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On the definition of homomorphism

by

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0. Introduction

Generalization of the concept of homomorphism to other than algebraic structures is not unambiguously determined. It is no problem how homomorphism has to be defined for groups, rings, lattices etc., in general for all algebraic structures, where the algebraic operations are total functions. Already in algebraic structures with partial operations it is not obvious how homomorphism has to be defined. As an example we take the case that there is one binary operation which is partially defined and written as a multiplication. For the definition of a homomorphic mapping $f: A \rightarrow B$ the following choice is often made:

If $a \in A$, $b \in A$ and $ab$ is defined, then $f(a)f(b)$ is defined and $f(a)f(b) = f(ab)$ (cf. Bruck [2], p. 3).

A partial binary operation may be considered as a ternary relation. The definition above corresponds with the following definition for an $n$-ary relation $R$:

1. If $R_A(x_1, \ldots, x_n)$, then $R_B(f(x_1), \ldots, f(x_n))$.

Some instances, where this definition may be found are: Bell and Slomson [1], p. 73, Cohn [3], p. 190, Grätzer [4], p. 81 for partial operations and p. 224 for relations.

A strange aspect of this definition is, that extension of the relation in $B$ has no influence on the validity of the homomorphism. This validity is strongly determined by the relation in $A$; therefore we call this homomorphism a domain homomorphism.

As a counterpart to this we may think of the following definition:

If $R_B(f(x_1), \ldots, f(x_n))$, then $R_A(x_1, \ldots, x_n)$,

which we could call a range homomorphism. This concept is not very useful. We can imagine this if we translate it to the case of a partial binary operation:

If $f(a)f(b)$ is defined and there exists a $c$ such that $f(a)f(b) = f(c)$, then $ab$ is defined and $ab = c$. 
This means that those elements of the image of \( f \), which are a product of elements of the image of \( f \), are image of only one element. The definition may be improved as follows:

(2) If \( R_B(f(x_1), \ldots, f(x_n)) \), then there exist \( x_1', \ldots, x_n' \) such that \( f(x_j) = f(x_j') \) for \( j = 1, \ldots, n \) and \( R_A(x_1', \ldots, x_n') \).

Grätzer ([4], p. 81) calls a mapping satisfying (1) and (2) for partial operations a full homomorphism. Tarski ([5], p. 574) restricts himself to surjective homomorphisms with a definition, which in short notation reads: \( R_B = fR_A \). For surjective mappings this coincides with (1) and (2). For general mappings it would give:

\[
R_B(y_1, \ldots, y_n) \text{ iff there exist } x_1, \ldots, x_n \text{ such that } y_j = f(x_j) \text{ for } j = 1, \ldots, n \text{ and } R_A(x_1, \ldots, x_n),
\]

but this implies that \( R_B(y_1, \ldots, y_n) \) may only hold for \( y_1, \ldots, y_n \) lying in the image of \( f \), which is not desirable if \( f \) is not surjective. We remark that for relations Grätzer only gives the definition of domain homomorphism, but nevertheless uses the same concept of homomorphic image as Tarski, and not the image of a surjective domain homomorphism.

In order to get a general concept of homomorphism we take as a starting point the homomorphism theorem for algebraic structures with total operations. We use it in the following form, formulated for groups:

If \( f: A \to B \) is a homomorphism, the fibering of \( f \) is a quotient group \( A/K \) of \( A \), the image of \( f \) is a subgroup \( f(A) \) of \( B \) and the induced mapping \( A/K \to f(A) \) is an isomorphism.

\( A \to A/K \to f(A) \to B \) is the canonical decomposition of \( f \) and the properties of the theorem in turn characterize homomorphic mappings. The only three concepts needed are subgroup, quotient group and isomorphism.

We shall develop a concept of structure, where three corresponding concepts are fundamental. We shall adopt a terminology, in which "structure" denotes the type of system we are going to discuss: structure of groups, of topological spaces etc. The groups, topological spaces etc. themselves are called the objects of the structure. We assume, that to every object there corresponds a set, called its carrier. Some bijective mappings between
carriers of objects make objects isomorphic, some subsets of carriers of objects are carriers of subobjects and some quotient sets of carriers of objects are carriers of quotient objects. These concepts have to obey some reasonable axioms.

There are many examples of such structures. Algebraic structures such as groups in the first place with the usual concepts of isomorphism, subgroup and quotient group. Also topological spaces form a structure, where isomorphism is topological homeomorphism, a subobject is a subset with the relative topology and a quotient object is a quotient set with the quotient topology. In the structure of totally ordered sets every subset of the carrier of an object is ordered by the order of the object, but only quotient sets, for which all equivalence classes are convex in the order of the object, are carrier of a quotient object. Totally ordered sets are an example of a relational structure, which will also be treated in general. Many other examples of structures may be imagined.

The mapping \( f: A \rightarrow B \), where \( A \) and \( B \) are carriers of objects, will be defined to be a homomorphism between these objects, if the image of \( f \) is carrier of a subobject in \( B \), the fibering of \( f \) is carrier of a quotient object in \( A \) and the induced bijective mapping is an isomorphism of these objects. For algebraic structures this concept coincides with the usual one; for topological spaces the requirement is stronger than continuity (cf. theorems 5.1 and 5.2); for compact Hausdorff spaces it coincides with continuity. For relational structures it depends on how subobjects and quotient objects are defined. For subobjects it is fairly obvious how this has to be done, but for quotient objects this is not the case. We choose the definition in which a relation holds in a quotient set iff there exists a system of representatives from the equivalence classes, for which the relation holds. This means that we take the greatest relation, which in a reasonable sense is compatible with the given relation. The resulting concept of homomorphism is just a mapping satisfying (1) and (2) above.

One can imagine other reasonable definitions of quotient object in a relational structure; as an example we mention the definition in which a relation holds in a quotient set iff for all choices of representatives from the equivalence classes the relation holds. One may even admit only those
quotient sets as carrier of a quotient object, in which the validity of the relation for a system of representatives implies the validity of that relation for all systems of representatives from the same equivalence class. Our definition, however, is better adapted to the transition from relations to operations. It is important to bear in mind, that what a structure is, in our formalism depends on the choice of the definition of subobject and quotient object.

Operations may be considered as special cases of relations; n-ary operations correspond to (n+1)-ary relations. Intermediate concepts between operations and general relations are partial operations and multi-operations, the latter being the dual of the former (cf. definition 8.2). It is trivial that if a relation induces a partial operation, the corresponding relation in a subobject also induces a partial operation and dually if a relation induces a multi-operation, the corresponding relation in a quotient object also induces a multi-operation. If we consider an object of an algebraic structure as an object of a corresponding relational structure, it is possible that a subobject of this object in the relational structure does not correspond to a subobject in the algebraic structure, but its relations induce partial operations and similarly for quotient objects, where the relations induce multi-operations. The subobject and the quotient object occurring in the definition of homomorphism are isomorphic and therefore in both objects the relations induce operations, if this is the case for the given objects, so it makes no difference whether one takes homomorphism with respect to the operational or with respect to the relational definition, provided both are defined. For further details we refer to section 8, in particular theorem 8.6.

The following two questions about homomorphism are important:
1. Is a bijective homomorphism an isomorphism?
2. Is a product of homomorphisms a homomorphism?

The first question has a positive answer for our concept of homomorphism (cf. theorem 3.4), but the answer to the second question is negative in general. In section 4 we exhibit necessary and sufficient conditions for a structure in order that the product property for homomorphisms holds. In
the algebraic case those conditions reduce to well-known Noetherian isomorphism theorems.

The concept of morphism in the theory of categories may be considered as another generalization of the classical concept of homomorphism, which differs essentially from ours. It is not the fact that morphisms need not to be mappings which is most important in this respect, but the fact that the fundamental properties are quite different. In category theory the product property of morphisms is included in the definition; on the other hand morphisms which are monomorphisms and epimorphisms are not necessarily isomorphisms.

In our concept of structure the carrier of an object is a set, isomorphism is related to bijective mappings of the carriers and subobjects and quotient objects are related to subsets and quotient sets of the carriers. In this way the category of sets and mappings is underlying our concept of structure. Perhaps it would be possible to replace this category of sets and mappings in the definition of structure by another category. We have not investigated this possibility.

We use a set theory in which a distinction of classes and sets is made, in order to include structures which are sufficiently large, such as the structure of all groups, all topological spaces and so on. On the other hand we accept the possibility that an element of a set is not a set itself.

In order to make terminology sufficiently clear we have felt the necessity to start our exposition with an enumeration of a number of well-known set-theoretical results. This is done in section 1. Section 2 gives the definition of structure and section 3 of homomorphism. In section 4 the product property of homomorphisms is discussed. Topological spaces are treated in section 5 to serve as an example. Algebraic and relational structures are discussed in detail in sections 6 and 7. Finally, in section 8 the correspondence between relations and operations and its consequences for the concept of homomorphism is investigated.
Our exposition is of a systematic nature and does not contain any genuine mathematical result. It is intended to provide only a new conceptual framework, of which the author hopes that it will be of some interest for the development of mathematics.

1. Sets and mappings

In order to fix notation and terminology, we state in this section some facts about sets and mappings.

The set of all subsets of \( A \) is denoted by \( P(A) \).

\( A \subseteq B \) means that \( A \) is a subset of \( B \); this includes the case \( A = B \).

