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LARGE DEVIATIONS FOR THE
ONE-DIMENSIONAL EDWARDS MODEL

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Abstract: In this paper we prove a large deviation principle for the empirical drift of a one-
dimensional Brownian motion with self-repulsion called the Edwards model. Our results extend
earlier work in which a law of large numbers, respectively, a central limit theorem were derived. In
the Edwards model a path of length $T$ receives a penalty $e^{-\beta H_T}$, where $H_T$ is the self-intersection
local time of the path and $\beta \in (0, \infty)$ is a parameter called the strength of self-repulsion. We
identify the rate function in the large deviation principle for the endpoint of the path as $\beta^2 I(\beta - \frac{1}{b})$,
with $I(\cdot)$ given in terms of the principal eigenvalues of a one-parameter family of Sturm-Liouville
operators. We show that there exist numbers $0 < b^* < 2 < \infty$ such that: (1) $I$ is linearly decreasing
on $[0, b^*)$; (2) $I$ is real-analytic and strictly convex on $(b^*, \infty)$; (3) $I$ is continuously differentiable
at $b^*$; (4) $I$ has a unique zero at $b^*$. (The latter fact identifies $b^*$ as the asymptotic drift of the
endpoint.) The critical drift $b^*$ is associated with a crossover in the optimal strategy of the path:
for $b \geq b^*$ the path assumes local drift $b$ during the full time $T$, while for $0 < b < b^*$ it assumes
local drift $b^*$ during time $\frac{b^* - b}{b^*} T$ and local drift $-b^*$ during the remaining time $\frac{b^* - b}{b^*} T$. Thus,
in the second regime the path makes an overshoot of size $\frac{b^* - b}{b^*} T$ in order to reduce its intersection
local time.

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1. Introduction and main results

1.1 The Edwards model

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion on $\mathbb{R}$ starting at the origin ($B_0 = 0$). Let $P$ be the Wiener measure and let $E$ be expectation with respect to $P$. For $T > 0$ and $\beta \in (0, \infty)$, define a probability law $\mathbb{Q}_T^\beta$ on paths of length $T$ by setting

$$ \frac{d\mathbb{Q}_T^\beta}{dP}[\cdot] = \frac{1}{Z_T^\beta} e^{-\beta H_T[\cdot]}, \quad Z_T^\beta = E(e^{-\beta H_T}), $$ \hspace{1cm} (1.1)

where

$$ H_T [(B_t)_{t \in [0,T]}] = \int_0^T du \int_0^T dv \delta(B_u - B_v) = \int_\mathbb{R} dx \, L(T, x)^2 $$ \hspace{1cm} (1.2)

is the Brownian intersection local time up to time $T$. The first expression in (1.2) is formal only. In the second expression the Brownian local times $L(T, x)$, $x \in \mathbb{R}$, appear. The law $\mathbb{Q}_T^\beta$ is called the $T$-polymer measure with strength of self-repulsion $\beta$. The Brownian scaling property implies that

$$ \mathbb{Q}_T^\beta [(B_t)_{t \in [0,T]} \in \cdot] = \mathbb{Q}_1^{\frac{\beta}{T}} _T \left( (\beta^{-\frac{1}{2}} B_{\beta^{\frac{1}{2}} T})_{t \in [0,T]} \in \cdot \right). $$ \hspace{1cm} (1.3)

It is known that under the law $\mathbb{Q}_T^\beta$, the endpoint $B_T$ satisfies the following central limit theorem:

Theorem 1.1 (Central limit theorem). There are numbers $a^*, b^*, c^* \in (0, \infty)$ such that for any $\beta \in (0, \infty)$:

(i) Under the law $\mathbb{Q}_T^\beta$, the distribution of the scaled endpoint $(|B_T| - b^* \beta^{\frac{1}{2}} T)/c^* \sqrt{T}$ converges weakly to the standard normal distribution.

(ii) $\lim_{T \to \infty} \frac{1}{T} \log Z_T^\beta = -a^* \beta^{\frac{3}{2}}$.

Theorem 1.1 is contained in [vdHdHK97, Theorem 2 and Proposition 1]. For the identification of $a^*, b^*, c^*$, see (2.4) below. Bounds on these numbers appeared in [vdH98, Theorem 3]. The numerical values are: $a^* \approx 2.19, b^* \approx 1.11, c^* \approx 0.63$. The law of large numbers corresponding to Theorem 1.1(i) was first obtained by Westwater [W84] (see also [vdHdHK95, Section 0.6]).

The main object of interest in the present paper is the rate function $I_\beta$ defined by \footnote{In fact, $I_\beta$ differs by a constant from what is usually called a rate function. This constant is $\lim_{T \to \infty} \frac{1}{T} \log Z_T^\beta = -a^* \beta^{\frac{3}{2}}$ (see Theorem 1.1(ii)). Hence, $I_\beta - a^* \beta^{\frac{3}{2}}$ is the true rate function.}

$$ -I_\beta(b) = \lim_{T \to \infty} \frac{1}{T} \log E \left( e^{-\beta H_T} 1_{\{B_T = b T\}} \right) $$ \hspace{1cm} (1.4)

where $B_T \approx b T$ is an abbreviation for $|B_T - b T| \leq \gamma_T$ for some $\gamma_T > 0$ such that $\gamma_T/T \to 0$ and $\gamma_T/\sqrt{T} \to \infty$ as $T \to \infty$. (We will see that the limit in (1.4) does not depend on the choice of $\gamma_T$.) It is clear from (1.3) that

$$ \beta^{-\frac{3}{2}} I_\beta(\beta^{\frac{1}{2}} b) = I_1(b), \quad b \geq 0, $$ \hspace{1cm} (1.5)

provided the limit in (1.4) exists for $\beta = 1$ and $b \geq 0$. Moreover,

$$ I_\beta(b) = I_\beta(-b), \quad b \leq 0. $$ \hspace{1cm} (1.6)

Therefore, we may restrict ourselves to $\beta = 1$ and $b \geq 0$. In the following we write $I = I_1$. 


