h]\}$$
and the modulus $| \cdot |_t'$ on $G_t$ by $|g|_t' = d_t'(g; e)$. Since $a_1, \ldots, a_d'$ is an algebraic basis for $(g, [\cdot, \cdot], t)$ it follows from a theorem of Carathéodory that $d_t'(g; h)$ is finite for all $g, h \in G_t$ (see also [NSW]). Moreover $d_t'(kg; kh) = d_t'(g; h)$ for all $g, h, k \in G_t$.

If $t = 0$ the modulus $| \cdot |_0'$ has the scaling property $|\gamma_t(g)|_0' = s |g|_0'$ for all $g \in G_0$. Therefore, if $| \cdot |$ is the homogeneous modulus on $G_0$ introduced in Section 5 then there exists a $c > 0$ such that $c^{-1}|g| \leq |g|_0' \leq c|g|$ for all $g \in G_0$.

Next, for $\delta > 0$ let

$$B_\delta^{(t)} = \{ g \in G_t : |g|_t' < \delta \}$$

be the ball in $G_t$ with radius $\delta$. We denote by $|B_\delta^{(t)}|_t$ the measure of $B_\delta^{(t)}$ with respect to the fixed (left-)Haar measure $\rho_t$ on $G_t$ (see (10)). If $t = 1$ we drop the subscript and superscript $t$ as before. Moreover, if confusion is possible, we write $|g|_{(a)} = |g|'$ to indicate the dependence of the modulus on the reduced weighted algebraic basis.

Finally, since $w_1, \ldots, w_d'$ have a common multiple it follows that $w_1, \ldots, w_d$ also have a common multiple. Set

$$\overline{w} = \min\{ x \in [1, \infty) : x \in w_i\mathbb{N} \text{ for all } i \in \{1, \ldots, d\} \} .$$

Then define a modulus $| \cdot |$ on $g$ by

$$\left| \sum_{i=1}^d \xi_i a_i \right|^{2\overline{w}} = \sum_{i=1}^d |\xi_i|^{2\overline{w}/w_i} .$$

It follows from the next proposition that the moduli $| \cdot |_t'$ are comparable locally.

**Proposition 6.1** Let $t \in [0, 1]$.

I. There exist $c \geq 1$ and $\epsilon \in (0, 1]$ such that

$$c^{-1}|a| \leq |\exp_t(a)|_t' \leq c|a|$$

for all $a \in g$ with $||a|| \leq \epsilon$, where $|| \cdot ||$ is a Euclidean norm on $g$.

II. There exists an $c \geq 1$ such that

$$c^{-1}\delta^{D'} \leq |B_\delta^{(t')}|_t \leq c \delta^{D'}$$

for all $\delta \in (0, 1]$.

**Proof** We may assume that all weights are integers. Indeed, if one multiplies all weights with a positive constant then the distance and modulus are replaced by the appropriate root of the old distance and modulus and the constants $c$ with the appropriate powers.

For all $n \in \mathbb{N}$ we define $C_n^{(t)} : g^n \to G_t$ by setting $C_1^{(t)}(b_1) = \exp_t(b_1)$ and

$$C_n^{(t)}(b_1, \ldots, b_n) = \exp_t(b_1) C_{n-1}^{(t)}(b_2, \ldots, b_n) \exp_t(-b_1) C_{n-1}^{(t)}(-b_2, \ldots, b_n)^{-1}$$

for $n > 1$. 

37
Lemma 6.2 For all $N \in \mathbb{N}$ there exist $\varepsilon_N > 0$ and $c_{N\beta} \in \mathbb{R}$ with $\beta \in J(N)$, $|\beta| \geq N + 1$ such that

$$C_N^{(\ell)}(b_1, \ldots, b_N) = \exp_t([b_1, [\ldots [b_{N-1}, b_N]_t]_t]_t + R_N(b_1, \ldots, b_N))$$

for all $b_1, \ldots, b_N \in \mathbb{N}$ with $\|b_i\| \leq \varepsilon_N$, where

$$R_N(b_1, \ldots, b_N) = \sum_{\beta \in J(N), |eta| \geq N + 1} c_{N\beta} b_{[\beta]}_t$$

and $b_{[\beta]}_t = [b_1, [\ldots [b_{i_m-1}, b_{i_m}]_t]_t$ if $\beta = (i_1, \ldots, i_m)$. Moreover, the sum in $R_N$ is absolutely convergent and

$$\sum_{\beta \in J(N), |eta| \geq N + 1} \|c_{N\beta} b_{[\beta]}_t\| \leq 1$$

Proof This follows from the Campbell–Baker–Hausdorff formula and induction on $N$ (see [NSW] Lemma 2.21 and [VSC] Section III.3).

In the next lemma we replace one low-order term in an element of the Lie algebra by several $a_i$ with $i \in \{1, \ldots, d'\}$ and another element of the Lie algebra which is not much larger than the original element. Since it is not possible to control all terms individually, we control all high-order terms together.

Lemma 6.3 For all $M \in \mathbb{N}$ and $\alpha_0 \in J^+_M(d')$ there exist $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, d'\}$ with the property that for each $C \geq 1$ there exist $C' \geq 1$ and $\varepsilon \in (0, 1]$ such that for $\delta \in (0, \varepsilon)$ and $c_\alpha \in \mathbb{R}$ with $\alpha \in J^+(d')$ the properties

1. $a = \sum_{\alpha \in J^+(d')} c_\alpha a_{[\alpha]}$ converges absolutely,
2. $|c_\alpha| \leq C\delta|\alpha|$ for $\alpha \in J^+_M(d')$,
3. $\|\sum_{\alpha \in J^+(d')} c_\alpha a_{[\alpha]}\| \leq C\delta^{M+1}$ and,
4. $c_\alpha = 0$ for all $\alpha \in J^+_{|\alpha|-1}(d')$

imply the existence of $c'_\alpha \in \mathbb{R}$, $\alpha \in J^+(d')$, and $s_1, \ldots, s_n \in \mathbb{R}$ such that

5. $b = \sum_{\alpha \in J^+(d')} c'_\alpha a_{[\alpha]}$, converges absolutely,
6. $|c'_\alpha| \leq C'\delta|\alpha|$ for $\alpha \in J^+_M(d')$,
7. $\|\sum_{\alpha \in J^+(d')} c'_\alpha a_{[\alpha]}\| \leq C'\delta^{M+1}$,
8. $|s_j| \leq C'\delta^{w_{ij}}$ for all $j \in \{1, \ldots, n\}$,
9. \begin{align*}
    c'_\alpha &= 0 \quad \text{for all } \alpha \in J^+_M(\alpha_0) \\
    c'_\alpha &= 0 \quad \text{if } |\alpha| = |\alpha_0| \text{ and } \alpha \neq \alpha_0, \text{ and},
\end{align*}

where $w_{ij}$ is the weight of the $i$th term in the expression $a_\alpha$.
10. \( \exp_t(a) = \exp_t(b) \exp_t(s_1a_{i_1}) \ldots \exp_t(s_na_{i_n}) \).

**Proof** There exists a \( u > 0 \) such that the Campbell–Baker–Hausdorff series converges absolutely with respect to \([ \cdot, \cdot ]_t\) on \( \{ a \in g : \| a \| \leq u \} \). We may assume that \( u = 2 \). Let \( M \in \mathbb{N} \) and \( a_0 \in J_M^+(d') \). Write \( a_0 = (j_1, \ldots, j_N) \) where \( N = |a_0| \). Let \( c_{N\beta} \in R \) and \( \varepsilon_N > 0 \) be as in Lemma 6.2. It is clear that there are \( n \in \mathbb{N} \) and \( i_1, \ldots, i_n \in \{1, \ldots, d'\} \) such that one has the identity

\[
C_M^{(t)}(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N}) = \exp_t(\omega_1a_{i_1}) \ldots \exp_t(\omega_na_{i_n})
\]

for all \( \tau_1, \ldots, \tau_N \in R \), where for each \( l \) there exists a \( k \) such that \( \omega_l a_{i_k} = \tau_l a_{j_k} \).

Let \( C \geq 1 \). Let \( \varepsilon \in (0,1] \) be such that \( \|a\| \leq 1 \) for all \( a = \sum c_\alpha a_{[\alpha]} \), where the \( c_\alpha \) satisfy 1., 2., 3. and 4. in the statement of the lemma for some \( \delta \in (0, \varepsilon) \) and, moreover, \( \varepsilon < \varepsilon_N C^{-1} d^{-1} \).

Now for \( \delta \in (0,\varepsilon) \) let \( a = \sum c_\alpha a_{[\alpha]} \in g \) and suppose that 1., 2., 3. and 4. are valid. Let \( \tau_1 = \text{sgn}(c_0) |c_{0\beta}|^{w_{j_1}/\|a_0\|} \) and \( \tau_l = |c_{\alpha l}|^{w_{j_l}/\|a_0\|} \) for all \( l \in \{2, \ldots, N\} \). Then \( |\tau_l| \leq C \delta^{|\alpha_l|} \) for all \( l \in \{1, \ldots, N\} \) and \( \tau_1 \ldots \tau_N = c_{0\beta} \). Let \( a'' = a - c_0 a_{[0\beta]} \). Then

\[
a = (c_{0\beta} a_{[0\beta]} + R_N(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N})) + (a'' - R_N(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N}))
\]

We estimate \( R_N(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N}) \). One has

\[
R_N(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N}) = \sum_{[\beta] \in J(N)} c_{N\beta} \tau^\beta b_{[\beta]} ,
\]

where \( b_k = a_{j_k} \) for all \( k \in \{1, \ldots, N\} \). But \( b_{[\beta]} = a_{[\alpha]} \) with \( \alpha = (j_{k_1}, \ldots, j_{k_m}) \) if \( \beta = (k_1, \ldots, k_m) \in J^+(N) \). Then

\[
|\tau^\beta| = \prod_{l=1}^m |\tau_{k_l}| \leq \prod_{l=1}^m C \delta^{|\tau_{k_l}|} = C^m \delta^{\|a\|} \leq C^M \delta^{\|a\|}
\]

if \( m = |\beta| = |\alpha| \leq M \). Similarly one deduces that

\[
\left\| \sum_{[\beta] \in J(N)} c_{N\beta} \tau^\beta b_{[\beta]} \right\| = \left\| \sum_{[\beta] \in J(N)} c_{N\beta} (d \varepsilon_N^{-1} \tau)^\beta (d^{-1} \varepsilon_N)^{|\beta|} b_{[\beta]} \right\|
\]

\[
\leq (C d \varepsilon_N^{-1})^{M+1} \sum_{[\beta] \in J(N)} \|c_{N\beta} (d^{-1} \varepsilon_N)^{|\beta|} b_{[\beta]} \|
\]

\[
\leq (C d \varepsilon_N^{-1})^{M+1} \delta^{M+1} ,
\]

where we have used \( C \delta < 1 \) and Lemma 6.2. So one can write

\[
a = (c_{0\beta} a_{[0\beta]} + R_N(\tau_1a_{j_1}, \ldots, \tau_Na_{j_N})) + a''
\]

with \( a'' = \sum c''_{[\alpha]} a_{[\alpha]} \), absolutely convergent. The coefficients \( c''_{[\alpha]} \) satisfy the estimates \( |c''_{[\alpha]}| \leq C_1 \delta^{\|a\|} \) for \( \alpha \in J_M^+(d') \) with a \( C_1 \geq 1 \) which depends only on \( M, a_0 \) and \( C \),

\[
\left\| \sum_{[\alpha] \in J(d')} c''_{[\alpha]} a_{[\alpha]} \right\| \leq C_1 \delta^{M+1}
\]

39
and, in addition, \( c''_\alpha = 0 \) for all \( \alpha \in J_{\geq 1}^+(d') \), \( c''_\alpha = 0 \) and \( c''_\alpha = c_\alpha \) if \( |\alpha| = |\alpha_0| \) and \( \alpha \neq \alpha_0 \). Then by the Campbell-Baker-Hausdorff formula one obtains as before that

\[
\exp_t (-c_{\alpha_0} a_{[\alpha_0]}) - R_N (\tau_1 a_{j_1}, \ldots, \tau_N a_{j_{\nu N}}) \exp_t a = \exp_t b
\]

where \( b = \sum_{\alpha \in J(d')} c'_\alpha a_{[\alpha]} \), for some \( c'_\alpha \in \mathbb{R} \) such that 5., 6., 7. and 9. in the statement of the lemma are valid for some \( C' \geq 1 \) which depends only on \( M, \alpha_0 \) and \( C \). Inverting the first element of the left hand side and using (26) gives 8. and 10.

Now we complete the proof of Proposition 6.1. Let \( d_2^{(2)} \) be the unweighted distance with respect to the algebraic basis \( a_1, \ldots, a_{d'} \) defined as follows. For \( \delta > 0 \) let \( C_2^{(2)}(\delta) \) be the set of all absolutely continuous functions \( \varphi : [0, 1] \to G_t \) which satisfy the differential equation

\[
\phi(s) = \sum_{i=1}^{d'} \varphi_i(s) B_i^{(1)}(\varphi(s))
\]

almost everywhere with

\[
|\varphi_i(s)| < \delta
\]

for all \( i \in \{1, \ldots, d'\} \) and \( s \in [0,1] \). Then the distance \( d_2^{(2)}(g; h) \) between two elements \( g, h \in G_t \) is defined by

\[
d_2^{(2)}(g; h) = \inf\{\delta > 0 : \exists \varphi \in C_2^{(2)}(\delta) [\varphi(0) = g \text{ and } \varphi(1) = h] \}
\]

Next let \( M = r \max(w_1, \ldots, w_{d'}) \), where \( r \) is the rank of the algebraic basis \( a_1, \ldots, a_{d'} \). Then

\[
C_t^{(2)}(\delta^{M/r}) \subseteq C_t(\delta) \subseteq C_t^{(2)}(\delta)
\]

for all \( \delta \in (0, 1] \). Hence \( d_2^{(2)}(g; h) \leq d'_t(g; h) \leq (d_2^{(2)}(g; h))^{r/M} \) for all \( g, h \in G_t \) with \( d'_t(g; h) \leq 1 \). It follows, however, from [NSW], Proposition 1.1 and Theorem 4, that there exist \( \varepsilon_1 \in (0, 1] \) and \( C_1 \geq 1 \) such that \( d_2^{(2)}(\exp_t a; e) \leq C_1 \|a\|^{1/r} \) for all \( a \in g \) with \( \|a\| \leq \varepsilon_1 \). Therefore

\[
|\exp_t a'_t| = d'_t(\exp_t a; e) \leq C_2 \|a\|^{1/M}
\]

for all \( a \in g \) with \( \|a\| \leq \varepsilon_1 \), where \( C_2 = C_t^{(2)}(\delta) \).

It follows by induction from Lemma 6.3 that there exist \( n \in \mathbb{N}, i_1, \ldots, i_n \in \{1, \ldots, d'\}, C_3 \geq 1 \) and \( \varepsilon \in (0, 1] \) such that for all \( \delta \in (0, \varepsilon) \) and all \( a = \sum_{\alpha \in J(d')} c_\alpha a_{[\alpha]} \) with \( c_\alpha \in \mathbb{R} \) and \( |c_\alpha| \leq \delta^{\|a\|} \), \( \alpha \in J_+^{(d')} \), there are \( c'_\alpha \in \mathbb{R} \), for \( \alpha \in J(d') \) with \( |\alpha| \geq M + 1 \) and \( s_1, \ldots, s_n \in \mathbb{R} \) such that \( \exp_t(a) = \exp_t(b) \exp_t(s_1 a_{i_1}) \ldots \exp_t(s_n a_{i_n}) \), where \( b = \sum_{\alpha \in J(d') : |\alpha| \geq M + 1} c'_\alpha a_{[\alpha]} \) converges absolutely. Moreover, \( \|b\| \leq C_3 \delta \) and \( |s_j| \leq C_3 \delta_{w_{i_j}} \) for all \( j \in \{1, \ldots, n\} \). But by the choice of \( a_{d'+1}, \ldots, a_d \) there exists for all \( j \in \{1, \ldots, d'\} \) an \( \alpha_j \in J(d') \) such that \( a_j = a_{[\alpha_j]} \) and \( \|a_j\| = w_{j} \).

Now let \( a \in g \) and suppose that \( |a| < \min(\varepsilon, C_3^{-1} \varepsilon_1) \). Write

\[
a = \sum_{\alpha \in J_+^{(d')}} c_\alpha a_{[\alpha]}
\]

with \( c_\alpha = 0 \) if \( \alpha \not\in \{\alpha_1, \ldots, \alpha_d\} \). Then \( |c_{\alpha_j}| \leq |a|^{w_j} = |a|^{|\alpha_j|} \) for all \( j \in \{1, \ldots, d\} \) and hence \( |c_\alpha| \leq |a|^{|\alpha|} \) for all \( \alpha \in J_+^{(d')} \). So for all \( \alpha \in J(d') \) with \( |\alpha| \geq M + 1 \) there
exist a $c'_\omega \in \mathbb{R}$ and $s_1, \ldots, s_n \in \mathbb{R}$ such that $\exp_t(a) = \exp_t(b) \exp_t(s_1a_{i_1}) \ldots \exp_t(s_na_{i_n})$, where $b = \sum_{\alpha \in J(\omega')} |\alpha| \geq M+1 c'_\omega a_{[\alpha]}$, converges absolutely, and, moreover, $\|b\| \leq C_3 |a|^{M+1}$ and $|s_j| \leq C_3 |a|^{M+1}$ for all $j \in \{1, \ldots, n\}$. Then

$$|\exp_t a'_t| \leq |\exp_t b'_t| + |\exp_t(s_1a_{i_1})|'_t + \ldots + |\exp_t(s_na_{i_n})|'_t \leq C_2 \|b\|^{1/M} + C_3 |a| + \ldots + C_3 |a| \leq C_2 C_3 |a|^{(M+1)/M} + n C_3 |a| \leq (n+1)C_2 C_3 |a| .$$

This proves the second inequality of Statement 1 of Proposition 6.1.

Next we prove the first inequality. Let $d_{t}^{(3)}$ be the weighted distance with respect to the full basis $a_1, \ldots, a_d$ defined as follows. For $\delta > 0$ let $C_{t}^{(3)}(\delta)$ be the set of all absolutely continuous functions $\varphi : [0, 1] \to G_t$ which satisfy the differential equation

$$\dot{\varphi}(s) = \sum_{i=1}^{d} \varphi_i(s) B_i^{(t)}{\big|}_{\varphi(s)}$$

almost everywhere with

$$|\varphi_i(s)| < \delta^{ui}$$

for all $i \in \{1, \ldots, d\}$ and $s \in [0, 1]$. Then the distance $d_{t}^{(3)}(g ; h)$ between two elements $g, h \in G_t$ is given by

$$d_{t}^{(3)}(g ; h) = \inf \{ \delta > 0 : \exists \varphi \in C_{t}^{(3)}(\delta) [\varphi(0) = g \text{ and } \varphi(1) = h] \} .$$

It follows immediately that $d_{t}^{(3)}(g ; h) \leq d_{t}^{(3)}(g ; h)$ for all $g, h \in G_t$.

Finally we need a fourth measure of distance which is this time a quasi-distance. For $\delta > 0$ let $C_{t}^{(4)}(\delta)$ be the set of all absolutely continuous functions $\varphi : [0, 1] \to G$ which satisfy the differential equation

$$\dot{\varphi}(s) = \sum_{i=1}^{d} \varphi_i(s) B_i^{(t)}{\big|}_{\varphi(s)}$$

almost everywhere with constant coefficients $\varphi_i$ satisfying $|\varphi_i| < \delta^{ui}$ for all $i \in \{1, \ldots, d\}$. The quasi-distance $d_{t}^{(4)}(g ; h)$ between two elements $g, h \in G$ is given by

$$d_{t}^{(4)}(g ; h) = \inf \{ \delta > 0 : \exists \varphi \in C_{t}^{(4)}(\delta) [\varphi(0) = g \text{ and } \varphi(1) = h] \} .$$

Then $d^{(4)}$ is locally equivalent to $d_{t}^{(3)}$ by [NSW], Theorem 2. So there exist $c, \varepsilon > 0$ such that $d_{t}^{(3)}(g ; h) \geq c d_{t}^{(4)}(g ; h)$ for all $g, h \in B_{t}^{(t)}$. In particular, $|g|'_t = d_{t}^{(4)}(g ; e) \geq c d_{t}^{(4)}(g ; e)$ for all $g \in B_{t}^{(t)}$.