If \( K \) is a set and all elements of \( K \) are sets, then:

\[
\cup K := \{ x \mid \exists y \in K \ x \in y \}.
\]

If \( K \in P(P(A)) \), then \( \cup K \in P(A) \).

If \( \equiv \) is an equivalence relation on a set \( A \), the corresponding partition of \( A \) is denoted by \( A/\equiv \) and called the quotient of \( A \) by \( \equiv \).

If \( Q \) is a partition of \( A \), we write \( Q \) quot \( A \). It means that there exists an equivalence relation \( \equiv \) on \( A \), such that \( Q = A/\equiv \).

The quotient set \( A/\sim \) by the equality relation on \( A \) is denoted by \( \bar{A} \). We have

\[
\bar{A} = \{ \{x\} \mid x \in A \}.
\]

If \( f: A \to B \) is a mapping and \( a \in A \), the image of \( a \) is denoted by \( af \).

Accordingly the composed mapping \( A \to C \) of \( f: A \to B \) and \( g: B \to C \) is denoted by \( fg \) (or \( f \circ g \)):

\[
a(f \circ g) = (af)g.
\]

We shall associate to the left, so \( afg \) without brackets will mean \( (af)g \).

If \( f: A \to B \) is a mapping, we call \( \text{Do } f := A \), \( \text{Ra } f := B \),

\( \text{Im } f := \{ xf \mid x \in A \} \).

If \( f: A \to B \) is a mapping, the equivalence relation \( \equiv_f \) on \( A \) defined by

\[
x \equiv_f y :\iff xf = yf \text{ for } x \in A, y \in A,
\]
is called the *fibering* of \( f \).

\[
\text{Coim } f := A/\equiv_f.
\]

A mapping \( f: A \to B \) is called *injective* if \( \text{Coim } f = \bar{A} \), *surjective* if \( \text{Im } f = B \), *bijective* if it is injective and surjective.

If \( D \subset A \), the mapping \( i: D \to A \), defined by \( x_i := x \) for \( x \in D \), is called the *embedding* \( D \to A \). It is injective.

If \( Q \simeq A \), the mapping \( \pi: A \to Q \), defined by

\[x\pi := "the element of Q containing x" \text{ for } x \in A,
\]

is called the *projection* \( A \to Q \). It is surjective.

If \( f: A \to B \) is a mapping, the *canonical decomposition* of \( f \) is

\[f = \pi_f \circ b_f \circ i_f,
\]

where

\[\pi_f \text{ is the projection } A \to \text{Coim } f,
\]

\[i_f \text{ is the embedding } \text{Im } f \to B,
\]

\[b_f \text{ is the bijective mapping } \text{Coim } f \to \text{Im } f, \text{ defined by } K b_f := xf \text{ for } K \in \text{Coim } f, x \in K.
\]

The canonical decomposition is unique in the following sense:
If \( f = \pi \circ b \circ i \), where \( \pi \) is a projection, \( b \) is a bijective mapping and \( i \) is an embedding, then \( \pi = \pi_f \), \( b = b_f \), \( i = i_f \).

If \( Q \simeq A \), the projection \( A \to Q \) is bijective iff \( Q = \bar{A} \). In that case it is called \( t_A \) (\( t_A(x) := \{x\} \) for \( x \in A \)).

If \( D \subset A \), the embedding \( D \to A \) is bijective, iff \( D = A \). In that case is is called \( l_A \) (\( l_A(x) := x \) for \( x \in A \)).

If \( f: A \to B \) is a mapping, the *surjectivization* of \( f \) is defined by

\[\text{surj } f := \pi_f \circ b_f; \text{ it is a surjective mapping } A \to \text{Im } f.
\]

\( f: A \to B \) is surjective, iff \( i_f = l_B \), iff \( \text{surj } f = f \).
If $f: A \to B$ is a mapping, the *injectivization* of $f$ is defined by $\text{inj } f := b_f \circ i_f$; it is an injective mapping $\text{Coim } f \to B$.

If $f: A \to B$ is injective, iff $\pi_f = t_A$, iff $t_A \circ \text{inj } f = f$.

If $f: A \to B$ is injective and $g: B \to C$ is injective, then $fg$ is injective. If $fg$ is injective, then $f$ is injective.

If $f$ is injective, there is a unique decomposition $f = gh$ with $g$ a bijective mapping and $h$ an embedding, viz. $g = \text{surj } f$, $h = i_f$.

If $f: A \to B$ is surjective and $g: B \to C$ is surjective, then $fg$ is surjective. If $fg$ is surjective, then $g$ is surjective.

If $f$ is surjective, there is a unique decomposition $f = hg$ with $g$ a bijective mapping and $h$ a projection, viz. $g = \text{inj } f$, $h = \pi_f$.

If $f$ is a mapping $A \to B$, $D \subseteq A$ and $i$ the embedding $D \to A$, then $i \circ f: D \to B$ is called the *restriction* of $f$ to $D$. If $f$ is injective, the restriction of $f$ to $D$ is injective too, so we may put $i \circ f = g \circ h$, where $g$ is bijective and $h$ is an embedding, and this decomposition is unique. We call $g = \text{surj } (if)$ the *bijective mapping subset induced to $f$ by $D$*; it is a mapping $D \to$ a subset of $B$. We shall apply this construction only if $f$ itself is bijective.

If $f$ is a mapping $A \to B$, $Q$ quot $B$ and $\pi$ the projection $B \to Q$, then $f \circ \pi: A \to Q$ is called the *clustering* of $f$ to $Q$. If $f$ is surjective, the clustering of $f$ to $Q$ is surjective too, so we may put $f \circ \pi = h \circ g$, where $g$ is bijective and $h$ is a projection, and this decomposition is unique. We call $g = \text{inj } (f\pi)$ the *bijective mapping quotient set induced to $f$ by $Q$*; it is a mapping of a quotient set of $A \to Q$. We shall apply this construction only if $f$ itself is bijective.

If $D_1 \subseteq D \subseteq A$, $i_1: D_1 \to D$ and $i: D \to A$ are embeddings, then $i_1 \circ i$ is the embedding $D_1 \to A$.

If $P$ is a set, all elements of $P$ are sets and $Q$ quot $P$, then we define $Q^* := \{UK \mid K \in Q\}$. 
Q_1 \text{quot } Q \text{ and } Q \text{quot } A \text{ do not imply } Q_1 \text{quot } A. \text{ If } \pi_1: Q \rightarrow Q_1 \text{ and } \pi: A \rightarrow Q \text{ are projections, } \pi_1 \circ \pi \text{ is not a projection, but it is surjective, so it may be put into the form } \pi_1 \circ \pi = \pi' \circ g \text{ with } \pi' \text{ a projection and } g \text{ bijective. } \pi' \text{ is the projection } A \rightarrow Q_1^* \text{ and } g \text{ is the inverse of the mapping } \\
\sigma: Q_1 \rightarrow Q_1^*, \text{ defined by } K \sigma := \cup K \text{ for } K \in Q_1. \text{ The special case } Q_1 = Q/\sim \text{ yields: } \\
\text{If } Q \text{ is a quotient set, then } (Q)^* = Q. \\

The canonical decomposition of \\
- a projection } \pi: A \rightarrow Q \text{ is } \pi_1 \circ \pi Q, \\
an embedding } i: D \rightarrow A \text{ is } t_0 \circ t^{-1}_{D} \circ i, \\
a bijective mapping } f: A \rightarrow B \text{ is } t_{A} \circ (t^{-1}_{A} f) \circ i_{B}. \\

\text{If } D \subseteq A, Q \text{quot } A, i \text{ is the embedding } D \rightarrow A, \pi \text{ is the projection } A \rightarrow Q, \text{ then } \\
\text{Coim}(i \circ \pi) = \{ K \cap D \mid K \in Q, K \cap D \neq \emptyset \}, \\
\text{Im}(i \circ \pi) = \{ K \mid K \in Q, K \cap D \neq \emptyset \}, \\
K_{i \circ \pi}^{-1} = K \cap D \text{ for } K \in \text{Im}(i \circ \pi). \\

2. \text{Structures} \\

We are going to define a structure. A structure will consist of objects; there will be the concepts of subobject, of quotient object and of isomorphism. An object has a carrier set. A subset of this carrier set may be the carrier set of a subobject; if it is, this subobject is uniquely determined. A similar state of affairs holds for quotient objects. Isomorphism will be reflexive, symmetric and transitive. Moreover an object will be transferable isomorphically by an arbitrary bijective mapping of its carrier set to another set as its carrier set. Isomorphic objects will have corresponding subobjects and quotient objects. For the concept of isomorphism the transfer to another carrier set will be chosen as a primitive notion.
Definition 2.1. A structure $S$ is a quintuple:

$$S = \langle \mathcal{O}, c, [], s, q \rangle,$$

where

- $\mathcal{O}$ is a class; its elements are called objects;
- $c$ assigns to every object $A$ a set $Ac$, called the carrier of $A$;
- $[]$ assigns to every pair consisting of an object $A$ and a bijective mapping $f$ with $Dof = Ac$ an object $[A,f]$;
- $s$ is a class of pairs of objects; if $<A,B> \in s$, $A$ is called a subobject of $B$;
- $q$ is a class of pairs of objects; if $<A,B> \in q$, $A$ is called a quotient object of $B$;

the following eleven axioms are to be satisfied:

A1. If $A \in \mathcal{O}$ and $f$ is a bijective mapping with $Dof = Ac$, then

$$[A,f]c = Ra f .$$

A2. If $A \in \mathcal{O}$, then

$$[A,1_{Ac}] = A .$$

A3. If $A \in \mathcal{O}$ and $f$ and $g$ are bijective mappings with $Dof = Ac$, $Dog = Ra f$, then

$$[[A,f],g] = [A,fg] .$$

A4. If $A \in \mathcal{O}$, $D \in \mathcal{O}$ and $<D,A> \in s$, then $Dc \subset Ac$.