1.2 Main results

Our first main result says that $I$ exists and has the shape exhibited in Fig. 1. 6

**Theorem 1.2** (Large deviations). Let $\beta = 1$.

(i) For any $b \geq 0$, the limit $I(b)$ in (1.4) exists and is finite.

(ii) $I$ is continuous and convex on $[0, \infty)$, and continuously differentiable on $(0, \infty)$.

(iii) There are numbers $a^{**} \in (a^*, \infty)$, $b^{**} \in (0, b^*)$ and $\rho(a^{**}) \in (0, \infty)$ such that $I(0) = a^{**}$, $I$ is linearly decreasing on $[0, b^{**}]$ with slope $-\rho(a^{**})$, is real-analytic and strictly convex on $(b^{**}, \infty)$, and attains its unique minimum at $b^*$ with $I(b^*) = a^*$ and $I''(b^*) = 1/c^2$.

(iv) $I(b) = \frac{1}{2}b^2 + O(b^{-1})$ as $b \to \infty$.

The linear piece of the rate function has the following intuitive interpretation. If $b \geq b^{**}$, then the best strategy for the path to realize the large deviation event $\{B_T \approx bT\}$ is to assume local drift $b$ during time $T$. In particular, the path makes no overshoot on scale $T$, and this leads to the real-analyticity and strict convexity of $I$ on $(b^{**}, \infty)$. On the other hand, if $0 \leq b < b^{**}$, then this strategy is too expensive, since too small a drift leads to too large an intersection local time. Therefore the best strategy now is to assume local drift $b^{**}$ during time $\frac{b^{**} - b}{\rho(b^{**})} T$ and local drift $-b^{**}$ during the remaining time $\frac{b^{**} - b}{\rho(b^{**})} T$. In particular, the path makes an overshoot on scale $T$, namely, $\frac{b^{**} - b}{2} T$, and this leads to the linearity of $I$ on $[0, b^{**}]$. At the critical drift $b = b^{**}$, $I$ is continuously differentiable.

For $b \to \infty$, $I(b)$ is determined by the Gaussian tail of $B_T$ because the intersection local time $H_T$ vanishes.

For the identification of $a^{**}, b^{**}, \rho(a^{**})$, see (2.5) below. The numerical values are: $a^{**} \approx 2.95$, $b^{**} \approx 0.85$, $\rho(a^{**}) \approx 0.78$. These estimates can be obtained with the help of the method in [vdH98].

There is an intimate connection between the rate function $I$ and the two moment generating functions $\Lambda^+, \Lambda^- : \mathbb{R} \to \mathbb{R}$ given by

$$
\Lambda^+(\mu) = \lim_{T \to \infty} \frac{1}{T} \log E(e^{-H_T e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}}),
$$

where

6In [MS87, Corollary 2.6 and Remark 2.7] it was proved that $\lim_{T \to \infty} \frac{1}{T} \log E(e^{-H_T} | B_T = 0) = -a^{**}$, which essentially gives the existence of $I(0)$ with value $a^{**}$. Furthermore, the existence of $I(b^*)$ with value $a^*$ follows from our earlier work [vdHdHK97, Proposition 1].
and the same formula for \( \Lambda^-(\mu) \) with \( \mathbb{1}_{\{B_T \geq 0\}} \) replaced by \( \mathbb{1}_{\{B_T \leq 0\}} \). Obviously, \( \Lambda^+(\mu) = \Lambda^-(\mu) \) for any \( \mu \in \mathbb{R} \), provided one limit exists.

Our second main result says that \( \Lambda^+ \) exists and has the shape exhibited in Fig. 2, and that its Legendre transform is equal to \( I \) on \([0, \infty)\).

**Theorem 1.3 (Exponential moments).** Let \( \beta = 1 \).

(i) For any \( \mu \in \mathbb{R} \), the limit \( \Lambda^+(\mu) \) in (1.7) exists and is finite.

(ii) \( \Lambda^+ \) equals \( -a^{**} \) on \((0, -\rho(a^{**})] \), is real-analytic and strictly convex on \((-\rho(a^{**}), \infty)\), and satisfies \( \lim_{\mu \to -\rho(a^{**})} (\Lambda^+)'(\mu) = b^{**} \).

(iii) \( \Lambda^+(\mu) = \frac{1}{2} \mu^2 + \mathcal{O}(\mu^{-1}) \) as \( \mu \to \infty \).

(iv) The restriction of \( I \) to \([0, \infty)\) is the Legendre transform of \( \Lambda^+ \), i.e.,

\[
I(b) = \max_{\mu \in \mathbb{R}} \left[ b\mu - \Lambda^+(\mu) \right], \quad b \geq 0. \tag{1.8}
\]

![Fig. 2. Qualitative picture of \( \mu \mapsto \Lambda^+(\mu) \).](image)

As a consequence of Theorem 1.3(ii), the maximum on the right-hand side of (1.8) is attained at some \( \mu > -\rho(a^{**}) \) if \( b > b^{**} \) and at \( \mu = -\rho(a^{**}) \) if \( 0 \leq b \leq b^{**} \). Analogous assertions hold for \( \Lambda^- \), in particular, the restriction of \( I \) to \((0, \infty)\) is the Legendre transform of \( \Lambda^- \). Since \( \Lambda^-(\mu) = \Lambda^+(\mu) \), the moment generating function equals

\[
\Lambda(\mu) = \lim_{T \to \infty} \frac{1}{T} \log E(e^{-H_T} e^{\mu B_T}) = \max \{ \Lambda^+(\mu), \Lambda^-(\mu) \} = \Lambda^+(|\mu|), \quad \mu \in \mathbb{R}, \tag{1.9}
\]

which is symmetric and strictly convex on \( \mathbb{R} \), and non-differentiable at 0, with \( \Lambda(0) = -a^* \) and \( \lim_{\mu \to 0} \Lambda'(\mu) = b^* \).