Now let $a = \sum_{i=1}^{d} \xi_i a_i \in B_{t}^{(t)}$ and for $\delta > 0$ let $\varphi : [0, 1] \to G$ be an absolutely continuous path such that $\varphi(0) = e$, $\varphi(1) = \exp_t(a)$ and

$$\dot{\varphi}(s) = \sum_{i=1}^{d} \varphi_i(s) B_i^{(t)}{\big|}_{\varphi(s)}$$

for $s \in [0, 1]$ with constant coefficients $\varphi_i$ satisfying

$$|\varphi_i| < \delta^{ui}$$
for all $i \in \{1, \ldots, d\}$. Let $b = \sum_{i=1}^{d} \varphi_{i}a_{i} \in \mathfrak{g}$ and $B^{(t)}$ be the left invariant vector field corresponding to $b$. Then $\varphi(s) = B^{(t)} \big|_{\varphi(s)}$ for all $s \in [0, 1]$. So $\varphi$ is a $C^\infty$-function and by the uniqueness theorem for integral curves (see, for example, [SaW], Theorem 2.37) it follows that $\varphi(s) = \exp_{t}(sb)$ for all $s \in [0, 1]$, if $\varepsilon$ is small enough. In particular, $\exp_{t}(a) = \varphi(1) = \exp_{t}(b)$. Hence, for small enough $\varepsilon$, it follows that $\xi_{i} = \varphi_{i}$ for all $i \in \{1, \ldots, d\}$. Thus $|\xi_{i}| < \delta_{w_{i}}$ for all $i \in \{1, \ldots, d\}$. But there is a $j \in \{1, \ldots, d\}$ such that $|\xi_{j}|^{2w_{j}/w_{i}} \geq d^{-1}|a|^{2w_{i}}$.

Therefore $\delta > |\xi_{j}|^{1/w_{j}} \geq d^{-1/(2w)}|a|$ and $d_{t}^{(k)}(\exp_{t}(a); e) \geq d^{-1/(2w)}|a|$. This completes the proof of Statement I.

The proof of Statement II follows easily from Statement I and (10).

**Remark** Although the proof of Proposition 6.1 suggests that the constants $c$ and $\varepsilon$ in Statement I (and hence also in II) depend on $t$, it can be shown that they can be chosen independent of $t$. A sketch of the proof is as follows. Suppose $G$ is also simply connected. Let $t \in (0, 1)$. Then the algebra isomorphism $\gamma_{t}$ from $(\mathfrak{g}, \cdot, \cdot, t)$ onto $(\mathfrak{g}, \cdot, \cdot)$ can be lifted to a Lie group isomorphism $F_{t}$ from $G_{t}$ onto $G_{t}$. Then $td_{t}^{(k)}(g; h) = d_{t}^{(k)}(F_{t}(g); F_{t}(h))$ for all $g, h \in G_{t}$ and hence $|\exp_{t}a_{i}^{t} = t^{-1}|\exp_{t}\gamma_{t}(a)|^{t}$ for all $a \in \mathfrak{g}$. Applying Proposition 6.1.1 (for the case $t = 1$) to the right hand side, together with the identity $|\gamma_{t}(a)| = t|a|$ gives the uniformity in $c$ and $\varepsilon$ if $t \in (0, 1)$. The general case then easily follows.

One immediately has the following equivalence conclusion.

**Corollary 6.4** There exists a $c > 0$ such that

$$c^{-1}|\exp_{0}a|_{0} \leq |\exp a|_{t} \leq c|\exp_{0}a|_{0}$$

for all $a \in \mathfrak{g}$ with $||a|| \leq 1$.

It is also straightforward to deduce that the moduli of different bases corresponding to the same filtration are equivalent.

**Corollary 6.5** Let $a_{1}, \ldots, a_{d}$ be a reduced weighted algebraic basis with weights $w_{1}, \ldots, w_{d}$ and $b_{1}, \ldots, b_{d}$ a second reduced weighted algebraic basis with weights $v_{1}, \ldots, v_{d}$ such that the filtrations with respect to the two weighted algebraic bases coincide. Then the corresponding moduli $|\cdot|_{(a)}$ and $|\cdot|_{(b)}$ are equivalent, i.e., there exists a $c \geq 1$ such that

$$c^{-1}|g|_{(b)}^{t} \leq |g|_{(a)}^{t} \leq c|g|_{(b)}^{t}$$

for all $g \in \mathfrak{g}$.

**Proof** Let $(\mathfrak{g}, \lambda)_{\lambda \geq 0}$ be a filtration of $\mathfrak{g}$ with weights $\lambda_{1} < \ldots < \lambda_{k}$ and let $a_{1}, \ldots, a_{d}$ and $b_{1}, \ldots, b_{d}$ be two bases of $\mathfrak{g}$ such that

$$\mathfrak{g}_{\lambda_{j}} = \text{span}\{a_{1}, \ldots, a_{\dim \mathfrak{g}_{\lambda_{j}}}\} = \text{span}\{b_{1}, \ldots, b_{\dim \mathfrak{g}_{\lambda_{j}}}\}$$

for all $j \in \{1, \ldots, k\}$. Set $w_{i} = \lambda_{j}$ if $a_{i} \in \mathfrak{g}_{\lambda_{j}} \setminus \mathfrak{g}_{\lambda_{j-1}}$. Define $|\cdot|_{(a)}$, $|\cdot|_{(b)} : \mathfrak{g} \to \mathbb{R}$ by

$$\sum_{i=1}^{d} \xi_{i}a_{i}^{2w_{i}} = \sum_{i=1}^{d} |\xi_{i}|^{2w_{i}/w_{i}}, \quad \sum_{i=1}^{d} \xi_{i}b_{i}^{2w_{i}} = \sum_{i=1}^{d} |\xi_{i}|^{2w_{i}/w_{i}}.$$
where
\[ \overline{w} = \min \{ x \in [1, \infty) : x \in \omega_i \mathbb{N} \text{ for all } i \in \{1, \ldots, d\} \} . \]

By Corollary 6.4 and [VSC], Proposition III.4.2, it suffices to show that there exist \( c \geq 1 \) and \( \delta \in (0,1] \) such that \( c^{-1} |a|_{(\omega)} \leq |a|_{(\omega)} \leq c |a|_{(\omega)} \) for all \( a \in \mathfrak{g} \) with \( \|a\| \leq \delta \). For all \( i, j \in \{1, \ldots, d\} \) there exist \( c_{ij} \in \mathbb{R} \) such that \( a_i = \sum_{j, w_j \leq w_i} c_{ij} b_j \) for all \( i \in \{1, \ldots, d\} \). Let \( a = \sum_{i=1}^{d} \xi_i a_i \in \mathfrak{g} \) and suppose that \( \|a\| \leq 1 \). Then
\[ a = \sum_{i=1}^{d} \sum_{j, w_j \leq w_i} \xi_i c_{ij} b_j = \sum_{j=1}^{d} \left( \sum_{i, w_i \geq w_j} \xi_i c_{ij} \right) b_j . \]

Therefore
\[ |a|_{(\omega)}^{2\overline{w}} = \sum_{j=1}^{d} \left( \sum_{i, w_i \geq w_j} |\xi_i c_{ij}|^{2\overline{w}/w_j} \right)^{\lambda_1} \leq \sum_{j=1}^{d} d^{2\overline{w}/\lambda_1} \max_{i, w_i \geq w_j} |\xi_i|^{2\overline{w}/w_i} \leq \left( \sum_{j=1}^{d} d^{2\overline{w}/\lambda_1} \max_{i, w_i \geq w_j} |c_{ij}|^{2\overline{w}/w_i} \right) |a|_{(\omega)}^{2\overline{w}} . \]

The converse inequality follows by a similar argument.

\[ \square \]

7 Kernels

In this section we extend the kernel theorem of Section 5 for homogeneous groups to general groups \( G \) by exploiting the homogeneous contraction \( G_0 \) of \( G \).

Let \((\mathcal{X}, G, U)\) be a continuous representation of a connected Lie group \( G \) and \( a_1, \ldots, a_d \) a reduced weighted algebraic basis of the Lie algebra \( \mathfrak{g} \) of \( G \). Extend the algebraic basis to a vector space basis \( a_1, \ldots, a_d, \ldots, a_d \) as in Remark 2.2 and adopt the notation of Section 3. In particular the Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_{(\omega)})\) of \( G_0 \) is obtained by contraction of the Lie algebra \( \mathfrak{g} \). Let \( m \in 2w\mathbb{N} \) and \( C : \mathcal{J}(d') \to \mathbb{C} \) be a form of order \( m \). In Proposition 4.6 we established that each \( G \)-weighted subcoercive form \( C \) is a \( G_0 \)-weighted subcoercive form. Therefore, throughout this section, we adopt the seemingly weaker assumption that \( C \) is a \( G_0 \)-weighted subcoercive form.

Let \( dU(C) \) be the operator on \( \mathcal{X} \) corresponding to the form \( C \). Our aim is to establish that the closure of \( dU(C) \) generates a continuous semigroup \( S \) with a kernel \( K \) satisfying Gaussian type bounds. We approach this problem by first constructing a family of functions \( K \) which formally corresponds to the semigroup kernel. This construction starts by local approximation with the kernel of the analogous operator on the homogeneous contraction \( G_0 \) of \( G \), i.e., the kernel which exists because of Proposition 5.5. In the next section we verify that the \( K \) do indeed have the properties of a semigroup kernel and the generator of the semigroup is the closure of the original operator \( dU(C) \).

The starting point of the construction is the observation that the kernel \( K \), if it exists, should be the fundamental solution for the heat operator \( \partial_t + dL(C) \). Precisely, if one defines \( K_t = 0 \) for \( t \leq 0 \) then \( (t,g) \mapsto K_t(g) \) from \( \mathbb{R} \times G \) into \( \mathbb{C} \) should be the fundamental solution for the heat operator \( \partial_t + dL(C) \), i.e.,
\[ ((\partial_t + dL(C))K_t)(g) = \delta(t) \delta(g) \] (27)
for all \( t \in \mathbb{R} \) and \( g \in G \). Now the parametrix method expresses \( K \) as a perturbation expansion in terms of a localized version of the corresponding kernel for \( G_0 \). The perturbation parameter is the 'time' variable \( t \) and the expansion is a direct analogue of 'time-dependent' perturbation theory. The surprise is that the perturbation expansion for the semigroup kernel is convergent for all \( t > 0 \).

The local approximation procedure starts with the exponential map.

Let \( W \) be the open neighbourhood of 0 in \( g \) as in Lemma 3.3.V and \( \Omega = \exp(W) \). For all \( \varphi: \Omega \to \mathbb{C} \) define \( \hat{\varphi}: W \to \mathbb{C} \) by \( \hat{\varphi} = \varphi \circ \exp \). Let \( X_i \) and \( X_i^{(0)} \) denote the vector fields on \( (g, [\cdot, \cdot]) \) and \( (g, [\cdot, \cdot]_0) \) as in Section 3. The key observation is the next lemma which states that the \( X_i^{(0)} \) and \( X_i \) are very similar.

**Lemma 7.1** For all \( i \in \{1, \ldots, d\} \) the differential operator \( \exp_\omega(X_i - X_i^{(0)}) \) is of actual order \( N_i \) with \( N_i < w_i \).

**Proof** Let \( M, \delta, c_{\varepsilon_1, \ldots, \varepsilon_n} \) be as in (9). Then

\[
X_i \bigg|_a - X_i^{(0)} \bigg|_a = \sum_{j=1}^d f_{ij}(a) X_j^{(0)} \bigg|_a ,
\]

for all \( a = \sum_{l=1}^d \xi_l a_l \in W \), where

\[
f_{ij}(a) = \sum_{n=1}^\infty \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} c_{\varepsilon_1, \ldots, \varepsilon_n} \pi^j \left( (\text{ad}_{\varepsilon_1} a) \cdots (\text{ad}_{\varepsilon_n} a) (a_i) - (\text{ad}_0 a)^n (a_i) \right)
\]

\[
= \sum_{n=1}^\infty \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} c_{\varepsilon_1, \ldots, \varepsilon_n} \pi^j \left( (\text{ad}_{\varepsilon_1} a) \cdots (\text{ad}_{\varepsilon_n} a) (\text{ad}_k a - \text{ad}_0 a) (\text{ad}_0 a)^{n-k} (a_i) \right)
\]

\[
= \sum_{n=1}^\infty \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} \sum_{k=1}^n \sum_{i_1, \ldots, i_n=1}^d c_{\varepsilon_1, \ldots, \varepsilon_n} \xi_{i_1} \cdots \xi_{i_n} \cdot \pi^j \left( (\text{ad}_{\varepsilon_1} a_{i_1}) \cdots (\text{ad}_{k-1} a_{i_{k-1}}) (\text{ad}_k a_{i_k} - \text{ad}_0 a_{i_k}) (\text{ad}_0 a_{i_{k+1}}) \cdots (\text{ad}_0 a_{i_n}) (a_i) \right).
\]

The difference of the two commutators is an element of \( g \), so there exist \( \mu_{l,k,i_1, \ldots, i_n} \in \mathbb{R} \), with \( |\mu_{l,k,i_1, \ldots, i_n}| \leq 2 c_1^2 \) for some \( c_1 > 0 \), such that

\[
(\text{ad}_{\varepsilon_1} a_{i_1}) \cdots (\text{ad}_{k-1} a_{i_{k-1}}) (\text{ad}_k a_{i_k} - \text{ad}_0 a_{i_k}) (\text{ad}_0 a_{i_{k+1}}) \cdots (\text{ad}_0 a_{i_n}) (a_i) = \sum_{l=1}^d \mu_{l,k,i_1, \ldots, i_n} a_l
\]

Then

\[
f_{ij}(a) = \sum_{n=1}^\infty \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} \sum_{k=1}^n \sum_{i_1, \ldots, i_n=1}^d c_{\varepsilon_1, \ldots, \varepsilon_n} \xi_{i_1} \cdots \xi_{i_n} \mu_{j,l,k,i_1, \ldots, i_n}.
\]

Now suppose \( j \in \{1, \ldots, d\} \) and \( w_j \geq w_i \). Since \( \{\|\alpha\| : \alpha \in J(d)\} \) is a discrete set, there exist \( \eta_j > 0 \) such that \( \|\alpha\| \geq w_j - w_i + \eta_j \) for all \( \alpha \in J(d) \) with \( \|\alpha\| > w_j - w_i \). Let \( (\mathbb{B}_\lambda)_{\lambda \geq 2} \) be the filtration corresponding to the weighted algebraic basis \( a_1, \ldots, a_d \). Consider the element

\[
(\text{ad}_{\varepsilon_1} a_{i_1}) \cdots (\text{ad}_{k-1} a_{i_{k-1}}) (\text{ad}_k a_{i_k} - \text{ad}_0 a_{i_k}) (\text{ad}_0 a_{i_{k+1}}) \cdots (\text{ad}_0 a_{i_n}) (a_i)
\]
of the Lie algebra \( g \). Now \( (\text{ad}_{0}a_{i_{k+1}}) \cdots (\text{ad}_{0}a_{i_{n}})(a_{i}) \in g_{w_{1}+w_{i_{k+1}}+\cdots+w_{i_{n}}} \). Therefore

\[
(\text{ad}_{\xi_{1}}a_{i_{1}} - \text{ad}_{0}a_{i_{1}}) \cdots (\text{ad}_{\xi_{n}}a_{i_{n}} - \text{ad}_{0}a_{i_{n}})(a_{i}) \in g_{(w_{1}+w_{i_{1}}+\cdots+w_{i_{n}})}.
\]

by Proposition 3.1.VI if \( \varepsilon_{k} = 1 \), and clearly also if \( \varepsilon_{k} = 0 \). Thus

\[
(\text{ad}_{\xi_{1}}a_{i_{1}}) \cdots (\text{ad}_{\xi_{k-1}}a_{i_{k-1}}) \cdots (\text{ad}_{\xi_{n}}a_{i_{n}} - \text{ad}_{0}a_{i_{n}})(a_{i}) \cdots (\text{ad}_{\xi_{k}}a_{i_{k}} - \text{ad}_{0}a_{i_{k}})(a_{i}) \in g_{(w_{1}+w_{i_{1}}+\cdots+w_{i_{n}})}.
\]

So if \( \mu_{j}, k, e_{1}, \ldots, e_{n} \neq 0 \) then \( a_{j} \in g_{(w_{1}+w_{i_{1}}+\cdots+w_{i_{n}})} \). Hence \( w_{j} < w_{i_{1}} + \cdots + w_{i_{n}} \) and therefore \( w_{i_{1}} + \cdots + w_{i_{n}} \geq w_{j} - w_{i} + \eta_{j} \). Moreover \( |\xi_{i}| \leq |a|_{\infty} \leq |a| \) if \( l \in \{1, \ldots, d\} \) and \( |a| \leq 1 \). Consequently

\[
f_{ij}(a) = \sum_{n=1}^{\infty} \sum_{e_{1}, \ldots, e_{n} \in \{0, 1\}} \sum_{k=1}^{n} \sum_{i_{1}, \ldots, i_{n}=1}^{d} |c_{e_{1}, \ldots, e_{n}}| |\xi_{i_{1}}| \cdots |\xi_{i_{n}}| |\mu_{j}, k, e_{1}, \ldots, e_{n}| \leq \sum_{n=1}^{\infty} \sum_{e_{1}, \ldots, e_{n} \in \{0, 1\}} \sum_{k=1}^{n} \sum_{i_{1}, \ldots, i_{n}=1}^{d} M\delta^{n}|a|^{w_{1}+\cdots+w_{n}} 2c_{1}^{n}.
\]

Let \( N \in \mathbb{N} \) be such that \( N > w_{j} - w_{i} + \eta_{j} \). Next, split the sum over \( n \) in two parts: the first over \( n \) with \( n \leq N \) and the second over \( n > N \). Then if \( |a| \leq 1 \)

\[
f_{ij}(a) \leq 2 \sum_{n=1}^{N} \sum_{e_{1}, \ldots, e_{n} \in \{0, 1\}} \sum_{k=1}^{n} \sum_{i_{1}, \ldots, i_{n}=1}^{d} M(c_{1}\delta)^{n}|a|^{w_{j}-w_{i}+\eta_{j}}
\]

\[
+ 2 \sum_{n=N+1}^{\infty} \sum_{e_{1}, \ldots, e_{n} \in \{0, 1\}} \sum_{k=1}^{n} \sum_{i_{1}, \ldots, i_{n}=1}^{d} M(c_{1}\delta|a|)^{n}
\]

\[
\leq 2 \sum_{n=1}^{N} M(2c_{1}\delta)^{n}|a|^{w_{j}-w_{i}+\eta_{j}} + 2 \sum_{n=N+1}^{\infty} M(n(2c_{1}\delta|a|))^{n}.
\]

Now the lemma follows if one takes \( a \in W, |a| \leq 1 \) and \( |a| < (3c_{1}\delta)^{-1} \). \( \square \)

Set \( H = dL(C) \) and let \( H_{X} \) be the elliptic operators constructed from \( H \) with the vector fields \( X_{i} \) of Section 3 replacing the generators \( A_{i} \). It follows immediately from this lemma together with the remark following Lemma 5.8 that

\[
H_{X} = P_{X(0)} + H'
\]

where \( P_{X(0)} = \sum_{\|\alpha\|=m} c_{\alpha}X^{(0)}_{\alpha} \) and

\[
\exp_{0\alpha}(H') = \sum_{\alpha} f_{\alpha}A^{\alpha}
\]

is an operator with actual order \( N \) with \( N < m \) and the \( f_{\alpha} \) are \( C^{\infty} \)-functions on \( \exp_{0}(W) \). Moreover, \( dL_{G_{0}}(P) = \exp_{0\alpha}(P_{X(0)}) \) where \( P \) is the principal part of \( C \). Then the results of Section 5 apply to \( dL_{G_{0}}(P) \).