A5. If $A \in \mathcal{O}$, $Q \in \mathcal{O}$ and $<Q,A> \in q$, then $Qc$ quot $Ac$.

A6. If $A \in \mathcal{O}$, $D_0 \in \mathcal{O}$, $D_1 \in \mathcal{O}$, $<D_0,A> \in s$, $<D_1,A> \in s$ and $D_0c = D_1c$, then $D_0 = D_1$.

A7. If $A \in \mathcal{O}$, $Q_0 \in \mathcal{O}$, $Q_1 \in \mathcal{O}$, $<Q_0,A> \in q$, $<Q_1,A> \in q$ and $Q_0c = Q_1c$, then $Q_0 = Q_1$.

A8. If $A \in \mathcal{O}$, then

$$<A,A> \in s .$$
A9. If $A \in \mathcal{O}$, then
$$< [A, t_{A_c}], A > \in q .$$

A10. If $A \in \mathcal{O}$, $D \in \mathcal{O}$, $f$ is a bijective mapping with $D \circ f = A_c$, $< D, A > \in s$ and $g$ is the bijective mapping subset induced to $f$ by $D_c$, then
$$< [D, g], [A, f] > \in s .$$

A11. If $A \in \mathcal{O}$, $Q \in \mathcal{O}$, $f$ is a bijective mapping with $R \circ f = A_c$, $< Q, A > \in q$ and $g$ is the bijective mapping quotient set induced to $f$ by $Q_c$, then
$$< [Q, g^{-1}], [A, f^{-1}] > \in q .$$

The axioms A4 and A5, A6 and A7, A8 and A9, A10 and A11 are dual in pairs.

In order to simplify notation we agree, that we shall write $A$, $B$, $C$, $D$, etc. for objects and $A$, $B$, $C$, $D$, etc. for the corresponding carriers. So if in a context $A$ and $A$ both are used, it is understood that $A = A_c$. This is also the case if subscripts are present: $D \circ f = D_c$. If in a certain context a set $A$ is given, and we introduce an object $A$ it is taken for granted that $A$ is the carrier of $A$ etc.

We now introduce the concept of isomorphism. Let a fixed structure be given.

**Definition 2.2.** If $A \in \mathcal{O}$ and $f$ is a bijective mapping with $D \circ f = A$, then we say that $f$ produces an isomorphic mapping $A \rightarrow [A, f]$.

**Definition 2.3.** If $A \in \mathcal{O}$ and $B \in \mathcal{O}$, then $A$ is called isomorphic with $B$ ($A \cong B$), iff there exists a mapping producing an isomorphic mapping $A \rightarrow B$.

**Theorem 2.1.** If $f$ produces an isomorphic mapping $A \rightarrow B$ and $g$ produces an isomorphic mapping $B \rightarrow C$, then $fg$ produces an isomorphic mapping $A \rightarrow C$.

$I_A$ produces an isomorphic mapping $A \rightarrow A$.

If $f$ produces an isomorphic mapping $A \rightarrow B$, then $f^{-1}$ produces an isomorphic mapping $B \rightarrow A$.

Isomorphism is an equivalence relation.
Proof. Trivial consequence of A1, A2, A3.

We will need later the concept of a substructure S' of a structure S. In introducing this concept we encounter the following difficulty of notation. In the symbols for concepts like [ ], c, s, q no explicit reference is made to the structure, to which they belong, which may cause confusion if more than one structure at a time are considered. We shall not make the structure explicit in those symbols, but help ourselves with additions like "in S" or "in S'".

Definition 2.4. A structure S' is called a substructure of a structure S, iff the following five conditions are satisfied:

1. The class of objects of S' is a subclass of the class of objects of S.
2. The carrier in S' of an object of S' is its carrier in S.
3. If A is an object of S' and f: A → B is a bijective mapping, then the object [A, f] in S is an object of S' and this object is the object [A, f] in S'.
4. If D and A are objects in S', then < D, A > ∈ S in S' iff < D, A > ∈ S in S.
5. If Q and A are objects in S', then < Q, A > ∈ q in S' iff < Q, A > ∈ q in S.

Remark 2.1. A substructure of S is determined by S and the class of its objects. A class of objects of S determines a substructure of S iff it is closed with respect to isomorphism.

Remark 2.2. A subset of the carrier of an object of S' may be carrier of a subobject in S, without being carrier of a subobject in S'; if it is carrier of a subobject in S', this object coincides with the object in S which is subobject with the same carrier. A similar remark applies to quotient objects.
3. **Homomorphism**

We now introduce the concept of homomorphism. Let a fixed structure be given.

**Definition 3.1.** If \( A \in \mathcal{O} \) and \( B \in \mathcal{O} \), then we say that a mapping \( f: A \rightarrow B \) produces a **homomorphic mapping** \( A \rightarrow B \), if the following three conditions are satisfied:

H1. There exists a \( D \in \mathcal{O} \) with \( < D, B > \in s \) and \( \text{Im} \ f = D \).

H2. There exists a \( Q \in \mathcal{O} \) with \( < Q, A > \in q \) and \( \text{Coim} \ f = Q \).

H3. \( b_f \) produces an isomorphic mapping \( Q \rightarrow D \).

**Remark 3.1.** In definition 3.1 \( D \) is determined uniquely by A6. We call it the **image object** of the homomorphic mapping and denote it by \( D_f \). Similarly \( Q \) is determined uniquely by A7. We call it the **coimage object** of the homomorphic mapping and denote it by \( Q_f \).

**Theorem 3.1.** If \( S' \) is a substructure of \( S \), \( A \) and \( B \) are objects in \( S' \) and \( f \) produces a homomorphic mapping \( A \rightarrow B \) in \( S' \), then \( f \) produces a homomorphic mapping \( A \rightarrow B \) in \( S \) with the same image object and coimage object.

**Proof.** Trivial.

**Remark 3.2.** The converse of theorem 3.1 does not hold.

We return to a fixed structure.

**Theorem 3.2.** If the surjective mapping \( f \) produces a homomorphic mapping \( A \rightarrow B \), then \( D_f = B \) and \( \text{inj} \ f \) produces an isomorphic mapping \( Q_f \rightarrow B \).

**Proof.** \( < D_f, B > \in s \) and \( D_f = \text{Im} \ f = B \) by H1. Moreover, \( < B, B > \in s \) by A8, so \( D_f = B \) by A6. Because \( f \) is surjective, \( \text{inj} \ f = b_f \) and \( b_f \) produces an isomorphic mapping \( Q_f \rightarrow B \) by H3.

**Theorem 3.3.** If the injective mapping \( f \) produces a homomorphic mapping \( A \rightarrow B \), then \( Q_f = [A, t_A] \) and \( \text{surj} \ f \) produces an isomorphic mapping \( A \rightarrow D_f \).
Proof. \(< Q_f, A > \in q \) and \( Q_f = \text{Coim } f = \overline{A} \) by H2. Moreover, \([A, t_A], A \) \( \in q \) by A9 and \([A, t_A], c = \text{Ra } t_A = \overline{A} \) by A1, so \( Q_f = \text{Coim } f = [A, t_A] \) by A7. Because \( f \) is injective, \( \text{surj } f = t_A b_f; t_A \) produces an isomorphic mapping \( A \to [A, t_A] \), \( b_f \) produces an isomorphic mapping \( [A, t_A] \to D_f \) by H3, so \( t_A b_f \) produces an isomorphic mapping \( A \to D_f \).

Theorem 3.4. If the bijective mapping \( f \) produces a homomorphic mapping \( A \to B \), then \( f \) produces an isomorphic mapping \( A \to B \).

Proof. Direct consequence of theorems 3.2 and 3.3 and of \( \text{surj } f = f \).

Theorem 3.5. If \( f \) produces an isomorphic mapping \( A \to B \), then \( f \) produces a homomorphic mapping \( A \to B \).

Proof. \(< B, B > \in s \) by A8 and \( \text{Im } f = B \).
\(< [A, t_A], A > \in q \) by A9 and \( \text{Coim } f = \overline{A} = [A, t_A], c \) by A1.
\( b_f = t^{-1} A f \) produces an isomorphic mapping \( [A, t_A] \to B \).

Remark 3.3. Roughly speaking we may say: a mapping is an isomorphism iff it is a bijective homomorphism.

Theorem 3.6. If \(< D, A > \in s \), then the embedding \( i: D \to A \) produces a homomorphic mapping \( D \to A \).

Proof. \(< D, A > \in s \) and \( \text{Im } i = D \).
\(< [D, t_D], D > \in q \) and \( \text{Coim } i = \overline{D} = [D, t_D], c \).
\( b_i = t^{-1} D i \) produces an isomorphic mapping \( [D, t_D] \to D \).

Theorem 3.7. If \( D \subset A \) and the embedding \( i: D \to A \) produces a homomorphic mapping \( D \to A \), then \( < D, A > \in s \).