The outline of the present paper is as follows. In Section 2 we introduce some preparatory material that will be needed in the sequel. Two basic propositions are presented in Section 3: a representation for the probabilities of certain events under the Edwards measure, and an integrable majorant under which the dominated convergence theorem can be applied. In Section 4 we carry out the proofs of Theorems 1.2-1.3. Some more refined results about the Edwards model (which will be needed in a
forthcoming paper [vdHdHK02]) appear in Section 5. Finally, Section 6 contains a technical proof of a result used in Section 5.

2. Preliminaries

In this section we provide some tools that are needed for the proofs of our main results in Section 1.2. These tools are taken from [vdHdH95], [vdHdHK97] and references cited therein. Section 2.1 introduces the Sturm-Liouville operators that determine the constants. Section 2.2 provides the ingredients that are needed for the formulation of the Ray-Knight Theorems (describing the joint distribution of the endpoint and the local times), and contains a mixing property. Section 2.3 gives a spectral decomposition of a function describing the “overshoots” of the path (i.e., the pieces outside the interval between the starting point and the endpoint) in terms of shifts of the Airy function, which plays an important role in various estimates.

2.1 Sturm-Liouville operators and definition of the constants

In [vdHdH95, Section 0.4] we introduced and analyzed a family of Sturm-Liouville operators $\mathcal{K}^a: L^2[0, \infty) \cap C^2[0, \infty) \to C[0, \infty)$, indexed by $a \in \mathbb{R}$, defined as

$$ (\mathcal{K}^a x)(h) = 2hx''(h) + 2x'(h) + (ah - h^2)x(h), \quad h \geq 0. \tag{2.1} $$

The operator $\mathcal{K}^a$ is symmetric and has a largest eigenvalue $\rho(a) \in \mathbb{R}$ with multiplicity one. The corresponding strictly positive (and $L^2$-normalized) eigenfunction $x_a: [0, \infty) \to (0, \infty)$ is real-analytic and vanishes faster than exponential at infinity, more precisely,

$$ \lim_{h \to \infty} h^{-\frac{3}{2}} \log x_a(h) = -\frac{\sqrt{2}}{3}. \tag{2.2} $$

The eigenvalue function $\rho: \mathbb{R} \to \mathbb{R}$ has the following properties:

(a) $\rho$ is real-analytic;

(b) $\rho$ is strictly log-convex, strictly convex and strictly increasing;

(c) $\lim_{a \to -\infty} \rho(a) = -\infty, \rho(0) < 0, \lim_{a \to \infty} \rho(a) = \infty.$

In terms of this object, the numbers $a^*, b^*, c^*$ appearing in Theorem 1.1 are defined as

$$ \rho(a^*) = 0, \quad b^* = \frac{1}{\rho'(a^*)}, \quad c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}, \tag{2.4} $$

while the numbers $a^{**}, b^{**}$ appearing in Theorem 1.2 are defined as

$$ a^{**} = 2^{\frac{1}{3}}(-a_0), \quad b^{**} = \frac{1}{\rho'(a^{**})}, \tag{2.5} $$

where $a_0 \approx -2.3381$ is the largest zero of the Airy function:

$$ y''(h) = hy(h) \text{ that vanishes at infinity.} \tag{2.6} $$

From [vdHdHK97, Lemma 6] we know that $a^* < -a_0$. Therefore $a^{**} > a^*$, which in turn implies that $b^{**} < b^*$.

2.2 Squared Bessel processes, a Girsanov transformation, and a mixing property

The basic tools in our study of the Edwards model are the Ray-Knight Theorems, which give a description of the joint distribution of the local time process $(L(T, x))_{x \in \mathbb{R}}$ and the endpoint $B_T$. These will be summarized in Proposition 3.1 below. The key objects entering into this description are introduced here.

The first key ingredients are:
(i) a squared two-dimensional Bessel process (BESQ²), \( X = (X_v)_{v \geq 0} \),
(ii) a squared zero-dimensional Bessel process (BESQ⁰), \( X^* = (X_v^*)_{v \geq 0} \),

and their additive functionals

\[
A(t) = \int_0^t X_v \, dv, \quad A^*(t) = \int_0^t X_v^* \, dv, \quad t \geq 0.
\] (2.7)

The respective generators of BESQ² and BESQ⁰ are given by \(^7\)

\[
Gf(h) = 2hf''(h) + 2f'(h), \quad G^*f(h) = 2hf''(h),
\] (2.8)

for sufficiently smooth functions \( f: [0, \infty) \to \mathbb{R} \). For \( h \geq 0 \), we write \( \mathbb{P}_h \) and \( \mathbb{P}_h^* \) to denote the probability law of \( X \) and \( X^* \) given \( X_0 = h \) and \( X^*_0 = h \), respectively. BESQ² takes values in \( C^+ = C^+[0, \infty) \), the set of non-negative continuous functions on \( [0, \infty) \). It has 0 as an entrance boundary, which is not visited in finite positive time with probability one. BESQ⁰ takes values in \( C_0^+ = C_0^+[0, \infty) \), the subset of those functions in \( C^+ \) that hit zero and afterwards stay at zero. It has 0 as an absorbing boundary, which is visited in finite time with probability one.