45
One has the following transformation property on $L^2(0)$, $L^2(W)$ and $L^2(\exp(W))$. Let $\psi, \varphi \in C_c^\infty(W)$. Then
\[
(H^*(\psi \circ \log), \varphi \circ \log) = (\psi \circ \log, H(\varphi \circ \log)) = (\sigma \psi, (P_{x(\varphi)} + H')\varphi)
\]
\[
= ((\sigma \psi) \circ \log_0, (dL_{G_0}(P) + \sum_\alpha f_\alpha A^\alpha)(\varphi \circ \log_0))
\]
\[
= (dL_{G_0}(P^t)((\sigma \psi) \circ \log_0), \varphi \circ \log_0)
\]
\[+ \sum_\alpha (-1)^{1\alpha}(A^\alpha(f_\alpha(\sigma \psi \circ \log_0)), \varphi \circ \log_0)
\]
with $\sigma$ as in (10). Hence by density
\[
(H^*(\psi \circ \log), \varphi \circ \log) = (dL_{G_0}(P^t)((\sigma \psi) \circ \log_0), \varphi \circ \log_0)
\]
\[+ \sum_\alpha (-1)^{1\alpha}(A^\alpha(f_\alpha(\sigma \psi \circ \log_0)), \varphi \circ \log_0)
\]
for all $\psi \in C_c^\infty(W)$ and $\varphi \in L_1(g)$ with supp $\varphi \subset W$. Let $\overline{K}$ be the kernel on $G_0$ corresponding to the operator $dL_{G_0}(P)$. If $t \leq 0$ we define $\overline{K}_t = 0$ as before. Now let $\chi, \chi' \in C_c^\infty(\Omega)$ be real with $\chi(e) = 1$ and $\chi' = 1$ on supp $\chi$. We will identify a function $\tau$ on $G$ with the function $1 \otimes \tau$ on $R \times G$. For $t \in R$ define the function $K_t^{(0)}$ on $G$ with compact support in $\Omega$, by
\[
K_t^{(0)} = (\overline{K}_t \circ \exp_0) \cdot \chi.
\]
Then the function $(t, g) \mapsto K^{(0)}(t, g) = K_t^{(0)}(g)$ is locally integrable, so $K^{(0)}$ is a distribution. We shall prove that it is an approximation of the fundamental solution of $\partial_t + H$. Let $\psi \in C_c^\infty(R \times G)$. Then with $\tau(t, g) = \psi(t, \exp \log_0(g)) \chi'(\exp \log_0(g)) \sigma(\log_0(g))$ for all $g \in \exp_0(W)$ it follows from (28) that
\[
(H^*(\psi \circ \log), \varphi \circ \log) = (dL_{G_0}(P^t)((\psi \circ \log_0), \varphi \circ \log_0)
\]
\[+ \sum_\alpha (-1)^{1\alpha}(A^\alpha(f_\alpha(\psi \circ \log_0)), \varphi \circ \log_0)
\]
for all $\psi \in C_c^\infty(W)$ and $\varphi \in L_1(g)$ with supp $\varphi \subset W$. We consider the two terms on the right hand side separately. For the first term one uses Proposition 5.6,
\[
\int_R dt \int_{G_0} dg \left((-\partial_t + dL_{G_0}(P^t))\tau\right)(t, g) \overline{K}_t(g)(\chi \circ \log_0)(g)
\]
\[+ \sum_\alpha (-1)^{1\alpha} \int_R dt \int_{G_0} dg \left(A^\alpha(f_\alpha \tau)\right)(t, g) \overline{K}_t(g)(\chi \circ \log_0)(g).
\]
where the sum is finite, the operators \( L_j \) are operators with actual order less than \( m \) and the \( \chi_i \in C^\infty_c(\Omega) \). Moreover, the first sum is over \( \alpha \) with \( \|\alpha\| = m \) and over \( \beta, \gamma \) with \( \|\beta\| < m \) where \( \gamma \) is a multi-index obtained from \( \alpha \) by omission of some indices and \( \beta \) is the multi-index formed by the omitted indices, i.e., the \( (\beta, \gamma) \) occurring are the pairs of multi-indices \( \text{Leibniz}(\alpha) \) in the Leibniz formula for the multi-derivative \( A^\alpha \) of a product but without the term \( \beta = \alpha \).

The second term can be handled similarly. Let \( |\cdot|' \) and \( |\cdot|_0 \) be the distances on \( G \) and \( G_0 \) with respect to the reduced weighted algebraic bases as defined in Section 6. Each \( C^\infty \)-function \( f_\alpha \) satisfies the estimate \( |f_\alpha(g)| \leq c(|\beta|_0(||\alpha||-N))^{\nu_0} \) for some \( N < m \). Then

\[
(-1)^{|\alpha|} \int_\mathbb{R} dt \int_{G_0} dg \left( A^\alpha(f_\alpha(t'))(t, g) \right) \overline{K_1}(g) (\chi \circ \log_0)(g) \\
= \sum_{(\beta, \gamma) \in \text{Leibniz}(\alpha)} \int_\mathbb{R} dt \int_{G_0} dg \left( f_\alpha(t') \right) (A^\beta \overline{K_1})(g) (A^\gamma(\chi \circ \log_0))(g)
\]

\[
= \sum_{(\beta, \gamma) \in \text{Leibniz}(\alpha)} (\psi, (L_\beta \overline{K_1} \circ \exp_0 \circ \log)) \cdot \chi_\gamma ,
\]

where \( L_\beta = \overline{f_\alpha} A^\beta \) is a differential operator of actual order at most \( N \), since \( ||\beta|| \leq ||\alpha|| \), and \( \chi_\gamma = (\chi' \circ \exp \circ \log_0) \cdot A^\gamma(\chi \circ \log_0) \in C^\infty_c(\Omega) \). So

\[
((\partial_t + H)(K^{(0)}_1))(g) = \delta(t) \delta(g) + M_t(g)
\]
as distributions on \( \mathbb{R} \times G \), where

\[
M_t = \sum_i (L_i \overline{K_1} \circ \exp_0 \circ \log) \cdot \chi_i ,
\]

the sum is finite, the operators \( L_i \) have actual order less than \( m \) and the \( \chi_i \in C^\infty_c(\Omega) \).

Proposition 5.5 gives estimates for \( \overline{K_1} \). Specifically, for all \( \alpha \in J(d') \) there exist \( b, c > 0 \) such that

\[
||A^\alpha \overline{K_1}(g)|| \leq c t^{-D'/2||\alpha||/m} e^{-b(||\alpha||^m t^{-1})^{1/(m-1)}}
\]

uniformly for all \( t > 0 \) and \( g \in G_0 \). Consequently, one has by Corollary 5.7

\[
||L \overline{K_1}(g)|| \leq c t^{-D'/2+|m|/m} e^{-b(||\alpha||^m t^{-1})^{1/(m-1)}}
\]

uniformly for all \( t > 0 \) and all \( g \) in a compact subset on which the operator \( L \) of actual order \( N \) is defined. It then follows from the estimates of Corollary 6.4 that for each \( \alpha \in J(d') \) there are \( b, c, \eta > 0 \) and \( \omega > 0 \) such that

\[
||A^\alpha K^{(0)}_1(g)|| \leq c t^{-D'/2||\alpha||/m} e^{\omega t e^{-b(||\alpha||^m t^{-1})^{1/(m-1)}}}
\]

\[
||A^\alpha M_t(g)|| \leq c t^{-D'/2+|m-\eta|/m} e^{\omega t e^{-b(||\alpha||^m t^{-1})^{1/(m-1)}}}
\]

uniformly for all \( t > 0 \) and \( g \in G \), since \( K^{(0)}_1 \) has compact support. For \( n \in \mathbb{N} \) define \( K^{(n)}_1 \) inductively by \( K^{(n)}_1 = -(K^{(n-1)}_1 \ast M_t) \), where the convolution product \( \ast \) is given on \( \mathbb{R} \times G \) by

\[
(\chi \ast \psi)_t(g) = \int_\mathbb{R} ds \int_{G} dh \chi_s(h) \psi_{t-s}(h^{-1}g) = \int_\mathbb{R} ds \int_{G} dh \chi_{t-s}(h) \psi_s(h^{-1}g)
\]

The main theorem of this section is the following.
Theorem 7.2 The series

\[ K_t = \sum_{n \geq 0} K_t^{(n)} \]

is \( L_p \)-convergent to a limit \( K_t \in L_p(\mathbb{R}) \) for all \( p \in [1, \infty] \) and \( t > 0 \). The limit \( K_t \) satisfies the heat equation (27), with the convention \( K_t = 0 \) for \( t \leq 0 \). Moreover, \( t \rightarrow K_t \) is continuous from \( (0, \infty) \) into \( L_p(\mathbb{R}) \) for all \( p \geq 0 \), where \( L_p(\mathbb{R}) = L_p(\mathbb{R}; e^{\|x\|_p} dx) \) is the weighted space with norm \( \|\varphi\|_p = \int e^{\|x\|_p} |\varphi(x)| \). Finally, for each \( \alpha \in J(d') \) there exist \( b, c > 0 \) and \( \omega \geq 0 \) such that

\[ |K_t(g)| \leq ct^{-D'/m} e^{\omega t} e^{-b((|g|)^{m-1})^{1/(m-1)}} \]

\[ |(A^\alpha K_t)(g)| \leq ct^{-(D'+\|\alpha\|)/m} e^{\omega t} e^{-b((|g|)^{m-1})^{1/(m-1)}} \]

for all \( g \in G \) and \( t > 0 \).

Proof Let \( \rho \geq 0 \) and define \( L_p^\rho \) and \( L_1^\rho \) to be the \( L_1 \)-spaces with respect to the measures \( dg e^{\|x\|_p} \) and \( dg e^{\|x\|_p} \), where \( dg \) is right invariant Haar measure on \( G \), with norms

\[ \|\varphi\|_p^\rho = \int_G dg e^{\|x\|_p} |\varphi(g)| , \quad \|\varphi\|_1^\rho = \int_G dg e^{\|x\|_p} |\varphi(g)| . \]

Similarly let \( L_\infty^\rho \) be the space of measurable functions for which \( g \mapsto e^{\|x\|_p} |\varphi(g)| \) is essentially bounded with norm

\[ \|\varphi\|_\infty^\rho = \operatorname{ess sup}_{g \in G} e^{\|x\|_p} |\varphi(g)| . \]

Then \( A^\alpha K_t^{(0)} \in L_p^\rho \) if \( \|\alpha\| < m \) and \( M_t \in L_p^\rho \) for all \( \rho \in \mathbb{R} \) by a quadrature estimate, using the volume estimates of Proposition 6.1.11. Specifically, one has bounds

\[ \|K_t^{(0)}\|_1^\rho \leq c e^{\omega(1+\rho m)t} , \quad \|M_t\|_1^\rho \leq c t^{-(m-\eta)/m} e^{\omega(1+\rho m)t} \]

for some \( c, \omega, \eta > 0 \), uniformly for all \( t > 0 \) and \( \rho \geq 0 \). It then follows by induction on \( n \) that

\[ \|K_t^{(n)}\|_1^\rho \leq c \frac{(bt)^{n\eta}}{\Gamma(n\eta + 1)} e^{\omega(1+\rho m)t} \]

for some \( b > 0 \), uniformly for all \( n \in \mathbb{N}_0 \), \( t > 0 \) and \( \rho \geq 0 \). Therefore the sum \( \sum K_t^{(n)} \) converges in \( L_1^\rho \) for all \( t > 0 \). Obviously the map \( t \mapsto K_t^{(n)} \) is continuous from \( (0, \infty) \) into \( L_1^\rho \) for all \( n \in \mathbb{N}_0 \), so the map \( t \mapsto K_t \) is also continuous from \( (0, \infty) \) into \( L_1^\rho \). By construction the distribution \( (t, g) \mapsto K_t(g) \) satisfies the heat equation (27).

Next we prove the Gaussian estimates on \( K \). We argue as in the appendix of [BrR] and in Section 4 of [EIR7]. The definition of \( K^{(n)} \) makes sense as an \( L_\infty \)-function and \( K_t^{(n)} \) is continuous for all \( t > 0 \) and \( n \in \mathbb{N}_0 \). The proof of the convergence of

\[ K_t = \sum_{n=0}^{\infty} K_t^{(n)} \]

in \( L_\infty \) is established by using the recursion relation \( K_t^{(n)} = -(K_t^{(n-1)} M_t) \) to estimate the norms \( \|K_t^{(n)}\|_\infty^\rho \). The starting point is the decomposition

\[ K_t^{(n)}(h) = -\int_0^{t/2} ds \int_G dk K_{t-s}^{(n-1)}(k) M_s(k^{-1} h) - \int_{t/2}^t ds \int_G dk K_{t-s}^{(n-1)}(k) M_s(k^{-1} h) \]

48
which leads to the iterative estimate
\[
\|K_t^{(n)}\|_\infty \leq \int_0^{t/2} ds \|K_t^{(n-1)}\|_\rho M_s \|_1 + \int_{t/2}^t ds \|K_t^{(n-1)}\|_1 M_s \|_\rho \, .
\]
But one has the initial estimates
\[
\|K_t^{(n)}\|_1 \leq c \frac{(bt)^{n\eta}}{\Gamma(n\eta + 1)} e^{\omega(1+\rho)^t} \quad \text{and} \quad \|K_t^{(0)}\|_\rho \leq c t^{-D'/m} e^{\omega(1+\rho)^t}
\]
together with the bounds
\[
\|M_t\|_1 \leq c t^{-(m-\eta)/m} e^{\omega(1+\rho)^t} \quad \text{and} \quad \|M_t\|_\rho \leq c t^{-(D'+m-\eta)/m} e^{\omega(1+\rho)^t} .
\]
The latter bounds follow from the Gaussian bounds of Proposition 5.5 and a quadrature argument. It then follows that
\[
\|K_t^{(n)}\|_\infty \leq c t^{-D'/m} e^{\omega(1+\rho)^t} \quad (29)
\]
for some \(b > 0\), uniformly for all \(n \in \mathbb{N}_0\), \(t > 0\) and \(\rho \geq 0\). One first establishes (29) for the finite number of \(n\) such that \(n\eta \leq D'\) and then by induction for all larger \(n\).

Therefore the sum for \(K_t\) converges in \(L^p_\infty\) and
\[
\|K_t\|_\infty \leq c t^{-D'/m} e^{\omega(1+\rho)^t}
\]
uniformly for all \(t > 0\) and \(\rho \geq 0\). One immediately obtains the pointwise bounds
\[
|K_t(g)| \leq c t^{-D'/m} e^{\omega(1+\rho)^t} e^{-\rho|g|^t}
\]
and minimizing over \(\rho \geq 0\) yields the Gaussian bounds
\[
|K_t(g)| \leq c t^{-D'/m} e^{\omega t} e^{-b'(\|g\|^{m-1})^{1/(m-1)}}
\]
uniformly for all \(t > 0\) and \(g \in G\). Similarly it follows that
\[
|K_t^{(n)}(g)| \leq c t^{-D'/m} e^{\omega t} \frac{(bt)^{n\eta}}{\Gamma(n\eta + 1)} e^{-b'(\|g\|^{m-1})^{1/(m-1)}} \quad (30)
\]
uniformly for all \(t > 0\) and \(g \in G\).

Before we can prove Gaussian estimates on the derivatives of \(K\) we need one more technical lemma which is a weighted version of Lemma 4.3 of [EIR7] or Lemma 4.3 of [EIR5].

Lemma 7.3 There exist \(M, \sigma > 0\) and \(N \in \mathbb{N}\) and for each \(i \in \{1, \ldots, d'\}\) and \(\alpha \in J_N(d')\) a function \(c_{i,\alpha}: G \to \mathbb{R}\) such that
\[
L(k^{-1})A_i L(k) - A_i = \sum_{\alpha \in J_N(d')} c_{i,\alpha}(k) A^\alpha
\]
and \(|c_{i,\alpha}(k)| \leq M(|k'|^{\|\alpha\|-\omega_i}v_0 e^{\sigma|k'|}\) for all \(k \in G\) and \(\alpha \in J_N(d')\).
Proof Since \( \text{Ad} \) is a continuous homomorphism from \( G \) into \( \mathcal{L}(g) \) and the algebraic basis has rank \( r \) the lemma is easily established if \( |k'| \geq 1 \) and \( N \geq r \). So let \( k \in G, \, |k'| \leq 1. \) We may assume that \( w_1, \ldots, w_d' \) are integers. Let \( N = r \max(w_1, \ldots, w_d') \). There exists an absolutely continuous path \( \gamma : [0,1] \to G \) such that \( \gamma(0) = e, \, \gamma(1) = k, \)

\[
\dot{\gamma}(t) = \sum_{i=1}^{d'} \gamma_i(t) Y_i \gamma(t)
\]

for \( t \in [0,1] \) almost everywhere, with \( Y_i \) the right invariant vector field corresponding to \( a_i \in g \) and such that

\[
|\gamma_i(t)| \leq 2(|k'|)^{w_i}
\]

for all \( i \in \{1, \ldots, d'\} \) and \( t \in [0,1] \). (One can easily swap from left invariant vector fields in Section 6 to right invariant vector fields by using the inverse on the Lie group.) Then for all \( i \in \{1, \ldots, d'\} \) one has

\[
\text{Ad}(k^{-1})a_i - a_i = \int_0^1 dt_1 \frac{d}{dt_1} \text{Ad}(\gamma(t)^{-1})a_i
\]

\[
= \int_0^1 dt_1 \sum_{i_1=1}^{d'} \gamma_{i_1}(t_1) \text{Ad}(\gamma(t_1)^{-1})[a_{i_1}, a_i]
\]

\[
= \sum_{i_1=1}^{d'} \int_0^1 dt_1 \gamma_{i_1}(t_1)[a_{i_1}, a_i] + \ldots
\]

\[
+ \sum_{i_1=1}^{d'} \sum_{i_2=1}^{d'} \int_0^1 dt_1 \ldots \int_0^1 dt_{N-1} \gamma_{i_1}(t_1) \ldots \gamma_{i_{N-1}}(t_1 \ldots t_{N-1}) \cdot t_1^{N-2} \ldots t_{N-2} [a_{i_{N-1}}, \ldots, [a_{i_1}, a_i]]
\]

\[
+ \sum_{i_1=1}^{d'} \sum_{i_2=1}^{d'} \int_0^1 dt_1 \ldots \int_0^1 dt_N \gamma_{i_1}(t_1) \ldots \gamma_{i_N}(t_1 \ldots t_N) \cdot t_1^{N-1} \ldots t_{N-1} \text{Ad}(\gamma(t_1 \ldots t_N)^{-1})[a_{i_N}, \ldots, [a_{i_1}, a_i]]
\]

Since \( \text{Ad}(g) \) maps \( g \) into \( g \) with uniform bounded norm, for \( g \) in a bounded set, and \( |\gamma_n(t)| \leq (2|k'|)^{w_n} \) for all \( n \in \{1, \ldots, d'\} \), the last term gives a contribution \( \sum_{j=1}^{d'} c_j^i(k) a_j \), with \( |c_{ij}^i(k)| \leq M(|k'|)^N \).