Proof. \( \text{surj } i = 1_D \). By theorem 3.3, \( 1_D \) produces an isomorphic mapping \( D \to D_f \), do \( D_f = D \) by A2 and \( < D, A > \in s \) by H1.

Theorem 3.8. If \(< Q, A > \in q \), then the projection \( \pi: A \to Q \) produces a homomorphic mapping \( A \to Q \).

Proof. \(< Q, Q > \in s \) and \( \text{Im } \pi = Q \).
\(< Q, A > \in q \) and \( \text{Coim } \pi = Q \).
\( b_\pi = 1_Q \) produces an isomorphic mapping \( Q \to Q \).
Theorem 3.9. If $\mathcal{Q}$ quot $A$ and the projection $\pi: A \rightarrow \mathcal{Q}$ produces a homomorphic mapping $A \rightarrow \mathcal{Q}$, then $< \mathcal{Q}, A > \in \mathcal{Q}$.

Proof. $\text{inj} \ \pi = 1_{\mathcal{Q}}$. By theorem 3.2, $1_{\mathcal{Q}}$ produces an isomorphic mapping $\mathcal{Q}_\pi \rightarrow \mathcal{Q}$, so $\mathcal{Q}_\pi = \mathcal{Q}$ by A2 and $< \mathcal{Q}, A > \in \mathcal{Q}$ by H2.

Theorem 3.10. If $f$ produces a homomorphic mapping $A \rightarrow B$, then

- $\pi_f$ produces a homomorphic mapping $A \rightarrow Q_f$,
- $b_f$ produces a homomorphic mapping $Q_f \rightarrow D_f$,
- $i_f$ produces a homomorphic mapping $D_f \rightarrow B$.

Proof. Trivial consequence of theorems 3.5, 3.6 and 3.8.

Theorem 3.11. If $f: A \rightarrow B$ is a mapping, $\pi_f$ produces a homomorphic mapping $A \rightarrow Q_f$, $b_f$ produces a homomorphic mapping $Q_f \rightarrow D_f$ and $i_f$ produces a homomorphic mapping $D_f \rightarrow B$, then $f$ produces a homomorphic mapping $A \rightarrow B$, $D_f = D$, $Q_f = Q$.

Proof. Trivial consequence of theorems 3.4, 3.7 and 3.9.

Remark 3.4. Roughly speaking we may say: a mapping is a homomorphism iff all the mappings in its canonical decomposition are homomorphisms.

The following theorem states that the concept of homomorphism is not affected by the replacement of an object by an isomorphic copy. We remind, that we do not yet know whether the product of homomorphic mappings is a homomorphic mapping; this question will be treated in the next section.

Theorem 3.12. If $f$ produces an isomorphic mapping $A \rightarrow B$, $g$ produces a homomorphic mapping $B \rightarrow C$ and $h$ produces an isomorphic mapping $C \rightarrow D$, then $fgh$ produces a homomorphic mapping $A \rightarrow D$.

Proof. $< D, C > \in s$ and $h$ is a bijective mapping $C \rightarrow D$.

Let $h'$ be the bijective mapping subset induced to $h$ by $D_g$, then, by A10,

- $< [D_g, h'], \mathcal{D} > \in s$.
- $< Q_g, B > \in \mathcal{Q}$ and $f$ is a bijective mapping $A \rightarrow B$. Let $f'$ be the bijective mapping quotient set induced to $f$ by $Q_g$, then, by A11, $< [Q_g, (f')^{-1}], A > \in \mathcal{Q}$.

There is an embedding $i'$ such that $i_g h = h' i'$ and a projection $\pi'$ such that $f\pi_g = \pi' f'$.

$fgh = f\pi_g b_i h = \pi' \circ (f' b_i h') \circ i'$; this delivers the canonical decomposition of $fgh$. 

\[ [[Q, (f')^{-1}], f' g h'] = [Q, b h'] = [[Q, b], h'] = [Q, h'] \]

\[ \text{Im}(fgh) = [D, h']c \text{ and } < [D, h'], D > \in s. \]

\[ \text{Coim}(fgh) = [Q, (f')^{-1}]c \text{ and } < [Q, (f')^{-1}], A > \in q. \]

This completes the proof.

4. Product property

In section 5 we shall exhibit an example of a structure and of mappings producing homomorphic mappings \( A \to B \) and \( B \to C \), whereas their product does not produce a homomorphic mapping \( A \to C \). We now give conditions which are necessary and sufficient in order that the product property for homomorphisms holds.

A12. If \( < D, A > \in s \) and \( < D_1, D > \in s \), then \( < D_1, A > \in s \).

A13. If \( < Q, A > \in q \) and \( < Q_1, Q > \in q \), then \( < [Q_1, \sigma], A > \in q \), where \( \sigma \) denotes the bijective mapping \( Q_1 \to Q_1^{*} \), defined by \( K \sigma := uK \) for \( K \in Q_1 \).

A14. If \( < D, A > \in s, < Q, A > \in q, \)

\( Q_1 := \{ K \cap D \mid K \in Q, K \cap D \neq \emptyset \}, \)

\( D_1 := \{ K \mid K \in Q, K \cap D \neq \emptyset \}, \)

\( \tau : D_1 \to Q_1 \) is defined by \( K \tau := K \cap D \) for \( K \in D_1 \),

then there exist objects \( Q_1 \) and \( D_1 \), such that

\( < Q_1, D > \in q, < D_1, Q > \in s, [D_1, \tau] = Q_1. \)

Theorem 4.1. In a structure the statement

"If \( f \) produces a homomorphic mapping \( A \to B \) and \( g \) produces a homomorphic mapping \( B \to C \), then \( fg \) produces a homomorphic mapping \( A \to C \)"

holds iff the structure satisfies A12, A13 and A14.

Proof. We first prove that the conditions A12, A13 and A14 are necessary, so suppose that the product property for homomorphisms holds.
Suppose \( \langle O,A \rangle \in s, \langle 0_1,0 \rangle \in s \). By theorem 3.6, the embeddings \( i : D \to A \) and \( i_1 : D_1 \to D \) produce homomorphic mappings \( D \to A \) and \( D_1 \to D \). By the product property \( i_1 i \) produces a homomorphic mapping \( D_1 \to A \), but \( i_1 i \) is the embedding \( D_1 \to A \), so, by theorem 3.7, \( \langle D_1,A \rangle > \in s \). This proves A12.

Suppose \( \langle Q,A \rangle \in q, \langle Q_1,Q \rangle \in q \), and \( \sigma \) defined as in A13. By theorem 3.8 the projections \( \pi : A \to Q \) and \( \pi_1 : Q \to Q_1 \) produce homomorphic mappings \( A \to Q \) and \( Q \to Q_1 \). By the product property \( \pi \pi_1 \) produces a homomorphic mapping \( A \to Q_1 \), but \( \pi \pi_1 = \pi' \sigma^{-1} \) where \( \pi' \) is the projection \( A \to Q_1^\ast \) and \( \sigma \) produces an isomorphic mapping \( Q_1 \to [Q_1,\sigma] \). By theorem 3.12, \( \pi' = (\pi \pi_1)^\circ \sigma \) produces a homomorphic mapping \( A \to [Q_1,\sigma] \) and, by theorem 3.9, \( \langle [Q_1,\sigma],A \rangle > \in q \). This proves A13.

Suppose \( \langle D,A \rangle \in s, \langle Q,A \rangle \in q, \) and \( Q_1, D_l \) and \( \tau \) defined as in A14. By theorem 3.6, the embedding \( i : D \to A \) produces a homomorphic mapping \( D \to A \) and, by theorem 3.8, the projection \( \pi : A \to Q \) produces a homomorphic mapping \( A \to Q \). By the product property \( i \pi \) produces a homomorphic mapping \( D \to Q \), so

\[
\begin{align*}
\langle D_1,\pi_1, D \rangle & \in S \\
\langle Q_1, \pi_1, D \rangle & \in Q \\
\langle Q_1, \pi_1, D \rangle & \in \text{Coim}(i) = Q_1
\end{align*}
\]

\( b_\pi \) produces an isomorphic mapping \( \pi_1 \to D_1 \), but \( b_\pi \tau = \tau^{-1} \), so \( [D_1,\tau] = \in \pi_1 \). This proves A14.

We now prove that the conditions A12, A13 and A14 are sufficient, so we suppose that they are satisfied.

Suppose \( f \) produces a homomorphic mapping \( A \to B \) and \( g \) produces a homomorphic mapping \( B \to C \).

\[
fg = b_f \pi b_i \pi b_i
\]

We apply A14 to \( \langle D_f,B \rangle \in s \) and \( \langle Q_g,B \rangle \in q \); take \( Q_1, D_1 \) and \( \tau \) as in A14, then we find objects \( Q_1 \) and \( D_1 \), such that \( \langle Q_1,D_f \rangle \in q \), \( \langle D_1,Q_g \rangle \in s \) and \( \tau \) produces an isomorphic mapping \( D_1 \to Q_1 \). Put \( h := f g \), then the projection \( \pi_h : D_f \to Q_1 \) produces a homomorphic mapping \( D_f \to Q_1 \) by theorem 3.8, the embedding \( \pi_h : D_f \to Q_1 \) produces a homomorphic mapping \( D_f \to Q_g \) by theorem 3.6, and \( b_h \tau = \tau^{-1} \) produces an isomorphic mapping \( Q_1 \to D_1 \).

\[
fg = b_f \pi b_h b_i \pi b_i
\]

The mapping \( k := b_f \pi \) is a surjective mapping, which, by theorem 3.12, produces a homomorphic mapping \( Q_f \to Q_1 \), but \( k = \pi_k \circ \text{inj} k \); by H2, \( \langle Q_k,Q_f \rangle \in q \) and, by theorem 3.2, \( \text{inj} k \) produces an isomorphic mapping \( Q_k \to Q_1 \).
The mapping \( m := i_h b g \) is an injective mapping, which, by theorem 3.12, produces a homomorphic mapping \( D_1 \to D_g \), but \( m = \text{surj} m \circ i_m \); by H1, \( < D_m, D_g > \in s \), and, by theorem 3.3, \( \text{surj} m \) produces an isomorphic mapping \( D_1 \to D_m \).

\[ fg = \pi'_k \circ \text{inj} k \circ \text{surj} m \circ i_m g. \]

We apply \( A12 \) to \( < D, C > \in s \) and \( < D_m, D_g > \in s \), yielding \( < \text{m}, C > \in s \); \( i_i g \) is the embedding \( D_m \to C \).