The second key ingredient is a certain Girsanov transformation, which turns BESQ² into a diffusion with strong recurrence properties. Namely, the process \( (D_y^{(a)})_{y \geq 0} \) defined by

\[
D_y^{(a)} = \frac{x_a(Y_v)}{x_a(Y_0)} \exp \left\{ - \int_0^y [(X_v)^2 - aX_v + \rho(a)] \, dv \right\}, \quad y \geq 0,
\] (2.9)

is a martingale under \( \mathbb{P}_h \) for any \( h \geq 0 \) and hence serves as a density with respect to a new Markov process in the sense of a Girsanov transformation. More precisely, the transformed process, which we also denote by \( X = (X_v)_{v \geq 0} \), has the transition density

\[
\mathbb{P}_h^a(h_1, h_2) \, dh_2 = \mathbb{E}_h \left( D_y^{(a)} 1_{\{Y_y \in dh_2\}} \right), \quad y, h_1, h_2 \geq 0.
\] (2.10)

We write \( \mathbb{P}_h^a \) to denote the probability law of the transformed process \( X \) given \( X_0 = h \). This transformed process possesses the invariant distribution \( x_a(h)^2 \, dh \), and so

\[
\mathbb{P}_h^a = \int_0^\infty dh \, x_a(h)^2 \, \mathbb{P}_h^a
\] (2.11)

is its probability law in equilibrium. The transformed process is reversible under \( \mathbb{P}_h^a \), since BESQ² is reversible with respect to the Lebesgue measure. Hence, \( x_a(h_1)^2 \, \mathbb{P}_h^a(h_1, h_2) \) is symmetric in \( h_1, h_2 \geq 0 \) for any \( y \geq 0 \).

The third key ingredient is the time-changed transformed process \( Y = X \circ A^{-1} = (X_{A^{-1}(t)})_{t \geq 0} \). We write \( \mathbb{P}_h^a \) to denote the probability law of \( Y \) given \( Y_0 = h \). This process possesses the invariant distribution \( \frac{1}{\rho'(a)} x_a(h)^2 \, dh \), and so

\[
\mathbb{P}_h^a = \frac{1}{\rho'(a)} \int_0^\infty dh \, h x_a(h)^2 \, \mathbb{P}_h^a
\] (2.12)

is its probability law in equilibrium. Both transformed processes \( X \) and \( Y = X \circ A^{-1} \) are ergodic.

The following mixing property will be used frequently in the sequel. By \( \langle \cdot, \cdot \rangle \) we denote the inner product on \( L^2 = L^2[0, \infty) \), and we write \( \langle f, g \rangle = \int_0^\infty dh \, h f(h) g(h) \) for the inner product on \( L^2 \) weighted with the identity. The latter space will be denoted by \( L^2, \bigcirc = L^{2, \bigcirc}[0, \infty) \).

\(^7\)BESQ⁰ is sometimes called Feller’s diffusion.
Proposition 2.1. Fix \(a \in \mathbb{R}\) and fix measurable functions \(f, g: [0, \infty) \to \mathbb{R}\) such that \(f/\text{id}, g \in L^2,\). For any family of measurable functions \(f_s, g_s: [0, \infty) \to \mathbb{R}, s \geq 0,\) such that \(f_s/\text{id}, g_s \in L^2, s \geq 0,\) and \(f_s \to f, g_s \to g\) as \(s \to \infty\) uniformly on compacts and in \(L^2,\) and for any family \(a_s, s \geq 0,\) such that \(a_s \to a\) as \(s \to \infty,\)

\[
\lim_{s \to \infty} \mathbb{E}^{a_s} \left( \frac{f_s(X_0)}{x_{a_s}(X_0)} \frac{g_s(Y_s)}{x_{a_s}(Y_s)} \right) = (f, x_a) \frac{1}{\rho(a)} (g, x_a). \tag{2.13}
\]

This proposition is a slight extension of Proposition 3 in [vdHdHK97]; we omit the proof.

2.3 BESQ\(^0\), the Airy function, and a spectral decomposition

For \(a < a^{**},\) introduce the function \(y_a: [0, \infty) \to (0, \infty]\) defined by

\[
y_a(h) = \mathbb{E}^h \left( e^{-\alpha_h} \right) \tag{2.14}
\]

(As a consequence of (2.17) and Proposition 2.2 below, the expectation on the right-hand side is infinite for \(a > a^{**}\).) It is known (see [vdHdHK97], Lemma 5) that \(y_a\) is equal to a normalized scaled shift of the Airy function \(\text{Ai}:\)

\[
y_a(h) = \frac{\text{Ai}(2^{-\frac{3}{4}}(h-a))}{\text{Ai}(-2^{-\frac{3}{4}}a)}, \quad h \geq 0, \tag{2.15}
\]

It is well-known (see [E56, p. 43] and (6.2) below) that \(y_a\) vanishes faster than exponential at infinity:

\[
\lim_{h \to \infty} h^{-\frac{3}{2}} \log y_a(h) = -\frac{\sqrt{2}}{3}. \tag{2.16}
\]

An important role is played in the sequel by the function \(w: [0, \infty)^2 \to [0, \infty]\) defined by

\[
w(h, t) \, dt = \mathbb{E}^h \left( e^{-\int_0^\infty (X_s^*)^2 \, ds} \right) \tag{2.17}
\]