Similarly one has for the \( n \)-th term the contribution

\[
\sum_{i_n=1}^{d'} \ldots \sum_{i_1=1}^{d'} \int_0^1 dt_1 \ldots \int_0^1 dt_n \gamma_{i_1}(t_1) \ldots \gamma_{i_n}(t_1 \ldots t_n) t_1^{n-1} \ldots t_{n-1} [a_{i_n}, \ldots, [a_{i_1}, a_i]]
\]

But

\[
|\gamma_{i_1}(t_1) \ldots \gamma_{i_n}(t_1 \ldots t_n)| \leq (2|k'|)^{w_1 + \ldots + w_n}.
\]

Moreover, one can expand in the obvious way

\[
[a_{i_n}, \ldots, [a_{i_1}, a_i]] = \sum_{\alpha} c_{\alpha} a^\alpha
\]

where the sum is over all \( \alpha \) with \( ||\alpha|| - w_i = w_{i_1} + \ldots + w_{i_n} \).
From these two observations it follows that there exist $c_{i,a}(k) \in \mathbb{R}$, such that
\[
L(k^{-1})A_i L(k) - A_i = dL(Ad(k^{-1})a_i - a_i) = \sum_{\alpha \in J^+_N(d')} c_{i,a}(k) A^\alpha.
\]
Moreover, there exists an $M > 0$ such that
\[
|c_{i,a}(k)| \leq M(|k'|)(|\sigma| - \omega_i)\nu_0
\]
for all $k \in G$ with $|k'| \leq 1$. This proves the lemma.

We continue with the proof of Theorem 7.2. Fix $\alpha \in J^+(d')$ and consider the decomposition
\[
(A^\sigma K^{(n)}_s)(h) = -\int_{t/2}^t ds \int_G dk (A^\sigma K^{(n-1)}_s)(k) M_s(k^{-1}h)
\]
\[
- \int_{t/2}^t ds \int_G dk K^{(n-1)}_s(k) (A^\sigma L(k) M_s)(h).
\]
We first estimate the second term with the aid of Lemma 7.3. Notice that
\[
e^{\rho|k^{-1}h|'}|(A^\sigma L(k) M_s)(h)| = e^{\rho|g|'}|(L(k^{-1})A^\sigma L(k) M_s)(g)|
\]
with $g = k^{-1}h$. By induction it follows from Lemma 7.3 that there exist $M, \sigma \geq 0, N \in \mathbb{N}$ and functions $c_\sigma: G \to \mathbb{R}$ for all $\sigma \in J^+_N(d')$, such that
\[
L(k^{-1})A^\sigma L(k) = \sum_{\beta \in J^+_N(d')} c_\beta(k) A^\beta
\]
and $|c_\beta(k)| \leq M(|\sigma|)(|\sigma| - |\alpha|)\nu_0 e^{\rho|\sigma|}$ for all $k \in G$ and $\beta \in J^+_N(d')$. Moreover, one has the estimates
\[
|(A^\beta M_s)(g)| \leq c_{s^{-D'+m-\eta}/m} e^{-b(|\sigma|)^{m_s-1} \eta / m} e^{\omega_{\eta}}
\]
for all $\beta \in J^+_N(d')$, for some $\eta > 0$. So
\[
e^{\rho|g|'}|(L(k^{-1})A^\sigma L(k) M_s)(g)| \leq \sum_{\beta \in J^+_N(d')} c_{s^{-D'+m-\eta}/m} e^{-b(|\sigma|)^{m_s-1} \eta / m} e^{\omega_{\eta}}
\]
Minimizing over $g$ then establishes that
\[
e^{\rho|k^{-1}h|'}|(A^\sigma L(k) M_s)(h)| \leq c_{s^{-D'+|\sigma|+m-\eta}/m} e^{\rho|k|'} \left(1 + (|\sigma|^{1/m})^N \right) e^{\omega(1+\rho m)^s}
\]
for some redefined values of $c$, $\omega$ and $N$. Using $s^{-1/m} \leq (t-s)^{-1/m}$ for all $s \in [t/2, t)$ it follows from (30) that
\[
e^{\rho|k|'} \int_{t/2}^t ds \int_G dk |K^{(n-1)}_s(k)| \cdot |(A^\sigma L(k) M_s)(h)|
\]
\[
\leq c_{s^{-D'+|\sigma|+m-\eta}/m} e^{\omega(1+\rho m)^s}.
\]

51
Now the integral over $G$ can be estimated as before and one obtains

\[
e^{\alpha|h|} \int_{t/2}^{t} ds \int_{G} |K_t^{(n-1)}(k)| \cdot |(A^\alpha L(k)M_r)(h)| \]

\[
\leq c'' \int_{t/2}^{t} ds \frac{(b(t - s))^{(n-1)\eta}}{\Gamma((n - 1)\eta + 1)} s^{-(D' + \|\alpha\| + m - \eta)/m e^{\omega''(1 + \rho^m)}}
\]

\[
\leq c t^{-(D' + \|\alpha\|)/m e^{\omega(1 + \rho^m)}} t^{-((bt)^{\eta})/\Gamma(n\eta + 1)}
\]

with the redefined values of $c$, $b$ and $\omega$ independent of $n$, $h$ and $\rho$ but dependent on $\alpha$. But these bounds together with the previous bounds

\[
\|M_r\|_t^p \leq c s^{-(m-n)/m e^{\omega(1 + \rho^m)}}
\]

allow one to solve the integral inequalities

\[
\|A^\alpha K_t^{(n)}\|_t^p \leq \int_{0}^{t/2} ds \|A^\alpha K_t^{(n-1)}\|_\infty^p \|M_r\|_t^p + \| \int_{t/2}^{t} ds \int_{G} dk K_t^{(n-1)}(k) (A^\alpha L(k)M_r)\|_\infty^p
\]

\[
\leq \int_{0}^{t/2} ds \|A^\alpha K_t^{(n-1)}\|_\infty^p \|M_r\|_t^p + c t^{-(D' + \|\alpha\|)/m e^{\omega(1 + \rho^m)}} \frac{t^{-((bt)^{\eta})/\Gamma(n\eta + 1)}}{t^{-((bt)^{\eta})/\Gamma(n\eta + 1)}}
\]

which are a direct consequence of the previous estimates. Arguing as before one deduces that

\[
|(A^\alpha K_t)(g)| \leq a t^{-(D' + \|\alpha\|)/m e^{\omega(1 + \rho^m)}} t^{-((bt)^{\eta})/\Gamma(n\eta + 1)}
\]

uniformly for all $t > 0$ and $g \in G$.

8 Weighted subcoercive operators

In this section we extend the generator theorem of Section 5 for homogeneous groups and left regular representations to general groups $G$ and arbitrary continuous representations by exploiting the homogeneous contraction $G_0$ of $G$. Moreover, we show that the ‘kernel’ $K$ constructed in Section 7 is indeed the kernel of the semigroup.

Adopt the notation of the previous section. So $C$ is a $G_0$-weighted subcoercive form. Let $U$ be a continuous representation of the Lie group $G$ by bounded operators $U(g)$ on the Banach space $\mathcal{X}$ and assume $U$ is weakly continuous, or weakly* continuous if $\mathcal{X}$ has a predual. Since the representation $U$ is continuous one has bounds

\[
\|U(g)\| \leq M e^{\rho |\lambda|'}
\]

with $M \geq 1$ and $\rho \geq 0$. But the kernel $K_t$ satisfies Gaussian bounds. Therefore $K_t \in L^p(G)$ and one can define bounded operators $S_t$ on $\mathcal{X}$ by

\[
S_t x = U(K_t)(g) x = \int_G dg K_t(g) U(g)x
\]

Note that $t \mapsto S_t x$ is continuous from $(0, \infty)$ into $\mathcal{X}$ for all $x \in \mathcal{X}$, since $t \mapsto K_t$ is continuous from $(0, \infty)$ into $L^p(G)$ (see Theorem 7.2). Because of the bounds $\|K_t^{(n)}\|_t^p \leq$
\[ c(b^n t^n / n!)^{n \epsilon (1 + \rho^m t)} \text{ it follows that } \lim_{t \to 0} S_t x = \lim_{t \to 0} U(K_t^{(0)}) x, \text{ if one of the two limits exists. But } (\overline{K}_t)_{t>0} \text{ is a bounded approximation of the identity (cf. the proof of Lemma 3.3 in [AER]) and hence} \]

\[ \lim_{t \to 0} U(K_t^{(0)}) x = \lim_{t \to 0} \int_{\mathcal{W}} \sigma(a) \overline{K}_t(\exp(a)) \dot{x}(a) U(\exp(a)) x = x. \]

Therefore \( \lim_{t \to 0} S_t x = x \) strongly if \( U \) is strongly continuous and weakly* if \( U \) is weakly* continuous.

We will first apply this to the \( L_p^\omega \), and \( L_{p,m}^\omega \), spaces with respect to the left regular representation. Then

\[ S_t \varphi = K_t \ast \varphi \]

and it follows that \( S_t L_p^\omega \subseteq L_p^{\infty} \subseteq D(H) \). Moreover, if \( p \in [1, \infty) \) and \( q \in (1, \infty] \) is conjugate to \( p \) then

\[-\int_{\mathbb{R}} dt (\partial_t \tau)(t) (\psi, S_t \varphi) + \int_{\mathbb{R}} dt \tau(t) (\psi, HS_t \varphi) = -\int_{\mathbb{R}} dt (\partial_t \tau)(t) (\psi, S_t \varphi) + \int_{\mathbb{R}} dt \tau(t) (H^* \psi, S_t \varphi) = 0 \]

for all \( \varphi \in L_p^\omega, \tau \in C^\infty_{\text{c}}([0, \infty)) \) and \( \psi \in C^\infty_{\text{c}}(G) \). But then by continuity and density ([ElR1] Theorem 2.4) it is valid for all \( \psi \in L_p^\omega \). On the other hand the map \( t \mapsto HS_t \varphi \) is continuous if \( \varphi \in L_{p,m}^\omega \). Therefore it follows from the lemma of Du Bois-Reymond that \( t \mapsto (\psi, S_t \varphi) \) is differentiable and \( \frac{d}{dt} (\psi, S_t \varphi) + (\psi, HS_t \varphi) = 0 \) for all \( \varphi \in L_{p,m}^\omega, \psi \in L_q^\omega \) and \( t > 0 \). Then

\[ \frac{d}{dt} S_t \varphi + HS_t \varphi = 0 \quad (31) \]

strongly for all \( \varphi \in L_{p,m}^\omega \) by an application of the mean value theorem and the continuity of \( t \mapsto HS_t \varphi \).

The family \( S = (S_t)_{t>0} \) forms a semigroup if, and only if, \( K \) is a convolution semigroup. But the definition of \( K \) seems unsuited for direct verification of this property and so we have to approach it indirectly. We will argue that it follows from the lower semiboundedness of \( \text{Re} \, H \) on \( L_2 \).

**Proposition 8.1** Each symmetric operator \( H = dL_G(C) \) on \( L_2(G) \), where \( C \) is a \( G_0 \)-weighted subcoercive form, is essentially self-adjoint and lower semibounded.

**Proof** It suffices to establish that the range of \( (\lambda I + H) \) is equal to \( L_2 \) and its inverse is bounded for all large positive \( \lambda \). For this we use a resolvent version of the foregoing parametrix techniques.

Let \( \chi, \chi' \in C^\infty_{\text{c}}(G), \supp \chi' \subset \Omega, \chi(e) = 1 \) and \( \chi' = 1 \) on \( \supp \chi \). Then for all \( \varphi \in C^\infty_{\text{c}}(G) \) and \( \psi \in L_2(G) \) one has for all \( r \in C^\infty_{\text{c}}(G) \) with \( \supp r \subseteq \supp \chi \)

\[ \int_G dg (r(g)(\psi, (\lambda I + H)L(g)\varphi) = (\psi, (\lambda I + H)(r \ast \varphi)) = 0, \quad (32) \]

\[ = \int_G dg ((\lambda I + H)r(g)(\psi, L(g)\varphi) \chi'(g) = 0, \quad (32) \]

\[ = \int_G dg r(g)((\lambda I + H)r(g)) = 0, \quad (32) \]

53
where \( r(g) = (\psi, L(g)\varphi) \chi'(g) \). Since \( C_\infty^\infty(G) \) is dense in \( L_1(G) \) it follows by continuity that (32) is valid for all \( r \in L_1(G) \) with \( \text{supp} \, r \subseteq \text{supp} \, \chi \). Now let \( r_\lambda \) be the function on \( G \) with support contained in \( \Omega \) such that \( \hat{r}_\lambda = (\tilde{R}_\lambda \circ \exp_0) \cdot \hat{\chi} \) where \( \tilde{R}_\lambda \) denotes the kernel of the resolvent \( (\lambda I + dL_G(P))^{-1} \) on \( G_0 \). Then

\[
(\psi, (\lambda I + H)(r_\lambda * \varphi)) = \int_G dg \, r_\lambda(g) (((\lambda I + H)\tau)(g) = \int_W da \sigma(a) ((\tilde{R}_\lambda \circ \exp_0)(a) \hat{\chi}(a) ((\lambda I + P_{X(0)} + H')\hat{\tau})(a) \nonumber \\
= \int_W da \sigma(a) (((\lambda I + P_{X(0)} + H')(\tilde{R}_\lambda \circ \exp_0)\hat{\chi}))(a) \hat{\tau}(a) \nonumber \\
= \int_W da \sigma(a) \hat{\delta}(a) \hat{\chi}(a) \hat{\tau}(a) + \int_W da \sigma(a) \hat{s}_\lambda(a) (\psi, L(\exp a)\varphi) ,
\]

in the sense of distributions where \( \hat{s}_\lambda \) has the form

\[
\hat{s}_\lambda = \sum_i ((L^{(i)}\tilde{R}_\lambda) \circ \exp_0) \cdot \hat{\chi}_i .
\]

Once again the \( \hat{\chi}_i \in C_\infty^\infty(W) \) and the \( L^{(i)} \) are operators of actual order less than \( m \). But the estimates of Lemma 5.9 imply that \( \| r_\lambda \|_1 \leq c \lambda^{-1} \) and \( \| s_\lambda \|_1 \leq c \lambda^{-n/m} \) for some \( \eta > 0 \) and large \( \lambda \). Therefore, if \( R_\lambda \) and \( S_\lambda \) denote the operators of convolution with \( r_\lambda \) and \( s_\lambda \), respectively, then \( \| R_\lambda \|_{2 \to 2} \leq c \lambda^{-1} \) and \( \| S_\lambda \|_{2 \to 2} \leq c \lambda^{-n/m} \). So

\[
(\psi, (\lambda I + H)(r_\lambda * \varphi)) = (\psi, \varphi) + (\psi, s_\lambda * \varphi)
\]

and

\[
(\lambda I + \overline{H})R_\lambda \varphi = \varphi + S_\lambda \varphi \tag{33}
\]

for all \( \varphi \in C_\infty^\infty(G) \). By density it follows that \( R_\lambda L_2 \subseteq D(H) \) and (33) is valid for all \( \varphi \in L_2 \). Thus if \( \lambda \) is sufficiently large that \( c \lambda^{-n/m} < 1 \) then \( (I + S_\lambda) \) has a bounded inverse and

\[
\varphi = (\lambda I + \overline{H})R_\lambda(I + S_\lambda)^{-1} \varphi
\]

for all \( \varphi \in L_2(G) \). This establishes that the range of \( (\lambda I + H) \) is equal to \( L_2(G) \) and hence \( H \) is self-adjoint. But it then follows that

\[
\varphi = (I + S_\lambda)^{-1}R_\lambda(\lambda I + \overline{H})\varphi
\]

and hence

\[
\| \varphi \|_2 \leq c \lambda^{-1}(1 - c \lambda^{-n/m} - 1)\| (\lambda I + \overline{H})\varphi \|_2 .
\]

Therefore \( (\lambda I + H) \) has a bounded inverse. Thus \( H \) is lower semibounded by spectral theory. \( \square \)

Now it is straightforward to prove that \( K \) is a convolution semigroup.

Since \( \Re H \) is a symmetric weighted subcoercive operator on \( L_2(G) \) it follows from Proposition 8.1 that it is lower semibounded on \( L_2 \), i.e., there is a \( \nu \geq 0 \) such that

\[
\Re(\varphi, H\varphi) \geq -\nu \| \varphi \|_2^2
\]

54
for all $\varphi \in L^r_{2,m}$. Next observe that if $\varphi_t \in D(H)$ satisfies the Cauchy equation

$$\frac{d}{dt} \varphi_t + H \varphi_t = 0$$

(34)

for all $t > 0$ then

$$\frac{d}{dt} \|\varphi_t\|_2^2 = -2 \text{Re}(\langle \varphi_t, H \varphi_t \rangle) \leq 2\nu \|\varphi_t\|_2^2.$$ 

Therefore $t \mapsto e^{-\nu t} \|\varphi_t\|_2$ is a decreasing function. Now suppose $\varphi_t^{(1)}$ and $\varphi_t^{(2)}$ both satisfy (34) and $\varphi_t^{(1)} \to \varphi$, $\varphi_t^{(2)} \to \varphi$ as $t \to 0$. Then $\varphi_t^{(1)} - \varphi_t^{(2)}$ also satisfies the equation but $\varphi_t^{(1)} - \varphi_t^{(2)} \to 0$ as $t \to 0$. Therefore, as a consequence of the foregoing decrease property, $\varphi_t^{(1)} = \varphi_t^{(2)}$, i.e., the solution of (34) is uniquely determined by the initial data $\varphi = \varphi_0$.

Now let $\varphi \in L^r_{2,m}$. Then $\varphi_t = S_{t+} \varphi = K_{t+} * \varphi$ satisfies (34) with initial data $\varphi_0 = S_t \varphi$ (see (31)). Moreover, $\varphi_t = S_t S_t^* \varphi$ satisfies the equation with the same initial data. Therefore

$$(S_{t+} - S_t S_t^*) \varphi = 0$$

for all $\varphi \in L^r_{2,m}$ and then, by continuity, for all $\varphi \in L_2$. This establishes that $S$ is a semigroup on $L_2$. But this implies that $K_t$ is a convolution semigroup. Therefore $S$ is also a semigroup on the other $L^r_\rho$-spaces or in any Banach space representation.

It follows from (31) that the generator $H_S$ of $S$ is an extension of $H$ on $L^r_\rho$. Now $L^r_{p,\infty}$ is a dense $S$-invariant subspace and hence a core of $H_S$. Therefore $H_S$ must be the closure of $H$.

At this point we have essentially established the main result for the left regular representation in the $L^r_\rho$-spaces if $p \in [1, \infty)$.

**Theorem 8.2** Let $(\mathcal{X}, G, U)$ be a continuous representation, $a_1, \ldots, a_d$ a reduced weighted algebraic basis in the Lie algebra $\mathfrak{g}$ of $G$ and $C$ a $G_0$-weighted subcoercive form of order $m$ where $G_0$ is the homogeneous contraction of $G$. Let $H = dU(C)$ be the associated operator. Then one has the following.

I. The closure $\overline{H}$ of $H$ generates a continuous semigroup $S$.

II. The semigroup $S$ is holomorphic in a sector $A(\theta) = \{z \in \mathbb{C} : |\arg z| < \theta\}$ where the angle of holomorphy $\theta$ satisfies the bounds $\theta \geq \theta_{C,G_0}$.

III. $\overline{H} = H^\dagger$, where $H^\dagger = dU_*(C^\dagger)$ is the dual operator.

**Proof** Since the kernel $K$ is a convolution semigroup it now follows that $(S_t)_{t \geq 0} = (U(K_t))_{t \geq 0}$ is a continuous semigroup. One then deduces as in Theorem 3.4 of [AER] that $\overline{H}$ is the generator and $S$ is holomorphic, with the holomorphy sector containing at least $\Lambda(\theta_{C,G_0})$. \hfill $\Box$

As a consequence of the bounds on the kernel we can compare the domain of powers of the operator $dU(C)$ and the differential structure of the representation associated with the weighted algebraic basis $a_1, \ldots, a_d$, i.e., the spaces $\mathcal{X}_n^\prime$.