We apply \( A13 \) to \( < Q_f, A > \in q \) and \( < Q_k, Q_f > \in q \), yielding \( < [Q_k, \sigma], A > \in q \) with \( \sigma \) as in \( A13 \), but \( \pi'_k = \pi' \sigma^{-1} \), where \( \pi' \) is the projection \( A \to Q^* k \) and \( Q_k = [Q_k, \sigma] c \).

\[ fg = \pi' \circ (\sigma^{-1} \circ \text{inj} k \circ \text{surj} m) \circ i_m g \] delivers the canonical decomposition of \( fg \) and \( \sigma^{-1} \circ \text{inj} k \circ \text{surj} m \) produces an isomorphic mapping \( [Q_k, \sigma] \to D_m \).

This concludes the proof of theorem 4.1.

5. Topological spaces

In the structure of topological spaces we take for an object a carrier set together with a set of subsets satisfying the usual axioms for open sets. Isomorphism is topological homeomorphism. A subobject of an object is an arbitrary subset of the carrier with the relative topology in the usual sense. A quotient object of an object is an arbitrary quotient set of the carrier with the quotient topology in the usual sense. It is easy to prove that the eleven axioms of a structure are satisfied. We remark that in this structure every subset of the carrier of an object is carrier of a subobject of that object and every quotient set of the carrier of an object is carrier of a quotient object of that object.

One could guess that homomorphic mappings should coincide with continuous mappings in this structure, but this is not the case, because there exist bijective continuous mappings which are not homeomorphic, whereas bijective homomorphic mappings are isomorphic by theorem 3.4. The following theorem characterizes the homomorphic mappings and may be proved easily.

**Theorem 5.1.** In the structure of topological spaces a mapping \( f: A \to B \) produces a homomorphic mapping \( A \to B \) iff it is continuous and for every subset \( D \) of \( B \), for which \( f^{-1}(D) \) is open in \( A \), there exists a subset \( D' \) of \( B \), which is open in \( B \), such that \( f^{-1}(D') = f^{-1}(D) \).
If the mapping is surjective the condition becomes simpler, because in that case $f^{-1}(D') = f^{-1}(D)$ implies $D' = D$. So we have:

**Theorem 5.2.** In the structure of topological spaces a surjective mapping $f: A \to B$ produces a homomorphic mapping $A \to B$ iff for every subset $D$ of $B$:

$$D \text{ is open in } B \iff f^{-1}(D) \text{ is open in } A.$$ 

The product property of homomorphic mappings does not hold in this structure. It is easy to prove that $A12$ and $A13$ are satisfied, but $A14$ is not, as is shown by the following example.

Let $A$ consist of $A = \{a,b,c\}$ with the topology $\{\emptyset,\{a\},\{a,b,c\}\}$, and take $D := \{a,c\}$, $Q := \{\{a,b\},\{c\}\}$. Then:

$$Q_1 = \{\{a\},\{c\}\}, \quad D_1 = Q.$$ 

The topology of $Q_1$ is $\{\emptyset,\{a\},\{a,c\}\}$, the topology of $D_1$ is $\{\emptyset,\{a,b\},\{c\}\}$; $Q_1$ and $D_1$ are not homeomorphic.

The topology of $A$ is rather pathological; it is not even $T_0$. The following example shows, that even in a space with a nice topology $A14$ may fail to hold.

Let $R$ consist of $R = [0,1]$, the closed unit interval on the real line with the usual topology and take $D := [0,\frac{1}{2}) \cup \{1\}$, $Q := \{[0,\frac{1}{2}],[\frac{1}{2},1]\}$. Then

$$Q_1 = \{[0,\frac{1}{2}],[1]\}, \quad D_1 = Q.$$ 

The topology of $Q_1$ is $\{\emptyset,[0,\frac{1}{2}]),\{1\},\{[0,\frac{1}{2}],[1]\}\}$, the topology of $D_1$ is $\{\emptyset,\{\frac{1}{2}\},\{0,\frac{1}{2}],[\frac{1}{2},1]\}\$; $Q_1$ and $D_1$ are not homeomorphic.

For future reference we remark that $D$ is not compact in the relative topology of $R$ and $Q$ is not Hausdorff in the quotient topology of $R$.

In general in the structure of topological spaces the mapping $\tau^{-1}$ of $A14$ is continuous and bijective.
We now consider the structure of compact Hausdorff topological spaces, which obviously is a substructure of the structure of topological spaces. In this structure subobjects are only those objects with a carrier which is a subset for which the relative topology is compact (it is always Hausdorff). Quotient objects are only those objects with a carrier which is a quotient set for which the quotient topology in Hausdorff (it is always compact).

In this structure the homomorphic mappings coincide with the continuous mappings, because of theorem 3.1, theorem 5.1 and the following well-known facts about continuous mappings of topological spaces in topological spaces:

1. The image of a continuous mapping of a compact space is compact in the relative topology.
2. The coimage of a continuous mapping in a Hausdorff space is Hausdorff in the quotient topology.
3. A continuous bijective mapping of a compact space in a Hausdorff space is a homeomorphic mapping.

Obviously, the continuous mappings have the product property and therefore the structure of compact Hausdorff topological spaces satisfies A12, A13 and A14. For A12 and A13 this is not surprising, because if A12 and A13 hold in a structure, they hold in every substructure. With respect to A14 we recall that for topological spaces $\tau^{-1}$ is continuous and bijective; if $\mathbb{D}$ and $\mathbb{Q}$ are compact and Hausdorff, then $\mathbb{Q}_1$ is compact and $\mathbb{D}_1$ is Hausdorff, so, by fact 3, $\tau^{-1}$ is homeomorphic and therefore $\mathbb{Q}_1$ and $\mathbb{D}_1$ both are compact Hausdorff.

6. Algebraic structures

Definition 6.1. A type is a pair $< V, \sigma >$, where $V$ is a set and $\sigma$ is a mapping of $V$ in the set $\mathbb{N}$ of natural numbers. We assume that $\sigma_0 = 0$ is a natural number and that every natural number is the set of its predecessors. Elements of $V$ are called names of operations and $\sigma_j$ is called the order of $j$ for $j \in V$.

Definition 6.2. The full algebraic structure of type $< V, \sigma >$ is defined as follows.

An object $A$ is a carrier set $A$ together with for every $j \in V$ a mapping $\sigma^{A,j}_A : A^{\sigma_j} \to A$. We call $\sigma^{A,j}_A$ the operation on $A$ of name $j$ and it is called a
jo-ary operation. The number jo is allowed to be zero; in that case $A^{jo}$ is the one element set $\{0\}$ and $\sigma_{A,j}^{o}$ is determined by $\emptyset \sigma_{A,j}$, which is an element of $A$. Therefore the operation is called a constant in that case. The carrier of an object is allowed to be empty; if, however, there exists a $j \in V$ for which $jo = 0$, there are no objects with empty carrier in the structure.

Before defining isomorphism we introduce the following notation: if $f: A \to B$ is a mapping and $n \in N$, then $f_n: A^n \to B^n$ is defined by $m(vf_n) := mvf$ for $v \in A^n$ and $m \in n$. If $f$ is bijective, $(f_n)^{-1} = (f^{-1})_n$; we then write $f_n^{-1}$.

Let $A$ be an object and $f: A \to B$ a bijective mapping; $[A,f]$ is defined to have carrier $B$ and for $j \in V$:

$$\sigma_{[A,f],j} := f_j^{-1} \sigma_{A,j}^f.$$ 

If $A$ is an object, $D \subseteq A$ and for all $j \in V$ and all $v \in D^{jo}$: $v \sigma_{D,j}^o \subseteq D$, then $D$ is defined to be carrier of a subobject $D$ of $A$ and $v \sigma_{D,j}^o := v \sigma_{A,j}^o$ for $j \in V$ and $v \in D^{jo}$. In that case $D$ is called closed with respect to all operations of $A$.

If $A$ is an object, $Q \verb|quot| A$, $\pi$ is the projection $A \to Q$ and for all $j \in V$, for all $v \in A^{jo}$ and all $w \in A^{jo}$, for which $v \pi_j = w \pi_j$, also $v \sigma_{A,j}^\pi = w \sigma_{A,j}^\pi$, then $Q$ is defined to be the carrier of a quotient object $Q$ of $A$ and $s \sigma_{Q,j}^\pi := v \sigma_{A,j}^\pi$ for $j \in V$, $s \in Q^{jo}$, $v \in A^{jo}$, $s = v \pi_j$. In that case the equivalence relation $\equiv$, for which $Q = A/\equiv$, is called a congruence relation with respect to all operations of $A$.