It is easily seen from (2.7) and (2.14) that \(\int_0^\infty dt \, e^{it} w(h, t) = y_a(h)\) for \(a < a^{**}\). We also have the following representation for \(w(h, t)\) derived in [vdHdHK97, Lemma 7]:

\[
w(h, t) = \mathbb{E}_h \left( e^{-\int_0^t B_s \, ds} \right) \varphi(t), \tag{2.18}
\]

\[
\varphi(t) = \frac{P_h(T_0 < t)}{t} = (8\pi)^{-\frac{1}{2}} t^{-\frac{3}{2}} e^{-\frac{2}{3}t^2}, \tag{2.19}
\]

with \(T_0 = \inf\{t > 0: B_t = 0\}\) the first time \(B\) hits zero. (We write \(P_h\) and \(E_h\) for probability and expectation with respect to standard Brownian motion \(B\) starting at \(h \geq 0,\) so that \(P = P_0, E = E_0.\))

We will need the following expansion of the function \(w\) in terms of shifts of the Airy function:

Proposition 2.2.

(i) For any \(\varepsilon > 0,\)

\[
w(h, t) = \sum_{k=0}^{\infty} e^{\alpha(k)(t-\varepsilon)} \langle w(\cdot, \varepsilon), e_k(\cdot) \rangle e_k(h), \quad h \geq 0, t \geq \varepsilon, \tag{2.20}
\]

where

\[
a^{(k)} = 2\frac{1}{2} a_k, \quad e_k(h) = c_k \text{Ai}(2^{-\frac{3}{4}}(h + a^{(k)})), \quad h \geq 0, \tag{2.21}
\]

with \(a_k\) the \(k\)-th largest zero of \(\text{Ai}\) and with \(c_k\) chosen such that \(\|e_k\|_2 = 1.\)
(ii) There exist constants $K_1, K_2, K_3 \in (0, \infty)$ such that
\begin{align}
-a^{(k)} &\sim K_1 k^{\frac{3}{2}}, \quad k \to \infty, \quad (2.22) \\
\int_0^\infty h e_k(h)^2 \, dh &\leq K_2 k^{\frac{3}{2}} \quad \forall k, \quad (2.23) \\
\int_0^\infty \frac{1}{h} e_k(h)^2 \, dh &\leq K_3 k^{\frac{1}{2}} \quad \forall k. \quad (2.24)
\end{align}

(Note that $a^{(0)} = -a^{**}$ by (2.5).)

Proof. (i) The proof comes in steps. We write $c$ for a generic constant in $(0, \infty)$ whose value may change from appearance to appearance.

1. Let $\mathcal{K}^*$ be the second-order differential operator on $C_0^\infty = C_0^\infty [0, \infty)$, the set of smooth functions $x : [0, \infty) \to \mathbb{R}$ that vanish at zero, defined by
\begin{equation}
(\mathcal{K}^* x)(h) = \begin{cases}
2x''(h) - hx(h) & \text{if } h > 0, \\
0 & \text{if } h = 0.
\end{cases} \quad (2.25)
\end{equation}

This operator is symmetric with respect to the $L^2$-inner product on $L_0^2 = L^2 \cap C_0^\infty$. Furthermore, we can identify all the eigenvalues and eigenfunctions of $\mathcal{K}^*$ in $L_0^2$ in terms of scaled shifts of the Airy function. Namely, a comparison of (2.6) and (2.25) shows that the $k$-th eigenspace is spanned by the eigenfunction $e_k : [0, \infty) \to \mathbb{R}$ given in (2.21) and the $k$-th eigenvalue is $a^{(k)}$, $k \in \mathbb{N}_0$.

2. We next show that $\mathcal{K}^*$ has a compact inverse on $L^2$. Therefore, this inverse has an orthonormal basis of eigenvectors in $L^2$, and hence the same is true for $\mathcal{K}^*$ itself. Consequently, $(e_k)_{k \in \mathbb{N}_0}$ is an orthonormal basis of $L^2$. This fact will be needed later.

We begin by identifying the inverse of $\mathcal{K}^*$. To do so, we follow [G81]. Let
\begin{equation}
y_1(u) = Bi(2^{\frac{3}{2}}u) - Bi(0) \frac{Ai(2^{\frac{3}{2}}u)}{Ai(0)}, \quad y_2(u) = Ai(2^{\frac{3}{2}}u), \quad (2.26)
\end{equation}
where $Ai$ is the Airy function and $Bi$ is another, linearly independent, solution to (2.6) (for the precise definitions of $Ai$ and $Bi$, see [AS70, 10.4.1–10.4.3]). Hence, both $y_1$ and $y_2$ solve $\mathcal{K}^* y = 0$, $y_1$ satisfies the boundary condition at zero ($y_1(0) = 0$), while $y_2$ satisfies the boundary condition at infinity ($y_2 \in L^2$). Let $G : [0, \infty)^2 \to \mathbb{R}$ (Green function) be defined by
\begin{equation}
G(u, v) = K y_1(u \wedge v) y_2(u \vee v) \quad \text{with} \quad K = -2y_1'(0)y_2(0). \quad (2.27)
\end{equation}
Let $\Gamma$ be the operator on $L^2$ defined by
\begin{equation}
(\Gamma y)(u) = \int_0^\infty G(u, v) y(v) \, dv. \quad (2.28)
\end{equation}

According to [G81, Proposition 2.15], $x = \Gamma y$ is a weak solution of the equation $\mathcal{K}^* x = y$ with boundary condition $x(0) = 0$, for any $y \in L^2$. In fact, we can adapt the proof of [G81, Proposition 9.12] to see that $\Gamma$ is the inverse of $\mathcal{K}^*$, since $\mathcal{K}^* x = 0$ does not have solutions in $L^2$ that satisfy the boundary condition $x(0) = 0$. Hence, we are done once we show that $\Gamma$ is a compact operator.