**Corollary 8.3** Let $(\mathcal{X}, G, U)$ be a continuous representation, $a_1, \ldots, a_d$ a reduced weighted algebraic basis in the Lie algebra $\mathfrak{g}$ of $G$ and $C$ a $G_0$-weighted subcoercive form of order $m$. Let $S$ be the semigroup generated by the closure of the operator $H = dU(C)$. 55
I. The semigroup $S$ maps into the smooth $C^\infty$-elements, i.e., $S_t\mathcal{X} \subseteq \mathcal{X}_\infty$ for all $t > 0$.

II. If $k \in [0, \infty)$ then there exist $c > 0$ and $\omega \geq 0$ such that

$$\|S_t x\|'_k \leq c t^{-k/m} \omega^t \|x\|$$

for all $t > 0$ and $x \in \mathcal{X}$.

III. If $n \in \mathbb{N}$ and $k \in [0, nm)$ then $D(\overline{H}^n) \subseteq \mathcal{X}'_k$ and there exists $c > 0$ such that

$$\|x\|'_k \leq c^{mn-k} \|\overline{H}^n x\| + c \epsilon^{-k} \|x\|$$

for all $x \in D(\overline{H}^n)$ and $\epsilon \in (0, 1]$. In particular

$$\mathcal{X}_\infty = \bigcap_{n=1}^\infty D(\overline{H}^n),$$

so the spaces of $C^\infty$-elements of $(\mathcal{X}, G, U)$ and of the operator $\overline{H}$ coincide.

**Proof** Statements I and II follow immediately from the fact that the kernel $K_t$ is smooth and, together with its derivatives, satisfies Gaussian bounds.

If $n \in \mathbb{N}$, $k \in [0, nm)$ and $\lambda > 0$ is large enough then

$$(\lambda I + \overline{H})^{-n} = (n-1)^{-1} \int_0^\infty dt \, e^{-\lambda t} t^{n-1} S_t.$$  

Therefore $D(\overline{H}) = R((\lambda I + \overline{H})^{-n}) \subseteq \mathcal{X}'_k$ by Statement II. Moreover,

$$\|A^\alpha (\lambda I + \overline{H})^{-n}\| \leq (n-1)^{-1} c \int_0^\infty dt \, e^{-(\lambda - \omega) t^{n-1} \|s\|/m}$$

$$= c'(\lambda - \omega)^{(mn-\|\alpha\|)/m} \leq c''(mn-\|\alpha\|)/m$$

if $\lambda$ is large enough. Taking $\epsilon$ proportional to $\lambda^{-1/m}$ and rearranging it follows that

$$\|A^\alpha x\| \leq c^{mn-\|\alpha\|} \|\overline{H}^n x\| + c'' \epsilon^{-\|\alpha\|} \|x\|$$

for all $x \in D(\overline{H}^n)$ and for small positive values of $\epsilon$. Statement III then follows. 

**Remark** Note that the constants $c$ in Corollary 8.3 depend on the kernel only through the constants $M$ and $\omega$ in the bounds $\|A^\alpha K_t\|_1 \leq M t^{-\|\alpha\|/m} \omega^t$ if $\rho \geq 0$ is such that $\|U(g)\| \leq Me^{\rho|s|'}$.

### 9 Regularity

The bounds on the semigroup in the previous section enable the deduction of several regularity results for the operators $dU(C)$ associated with a representation $(\mathcal{X}, G, U)$, a reduced weighted algebraic basis and a $G_0$-weighted subcoercive form of order $m$, where $G_0$ is the homogeneous contraction of $G$. Recall

$$w = \min\{x \in [1, \infty) : x \in w_i \mathbb{N} \text{ for all } i \in \{1, \ldots, d'\}\}.$$
We adopt the notation of [BuB] for the real interpolation spaces. We need two special interpolation spaces associated with the representation $U$ and the distance corresponding to the weighted algebraic basis.

Let $\mathcal{O}$ be a bounded open neighbourhood of the identity $e$ of $G$, $p \in [1, \infty]$ and $n \in \mathbb{N}$. Then for each $\gamma \in (0, n \lambda_1)$, with $\lambda_1$ the smallest weight of the algebraic basis, define $\| \cdot \|_{\gamma, \nu, U} : \mathcal{X} \to [0, \infty]$ by

$$
\| x \|_{\gamma, \nu, U} = \| x \| + \left( \int_{\mathcal{O}_n} d\mu_n(g) \left( |g|^{-\gamma} \| (I - U(g_1)) \cdots (I - U(g_n)) x \| \right)^p \right)^{1/p},
$$

where $g = (g_1, \ldots, g_n)$ and $|g| = |g_1'| + \cdots + |g_n'|$. Moreover, $\mu_n$ is the absolutely continuous measure with respect to the left Haar measure on $G^n$ with density $g \mapsto |g|^{-nD'}$. The usual changes are needed in the case $p = \infty$. Then the Lipschitz space $\mathcal{X}_{\gamma, \nu}^n(U)$ is defined by

$$
\mathcal{X}_{\gamma, \nu}^n(U) = \{ x \in \mathcal{X} : \| x \|_{\gamma, \nu, U} < \infty \}.
$$

It is a Banach space with respect to the norm $\| \cdot \|_{\gamma, \nu, U}$. Note that as the space is independent of the choice of $\mathcal{O}$, up to equivalence of norms, we have omitted it from the notation.

Next we introduce a uniform version of the Lipschitz spaces. First, for each $x \in \mathcal{X}$ and $n \in \mathbb{N}_0$ define $\omega_x^{(n)} : (0, \infty) \to [0, \infty)$ by $\omega_x^{(0)}(t) = \| x \|$ and

$$
\omega_x^{(n)}(t) = \sup_{g_1, \ldots, g_n \in G, |g_1| \leq t} \| (I - U(g_1)) \cdots (I - U(g_n)) x \|
$$

for $n \in \mathbb{N}$. Secondly, for $\gamma \in (0, n \lambda_1)$ define $\| \cdot \|_{\gamma, \nu, w} : \mathcal{X} \to [0, \infty]$ by

$$
\| x \|_{\gamma, \nu, w} = \| x \| + \left( \int_0^t dt t^{-1} \left( t^{-\gamma} \omega_x^{(n)}(t) \right)^p \right)^{1/p}.
$$

Then the space

$$
\mathcal{X}_{\gamma, \nu, w}^n = \{ x \in \mathcal{X} : \| x \|_{\gamma, \nu, w} < \infty \}
$$

is a Banach space with respect to the norm $\| \cdot \|_{\gamma, \nu, w}$.

Finally we also use $\| \cdot \|_Y$ to denote the norm on a Banach space $Y$.

**Theorem 9.1** Let $(\mathcal{X}, G, U)$ be a continuous representation, $a_1, \ldots, a_d$ a reduced weighted algebraic basis in the Lie algebra $g$ of $G$, $\lambda_1$ the smallest weight and $C$ a $G_0$-weighted subcoercive form of order $m$, where $G_0$ is the homogeneous contraction of $G$. Let $S$ be the semigroup generated by the closure of the operator $H = du(C)$.

I. If $p \in [1, \infty]$, $\gamma > 0$, $n = \min\{ n \in \mathbb{N} : \gamma < nw \}$, $k = \min\{ k \in \mathbb{N} : k \geq nw/\lambda_1 \}$ and $k, n \in \mathbb{N}$ are such that $k \geq k$, $n \geq n$ then

$$
(\mathcal{X}, \mathcal{X}_{nw}^\gamma)_{\gamma/(nw), p; R} = (\mathcal{X}, D(H^n))_{\gamma/(nm), p; R} = \mathcal{X}_{\gamma}^{k, p, w} = \mathcal{X}_{\gamma}^{k, p}(U)
$$

as Banach spaces.

II. Let $p \in [1, \infty]$. If $n_1, n_2 \in \mathbb{N}$ and $0 < \gamma < n_1 \wedge n_2$ then

$$
(\mathcal{X}, \mathcal{X}_{n_1w}^\gamma)_{\gamma/(n_1w), p; R} = (\mathcal{X}, \mathcal{X}_{n_2w}^\gamma)_{\gamma/(n_2w), p; R}.
$$
III. If $l, n \in \mathbb{N}_0$ and $k \in \{lw, nw\}$ then there exists $c > 0$ such that

$$\|x\|_{\omega}^k \leq \epsilon^{nw-k} \|x\|_{nw} + c \epsilon^{-(k-lw)} \|x\|_{lw}$$

for all $\epsilon > 0$ and $x \in X'_{nw}$.

IV. If $l, n \in \mathbb{N}_0$ and $k \in \{lw, nw\}$ then there exists $c > 0$ such that

$$N'_k(x) \leq \epsilon^{nw-k} N'_{nw}(x) + c \epsilon^{-(k-lw)} \|x\|_{lw}$$

for all $\epsilon > 0$ and $x \in X'_{nw}$.

V. If $n, k \in \mathbb{N}$, $\gamma \in (0, nw)$ and $p \in [1, \infty]$ then

$$\{x \in D(H^k) : H^k x \in (\mathcal{X}^{\prime}, X')_{\gamma/(nw), p; K}\} \subseteq (\mathcal{X}^{\prime}, X')_{\gamma/(nw), p; K; km}.$$  

Moreover, if $\lambda$ is large enough then there exists $c > 0$ such that

$$\|x\|_{(\mathcal{X}, X')_{\gamma/(nw), p; K, km}} \leq c \|(H + \lambda I)^k x\|_{(\mathcal{X}, X')_{\gamma/(nw), p; K}}$$

for all $x \in \{x \in D(H^k) : H^k x \in (\mathcal{X}^{\prime}, X')_{\gamma/(nw), p; K}\}$ where $(\mathcal{X}^{\prime}, X')_{\gamma/(nw), p; K; km}$ denotes the space of (weighted) km-times differentiable vectors for the Lipschitz space $(\mathcal{X}, X')_{\gamma/(nw), p; K}$.

Proof The proofs are very similar to those in Section 5 of [EIR5], so we only indicate the differences. The equality $(\mathcal{X}, X')_{\gamma/(nw), p; K} = (\mathcal{X}, D(H^\gamma))_{\gamma/(nw), p; K}$ follows as in Proposition 5.1 of [EIR5] and therefore Statement II is valid. So $(\mathcal{X}, X')_{\gamma/(nw), p; K} = (\mathcal{X}, X'_{nw})_{\gamma/(nw), p; K}$. Now it follows as on pp. 581-582 in [EIR5] that

$$\omega_{\alpha \in (d)} \|A^{\alpha} x_{\infty}\|$$

for some $c > 0$, uniformly for all $t \in (0,1]$ and $x_{\infty} \in X_{\infty}$. Since $\|\alpha\| \geq \lambda_{1} k \geq nw$ for all $\alpha$ with $|\alpha| = k$ one can argue as in the proof of Theorem 5.7 in [EIR5] to deduce that $(\mathcal{X}, X'_{nw})_{\gamma/(nw), p; K} \subseteq \mathcal{X}_{\gamma}^{kp; \omega}$. The inclusion $\mathcal{X}_{\gamma}^{kp; \omega} \subseteq \mathcal{X}_{\gamma}^{kp; \omega}$ follows by definition and the local boundedness of the representation. Next the inclusions $\mathcal{X}_{\gamma}^{kp; \omega} \subseteq \mathcal{X}_{\gamma}^{kp; \omega}(U) \subseteq (\mathcal{X}, D(H^\gamma))_{\gamma/(km), p; K}$ can be proved precisely as in Steps 2 and 3 of the proof of Theorem 3.2 in [EIR2]. Statement I follows by an application of the reiteration theorem (see [BuB]).

Next we turn to unitary representations.

Theorem 9.2 Let $(\mathcal{X}, G, U)$ be a unitary representation, $a_1, \ldots, a_d$ a reduced weighted algebraic basis in the Lie algebra $\mathfrak{g}$ of $G$ and $C$ a $G_0$-weighted subcoercive form of order $m$, where $G_0$ is the homogeneous contraction of $G$.

I. The operator $H = dU(C)$ is closed.
II. For all \( n \in \mathbb{N} \) and all large \( \lambda > 0 \)
\[
D((\lambda I + H)^{nw/m}) = X'_{nw}
\]
with equivalent norms.

III. For each \( \varepsilon > 0 \) there exists a \( \nu \in \mathbb{R} \), independent of the representation \( U \), such that
\[
\text{Re}(x, Hx) \geq (\mu_{C,G_0} - \varepsilon)(\|x\|_{m/2}'^2) - \nu \|x\|^2
\]
for all \( x \in \mathcal{X}_\infty \).

IV. If \( n \in \mathbb{N} \) then
\[
X'_{nw} = \bigcap_{i=1}^{d'} D(A^{nw/\omega}_i)
\]

V. For each \( \theta \in (0, \theta_{C,G_0}) \) there exists an \( \omega > 0 \) such that \( \|S_z\| \leq e^{\omega|z|} \) uniformly for all \( z \in \Lambda(\theta) \) where \( S \) is the holomorphic semigroup generated by \( H \).

**Proof** The proofs of Statements I, II and IV are as in the proof of Theorem 5.8 in [EIR5] and the proof of Statement V is similar to the proof in [BGJR]. Since Statement III is stronger than Statement III in [EIR5] we give a new proof.

Let \( C_0 \) be the weighted subcoercive form such that
\[
dV(C_0) = \sum_{\alpha \in J(d')} (-1)^{|\alpha|} A^{\alpha} \cdot A^\alpha
\]
in any continuous representation \((\mathcal{Y}, G, V)\). Let \( M \) be the number of multi-indices \( \alpha \in J(d') \) with \( \|\alpha\| = m/2 \). Then
\[
(N'_{\nu,m/2}(x))^2 \leq (x, dV(C_0)x) \leq M(N'_{\nu,m/2}(x))^2
\]
for all \( x \in \mathcal{Y}_\infty(V) \) if \( V \) is unitary. Next, let \( \alpha_0 \in J(d') \) with \( \|\alpha_0\| = m/2 \) and \( \varepsilon \in (0, (2M)^{-1} \mu_{C,G_0}) \). Further let \( C_1 \) be the homogeneous form such that
\[
dV(C_1) = (\mu_{C,G_0} - 2M\varepsilon)(-1)^{n_0}|A^{\alpha_0}| \cdot A^{\alpha_0} + \varepsilon \frac{dV(C_0)}{}
\]
Then
\[
\text{Re}(\varphi, dL_{G_0}(RC - C_1)\varphi) \geq M\varepsilon (N'_{2}L_{G_0}(\varphi))^2
\]
for all \( \varphi \in L^{2\infty}(G_0) \). So \( RC - C_1 \) is a \( G_0 \)-weighted subcoercive form for which the corresponding operator is essentially self-adjoint and its closure generates a semigroup. Hence \( dU(RC - C_1) \) is lower semibounded by spectral theory, with lower bound \(-\nu \leq 0 \). Therefore,
\[
\text{Re}(dU(C)x, x) = (dU(RC)x, x)
\geq (x, dU(C_1)x) - \nu \|x\|^2 \geq (\mu_{C,G_0} - 2M\varepsilon)\|A^{\alpha_0}x\|^2 - \nu \|x\|^2
\]
Since the number of multi-indices \( \alpha_0 \) with \( \|\alpha_0\| = m/2 \) is finite the theorem follows. \( \square \)

It is also possible to obtain regularity results for the left regular representation on the \( L_p \)-spaces with respect to left Haar measure if \( p \in (1, \infty) \). These are basically a result of the good kernel bounds and the regularity on \( L_2 \).
Corollary 9.3 Let $G$ be a connected Lie group, $a_1, \ldots, a_d$ a reduced algebraic basis of the Lie algebra $g$ of $G$ and $C$ a $G_0$-weighted subcoercive form of order $m$. Let $L$ be the left regular representation on $L_p$, where $p \in (1, \infty)$. Then

I. The operator $H = dL(C)$ is closed.

II. For all $n \in \mathbb{N}$ one has

$$D((\lambda I + H)^{nw/m}) = L_{p,nw}'$$

with equivalent norms, if $\lambda > 0$ is large enough.

III. If $n \in \mathbb{N}$ then

$$L_{p,nw}' = \bigcap_{i=1}^{d'} D(A_{i, nw}^{r/nw})$$

Similar statements are valid on the space $L_p$-spaces with respect to right Haar measure, $L_p$.

Proof The proof is precisely the same as for the unweighted operators in [BER]. □

Corollary 9.4 Let $G$ be a connected Lie group, $a_1, \ldots, a_d$ a reduced algebraic basis of the Lie algebra $g$ of $G$ and $C$ a $G_0$-weighted subcoercive form of order $m$. Let $L$ be the left regular representation on $L_p$, where $p \in (1, \infty)$, and $H = dL(C)$. If $\theta \in (0, \theta_C)$ then there is a $\nu_0 \geq 0$, independent of $p$, such that the operators $\nu I + H$, $\nu > \nu_0$, have a bounded functional calculus over the bounded functions holomorphic in a sector $\Lambda(\varphi)$ with $\varphi \in (\pi/2 - \theta, \pi]$.

Proof The proof is precisely the same as in [ElR4]. □

Note that in the next section we establish that $C$ is $G_0$-weighted subcoercive if and only if it is $G$-weighted subcoercive so the last two results could be phrased entirely in terms of $G$.

10 Weighted subcoercive forms. Part II

In this section we prove that all conditions of Proposition 4.6 concerning the Gårding inequality are equivalent by establishing the missing implications. Moreover, we give other characterizations in the spirit of the characterization of hypoelliptic operators by Rockland operators on a homogeneous group.

Theorem 10.1 Let $G$ be a connected Lie group, $a_1, \ldots, a_d$ a reduced weighted algebraic basis of the Lie algebra $g$ of $G$ and $G_0$ the corresponding homogeneous contraction of $G$. Further let $m \in 2w\mathbb{N}$ and $C$ be an $m$-th order form with principal part $P$. The following conditions are equivalent.

I. The form $C$ is $G$-weighted subcoercive.

II. The form $C$ is $G_0$-weighted subcoercive.

III. For all non-trivial irreducible unitary representations $(\mathcal{X}, G_0, U)$ of $G_0$ one has

$$\text{Re}(x, dU(P)x) > 0$$

for all $x \in \mathcal{X}_\infty(U)$ with $x \neq 0$, where $P$ is the principal part of $C$. 60
IV. The operator \( dL_{G_0}(\mathcal{R}P) \) is a positive Rockland operator.

Moreover, if any one of the four equivalent conditions is valid then \( \mu_{C,G} = \mu_{C,G_0} \) and 
\( \theta_{C,G} = \theta_{C,G_0} \).

Proof. The implication \( I \Rightarrow II \) has been established in Proposition 4.6 together with the inequality 
\( \mu_{C,G} \leq \mu_{C,G_0} \). The converse implication \( II \Rightarrow I \) follows from Theorem 9.2.III and 
\( \mu_{C,G} \geq \mu_{C,G_0} \). Therefore I and II are equivalent and \( \theta_{C,G} = \theta_{C,G_0} \).

The implication \( II \Rightarrow III \) is trivial since \( N_{U,m/2}(x) \neq 0 \) if \( U \) is a non-trivial
irreducible unitary representation and \( x \in \mathcal{X}_\infty(U) \) is non-zero.

If III is valid then \( dL_{G_0}(\mathcal{R}P) \) is hypoelliptic by the Helffer–Nourrigat theorem. Moreover, the
Plancherel formula, [Kir] Proposition 4, gives

\[
(dL_{G_0}(\mathcal{R}P)\varphi, \varphi) \geq 0
\]

for all \( \varphi \in C_c^\infty(G_0) \), and hence by continuity for all \( \varphi \in L_{2;\infty}(G_0) \). So \( \mathcal{R}P \) is a positive
Rockland form.

The implication \( IV \Rightarrow II \) follows from Theorem 2.5 of [EIR6].

It now follows that all conditions of Proposition 4.6 are equivalent.

Remark. If \( G_0 = \mathbb{R}^d \) then the equivalence \( II \Leftrightarrow III \) in Theorem 10.1 states that a form 
\( C \) is \( G_0 \)-weighted subcoercive if, and only if, \( \text{Re} \sum_{\|\alpha\|=m} c_\alpha (i\xi)^\alpha > 0 \) for all \( \xi \in \mathbb{R}^d \)
with \( \xi \neq 0 \). This gives new proofs for Examples 4.1 and 4.2.