It is easy to prove that the eleven axioms of a structure are satisfied.

We make a change of notation in order to adapt it to the usual one in universal algebra.

We write $(v_0, \ldots, v_{n-1})$ for $v$ if $v \in A^n$ and $v_m$ for $m$ if $v \in A^n$ and $m \in n$. The definition of $f_n$ then reads as follows:

$$(v_0, \ldots, v_{n-1})f_n := (v_0^f, \ldots, v_{n-1}^f) \quad \text{for} \quad (v_0, \ldots, v_{n-1}) \in A^n.$$

In the new notation the bijective mapping $f: A \to B$ produces an isomorphic mapping $A \to B$ iff

$$(v_0^f, \ldots, v_{jo-1}^f) \sigma_{D,j} = (v_0, \ldots, v_{jo-1}) \sigma_{A,j}^f$$
for all $j \in V$ and $(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}$.

For a quotient object $Q$ of $A$ with projection $\pi: A \rightarrow Q$ we have

$$(v_{0}^{\pi}, \ldots, v_{j_0-1}^{\pi})_{Q,j} = (v_0, \ldots, v_{j_0-1})_{A,j}^{\pi}$$

for all $j \in V$ and $(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}$.

The concept of homomorphism coincides with the usual one in universal algebra on account of the following theorem.

**Theorem 6.1.** In the full algebraic structure of type $< V, O >$ a mapping $f: A \rightarrow B$ produces a homomorphic mapping $A \rightarrow B$ iff

$$(*) \quad (v_0, \ldots, v_{j_0-1})_{A,j} \overset{f}{\rightarrow} = (v_0^f, \ldots, v_{j_0-1}^f)_{B,j}$$

for all $j \in V$ and $(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}$.

**Proof.** The if-part is well-known from universal algebra (cf. [4], p. 57, theorem 1, p. 36, lemma 3 and p. 37, lemma 6). For the sake of completeness we give a proof.

Suppose $(*)$ is satisfied. Suppose $j \in V$ and $(w_0, \ldots, w_{j_0-1}) \in (\text{Im } f)^{j_0}$, then there exists $(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}$ such that $w_m = v_m f$ for $m = 0, \ldots, j_0-1$.

Therefore

$$(w_0, \ldots, w_{j_0-1})_{B,j} = (v_0^f, \ldots, v_{j_0-1}^f)_{B,j} = (v_0, \ldots, v_{j_0-1})_{A,j} f \in \text{Im } f,$$

so $\text{Im } f$ is carrier of a subobject $B$ of $B$. We now suppose $j \in V$,

$$(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}, \quad (w_0, \ldots, w_{j_0-1}) \in \mathcal{A}^{j_0} \quad \text{and } v_m f = w_m f \text{ for } m = 0, \ldots, j_0-1,$$

then

$$(v_0, \ldots, v_{j_0-1})_{A,j} f = (v_0^f, \ldots, v_{j_0-1}^f)_{B,j} = (w_0, \ldots, w_{j_0-1})_{A,j} f,$$

so $\text{Coim } f$ is carrier of a quotient object $Q$ of $A$. Finally, suppose $j \in V$ and

$$(s_0, \ldots, s_{j_0-1}) \in (\text{Coim } f)^{j_0},$$

then there exists $(v_0, \ldots, v_{j_0-1}) \in \mathcal{A}^{j_0}$ such that $s_m = v_m f$ for $m = 0, \ldots, j_0-1$ and therefore $s_m b_f = v_m f b_f = v_m f$ for $m = 0, \ldots, j_0-1$. Therefore
so \( b_f \) produces an isomorphic mapping \( Q \rightarrow B \). This completes the proof, that \( f \) produces a homomorphic mapping \( A \rightarrow B \).

We now prove the converse and therefore suppose that \( f \) produces a homomorphic mapping \( A \rightarrow B \). Suppose \( j \in V \) and \( (v_0, \ldots, v_{j_0 - 1}) \in A^{j_0} \), then

\[
(v_0, \ldots, v_{j_0 - 1})\sigma_{A,j}f = (v_0, \ldots, v_{j_0 - 1})\sigma_{A,j}^f = (v_0\bar{f}, \ldots, v_{j_0 - 1}\bar{f})\sigma_{Q,j}^f = (v_0\bar{f}, \ldots, v_{j_0 - 1}\bar{f})\sigma_{Q,j}^{b_f} =
\]

so \((*)\) is satisfied.

Remark 6.1. It is a trivial consequence of theorem 6.1 that in a full algebraic structure homomorphic mappings have the product property. Therefore A12, A13 and A14 are satisfied. This again is a well-known result from universal algebra, where usually \((*)\) is taken as definition for homomorphism and the translations into algebraic terms of A12, A13 and A14 constitute well-known theorems. In particular A13 and A14 correspond to Noetherian isomorphism theorems (cf. [4], p. 62, theorem 4 for A13 and p. 58, theorem 2 for A14).

Definition 6.3. An algebraic structure of type \(< V, \circ \>\) is a substructure of the full algebraic structure of type \(< V, \circ >\).

Theorem 6.2. In an algebraic structure of type \(< V, \circ >\) a mapping \( f: A \rightarrow B \) produces a homomorphic mapping \( A \rightarrow B \) iff the following two requirements are satisfied:

1. \( \text{Im } f \) is carrier of a subobject of \( B \) or \( \text{Coim } f \) is carrier of a quotient object of \( A \).
This theorem follows easily from theorems 3.1 and 6.1; note that if \( f \) produces a homomorphic mapping in the full algebraic structure, then \( Q_f \) and \( D_f \) are isomorphic, so both or none of them are objects in the substructure.

**Remark 6.2.** If an algebraic structure \( S \) satisfies the condition that for every object \( A \) in \( S \) all subobjects of \( A \) in the corresponding full algebraic structure are objects of \( S \) and therefore also subobjects of \( A \) in \( S \), then requirement 1 in theorem 6.2 is redundant and the theorem gets the same shape as theorem 6.1. This is also true if a corresponding condition for quotient objects holds. It is well-known that both conditions are satisfied if the algebraic structure is equational. Homomorphic mappings also have the product property in these cases.

**Remark 6.3.** We consider what happens with constants. If \( j \in V \) and \( j_0 = 0 \), we write \( \gamma_{A,j} := \emptyset \). The isomorphism requirement for constants reads:
\[
\gamma_{[A,f],j} = \gamma_{A,j}^f.
\]
The subobject requirement is \( \gamma_{A,j} \in D \) and \( \gamma_{D,j} := \gamma_{A,j} \).

For quotient objects no requirement is made and \( \gamma_{Q,j} := \gamma_{A,j}^{\gamma}. \) Requirement (*) in theorem 6.1 gets the form \( \gamma_{A,j}^f = \gamma_{B,j} \) for this \( j \).

7. Relational structures

We use the same concept of type as for algebraic structures, except that we now call the elements of \( V \) names of relations.

**Definition 7.1.** The full relational structure of type \( < V,0 > \) is defined as follows.

An object \( A \) is a carrier set \( A \) together with for every \( j \in V \) a subset \( R_{A,j} \) of \( A^{j_0} \). We call \( R_{A,j} \) the relation (or the predicate) on \( A \) of name \( j \) and it is called a \( j \)-ary relation. If \( j_0 = 0 \), there are two possibilities for \( R_{A,j} \), viz. \( R_{A,j} = \{ \emptyset \} \) or \( R_{A,j} = \emptyset \). We call \( R_{A,j} \) a proposition in that case, which is true if it is \( \{ \emptyset \} \) and false if it is \( \emptyset \). The carrier of an object is allowed to be empty.

Let \( A \) be an object and \( f: A \to B \) a bijective mapping; \([A,f]\) is defined to have carrier \( B \) and for \( j \in V \):
\[
R_{[A,f],j} := \{ v f j_0 \mid v \in R_{A,j} \}.
\]
If $A$ is an object and $D \subseteq A$, then $D$ is carrier of a subobject $D$ of $A$ with

$$R_{D,j} := R_{A,j} \cap D^{j\circ} \quad \text{for } j \in V.$$  

If $A$ is an object, $Q$ quot $A$ and $\pi$ is the projection $A \rightarrow Q$, then $Q$ is carrier of a quotient object $Q$ of $A$ with

$$R_{Q,j} := \{v^{\pi,j} \mid v \in R_{A,j}\} \quad \text{for } j \in V;$$

this means that $R_{Q,j}$ holds iff there exists a sequence of representatives from the equivalence classes, for which $R_{A,j}$ holds.

It is easy to prove that the eleven axioms of a structure are satisfied. In this structure every subset (quotient set) of the carrier of an object is carrier of a subobject (quotient object).