3. By [G81, Theorem 8.54], it suffices to show that $\Gamma$ is a Hilbert-Schmidt operator, i.e., $G$ is square-integrable on $[0, \infty)^2$. In order to show this, we first note that (2.27) gives
\begin{equation}
\int_0^\infty du \int_0^\infty dv \, G^2(u, v) = 2K^2 \int_0^\infty du \int_0^u dv \, y_2(u)^2 y_1(v)^2. \quad (2.29)
\end{equation}
Substitute (2.26) to see that, since \( \text{Ai} \in L^2 \), it suffices to show that
\[
\int_0^\infty du \int_0^u dv \, \text{Ai}(u)^2 \text{Bi}(v)^2 < \infty. \tag{2.30}
\]
Since \( \text{Bi} \) is locally bounded and \( \text{Ai} \in L^2 \), the latter amounts to
\[
\int_1^\infty du \int_1^u dv \, \text{Ai}(u)^2 \text{Bi}(v)^2 < \infty. \tag{2.31}
\]
We next use [AS70, 10.4.59 and 10.4.63], which shows that
\[
\text{Ai}(u) \leq ce^{-\frac{1}{3}e^\frac{2}{3}u^{\frac{3}{2}}}, \quad \text{Bi}(v) \leq ce^{-\frac{1}{3}e^\frac{2}{3}v^{\frac{3}{2}}}, \quad u, v \geq 1. \tag{2.32}
\]
Hence
\[
\int_1^\infty du \int_1^u dv \, \text{Ai}(u)^2 \text{Bi}(v)^2 \leq c^4 \int_1^\infty du \int_1^u dv \, v^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-\frac{1}{3}e^\frac{2}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} \tag{2.33}
\]
Use partial integration to see that
\[
\int_1^u dv \, v^{-\frac{1}{2}} e^{-\frac{1}{3}e^\frac{2}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} = \frac{1}{2} \int_1^u dv \, v^{-\frac{1}{2}} \frac{d}{dv} \left( e^{-\frac{1}{3}e^\frac{2}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} \right) \leq \frac{1}{2} \left[ v^{-\frac{1}{2}} e^{-\frac{1}{3}e^\frac{2}{3}(u^{\frac{3}{2}} - v^{\frac{3}{2}})} \right]_{v=1} \leq \frac{1}{2} u^{-1}, \quad u \geq 1. \tag{2.34}
\]
Hence
\[
\int_1^\infty du \int_1^u dv \, \text{Ai}(u)^2 \text{Bi}(v)^2 \leq \frac{1}{2} c^4 \int_1^\infty du \, u^{-\frac{3}{2}} < \infty. \tag{2.35}
\]
This proves that \( \Gamma \) is a compact operator, so that \( (e_k)_{k \in \mathbb{N}_0} \) is an orthonormal basis of \( L^2 \).

4. To prove the expansion in (2.20), we now need the following:

**Lemma 2.3.** For any \( \varepsilon > 0 \), the function \( w \) is a solution of the initial-boundary-value problem
\[
\begin{align*}
\partial_t w(h,t) &\quad = \mathcal{K}^*(w(\cdot,t))(h), & h \geq 0, t > \varepsilon, \\
w(0,t) &\quad = 0, & t \geq \varepsilon, \tag{2.36}
\end{align*}
\]
and the initial value \( w(\cdot, \varepsilon) \) lies in \( C_0^\infty \).

**Proof.** Use the Markov property at time \( s > 0 \) in (2.18) to see that, for any \( h > 0 \) and \( t > s \),
\[
w(h,t) = E_\frac{1}{2} \left( e^{-\frac{1}{2} h^2 (2B_s - 4d \{ I_{s > \varepsilon} \} \w(2B_s, t - s))} \right). \tag{2.37}
\]
Now differentiate with respect to \( s \) at \( s = 0 \), to obtain
\[
0 = -hw(h,t) + 2(\partial_t)^2 w(h,t) - \partial_t w(h,t) = \mathcal{K}^*(w(\cdot,t))(h) - \partial_t w(h,t). \tag{2.38}
\]
This shows that the partial differential equation in (2.36) is satisfied on \( (0, \infty)^2 \). It is clear that it is also satisfied at the boundary where \( h = 0 \), since \( w(0,t) = 0 \) for all \( t > 0 \) (recall (2.17–2.18)). \( \square \)

5. From (2.18) it follows that \( w(\cdot, \varepsilon) \in C_0^\infty \) for any \( \varepsilon > 0 \). A spectral decomposition in terms of the eigenvalues \( (\alpha(k))_{k \in \mathbb{N}_0} \) and the eigenfunctions \( (e_k)_{k \in \mathbb{N}_0} \) of \( \mathcal{K}^* \) shows that (2.36) has the solution given in (2.20).