The implication \( I' \Rightarrow 4 \) in Proposition 4.6 states that

\[
\text{Re}(\varphi, dL_{G_0}(P)\varphi) \geq \mu \left( N_{2;m/2}''(\varphi) \right)^2 - \nu \| \varphi \|_2^2
\]

for all \( \varphi \in L_{2;\infty}(G_0) \) if \( C \) is a \( G \)-weighted subcoercive form, where \( P \) is the principal part
of \( C \). This clearly implies that for all \( \gamma \in (0, m/2) \), \( p \in [1, \infty] \) and \( n \in \mathbb{N} \) with \( n > m \) there exist
\( \mu > 0 \) and \( \nu \in \mathbb{R} \) such that

\[
\text{Re}(\varphi, dL_{G_0}(P)\varphi) \geq \mu \left( \| \varphi \|_{p,L_{\mathcal{R}G_0}}^n \right)^2 - \nu \| \varphi \|_2^2
\]

for all \( \varphi \in L_{2;\infty}(G_0) \). We next show that this seemingly weaker inequality also characterizes
weighted subcoercivity.

Proposition 10.2. Let \( G \) be a connected Lie group, \( a_1, \ldots, a_d \) a reduced weighted algebraic
basis of the Lie algebra \( \mathfrak{g} \) of \( G \) and \( G_0 \) the corresponding homogeneous contraction of \( G \).
Further let \( m \in 2w\mathbb{N} \) and \( C \) be an \( m \)-th order form with principal part \( P \). The following
conditions are equivalent.

I. The form \( C \) is \( G \)-weighted subcoercive.

II. There exist \( \gamma \in (0, m/2) \), \( p \in [1, \infty] \), \( n \in \mathbb{N} \) with \( n > m \), \( \mu > 0 \) and \( \nu \in \mathbb{R} \) such that

\[
\text{Re}(\varphi, dL_{G_0}(P)\varphi) \geq \mu \left( \| \varphi \|_{\gamma,p,L_{\mathcal{R}G_0}}^n \right)^2 - \nu \| \varphi \|_2^2
\]

for all \( \varphi \in L_{2;\infty}(G_0) \).
Proof. We only need to prove the implication $II \Rightarrow I$. The proof is a modification of the reduction theorem in Section 2 in [HeN]. We will show that Condition III of Theorem 10.1 is valid using a scaling argument and a refinement of the proof of Lemma 5.1. We may assume that $p = 2$ and $\gamma < 1$ by an application of the reiteration theorem [BuB] Proposition 3.2.18. Moreover, we may assume that $G = G_0$.

If $U$ is a bounded representation in $\mathcal{X}$ on $G$ we define $N_\gamma^U : \mathcal{X} \to [0, \infty]$ by

$$N_\gamma^U(x) = \left( \int_{G^n} d\mu_n(g) \left( |g|^{-\gamma} \|I - U(g_1) \ldots (I - U(g_n))x\| \right)^2 \right)^{1/2},$$

where $g = (g_1, \ldots, g_n)$, $|g| = |g_1| + \ldots + |g_n|$ and $\mu_n$ is the absolutely continuous measure with respect to the left Haar measure on $G^n$ with density $g \mapsto |g|^{-n\delta'}$ as before. Then there exists a constant $c > 0$ such that

$$\|x\|_{n,2,U} \leq \|x\| + N_\gamma^U(x) \leq \|x\|_{n,2,U}(x) + c \|x\|$$

for all $x \in \mathcal{X}$. So one has

$$\mathrm{Re}(\varphi, dL_G(P)\varphi) \geq \mu (N_\gamma^L(G)) - \nu \|\varphi\|_2^2$$

for all $\varphi \in L_{2,\infty}(G)$. Therefore, by scaling,

$$\delta^m \mathrm{Re}(\varphi, dL_G(P)\varphi) \geq \mu \delta^{2\gamma} (N_\gamma^L(G)) - \nu \|\varphi\|_2^2$$

uniformly for all $\delta > 0$ and $\varphi \in L_{2,\infty}(G)$.

Next we need some details about standard induced representations of $G$. We follow [HeN] Section 2 and [CoG]. Let $m$ be subalgebra of $g$ and let $b_1, \ldots, b_k \in g$ be such that $k = \text{codim} \ m$ and $m + \text{span}\{b_1, \ldots, b_i\}$ is a subalgebra of $g$ for all $i \in \{1, \ldots, k\}$. Such elements exist by [CoG] Theorem 1.1.13. Define $\alpha : \mathbb{R}^k \to G$ by

$$\alpha(s_1, \ldots, s_k) = \exp(s_1b_1) \ldots \exp(s_kb_k).$$

For every $g \in G$ there exist (unique) $E_m(g) \in m$ and $F_m(g) \in \mathbb{R}^k$ such that

$$g = \exp(E_m(g) \alpha(F_m(g))$$

(see [CoG] Theorem 1.2.12). We will always assume that the elements $b_1, \ldots, b_k$ are normalized such that

$$\int_{G} dg \varphi(g) = \int_{m} dm \int_{\mathbb{R}^k} ds \varphi(\exp m \alpha(s))$$

for all $\varphi \in C_c(G)$. Let $l \in g^*$ and suppose that $l([m, m]) = \{0\}$. Then $U_{l,m} : L_2(\mathbb{R}^k) \to L_2(\mathbb{R}^k)$ defined by

$$(U_{l,m}(g)\varphi)(s) = e^{it(E_m(\alpha(s))\varphi(F_m(\alpha(s))g))}$$

is unitary and $U_{l,m}$ is a unitary representation of $G$ in $L_2(\mathbb{R}^k)$. Although the representation $U_{l,m}$ depends on the choice of $b_1, \ldots, b_k$, we have omitted it from the notation. If $m$ is a polarizing subalgebra for $l$ then the representation $U_{l,m}$ is irreducible, and all irreducible unitary representations of $G$ are of this form, up to unitary equivalence (see [CoG] Chapter 2).
We also need some results on reduction of variables. Let $n \subseteq m$ be subalgebras with $[m, m] \subseteq n$ and let $b_1, \ldots, b_p, \ldots, b_q \in g$, where $q = \text{codim} n$, such that $n + \text{span}\{b_1, \ldots, b_i\}$ is a subalgebra of $g$ for all $i \in \{1, \ldots, q\}$ and $m = n + \text{span}\{b_1, \ldots, b_p\}$. Set $k = q - p = \text{codim} m$. Now we define $\alpha : \mathbb{R}^q \to G$ by

$$\alpha(s_1, \ldots, s_q) = \exp(s_1 b_1) \ldots \exp(s_q b_q)$$

and also introduce $\beta : \mathbb{R}^k \to G$ by

$$\beta(s_1, \ldots, s_k) = \exp(s_1 b_{p+1}) \ldots \exp(s_k b_{p+1})$$

For $\xi \in \mathbb{R}^p$ define $l_\xi \in g^*$ by

$$l_\xi(a + \sum_{i=1}^p t_i b_i + \sum_{i=1}^k s_i b_{p+i}) = \sum_{i=1}^p \xi_i s_i$$

for all $a \in n$, $t \in \mathbb{R}^p$ and $s \in \mathbb{R}^k$. Let $l \in g^*$ and suppose that $l([m, m]) = \{0\}$ and, moreover, $l(b_i) = 0$ for all $i \in \{1, \ldots, p\}$. We will give a relation between $U_{l,n}$ and $U_{l+1, m}$. Note that $U_{l+1, n} = U_{l, n}$. Let $F$ denote the (partial) Fourier transform on $L^2(\mathbb{R}^p \times \mathbb{R}^k)$ with respect to the first $p$ variables. If $\varphi \in \mathcal{S}(\mathbb{R}^p \times \mathbb{R}^k)$ and $\xi \in \mathbb{R}^p$ define $(F\varphi)_\xi \in \mathcal{S}(\mathbb{R}^k)$ by $(F\varphi)_\xi(s) = (F\varphi)(\xi, s)$.

Lemma 10.3 If $\varphi \in \mathcal{S}(\mathbb{R}^p \times \mathbb{R}^k)$ then

$$(FU_{l,n}(g)\varphi)_\xi = U_{l+1, m}(g)(F\varphi)_\xi$$

for all $\xi \in \mathbb{R}^p$ and $g \in G$.

Proof Let $s \in \mathbb{R}^k$ and $t \in \mathbb{R}^p$. Then

$$(U_{l,n}(g)\varphi)(t, s) = e^{i(l(E_n(\alpha(t, s)) g))}(F_n(\alpha(t, s) g))$$

and

$$(U_{l+1, m}(g)(F\varphi)_\xi)(s) = e^{i((l+1)(E_m(\beta(s)) g))}(F\varphi)_\xi(F_m(\beta(s) g))$$

Now

$$\alpha(t, s) g = \alpha(t, 0) \beta(s) g = (\alpha(t, 0) \exp E_m(\beta(s) g))\beta(F_m(\beta(s) g))$$

$$= \exp E_n(\alpha(t, 0) \exp E_m(\beta(s) g)) \cdot \alpha(F_n(\alpha(t, 0) \exp E_m(\beta(s) g))) \cdot \beta(F_m(\beta(s) g))$$

So

$$E_n(\alpha(t, s) g) = E_n(\alpha(t, 0) \exp E_m(\beta(s) g))$$

and

$$F_n(\alpha(t, s) g) = (\pi_1(F_n(\alpha(t, 0) \exp E_m(\beta(s) g))), F_m(\beta(s) g))$$

where $\pi_1$ is the projection from $\mathbb{R}^p \times \mathbb{R}^k$ onto $\mathbb{R}^p$. Since $l([m, m]) = \{0\}$ it follows from the Campbell–Baker–Hausdorff formula that $l(\log(\exp a \exp b)) = l(a) + l(b)$ for all $a, b \in m$. Therefore

$$l(E_n(\alpha(t, s) g)) = l(t_1 b_1) + \ldots + l(t_p b_p) + l(E_m(\beta(s) g)) = l(E_m(\beta(s) g))$$

63
If one uses \([m, m] \subseteq n\) and the Campbell–Baker–Hausdorff formula once again one sees that \(F_n(\exp a \exp b) = F_n(\exp a) + F_n(\exp b)\) for all \(a, b \in m\). So

\[
F_n(\alpha(t, 0) \exp E_m(\beta(s) g)) = (t, 0) + F_n \exp E_m(\beta(s) g).
\]

Therefore

\[
\varphi(F_n(\alpha(t, s) g)) = \varphi(t + \pi_t F_n \exp E_m(\beta(s) g), F_m(\beta(s) g))
\]

for all \(t \in \mathbb{R}^p\). Using the identity \(l_\xi(a) = \xi \cdot \pi_t F_n \exp a\) for all \(a \in m\) one establishes that

\[
(FU_{i,n}(g) \varphi)(\xi, s) = e^{i(l(E_m(\beta(s) g)))} e^{i\pi_t F_n \exp E_m(\beta(s) g)} \varphi(\xi, F_m(\beta(s) g))
\]

\[
= e^{i(l+1)(E_m(\beta(s) g))} \varphi(\xi, F_m(\beta(s) g))
\]

\[
= \left(U_{i+1,t,m}(g)(F \varphi)(\xi)\right)(s)
\]

and the lemma has been proved. \(\square\)

This relation is the key to obtaining a connection between \(N^U_{i+1,t,m}\) and \(N^U_{i,n}\).

**Lemma 10.4** Let \(\xi_0 \in \mathbb{R}^p\) and \(\tau \in C_c(\mathbb{R}^p)\) be positive with \(\|\tau\|_2 = 1\). For \(j \in \mathbb{N}\) define \(\tau_j \in S(\mathbb{R}^p)\) by \(\tau_j(\xi) = j^{p/2} \tau(j(\xi - \xi_0))\). Let \(\psi \in S(\mathbb{R}^k)\) and set \(\varphi_j = (F^{-1} \tau_j) \otimes \psi\). Then one has the following.

**I.** \(N^U_{i+1,t_0,m}(\psi) = \lim_{j \to \infty} N^U_{i,n}(\varphi_j)\).

**II.** \((\psi, dU_{i+1,t_0,m}(a^\alpha) \psi) = \lim_{j \to \infty} (\varphi_j, dU_{i,n}(a^\alpha) \varphi_j)\) for all \(\alpha \in J(d)\).

**Proof** For all \(j \in \mathbb{N}\) one has

\[
\left(N^U_{i,n}(\varphi_j)\right)^2 = \int G^n d\mu_n(g) \left(|g|^{-\gamma} \|I - U_{i,n}(g_1)\| \cdots \|I - U_{i,n}(g_n)\|\varphi_j\|^2\right).
\]

\[
= \int G^n d\mu_n(g) \left(|g|^{-\gamma} \|F(I - U_{i,n}(g_1))\| \cdots \|I - U_{i,n}(g_n)\|\varphi_j\|^2\right)
\]

\[
= \int G^n d\mu_n(g) \left(|g|^{-\gamma} \|\tau_j \otimes (I - U_{i+1,t,m}(g_1)) \cdots (I - U_{i+1,t,m}(g_n))\psi\|^2\right)
\]

\[
= \int G^n d\mu_n(g) \int_{\mathbb{R}^k} ds \int_{\mathbb{R}^p} d\xi |g|^{-2\gamma} |\tau_j(\xi)|^2 \cdot |(I - U_{i+1,t,m}(g_1)) \cdots (I - U_{i+1,t,m}(g_n))\psi\|^2(s)^2
\]

\[
= \int G^n d\mu_n(g) \int_{\mathbb{R}^k} ds |g|^{-2\gamma} \psi_j(g, s)^2,
\]

where

\[
\psi_j(g, s) = \int_{\mathbb{R}^p} d\xi |\tau_j(\xi)|^2 \left|(I - U_{i+1,t,m}(g_1)) \cdots (I - U_{i+1,t,m}(g_n))\psi\right|^2(s)^2
\]

Obviously

\[
\lim_{j \to \infty} \psi_j(g, s) = \left|(I - U_{i+1,t_0,m}(g_1)) \cdots (I - U_{i+1,t_0,m}(g_n))\psi\right|^2(s)^2
\]

for all \(g \in G^n\) and \(s \in \mathbb{R}^k\), by (35), so if we can show that \(\int_{G^n} d\mu_n(g) \int_{\mathbb{R}^k} ds |g|^{-2\gamma} \psi_j(g, s)\) is uniformly bounded in \(j\) then the first statement follows from the Lebesgue dominated convergence theorem.
Clearly
\[
\int_{\mathbb{R}^k} ds \psi_j(g, s) = \int_{\mathbb{R}^p} d\xi |\tau_j(\xi)|^2 \|(I - U_{t, \xi, m}(g_1)) \cdots (I - U_{t, \xi, m}(g_n))\psi\|^2 \\
\leq \int_{\mathbb{R}^p} d\xi |\tau_j(\xi)|^2 2^{2n}\|\psi\|^2 = 2^{2n}\|\psi\|^2
\]
for all \( g \in G^n \). So
\[
\int_{\{g:|g| \geq 1\}} d\mu_n(g) |g|^{-2\gamma} \psi_j(g, s) \leq 2^{2n}\|\psi\|^2 \int_{\{g:|g| \geq 1\}} d\mu_n(g) |g|^{-2\gamma} < \infty
\]
for all \( j \in \mathbb{N} \). Finally, let \( g = (g_1, \ldots, g_n) \in G^n \). Then
\[
\int_{\mathbb{R}^k} ds \psi_j(g, s) \leq 2^{2(n-1)} \int_{\mathbb{R}^p} d\xi |\tau_j(\xi)|^2 \|(I - U_{t, \xi, m}(g_n))\psi\|^2
\]
Now suppose \( g_n = \exp(a) \). Then
\[
\|(I - U_{t, \xi, m}(g_n))\psi\| \leq \|a\| \left( \sum_{i=1}^d \|dU_{t, \xi, m}(a_i)\psi\|^2 \right)^{1/2} \leq d^{1/2}\|a\| \sum_{i=1}^d \|dU_{t, \xi, m}(a_i)\psi\|
\]
For all \( i \in \{1, \ldots, d\} \) and \( s \in \mathbb{R}^k \) let \( P_i(s) \in \text{span}\{b_1, \ldots, b_p\} \) be such that
\[
\frac{d}{dt} E_m(\beta(s) \exp(ta_i))|_{t=0} = P_i(s) + b
\]
for some \( b \in \mathbb{n} \). Then \( P_i \) is a polynomial function and
\[
(dU_{t, \xi, m}(a_i)\psi)(s) = (dU_{t, \xi, m}(a_i)\psi)(s) + i \sum_{i=1}^d \xi(P_i(s))\psi(s)
\]
for all \( \xi \in \mathbb{R}^d \). Since \( \int d\xi |\tau_j(\xi)|^2 |\xi_1, \xi_2|^2 \) is uniformly bounded for all \( i_1, i_2 \in \{1, \ldots, d\} \) one deduces that
\[
\int_{\mathbb{R}^k} ds \psi_j(g, s) \leq c\|a\|^2
\]
for some \( c > 0 \), uniformly for all \( j \in \mathbb{N} \) and \( g \in G^n \), where \( a = \log g_n \). Now \( \|a\| \leq c'|g_n| \) if \( |g_n| \leq 1 \). Therefore
\[
\|a\| \leq c'|g_n| \leq c'|g|
\]
if \( |g| \leq 1 \). Since \( \gamma < 1 \) one then establishes that
\[
\int_{\{g:|g| \leq 1\}} d\mu_n(g) |g|^{-2\gamma} \psi_j(g, s) \leq c(c')^2 \int_{\{g:|g| \leq 1\}} d\mu_n(g) |g|^{2(1-\gamma)} < \infty
\]
uniformly for all \( j \in \mathbb{N} \) and Statement I follows.

One can establish Statement II by a similar argument (see also the proof of Lemma 2.2 in [HeN]).

Corollary 10.5 If
\[
\delta^m \text{Re}(\varphi, dU_{t, n}(P)\varphi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_{t, n}}(\varphi))^2 - \nu \|\varphi\|^2
\]
for all \( \varphi \in S(\mathbb{R}^k) \) then
\[
\delta^m \text{Re}(\psi, dU_{t, t_0, m}(P)\psi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_{t, t_0, m}}(\psi))^2 - \nu \|\psi\|^2
\]
for all \( \psi \in S(\mathbb{R}^k) \) and \( \xi_0 \in \mathbb{R}^n \).
Corollary 10.6 If \( n \subseteq m \) are subalgebras of \( g \) with codimensions \( k \) and \( q \) and \( [m, m] \subseteq n \) and \( l \in g^* \) is such that \( l([m, m]) = \{0\} \) then

\[
\delta^m \Re(\varphi, dU_{l,n}(P)\varphi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_l,m}(\varphi))^2 - \nu \|\varphi\|^2
\]

for all \( \varphi \in \mathcal{S}(\mathbb{R}^k) \) implies

\[
\delta^m \Re(\psi, dU_{l,m}(P)\psi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_l,m}(\psi))^2 - \nu \|\psi\|^2
\]

for all \( \psi \in \mathcal{S}(\mathbb{R}^k) \).