Just as for algebraic structures we switch to the notation $(v_0, \ldots, v_{n-1})$ for $v \in A^n$. In this notation the formulae in the definition of isomorphism and of quotient object read:

$$R_{[A,f],j} := \{(v_0 f, \ldots, v_{j\circ -1} f) \mid (v_0, \ldots, v_{j\circ -1}) \in R_{A,j}\},$$

$$R_{[Q,f],j} := \{(v_0^{\pi}, \ldots, v_{j\circ -1}^{\pi}) \mid (v_0, \ldots, v_{j\circ -1}) \in R_{A,j}\}.$$  

**Theorem 7.1.** In the full relational structure of type $<V,o>$ a mapping $f: A \rightarrow B$ produces a homomorphic mapping $A \rightarrow B$ iff

$$\text{(**) for all } j \in V \text{ and } (v_0, \ldots, v_{j\circ -1}) \in A^{j\circ}:$$

$$(v_0 f, \ldots, v_{j\circ -1} f) \in R_{B,j} \text{ iff there exists } (v'_0, \ldots, v'_{j\circ -1}) \in R_{A,j} \text{ such that } v'_m f = v_m f \text{ for } m = 0, \ldots, j\circ -1.$$  

**Remark 7.1.** Condition (***) is equivalent to:

for all $j \in V$ and $(v_0, \ldots, v_{j\circ -1}) \in A^{j\circ}$:

if $(v_0, \ldots, v_{j\circ -1}) \in R_{A,j}$, then $(v_0 f, \ldots, v_{j\circ -1} f) \in R_{B,j}$ and

if $(v_0 f, \ldots, v_{j\circ -1} f) \in R_{B,j}$, then there exists $(v'_0, \ldots, v'_{j\circ -1}) \in R_{A,j}$

such that $v'_m f = v_m f \text{ for } m = 0, \ldots, j\circ -1.$
Proof. Suppose (**) is satisfied. Let \( D \) be the subobject of \( B \) with carrier \( \text{Im} \, f \) and \( Q \) the quotient object of \( A \) with carrier \( \text{Coim} \, f \). Suppose \( j \in V \), then

\[
R_{D,f,j} = R_{B,j} \cap (\text{Im} \, f)^{j_0} =
\]

\[
= \{ (v_0, \ldots, v_{j_0-1}) \mid (v_0, \ldots, v_{j_0-1} \in A, (v_0, \ldots, v_{j_0-1} \in R_{B,j} \}
\]

\[
= \{ (v_0, \ldots, v_{j_0-1}) \mid (v_0, \ldots, v_{j_0-1} \in A, \text{there exists}
\]

\[
(v_0', \ldots, v_{j_0-1}') \in R_{A,j} \text{ such that } v'_m = v_m \text{ for } m = 0, \ldots, j_0-1 \}
\]

\[
= \{ (v_0, \ldots, v_{j_0-1}) \mid (v_0, \ldots, v_{j_0-1} \in R_{A,j} \}
\]

\[
= \{ (v_0^{b_f}b_f, \ldots, v_{j_0-1}^{b_f}b_f) \mid (v_0, \ldots, v_{j_0-1}) \in R_{A,j} \}
\]

\[
= \{ (s_0b_f, \ldots, s_{j_0-1}b_f) \mid (s_0, \ldots, s_{j_0-1}) \in R_{Q,j} \}
\]

so \( b_f \) produces an isomorphic mapping \( Q \to D \) and therefore \( f \) produces a homomorphic mapping \( A \to B \).

We now prove the converse and therefore suppose that \( f \) produces a homomorphic mapping \( A \to B \). Suppose \( j \in V \) and \( (v_0, \ldots, v_{j_0-1}) \in A^{j_0} \), then

\[
(v_0, \ldots, v_{j_0-1}) \in R_{B,j} \iff (v_0, \ldots, v_{j_0-1} \in R_{D,f,j} \iff \text{there exists }
\]

\[
(s_0, \ldots, s_{j_0-1}) \in R_{Q,f,j} : v_{m} = s_{m}b_f \text{ for } m = 0, \ldots, j_0-1 \iff \text{there exists }
\]

\[
(v_0', \ldots, v_{j_0-1}') \in R_{A,j} : v_{m} = v'_m \text{ for } m = 0, \ldots, j_0-1, \text{ so (**) is satisfied.}
\]

Remark 7.2. It is easy to show that, if there exists a \( j \in V \) for which \( j_0 \neq 0 \), homomorphic mappings do not have the product property. On the other hand obviously A12 and A13 hold, so A14 must fail to hold.

Definition 7.2. A relational structure of type \( < V, o > \) is a substructure of the full relational structure of type \( < V, o > \).

Theorem 7.2. In a relational structure of type \( < V, o > \) a mapping \( f: A \to B \) produces a homomorphic mapping \( A \to B \) iff the following two requirements are satisfied:
1. \( \text{Im} f \) is carrier of a subobject of \( B \) or Coim \( f \) is carrier of a quotient object of \( A \).

2. Requirement (**\( \)) of theorem 7.1.

This theorem follows in the same way from theorem 7.1 as theorem 6.2 from theorem 6.1. Similar remarks about the redundancy of requirement 1 may be made here.

**Remark 7.3.** We consider what happens with propositions. We write \( 0 := \emptyset \) (false) and \( 1 := \{\emptyset\} \) (true). If \( j \in V \) and \( jo = 0 \), then \( R_A,j = 0 \) or \( R_A,j = 1 \). The isomorphism requirement for propositions reads: \( R_{[A,f]}j = R_A,j \). For subobjects we have \( R_D,j = R_A,j \) and for quotient objects \( R_Q,j = R_A,j \). Requirement (**\( \)) in theorem 7.1 gets the form \( R_{B,j} = R_A,j \) for this \( j \).

**Example 7.1.** We consider totally ordered sets. In order to get them we take the type \(<V,o>\) with \( V = \{\leq\} \) and \( o = 2 \) and the relational structure of the objects \( A \) which are totally ordered by the binary relation \( R_A,\leq \). As usual we switch to infix notation, so we write \( v_0 \leq_A v_1 \) instead of \( (v_0,v_1) \in R_A,\leq \). In this structure a quotient set of the carrier of an object is carrier of a quotient object iff every element of the quotient set is convex in the object. A subset of the carrier of an object is always carrier of a subobject. Therefore a homomorphic mapping is characterized by requirement (**\( \)) of theorem 7.1. In fact this requirement may be simplified further in the following way:

for all \( v_0 \in A \) and \( v_1 \in A \): if \( v_0 \leq_A v_1 \), then \( v_0f \leq_B v_1f \).

We show that this requirement implies (**\( \)). Assume \( v_0f \leq_B v_1f \). If \( v_0 \leq_A v_1 \), the choice \( v'_0 := v_0 \), \( v'_1 := v_1 \) is a good one for (**\( \)). If, however, \( v_0 \leq_A v_1 \) does not hold, we have \( v_1 \leq_A v_0 \) and therefore \( v_1f \leq_B v_0f \), which, together with \( v_0f \leq_B v_1f \), implies \( v_1f = v_0f \), so the choice \( v'_0 := v_0 \), \( v'_1 := v_0 \) now is a good one for (**\( \)). We remark, that if one chooses \(<\) instead of \( \leq \) as a basic relation for totally ordered sets, our formalism does not give useful results. In that case, if \( A \) is an object, the only quotient set of \( A \), which is carrier of a quotient object of \( A \), is \( \overline{A} \) and all homomorphic mappings are injective.
Example 7.2. We proceed as in example 7.1, but now for partially ordered sets. Again every subset of the carrier of an object is carrier of a subobject and (**) is a characterization for homomorphic mappings. This condition, however, may not be replaced by that of example 7.1, as is seen from the following example:

\[ A = \{a, b, c, d\}, \quad R_A \leq = \{(a,a), (a,b), (b,b), (c,c), (c,d), (d,d)\} ; \]
\[ B = \{\alpha, \beta, \gamma\}, \quad R_B \leq = \{(\alpha,\alpha), (\alpha,\beta), (\beta,\beta), (\beta,\gamma), (\gamma,\gamma)\} . \]

f: A \to B is defined by af = a, bf = cf = \beta, df = \gamma. It satisfies: "if \( v_0 \leq_A v_1 \) then \( v_0 f \leq_B v_1 f \)" , but it does not produce a homomorphic mapping \( A \to B \), because the relation induced on \( \text{Coim} f \) by \( A \) is not transitive, so \( \text{Coim} f \) is not carrier of a quotient object of \( A \).

8. Relations and operations

Definition 8.1. If \( A \) is a set and \( R \) is an \( n \)-ary relation on \( A \), then \( R \) is said to induce a partial operation if \( n > 0 \) and for all \( (v_0, \ldots, v_{n-1}) \in R \), \( (v'_0, \ldots, v'_{n-1}) \in R: v_m = v'_m \) for \( m = 0, \ldots, n-2 \) implies \( v_{n-1} = v'_{n-1} \).

Theorem 8.1. If \( A \) is an object in the full relational structure of type \( \langle V, o \rangle \), \( D \) is a subobject of \( A \), \( f \) is a bijective mapping \( A \to B \), \( j \in V \) and \( R_{A,j} \) induces a partial operation, then \( R_D,j \) and \( R_{[A,f],j} \) induce partial operations.

Proof. Trivial.

Definition 8.2. If \( A \) is a set and \( R \) is an \( n \)-ary relation on \( A \), then \( R \) is said to induce a multi-operation if \( n > 0 \) and for all \( (w_0, \ldots, w_{n-2}) \in A^{n-1} \), there exists \( (v_0, \ldots, v_{n-1}) \in R \) such that \( w_m = v_m \) for \( m = 0, \ldots, n-2 \).