(ii) In [AS70, 10.4.94,10.4.96,10.4.97,10.4.105] the following asymptotics for the Airy function can be found. As \( k \to \infty \),
\[
-a_k \sim ck^{\frac{3}{2}}, \quad a_{k-1} - a_k \sim ck^{-\frac{1}{4}}, \quad \max_{[a_k, a_{k-1}]} |\text{Ai}| \sim ck^{-\frac{1}{6}}, \quad |\text{Ai}'(a_k)| \sim ck^{\frac{1}{6}}. \tag{2.39}
\]
We will use these in combination with the observation that, by (2.6), \( \text{Ai} \) is convex (concave) between any two successive zeroes where it is negative (positive).
The first assertion in (2.39) is (2.22). To prove (2.23–2.24), we write the recursion
\[
c_k^{-2} = \int_0^\infty \text{Ai}(2^{-\frac{1}{2}}(h+a^{(k)}))^2 \, dh = c_{k-1}^{-2} + 2\frac{1}{h} \int_{a_k}^{a_{k-1}} \text{Ai}(h)^2 \, dh. \tag{2.40}
\]
Using the second and third assertion in (2.39), we find that \(\int_{a_k}^{a_{k-1}} \text{Ai}(h)^2 \, dh \asymp k^{-\frac{2}{3}}\) and hence that \(c_k^{-2} \asymp k^{\frac{1}{3}}\). In a similar way, we find that
\[
\int_0^\infty h\text{Ai}(2^{-\frac{1}{2}}(h+a^{(k)}))^2 \, dh \leq c_k, \quad \int_0^\infty \frac{1}{h} \text{Ai}(2^{-\frac{1}{2}}(h+a^{(k)}))^2 \, dh \leq c_k^{\frac{2}{3}}. \tag{2.41}
\]
Combining (2.41) with (2.21) and \(c_k^{-2} \asymp k^{\frac{1}{3}}\), we obtain (2.23–2.24). \(\square\)

3. Two Basic Propositions

In this section we present the basic tools of our proofs. Section 3.1 introduces the Ray-Knight Theorems, which give a flexible representation for the probabilities of certain events under the Edwards measure. Section 3.2 exhibits an integrable majorant under which limits may be interchanged with integrals.

3.1 Ray-Knight representation

In this section we formulate the Ray-Knight Theorems that were already announced in Section 2.2. We do this in the compact form derived in [vdHdHK97, Section 1.2], which is best suited for the arguments in the sequel.

For any measurable set \(G \subset C_0\), define \(w_G \colon [0, \infty) \times [0, \infty) \to \mathbb{R}\) by
\[
w_G(h,t) \, dt = \mathbb{E}_h^\tau\left( e^{-\int_0^\tau (X_s^*)^2 \, ds} \mathbb{1}_{\{X^* \in G\} \mathbb{1}_{\{A^*(\infty) \in dt\}} \right). \tag{3.1}
\]
It is clear that \(w_G\) is increasing in \(G\). For \(G = C_0\), \(w_{C_0}\) is identical to \(w\) defined in (2.17).

For \(y \geq 0\), denote by \(C^+[0,y]\) the set of non-negative continuous functions on \([0,y]\). Then the set \(C^+ = \bigcup_{y \geq 0} \{(y) \times C^+[0,y]\}\) is the appropriate state space of the pair \((B_T, L(T, B_{T-t} \cdots )_{|0,B_T}|)\) consisting of the endpoint \(B_T(\geq 0)\) and the local time process between the endpoint \(B_T\) and the starting point \(0\).

Proposition 3.1 (Ray-Knight representation). Fix \(a \in \mathbb{R}\). Then, for any \(T > 0\) and any measurable sets \(G^+, G^- \subset C_0\) and \(F \subset C\),
\[
\qquad e^{\alpha T} \mathbb{E}(e^{-H_T} e^{-\rho(a)B_T}) \mathbb{1}_{\{L(T,B_T+\cdots ) \in G^+\}} \mathbb{1}_{\{(B_T,L(T,B_{T-t} \cdots )_{|0,B_T}) \in F\}} \mathbb{1}_{\{(L,-\cdots ) \in G^-\}}
\quad = \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{1}_{\{t_1 + t_2 \leq T\}} e^{\alpha(t_1 + t_2)}
\quad \times \mathbb{E}_a\left( \mathbb{1}_{\{(A-1(t_1-t_2),X|_{0,A-1(T-t_1-t_2)}) \in F\}} \frac{w_{G^+}(X_0,t_1) w_{G^-}(Y_{T-t_1-t_2},t_2)}{x_a(X_0) x_a(Y_{T-t_1-t_2})} \right). \tag{3.2}
\]

Proof. We briefly indicate how (3.2) comes about. Details can be found in [vdHdHK97, Section 1.2]. Recall the notation in Section 2.2. Fix \(T > 0\). Then, according to the Ray-Knight Theorems, for any \(t_1, t_2, h_1, h_2 \geq 0\) and \(y > 0\), conditioned on the event
\[
\{B_T = y\} \cap \{L(T,B_T) = h_1\} \cap \{L(T,0) = h_2\} \cap \left\{ \int_{B_T}^\infty L(T,x) \, dx = t_1 \right\} \cap \left\{ \int_0^\infty L(T,-x) \, dx = t_2 \right\}, \tag{3.3}
\]
the joint distribution of the processes

\[ L(T, B_T + \cdot), \quad L(B_T - \cdot)|_{[0,y]}, \quad L(T, -\cdot), \]

on \( C_0^+ \times C^+[0,y] \times C_0^+ \) is equal to the joint distribution of the processes

\[ X^{*,1}(\cdot), \quad X(\cdot)|_{[0,y]}, \quad X^{*,2}(\cdot), \]

under

\[ \mathbb{P}_{h_1}^+ (\cdot | A^*(\infty) = t_1) \otimes \mathbb{P}_{h_2}^+ (\cdot | A(y) = T - t_1 - t_2, X_y = h_2) \otimes \mathbb{P}_{h_2}^+ (\cdot | A^*(\infty) = t_2), \]

where \( X \) is BESQ and \( X^{*,1}, X^{*,2} \) are independent copies of BESQ. In particular, the intersection local time in (1.2) has the representation

\[ H_T^{law} = \int_0^\infty (X_v^{*,1})^2 dv + \int_0^y (X_v)^2 dv + \int_0^\infty (X_v^{*,2})^2 dv. \]