Now we finish the proof of Proposition 10.2. Let \((g_\lambda)_{\lambda \geq 0}\) be the filtration of \( g \) and \( \lambda_1 < \ldots < \lambda_k \) the weights of the filtration. Let \((\gamma_i)_{i \geq 0}\) be the family of dilations on the homogeneous Lie algebra \( g \). Let \( l \in g^* \) and \( m \) a polarizing subalgebra of \( g \) for \( l \). For \( j \in \{1, \ldots, k\} \) set

\[
m_j = m \cap \text{span}\{a \in g : \exists \lambda_j, \forall t > 0[\gamma_t(a) = t^j a]\}
\]

and set \( m_{k+1} = \{0\} \). Then \( m_{k+1} \subset m_k \subset \ldots \subset m_2 \subset m_1 = m \) are subalgebras of \( g \) and \( [m_j, m_j] \subset m_{j+1} \) for all \( j \in \{1, \ldots, k\} \). The representation \( U_{l,m_{k+1}} \) is unitarily equivalent with the left regular representation \( L_G \) of \( G \) in \( L^2(G) \), so

\[
\delta^m \Re(\varphi, dU_{l,m_{k+1}}(P)\varphi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_l,m_{k+1}}(\varphi))^2 - \nu \|\varphi\|^2
\]

for all \( \varphi \in \mathcal{S}(\mathbb{R}^k) \) and \( \delta > 0 \). Hence by downward induction on \( j \) it follows from Corollary 10.6 that

\[
\delta^m \Re(\varphi, dU_{l,m_{j}}(P)\varphi) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U_l,m_{j}}(\varphi))^2 - \nu \|\varphi\|^2
\]

for all \( j \in \{1, \ldots, k\} \) and \( \varphi \in \mathcal{S}(\mathbb{R}^{n_j}) \), where \( n_j = \text{codim} m_j \). But \( U_{l,m_j} = U_{l,m} \), so

\[
\delta^m \Re(x, dU(P)x) \geq \mu \delta^{2\gamma}(N_{\gamma}^{U}(x))^2 - \nu \|x\|^2
\]

for any irreducible unitary representation \( U \) of \( G \), \( x \in X_{\infty}(U) \) and \( \delta > 0 \). Now suppose \( U \) is a non-trivial irreducible unitary representation of \( G \) and \( x \in X_{\infty}(U) \) is non-trivial. Then \( N_{\gamma}^{U}(x) \neq 0 \) since otherwise \((I-U(g_1))\ldots(I-U(g_n))x = 0\) for all \( g_1, \ldots, g_n \in G \setminus \{e\} \) and therefore \( A^\alpha x = 0 \) for all \( \alpha \in J(d) \) with \( |\alpha| = m \). Choose \( \delta > 0 \) so large that \( \mu \delta^{2\gamma}(N_{\gamma}^{U}(x))^2 - \nu \|x\|^2 > 0 \). Then \( \delta^m \Re(x, dU(P)x) > 0 \) and \( \Re(x, dU(P)x) > 0 \). Now the proposition follows from Theorem 10.1.III. \( \Box \)

The next proposition gives a necessary and sufficient condition for a form \( P \) to be a Rockland form.

Proposition 10.7 Let \( G \) be a connected Lie group, \( a_1, \ldots, a_d \) a reduced weighted algebraic basis of the Lie algebra \( g \) of \( G \) and \( m \in 2wN \). Let \( C \) a form of order \( m_0 \) with \( m_0 \leq m \). The following conditions are equivalent.

I. The order of the form \( C \) equals \( m \) and the operator \( dL_{G_0}(P) \) is hypoelliptic, where \( P \) is the principal part of \( C \).
II. There exists a $c > 0$ such that
\[ \| A_i \varphi \|_2 \leq \varepsilon^{m-w_i} \| dL_G(C) \varphi \|_2 + c \varepsilon^{-w_i} \| \varphi \|_2 \]
uniformly for all $\varepsilon \in (0,1]$, $\varphi \in C^\infty_c(G)$ and $i \in \{1, \ldots, d'\}$.

III. There exist $c > 0$ and a neighbourhood $V$ of the identity of $G$ such that
\[ \| A_i \varphi \|_2 \leq \varepsilon^{m-w_i} \| dL_G(C) \varphi \|_2 + c \varepsilon^{-w_i} \| \varphi \|_2 \]
uniformly for all $\varepsilon \in (0,1]$, $\varphi \in C^\infty_c(V)$ and $i \in \{1, \ldots, d'\}$.

**Proof** $I \Rightarrow II.$ Suppose $dL_{G_0}(P)$ is hypoelliptic and $m = m_0$. Consider the form $C_1 = C_i^t C$. The principal part of $C_1$ is $P^t P$ and clearly $dL_{G_0}((P^t P)) = dL_{G_0}(P^t P)$ is a positive Rockland operator on $L^2(G_0)$. Hence the form $C_1$ is a weighted subcoercive form by Theorem 10.1. So by Theorems 9.1.III and 9.2.II there exist $c, \lambda > 0$ such that
\[ \| A_i \varphi \|_2 \leq \varepsilon^{m-w_i} \| (\lambda I + dL_G(C_1))^{1/2} \varphi \|_2 + c \varepsilon^{-w_i} \| \varphi \|_2 \]
uniformly for all $\varepsilon > 0$, $\varphi \in C^\infty_c(G)$ and $i \in \{1, \ldots, d'\}$. Since $dL_G(C_1)$ is the generator of a bounded semigroup it follows from [Rob2], Lemma II.3.2, that there exists a $c' > 0$ such that $\| (\lambda I + dL_G(C_1))^{1/2} \varphi \|_2 \leq \| (dL_G(C_1))^{1/2} \varphi \|_2 + c' \| \varphi \|_2$, uniformly for all $\varphi \in C^\infty_c(G)$. Then
\[ \| A_i \varphi \|_2 \leq \varepsilon^{m-w_i} \| dL_G(C_1) \varphi \|_2 + (c \varepsilon^{-w_i} + c' \varepsilon^{-w_i}) \| \varphi \|_2 \]
\[ = \varepsilon^{m-w_i} \| dL_G(C) \varphi \|_2 + (c \varepsilon^{-w_i} + c' \varepsilon^{-w_i}) \| \varphi \|_2 \]
from which Condition II follows.

The implication $II \Rightarrow III$ is trivial, so it remains to prove $III \Rightarrow I$. Temporarily, define the form $P: J(d') \rightarrow C$ by
\[ P(\alpha) = \begin{cases} C(\alpha) & \text{if } \| \alpha \| = m, \\ 0 & \text{if } \| \alpha \| < m. \end{cases} \]
Then $P$ is the principal part of the form $C$ if $m = m_0$, but $P = 0$ if $m_0 < m$. We use the notation of Section 3. In particular, $W$ is the set constructed in Lemma 3.3.V. We may assume that $\exp W \subset V$.

First, the bounds in Condition III can be rephrased as
\[ \varepsilon^{w_i} \left( \int_W da (a) |(X_i \psi)(a)|^2 \right)^{1/2} \leq \varepsilon^m \left( \int_W da (a) \sum_{\| \alpha \| \leq m} c_{\alpha}(X^\alpha \psi)(a)^2 \right)^{1/2} + c \left( \int_W da (a) |\psi(a)|^2 \right)^{1/2} \]
for all $\psi \in C^\infty_c(W)$, $\varepsilon \in (0,1]$ and $i \in \{1, \ldots, d'\}$. Next fix $\psi \in C^\infty_c(W)$. Let $t \in (0,1]$. Replacing $\varepsilon$ by $ct$ and $\psi$ by $\psi_{t^{-1}}$ in the previous inequality gives
\[ \varepsilon^{w_i} \left( \int_W da (a) t^{w_i} |(X_i \psi_{t^{-1}})(a)|^2 \right)^{1/2} \leq \varepsilon^m \left( \int_W da (a) \sum_{\| \alpha \| \leq m} t^{-\| \alpha \|} c_{\alpha} t^{\| \alpha \|}(X^\alpha \psi_{t^{-1}})(a)^2 \right)^{1/2} + c \left( \int_W da (a) |\psi_{t^{-1}}(a)|^2 \right)^{1/2} \]
67
for all \( \epsilon \in (0,1] \) and \( i \in \{1, \ldots, d'\} \). (Note that the integrals only need to be carried out over \( \gamma_i(W) \).) Changing variables and dividing by \( t^{D/2} \) then gives the estimates

\[
\epsilon^{m_n} \left( \int_W da \sigma(\gamma_i(a)) |t^{D/2} \psi_{i-1}(\gamma_i(a))|^2 \right)^{1/2}
\leq \epsilon^m \left( \int_W da \sigma(\gamma_i(a)) \sum_{|\alpha| \leq m} \|c_\alpha t^{D/2} \psi_{i-1}(X^\alpha \psi_{i-1})(\gamma_i(a)) |^2 \right)^{1/2}
+ c \left( \int_W da \sigma(\gamma_i(a)) |t^{D/2} \psi_{i-1}(\gamma_i(a))|^2 \right)^{1/2}
\]

for all \( \epsilon \in (0,1] \) and \( i \in \{1, \ldots, d'\} \). Therefore by Corollary 3.7 one deduces that

\[
\epsilon^{m_n} \left( \int_{\mathfrak{g}} da |(X_i^{(0)} \psi)(a)|^2 \right)^{1/2}
\leq \epsilon^m \left( \int_{\mathfrak{g}} da \sum_{|\alpha| = m} c_\alpha (X^{(0)^\alpha} \psi)(a)|^2 \right)^{1/2} + c \left( \int_{\mathfrak{g}} da |\psi(a)|^2 \right)^{1/2}
\]

for all \( \psi \in C^\infty_c(W) \), \( \epsilon \in (0,1] \) and \( i \in \{1, \ldots, d'\} \).

Next let \( \psi \in C^\infty_c(\mathfrak{g}) \). There exists \( r \geq 1 \) such that \( \text{supp} \psi_r \subset W \). Then applying the previous inequality to \( \psi_r \) gives

\[
(\epsilon r)^{m_n} \left( \int_{\mathfrak{g}} da |(X_i^{(0)} \psi)(a)|^2 \right)^{1/2}
\leq (\epsilon r)^m \left( \int_{\mathfrak{g}} da \sum_{|\alpha| = m} c_\alpha (X^{(0)^\alpha} \psi)(a)|^2 \right)^{1/2} + c \left( \int_{\mathfrak{g}} da |\psi(a)|^2 \right)^{1/2}
\]

and choosing \( \epsilon = r^{-1} \) finally gives

\[
\|A_i^{(0)} \varphi\| \leq \|dL_{G_0}(P)\varphi\| + c \|\varphi\|
\]

uniformly for all \( \varphi \in C^\infty_c(G) \) and by density, for all \( \varphi \in L_{2;\infty}(G_0) \).

Now one can argue as in the proof of Proposition 10.2. By scaling one has

\[
\delta^{m_n} \|A_i^{(0)} \varphi\| \leq \delta^n \|dL_{G_0}(P)\varphi\| + c \|\varphi\|
\]

for all \( \delta > 0 \) and \( \varphi \in L_{2;\infty}(G_0) \) and by reduction (Lemma 10.4.II)

\[
\delta^{m_n} \|dU(a_i) x\| \leq \delta^n \|dU(P)x\| + c \|x\|
\]

for each irreducible unitary representation \( U \) of \( G_0, x \in X_{\infty}(U) \) and \( \delta > 0 \). Now suppose \( U \) is non-trivial, \( dU(P)x = 0 \) and \( x \neq 0 \). Then \( \delta^{m_n} \|dU(a_i) x\| \leq c \|x\| \) and hence \( \|dU(a_i) x\| = 0 \) for all \( i \in \{1, \ldots, d'\} \). Since \( a_1, \ldots, a_{d'} \) is an algebraic basis, this implies that \( x = 0 \), which is a contradiction. So \( dL_{G_0}(P) \) is a Rockland operator and therefore hypoelliptic by the Helffer–Nourrigat theorem. In particular, \( P \neq 0 \) and hence the order of the form \( C \) equals \( m \).

This proposition has immediate implications for subcoercive forms.
Corollary 10.8 Let $G$ be a connected Lie group, $a_1, \ldots, a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of $G$. Suppose $m \in 2\omega\mathbb{N}$ and let $C$ be an $m$-th order form with principal part $P$. Then the following conditions are equivalent.

I. The form $C$ is $G$-weighted subcoercive.

II. There are $c, \mu > 0$ and an open neighbourhood $V$ of the identity of $G$ such that

\[ \mu \varepsilon^{2\omega \mu} \| A_i \varphi \|_2^2 \leq \varepsilon^m \Re \langle \varphi, dL_G(C)\varphi \rangle + c \| \varphi \|_2^2 \]

for all $\varphi \in C_c^\infty(V)$, all $\varepsilon \in (0, 1]$ and all $i \in \{1, \ldots, d'\}$.

Proof The implication $I \Rightarrow II$ follows from the inequalities

\[ \mu \varepsilon^{2\omega \mu} \| A_i \varphi \|_2^2 \leq \varepsilon^m \| A_i^{(2\omega \mu)} \varphi \|_2^2 + c \| \varphi \|_2^2 \]

which are valid for all $\varepsilon \in (0, 1]$ (see, for example [Rob2], Lemma III.3.3). The converse implication $II \Rightarrow I$ follows from

\[
\mu \varepsilon^{2\omega \mu} \| A_i \varphi \|_2^2 \leq \varepsilon^m \| A_i^{(2\omega \mu)} \varphi \|_2^2 + c \| \varphi \|_2^2 \\
\leq \varepsilon^m (\varepsilon^{-m} \| \varphi \|_2^2 + \varepsilon^m \| dL_G(\mathbb{R}C)\varphi \|_2^2) + c \| \varphi \|_2^2 \\
\leq (\varepsilon^m \| dL_G(\mathbb{R}C)\varphi \|_2 + (1 + c) \| \varphi \|_2^2)^2.
\]

One then deduces from Proposition 10.7 that $dL_G(\mathbb{R}P)$ is hypoelliptic, where $P$ is the principal part of $C$. But the contraction process also shows that $dL_G(\mathbb{R}P)$ is a positive operator. Therefore $C$ is $G$-weighted subcoercive by Theorem 10.1.

This corollary shows that Conditions I and II in Theorem 1.1 are equivalent in case of a reduced weighted algebraic basis. The second implication of Proposition 10.7 is the equivalence of Conditions I and III in Theorem 1.1 for reduced bases.

Theorem 10.9 Let $G$ be a connected Lie group, $a_1, \ldots, a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of $G$. Suppose $m \in 2\omega\mathbb{N}$ and let $C$ be an $m$-th order form with principal part $P$. Then the following conditions are equivalent.

I. The form $C$ is $G$-weighted subcoercive.

II. The closure of $dL_G(C)$ generates a holomorphic semigroup $S$ on $L^2(G)$ which is quasi-contractive in an open sector $\Lambda(\theta) \subset \mathbb{C}$ with $\theta \in (0, \pi/2)$. Moreover, $S_t$ maps $L^2(G)$ into $D(A_i)$ for all $i \in \{1, \ldots, d'\}$ and there exist $c, \omega > 0$ such that

\[ \| A_i S_t \|_{2 \rightarrow 2} \leq c t^{-\omega/m} e^{\omega t} \]

for all $t > 0$.

Proof We only need to prove $II \Rightarrow I$. It follows from the bounds on the derivatives of the semigroup, by Laplace transformation, that there exists a $c > 0$ such that

\[ \| A_i \varphi \|_2 \leq \varepsilon^{-m \omega} \| dL_G(C)\varphi \|_2 + c e^{-\omega \epsilon} \| \varphi \|_2 \]

uniformly for all $\varepsilon \in (0, 1]$, $\varphi \in C_c^\infty(G)$ and $i \in \{1, \ldots, d'\}$ Hence the operator $dL_G(\mathbb{R}P)$ is hypoelliptic by Proposition 10.7. But it follows from quasi-contractivity that $e^{i\alpha (dL_G(C) - \omega \mathbb{I})}$ generates a contraction semigroup, if $\omega$ is large enough, uniformly for all $\alpha \in (-\theta, \theta)$,
and hence, by the Lumer-Phillips theorem, that \( \Re(\varphi, e^{i\alpha}(dL_G(C) - \omega I)\varphi) \geq 0 \) for all \( \varphi \in L^2_{2m}(G) \). Then applying the contraction process it follows that \( \Re(\varphi, e^{i\alpha}dL_{G_0}(P)\varphi) \geq 0 \) for all \( \varphi \in L^2_{2m}(G_0) \) and \( \alpha \in (-\theta, \theta) \). The proof of this implication is a variation of the proofs used in Propositions 4.6 and 10.7. Then \( |(\varphi, dL_{G_0}(P)\varphi)| \leq M(\varphi, dL_{G_0}(\mathcal{R} P)\varphi) \) for all \( \varphi \in L^2_{2m}(G_0) \), where \( M = \cot \theta \). Hence, by the reduction theorem, it follows that

\[
|(x, dU(\mathcal{R} P)x)| \leq M(x, dU(\mathcal{R} P)x)
\]

for each unitary irreducible representation \( U \) of \( G_0 \) and all \( x \in \mathcal{X}_\infty(U) \), and then, by density, for all \( x \in \mathcal{X}'_m(U) \). But it then follows from [Kat] p. 310 that

\[
|(y, dU(\mathcal{R} P)x)| \leq M(x, dU(\mathcal{R} P)x)^{1/2}(y, dU(\mathcal{R} P)y)^{1/2}
\]  
(36)

for all \( x, y \in \mathcal{X}'_m(U) \).

We shall prove that \( \mathcal{R} P \) is a Rockland form. Let \( U \) be a non-trivial irreducible unitary representation of \( G_0 \), \( x \in \mathcal{X}_\infty(U) \) and suppose \( dU(\mathcal{R} P)x = 0 \). Then it follows from (36) that \( (y, dU(\mathcal{R} P)x) = 0 \) for all \( y \in \mathcal{X}'_m(U) \). Since \( \mathcal{X}'_m(U) \) is dense in \( \mathcal{X} \) one establishes that \( dU(\mathcal{R} P)x = 0 \). Therefore \( dU(P)x = 0 \) and thus \( x = 0 \) since \( P \) is a Rockland form. Hence \( \mathcal{R} P \) is positive Rockland form and \( C \) is \( G \)-weighted subcoercive by Theorem 10.1. This completes the proof of the theorem.

If the principal part of the form \( C \) is symmetric one can weaken the assumptions of the previous theorem. One only needs quasi-contractivity of the semigroup on the positive real line.

**Theorem 10.10** Let \( G \) be a connected Lie group, \( a_1, \ldots, a_d' \) a reduced weighted algebraic basis of the Lie algebra \( \mathfrak{g} \) of \( G \) and \( G_0 \) the corresponding homogeneous contraction of \( G \). Suppose \( m \in 2wN \) and let \( C \) be an \( m \)-th order form with symmetric principal part \( P \), i.e., \( P = Pt \). The following conditions are equivalent.

I. The form \( C \) is \( G \)-weighted subcoercive.

II. The closure of \( dL_G(C) \) generates a continuous, quasi-contraction, semigroup \( S \) on \( L^2(G) \) which maps into \( D(A_i) \) for all \( i \in \{1, \ldots, d'\} \). Moreover, there exist \( c, \omega > 0 \) such that \( \|A_iS_t\|_{L^2} \leq ct^{-w_i/m}e^{\omega t} \) for all \( t > 0 \).

**Proof** It follows as in the proof of Theorem 10.9 that \( \mathcal{R} P = P \) is hypoelliptic and \( \text{Red}L_{G_0}(P) \geq 0 \). So \( \text{Red}L_{G_0}(P) \) is a positive Rockland operator.

### 11 General algebraic bases

In Section 2 we passed from a weighted algebraic basis to a reduced weighted algebraic basis and the subsequent results have been largely formulated in terms of reduced bases. In this section we examine the passage from the reduced basis back to the original basis and the extension of the foregoing results to general weighted bases.

Let \( a_1, \ldots, a_{d'} \) be a weighted algebraic basis with weights \( w_1, \ldots, w_{d'} \) and filtration \( (\mathcal{G}_\lambda)_{\lambda \geq 0} \). Assume \( \bigcap_{i=1}^{d'} w_i\mathbb{N} \neq \emptyset \). We can define a distance \( d(\cdot ; \cdot) \) and modulus \( \|\cdot\|_\omega = \|\cdot\|' \)
on $G$ similarly to the definitions with respect to a reduced weighted algebraic basis in the beginning of Section 6.