Theorem 8.2. If \( A \) is an object in the full relational structure of type \( \langle V, o \rangle \), \( Q \) is a quotient object of \( A \), \( f \) is a bijective mapping \( A \to B \), \( j \in V \) and \( R_{A,j} \) induces a multi-operation, then \( R_Q,j \) and \( R_{[A,f],j} \) induce multi-operations.

Proof. Trivial.
Definition 8.3. If A is a set and the n-ary relation R on A induces a partial operation and a multi-operation, then R is called operational. In that case the operation on A corresponding to R is the (n-1)-ary operation \( \sigma \) on A defined by: for all \( (v_0, \ldots, v_{n-1}) \in A^n \) and \( (w_0, \ldots, w_{n-2}) \in A^{n-1} \) for which \( w_m = v_m \) for \( m = 0, \ldots, n-2 \): \( (w_0, \ldots, w_{n-2}) \sigma = v_{n-1} \) iff \( (v_0, \ldots, v_{n-1}) \in R \).

Definition 8.4. If in a relational structure S of type \( < V, o > \) for all objects A and all \( j \in V \), \( R_{A,j} \) is operational, then S is called operational. In that case we define a corresponding algebraic structure T as follows.

We define \( o' : V \to N \) by \( j o' := j o - 1 \) for \( j \in V \); the type of T is \( < V, o' > \).

For every object A of S we define an object \( A_\Phi \) of T with the same carrier as A and for \( j \in V \), \( \sigma_{A_\Phi, j} \) is the operation on A corresponding to \( R_{A,j} \). The class of objects of T is \( \{ A_\Phi \mid A \text{ is object of } S \} \). In order to prove that actually an algebraic structure is defined this way, we have to show that if B is an object of T and C is isomorphic with B in the full algebraic structure of type \( < V, o' > \), then C is an object of T. Suppose A is an object of S, such that \( B = A_\Phi \) and \( f : A \to C \) is a bijective mapping such that \( C = [B, f] \) in the full algebraic structure. For \( [A, f] \), taken in S, obviously \( [A, f]_\Phi = C \) holds, which implies that C is an object of T. Apart from this we have proved the following theorem.

Theorem 8.3. If S is an operational relational structure, T is the algebraic structure corresponding to S, A and B are objects of S and f is a bijective mapping \( A \to B \), then f produces an isomorphic mapping \( A \to B \) in S iff f produces an isomorphic mapping \( A_\Phi \to B_\Phi \) in T.

The concepts of isomorphism correspond mutually in both structures. It is easy to prove, that the same holds for the concepts of subobject and of quotient object; this is expressed in the following theorem.

Theorem 8.4. If S is an operational relational structure, T is the algebraic structure corresponding to S, A and B are objects of S, then

\[ < A, B > \in S \iff < A_\Phi, B_\Phi > \in S \text{ in } T, \]

\[ < A, B > \in q \text{ in } S \iff < A_\Phi, B_\Phi > \in q \text{ in } T. \]
On account of theorems 8.3 and 8.4 also the concepts of homomorphism correspond:

**Theorem 8.5.** If $S$ is an operational relational structure, $T$ is the algebraic structure corresponding to $S$ and $f$ is a mapping $A \rightarrow B$, then $f$ produces a homomorphic mapping $A \rightarrow B$ in $S$ iff $f$ produces a homomorphic mapping $A \varphi \rightarrow B \varphi$ in $T$.

This result may be strengthened in the following way.

**Theorem 8.6.** If $S$ is a relational structure, $S'$ is the substructure of $S$ consisting of those objects of $S$ for which all relations are operational, $T$ is the algebraic structure corresponding to $S'$, $A$ and $B$ are objects of $S'$ and $f$ is a mapping $A \rightarrow B$, then $f$ produces a homomorphic mapping $A \rightarrow B$ in $S$ iff $f$ produces a homomorphic mapping $A \varphi \rightarrow B \varphi$ in $T$.

**Proof.** If $f$ produces a homomorphic mapping $A \varphi \rightarrow B \varphi$ in $T$, it produces a homomorphic mapping $A \rightarrow B$ in $S'$ by theorem 8.5 and then also in $S$ by theorem 3.1. Conversely suppose $f$ produces a homomorphic mapping $A \rightarrow B$ in $S$ with image object $D_f$ and coimage object $Q_f$ in $S$. Because $A$ and $B$ are objects of $S'$ all relations in $A$ and $B$ are operational and therefore, by theorem 8.1, all relations in $D_f$ induce partial operations and, by theorem 8.2, all relations in $Q_f$ induce multi-operations. Moreover, $D_f = [Q_f, b_f]$ and $Q_f = [D_f, b_f^{-1}]$ in $S$. Again by theorems 8.1 and 8.2, all relations in $Q_f$ induce partial operations and all relations in $D_f$ induce multi-operations. We have found that all relations in $D_f$ and $Q_f$ are operational and therefore $D_f$ and $Q_f$ are objects in $S'$ and $b_f$ produces an isomorphic mapping $Q_f \rightarrow D_f$ in $S'$, so $f$ produces a homomorphic mapping $A \rightarrow B$ in $S'$ and therefore, by theorem 8.5, $f$ produces a homomorphic mapping $A \varphi \rightarrow B \varphi$ in $T$.

**Remark 8.1.** Roughly speaking we may say that in those cases where homomorphism with respect to the relational and with respect to the algebraic definition both are defined, they coincide, even if in the relational structure subobjects and quotient objects are admitted, whose relations are not operational.
Algebraic structures with partial operations may be treated in a similar way as algebraic structures with total operations. We choose the concepts in such a way that they correspond to the concepts for relational structures in which all relations induce partial operations. We are not going to discuss this correspondence in detail, but only give the results for the algebraic structures.

**Definition 8.5.** The full algebraic structure with partial operations of type \(< V, o >\) is defined as follows.

An object \(A\) is a carrier set \(A\) together with for every \(j \in V\) a set \(E_{A,j} < A^o\) and a mapping \(\sigma_{A,j} : E_{A,j} \to A\). We call \(\sigma_{A,j}\) the partial operation on \(A\) of name \(j\).

Let \(A\) be an object and \(f : A \to B\) a bijective mapping; \([A,f]\) is defined to have carrier \(B\) and for \(j \in V:\)

\[E_{[A,f],j} := \{(v_0, \ldots, v_{j_0-1}, v) \mid (v_0, \ldots, v_{j_0-1}) \in E_{A,j}\}\]

and

\[\sigma_{[A,f],j} := \sigma_{A,j}^{-1} \circ \sigma_{A,j} \circ f,\]

where \(\sigma_{A,j}\) is the bijective mapping subset induced to \(f_{j_0}\) by \(E_{A,j}\); it is a mapping \(E_{A,j} \to E_{[A,f],j}\).

If \(A\) is an object and \(D \subseteq A\), then \(D\) is defined to be the carrier of a sub-object \(D\) of \(A\) and for \(j \in V:\)

\[E_{D,j} := \{(v_0, \ldots, v_{j_0-1}) \mid (v_0, \ldots, v_{j_0-1}) \in D^o \cap E_{A,j}, (v_0, \ldots, v_{j_0-1}) \sigma_{A,j} \in D\}\]

and

\[(v_0, \ldots, v_{j_0-1}) \sigma_{D,j} := (v_0, \ldots, v_{j_0-1}) \sigma_{A,j} \text{ for } (v_0, \ldots, v_{j_0-1}) \in E_{D,j}.\]

If \(A\) is an object, \(Q\) quot \(A, \pi\) is the projection \(A \to Q\) and for all \(j \in V\), for all \((v_0, \ldots, v_{j_0-1}) \in E_{A,j}\) and all \((w_0, \ldots, w_{j_0-1}) \in E_{A,j}\), for which \(v_m \pi = w_m \pi\) for \(m = 0, \ldots, j_0-1\), also \((v_0, \ldots, v_{j_0-1}) \sigma_{A,j} \pi = (w_0, \ldots, w_{j_0-1}) \sigma_{A,j} \pi\), then \(Q\) is defined to be the carrier of a quotient object \(Q\) of \(A\) and for \(j \in V:\)

\[E_{Q,j} := \{(v_0^{\pi}, \ldots, v_{j_0-1}^{\pi}) \mid (v_0, \ldots, v_{j_0-1}) \in E_{A,j}\}\]

and
Theorem 8.7. In the full algebraic structure with partial operations of type \( <V,o> \) a mapping \( f: A \to B \) produces a homomorphic mapping \( A \to B \) iff:

for all \( j \in V \) and \( (v_0, \ldots, v_{j-1}) \in A^j \):

if \( (v_0, \ldots, v_{j-1}) \in A^j, \) then \( (v_0f, \ldots, v_{j-1}f) \in B^j \) and

\[
(v_0f, \ldots, v_{j-1}f)\sigma_B^j = (v_0, \ldots, v_{j-1})\sigma_A^j f,
\]

and

if \( (v_0f, \ldots, v_{j-1}f) \in B^j, \) \( (v_0f, \ldots, v_{j-1}f)\sigma_B^j \in \text{Im} f, \)

then there exists \( (v'_0, \ldots, v'_{j-1}) \in A^j, \) such that \( v'_m f = v_m f \) for \( m = 0, \ldots, j-1. \)

In an algebraic structure, which is a substructure of a full algebraic structure, the condition of theorem 8.7 must be supplemented by the condition that \( \text{Im} f \) is carrier of a subobject of \( B \) or \( \text{Coim} f \) is carrier of a quotient object of \( A. \)

References