Next, Proposition 2.1 established that there exists a reduced weighted algebraic basis $b_1, \ldots, b_{d''}$ with weights $v_1, \ldots, v_{d''}$ such that $\{b_1, \ldots, b_{d''}\} \subseteq \{a_1, \ldots, a_{d'}\}$ and $v_i = w_j$ if $b_i = a_j$. Moreover, the filtrations corresponding to the algebraic basis $a_1, \ldots, a_{d'}$ and the reduced basis $b_1, \ldots, b_{d''}$ coincide. The reduced basis is a subset of the original basis obtained by eliminating those directions $a_j$ such that $a_j \in g_m$. But the moduli $| \cdot |'(a_j)$ and $| \cdot |'(b_j)$ are equivalent.

Lemma 11.1 There exists a $c \geq 1$ such that

$$c^{-1} | g \cdot |'(b) \leq | g \cdot |'(a) \leq c | g \cdot |'(b)$$

for all $g \in G$.

Proof Obviously $| g \cdot |'(a) \leq | g \cdot |'(b)$ for all $g \in G$. Next, for all $i \in \{1, \ldots, d'\}$ let $w'_i = \min\{\lambda > 0 : a_i \in g_\lambda\}$. Then one easily proves by induction on the weights of the filtration that the filtration $(g_\lambda)_{\lambda \geq 0}$ equals the filtration corresponding to the weighted algebraic basis $a_1, \ldots, a_{d'}$ with weights $w'_1, \ldots, w'_{d'}$. So $a_1, \ldots, a_{d'}$ with weights $w'_1, \ldots, w'_{d'}$ is a reduced weighted algebraic basis. Let $| \cdot |'(a)$ denote the modulus with respect to this weighted algebraic basis. Then obviously $| g \cdot |'(a) \leq | g \cdot |'(a)$ for all $g \in G$ with $| g \cdot |'(a) < 1$. But the moduli $| \cdot |'(a)$ and $| \cdot |'(b)$ are equivalent by Corollary 6.5. Therefore the lemma follows for small $g$. For large $g$ the distances are comparable by [VSC] Proposition III.4.2.

Corollary 11.2 There exists a $c \geq 1$ such that

$$c^{-1} \delta^{D'} \leq | B'_\delta | \leq c \delta^{D'}$$

for all $\delta \in (0, 1]$, where

$$D' = \sum_{\lambda > 0} \lambda \dim (g_\lambda / g_\lambda)$$

is the local dimension and $B'_\delta$ is the ball with radius $\delta$.

Proof This follows from Proposition 6.1.II and the previous lemma.

The reduced weighted algebraic basis $b_1, \ldots, b_{d''}$ is constructed from the weighted algebraic basis by deleting the 'overweight' directions (see Proposition 2.1). But these directions have a representation

$$a_j = \sum_{a \in J^+(d'')} c_{ja} b[a]$$

where we have used $\| \cdot \|_v$ to denote the length of the multi-index with respect to the weights $v_i$ of the reduced basis. On the other hand, such a representation also exists if $a_j$ is an element of $\{b_1, \ldots, b_{d''}\}$. Hence in a continuous representation $(X, G, U)$ of the group

$$A_j = \sum_{a \in J^+(d'')} c_{ja} B[a]$$

with $\| a \|_v \leq w_j$,
where \( B_j = dU(b_j) \). Therefore, expanding the commutators,
\[
A_j = \sum_{\alpha \in J^j(d'')} c^j_\alpha B^\alpha
\]
for suitable \( c^j_\alpha \in \mathbb{R} \).

If \( C: J(d') \to C \) is an \( m \)-th order form then it follows from (37) that there exist \( c^\beta_\rho \in C \) such that
\[
dU(C) = \sum_{\alpha \in J(d')} \sum_{\beta \in J(d'')} c_\alpha A^\alpha = \sum_{\beta \in J(d'')} c^\beta_\rho B^\beta
\]
where we have now used \( \| \cdot \|_w \) to denote the weighted length of the multi-indices with respect to the weights \( w_i \) of the original basis. The form \( C \) is an \( m \)-th order form with respect to the weighted algebraic basis \( a_1, \ldots, a_d \) and we use the notation \( C = C_a \) to denote the dependence on the basis. Further let \( C_b: J(d'') \to C \) be the \( m \)-th order form with the coefficients \( c^\beta_\rho \) entering on the right hand side of (38). Then (38) states that \( dU(C_a) = dU(C_b) \). The form \( C_b \) has order less than or equal to \( m \) with respect to the weighted algebraic basis \( b_1, \ldots, b_d \) and weights \( v_1, \ldots, v_d \).

We temporarily add a subscript \( a \) and \( b \) to the spaces \( \mathcal{X}_{n}^j(U) \) and the (semi)norms \( \| \cdot \|_{u,n} \) and \( N_{u,n} \) to denote the dependence of the weighted algebraic basis. Obviously \( \mathcal{X}_{a,n}^j(U) \subseteq \mathcal{X}_{b,n}^j(U) \), \( N_{b,U,n}(x) \leq N_{a,U,n}(x) \) and \( \| x \|_{b,U,n} \leq \| x \|_{a,U,n} \) for all \( n \in [0, \infty) \) and \( x \in \mathcal{X}_{a,n}^j(U) \), since the \( b_i \) are a subset of the \( a_i \) with the same weight. Next suppose \( m \in w, \forall \) for all \( i \in \{1, \ldots, d\} \) and set
\[
v = \min \{ x \in [1, \infty) : x \in v_i N \text{ for all } i \in \{1, \ldots, d\} \}
\]
\[
w = \min \{ x \in [1, \infty) : x \in w_i N \text{ for all } i \in \{1, \ldots, d\} \}.
\]
Then \( w \in vN \). Let \( k \in \mathbb{N} \). It follows from (38) that \( \mathcal{X}_{b,kv}^j(U) \subseteq \mathcal{X}_{a,kv}^j(U) \), \( N_{b,U,kv}(x) \leq c \| x \|_{b,U,kv} \) and hence \( \| x \|_{a,U,kv} \leq c' \| x \|_{b,U,kv} \) for some \( c, c' > 0 \), uniformly for all \( x \in \mathcal{X}_{a,kv}^j(U) \). So the spaces \( \mathcal{X}_{a,kv}^j(U) \) and \( \mathcal{X}_{b,kv}^j(U) \) are equal, with equivalent norms. Moreover, it follows from Theorem 9.1.17 that there exists a \( c > 0 \) such that
\[
N_{a,U,kv}(x) \leq c (N_{b,U,kv}(x) + \| x \|_v)
\]
for all \( x \in \mathcal{X}_{a,kv}^j(U) \).

Now Theorem 1.1 of the introduction follows as a corollary of the results we have established for reduced weighted bases.

**Proof of Theorem 1.1** Let \( C \) be an \( m \)-th order form and assume that the weights \( w_i \) satisfy \( m/w_i \in 2\mathbb{N} \).

If the weighted algebraic basis \( a_1, \ldots, a_d \) is a reduced weighted algebraic basis then the theorem follows from Proposition 4.6, \( \text{prop} \Rightarrow 3 \), Theorems 7.2, 8.2, 10.9 and Corollary 10.8.

If, however, \( a_1, \ldots, a_d \) is not a reduced weighted basis one can proceed as above and introduce the reduced weighted subbasis \( b_1, \ldots, b_d \). Then \( C = C_a \) is the given \( m \)-th order form. Let \( C_b \) be the associated form of order less than or equal to \( m \) with respect to the weighted algebraic basis \( b_1, \ldots, b_d \). We say that \( C_b \) satisfies Condition I of Theorem 1.1 if there are \( \mu, \nu > 0 \) and an open neighbourhood \( V \) of the identity of \( G \) such that
\[
\text{Re}(\varphi, dL_G(C_b)\varphi) \geq \mu (N'_{b,2m/2}(\varphi))^2 - \nu \| \varphi \|^2_2
\]
for all \( \varphi \in C^\infty_c(V) \). Similarly, we say that the form \( C_b \) satisfies Conditions II, III or IV of Theorem 1.1 if the particular condition is valid for the form \( C_b \), the algebraic basis \( b_1, \ldots, b_{d''} \), weights \( v_1, \ldots, v_{d''} \) and infinitesimal generators \( B_1, \ldots, B_{d''} \).

We first show that the order of the form \( C_b \) equals \( m \) if the form \( C_b \) satisfies one of the Conditions I-IV of Theorem 1.1. Obviously one has the implications I \( \Rightarrow \) II and IV \( \Rightarrow \) III for the form \( C_b \). The proof is the same as in Section 1. But if the form \( C_b \) satisfies Condition II or III then there exist \( c > 0 \) and a neighbourhood \( V \) of the identity of \( G \) such that

\[
\|B_i \varphi\|_2 \leq e^{m-w_i} \|d\lambda(C_b)\varphi\|_2 + c e^{-w_i} \|\varphi\|_2
\]

uniformly for all \( \varepsilon \in (0,1] \), \( \varphi \in C^\infty_c(V) \) and \( i \in \{1, \ldots, d''\} \). This follows as in the proof of Corollary 10.8 and Theorem 10.9. Therefore the order of the form \( C_b \) equals \( m \) by Proposition 10.7. Hence the Conditions I-IV are all equivalent for the form \( C_b \).

Now we prove Theorem 1.1 for the form \( C_a \). If \( C_a \) satisfies Condition I, i.e., \( C_a \) is a \( G \)-weighted subcoercive form, then in the left regular representation on \( L^2(G) \) one has

\[
\text{Re}(\varphi, d\lambda(C_b)\varphi) = \text{Re}(\varphi, d\lambda(C_a)\varphi) \\
\geq \mu (N'_{a,2,m/2}(\varphi))^2 - \nu \|\varphi\|^2_2 \geq \mu (N'_{b,2,m/2}(\varphi))^2 - \nu \|\varphi\|^2_2
\]

for all \( \varphi \in C^\infty_c(V) \), with \( V \) the open neighbourhood of the identity occurring in the definition of the subcoercivity of \( C_a \). Hence \( C_b \) satisfies Condition I and \( C_b \) is an \( m \)-th order weighted subcoercive form. Since the \( b_i \) are a subset of the \( a_i \) with the same weight Conditions II, III and IV for \( C_a \) obviously imply the same condition for the form \( C_b \).

Conversely, if \( C_b \) is an \( m \)-th order weighted subcoercive form then it follows from (39) that the form \( C_a \) is weighted subcoercive. Then Condition II is also valid for \( C_a \), as we have proved already in Section 1. In any representation \((\mathcal{X}, G, U)\) the closure of \( dU(C_b) \) generates a semigroup which is holomorphic in an open sector containing \( \Lambda(\theta_{C_b,G}) \). Moreover, it has a representation independent kernel. Since \( dU(C) = dU(C_a) = dU(C_b) \) this establishes the generator property for \( dU(C) \). The Gaussian bounds for the semigroup kernel follow from (37) and the bounds on the derivatives of the kernel with respect to the \( B^\beta \). The bounds on the derivatives of the semigroup in Condition III for \( C_a \) follow again by a quadrature estimate. This completes the proof of Theorem 1.1.

It should again be emphasized that Theorem 1.1 is valid for any Lie group \( G \) and any weighted algebraic basis of the Lie algebra \( \mathfrak{g} \) of \( G \). Although most of the foregoing material involves reduced weighted algebraic bases and the corresponding homogeneous contraction \( G_0 \) the final result is independent of these concepts.

In Section 1 we defined \( \|x\|_n = 0 \) if \( n \notin \{\|a\|_w : a \in J(d')\} \) to avoid complications in various proofs. We now drop this condition for the weighted algebraic basis \( a_1, \ldots, a_{d''} \).

For \( n \in [0, \infty) \) define \( \|\cdot\|_n : \mathcal{X}'_{b,n}(U) \rightarrow [0, \infty) \) by

\[
\|\cdot\|_n = \max_{\alpha \in J(d')} \|A^\alpha x\|_{n\alpha}
\]

Then \( (\mathcal{X}'_{b,n}(U), \|\cdot\|_n) \) is a normed space and the two spaces \( (\mathcal{X}'_{b,n}(U), \|\cdot\|_n) \) and \( (\mathcal{X}'_{a,n}(U), \|\cdot\|_n) \) are equal, with equivalent norms, if \( n \in vN \). Hence all conclusions of Theorems 9.1 and 9.2 and Corollaries 8.3, 9.3 and 9.4 are valid if \( C \) is a \( G \)-weighted
sub coercive form with respect to the weighted algebraic basis and the norms \[ \| \cdot \|_n \] on the space \( \chi'_{a,n}(U) \). Most statements follow directly from the comparable statement for the reduced weighted algebraic basis \( b_1, \ldots, b_d \), so we indicate the differences. In Corollary 8.3.III one fixes \( k \in [0, \infty) \). Let \( k_0 = \max \{ \| \alpha \|_n : \alpha \in J(d'), \| \alpha \|_n \leq k \} \). Then it follows from (38) that \( \chi'_{b,k_0}(U) \subseteq \chi'_{a,k}(U) \) and \[ \| x \|_{b, U, k_0} \leq c' \| x \|_{b, U, k_0} \] for some \( c' > 0 \). Therefore,

\[ \| S \|_{b, U, k_0} \leq c' \| S \|_{b, U, k_0} \leq c \| x \| \leq c \| x \| \leq c' \| x \|_{b, U, k_0} \]

for a suitable \( \omega' \geq \omega \). Next, in Corollary 8.3.III one has \[ \| x \|_{b, U, k_0} \leq e^{mn-k_0} \| H^n x \| + \| x \|_{b, U, k_0} \] which is equivalent to the J-interpolation inclusion \( (\chi', D(H^n))_{k_0/\langle mn \rangle, 1; \chi, D(H^n))_{k_0/\langle mn \rangle, 1} \subseteq \chi'_{b,k_0}(U) \) (see [Tri] Lemma 1.10.1(a)). Therefore one has the following continuous inclusions

\[ \chi'_{b,k_0}(U) \subseteq \chi'_{a,k}(U) \]

from which the new version of Corollary 8.3.III for the algebraic basis \( a_1, \ldots, a_d \) and the new norm follows. Theorem 9.1.III can be proved similarly.

Earlier work [ElR7] on unweighted bases and subcoercive operators was based on the assumption of \( \bar{G} \)-coercivity where \( \bar{G} \) denotes the Rothschild–Stein local approximant of \( G \), i.e., \( \bar{G} \) is the nilpotent Lie group with \( d' \) generators which is free of step \( r \) where \( d' \) and \( r \) are the number of elements and the rank of the algebraic basis, respectively. Thus \( \bar{G} = G(d', r, 1, \ldots, 1) \).

The next proposition allows one to deduce that the earlier results [ElR7] are a corollary of Theorem 1.1.

**Proposition 11.3** Let \( G \) be a connected nilpotent Lie group of rank \( r \) and \( a_1, \ldots, a_{d'} \) a weighted algebraic basis of the Lie algebra \( g \) of \( G \) with weights \( w_1, \ldots, w_{d'} \). Let \( C : J(d') \rightarrow C \) be a form of order \( m \) with \( m \in 2wN \). If \( C \) is a \( G(d', r, w_1, \ldots, w_{d'}) \)-weighted subcoercive form then \( C \) is a \( G \)-weighted subcoercive form.

**Proof** Let \( \tilde{g} = g(d', r, w_1, \ldots, w_{d'}) \) be the weighted nilpotent Lie algebra with generators \( a_1, \ldots, a_{d'} \) and weights \( w_1, \ldots, w_{d'} \) which is free of step \( r \) and \( \tilde{G} = G(d', r, w_1, \ldots, w_{d'}) \) the corresponding connected simply connected Lie group (see Example 2.9). Then the Lie algebra generated by the operators \( dL_C(a_i)|_{L_2(G)} \), \( dL_C(a_i)|_{L_2(x\in G)} \) is nilpotent of step \( r \) and has \( d' \) generators. So there exists an algebra homomorphism \( T : \tilde{g} \rightarrow \text{Hom}(L_2(G)) \) such that \[ T(a_i) = dL_C(a_i)|_{L_2(G)} \] . Then \( T \) is a representation of \( \tilde{g} \) in the Hilbert space \( L_2(G) \) by skew-adjoint operators such that the (dense) set of analytic vectors for \( L_2 \) is a set of analytic vectors for \( T(a_i) \) for all \( i \in \{ 1, \ldots, d' \} \). So, by [Sim] Corollary 2, there exists a unitary representation \( U \) of \( \tilde{G} \) in \( L_2(G) \) such that \( L_2(G) \subseteq \langle (L_2(G)) \rangle \). Then, by Theorem 9.2.III, it follows that there exists \( \mu, \nu > 0 \) such that

\[ \text{Re}(\varphi, dL_C(C)\varphi) = \text{Re}(\varphi, dU(C)\varphi) \geq \mu (N^2_{L_2(G), m/2})^2 - \nu \| \varphi \|_2^2 \]

for all \( \varphi \in \langle (L_2(G)) \rangle \) and, in particular, for all \( \varphi \in L_2(G) \). One can now immediately recover all the main results of [ElR7].
Corollary 11.4 Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \) and \( a_1, \ldots, a_d \) an (unweighted) algebraic basis of rank \( r \) of \( \mathfrak{g} \). Let \( C: J(d') \to C \) be a subcoercive form of order \( m \) and step \( r \) (see [EIR3]). Let \((X,G,U)\) be a representation of \( G \). Then the closure of the operator \( dU(C) \) generates a holomorphic semigroup \( S \) which is holomorphic in an open sector containing \( \Lambda(\theta_{\mathbb{G},d'}) \), where \( \mathbb{G} = G(d',r,1,\ldots,1) \). Moreover, \( S \) has a representation independent kernel which satisfies Gaussian type bounds of order \( m \).

Proof If \( C \) is a subcoercive form of order \( m \) and step \( r \) then \( C \) is a \( G(d',r,1,\ldots,1) \)-weighted subcoercive form (see Example 4.4). Since \( G_0 \) is a homogeneous (nilpotent) Lie group of rank \( r \) by Lemma 3.13 it follows from Proposition 11.3 that \( C \) is a \( G_0 \)-weighted subcoercive form. Hence it is a \( G \)-weighted subcoercive form. The corollary follows immediately.

The final example shows that the assumptions of [EIR7] are strictly stronger than those of weighted subcoercivity.

Example 11.5 Reconsider the five-dimensional Heisenberg group \( G \) with Lie algebra \( \mathfrak{h}_2 \) from Example 3.12. Thus one has a basis \( a_1, \ldots, a_5 \) with \([a_1, a_2] = [a_3, a_4] = a_5 \). Take the weighted algebraic basis \( a_1, \ldots, a_4 \) with all weights equal to one. Then \( \mathbb{G} = G(4,2,1,1,1,1) \) has dimension 10 and the Lie algebra \( \mathfrak{g} \) has a basis \( \{\tilde{a}_1, \ldots, \tilde{a}_4\} \cup \{\tilde{a}_{ij} : 1 \leq i < j \leq 4\} \). The commutation relations are \([\tilde{a}_i, \tilde{a}_j] = \tilde{a}_{ij} \) if \( 1 \leq i < j \leq 4 \). Let \( C_1 \) be the form such that

\[
dU(C_1) = -A_1^2 - A_2^2 - A_3^2 - A_4^2
\]

for any representation. Further let \( C_2 \) be the form such that

\[
dU(C_2) = dU(C_1) - \lambda i^{-1}([A_1, A_2] - [A_3, A_4])
\]

where \( \lambda \) is an eigenvalue of the operator \( dV(C_1) \), and \( V \) is an irreducible unitary representation of \( \mathbb{G} \) with \( dV(\tilde{a}_{12} - \tilde{a}_{34}) = iI \). Then \( C_2 \) is not a subcoercive form of step 2 since \( (x_\lambda, dV(C_2)x_\lambda) = 0 \), where \( x_\lambda \) is an eigenvector of \( dV(C_1) \) with eigenvalue \( \lambda \) (see [EIR7] Corollary 3.5). On the other hand, \( dL_G(C_2) = dL_G(C_1) \), so \( C_2 \) is a \( G \)-weighted subcoercive form.

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