Use (2.10) for \( y = A^{-1}(T-t_1-t_2) \) and note that, on the event \( \{ A(T-t_1-t_2) = y \} \cap \{ X_0 = h_1, X_y = h_2 \} \), (2.9) becomes

\[ D_y^{(a)} = x_a(h_2) \exp \left\{ - \int_0^y (X_v)^2 dv \right\} e^{a(T-t_1-t_2)} e^{-\rho(a)y}, \]

which implies that

\[ e^{aT} e^{-\rho(a)B_T^{law}} \overset{law}{=} \frac{x_a(h_1)}{x_a(h_2)} D_y^{(a)} e^{a(t_1+t_2)} e^{-f_0^{*}(X_v^{*,1})^2 dv} e^{-f_0^{*}(X_v^{*,2})^2 dv}. \]

Integrate the left-hand side with respect to \( P \) and the right-hand side with respect to the measure in (3.6), and absorb the term \( D_y^{(a)} \) into the notation of the transformed diffusion. Integrate over \( h_1, h_2 \geq 0 \) and note that \( X_0 \) has the distribution \( x_a(h_1)^2 dh_1 \) under \( \widehat{E}_a^* \). Finally, use the notation in (3.1), to obtain (3.2).

\[ \square \]

3.2 Domination

In order to perform the limit \( T \to \infty \) on the right-hand side of (3.2), we will need the dominated convergence theorem to interchange this limit with the integrals over \( t_1 \) and \( t_2 \). The following proposition provides the required domination.

**Proposition 3.2 (Domination).** For any \( a_s, s \geq 0 \), in a compact subset of \( (-\infty, a^{**}) \), the map

\[ (t_1, t_2) \mapsto \sup_{s \geq 0} e^{a_s(t_1+t_2)} \widehat{E}_a^* \left( \frac{w(X_0, t_1)}{x_a(Y_0)} \frac{w(Y_s, t_2)}{x_a(Y_s)} \right) \]

is integrable over \( (0, \infty)^2 \).

**Proof.** Under the expectation in (3.10) we make a change of measure from the invariant distribution of \( X \) to the invariant distribution of \( Y \), i.e., we replace \( \widehat{E}_a^* \) by \( \widehat{E}_a^* \) and add a factor of \( \rho'(a_s)/Y_0 \). Fix \( 1 < p \leq q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), apply Hölder’s inequality and use the stationarity of \( Y \) under \( \widehat{E}_a^* \). This gives, for any \( t_1, t_2 > 0 \), the bound

\[ \frac{w(Y_0, t_1)}{Y_0 x_a(Y_0)} \frac{w(Y_s, t_2)}{x_a(Y_s)} \leq W_p^{(1)}(t_1) W_q^{(2)}(t_2), \]

where the functions \( W_p^{(1)}, W_q^{(2)} : (0, \infty) \to (0, \infty) \) are defined by

\[ W_p^{(1)}(t) = \widehat{E}_a^* \left( \left( \frac{w(Y_0, t)}{Y_0 x_a(Y_0)} \right)^p \right)^{1/p}, \quad W_q^{(2)}(t) = \widehat{E}_a^* \left( \left( \frac{w(Y_0, t)}{x_a(Y_0)} \right)^q \right)^{1/q}. \]
Hence, it suffices to show that the maps

\[ t \mapsto e^{a_s t} W_p^{(1)}(t), \quad t \mapsto e^{a_s t} W_q^{(2)}(t), \]  

are integrable at zero and at infinity, uniformly in \( s \), for a suitable choice of \( p \) and \( q \). In the proof of Proposition 4 in \cite{vdHdHK97} we showed that \( W_p^{(1)} \) and \( W_q^{(2)} \), with \( a_s \) replaced by \( a^* \), are integrable at zero when \( p < q \) with \( p,q \) sufficiently close to 2. An inspection of the proof shows that they are actually integrable at zero uniformly in \( s \).

We will show that \( t \mapsto e^{a_s t} W_2^{(1)}(t) \) and \( t \mapsto e^{a_s t} W_2^{(2)}(t) \) are integrable at infinity uniformly in \( s \). This will complete the proof because the left-hand side of (3.11) does not depend on \( p,q \).

We use Proposition 2.2 with \( \varepsilon = 1 \) together with the representations (recall (2.12))

\[ W_2^{(1)}(t) = \frac{1}{\sqrt{\rho'(a_s)}} \left( \int_0^\infty dh \, \frac{1}{h} w(h,t) \right)^{\frac{1}{2}}, \quad W_2^{(2)}(t) = \frac{1}{\sqrt{\rho'(a_s)}} \left( \int_0^\infty dh \, h w(h,t) \right)^{\frac{1}{2}}. \]  

Using (2.20), the Cauchy-Schwarz inequality and the fact that \( \|e_k\|_2 = 1 \), we estimate

\[ W_2^{(1)}(t) \leq \frac{1}{\sqrt{\rho'(a_s)}} \left( \left\| w(\cdot, 1) \right\|^2_2 \sum_{k_1, k_2 = 0}^\infty e^{a^{(k_1)}(t-1)} \left( \int_0^\infty \frac{1}{h} e_{k_1}(h) e_{k_2}(h) \right)^{\frac{1}{2}} dh \right)^{\frac{1}{2}}, \quad t \geq 1. \]  

Using the Cauchy-Schwarz inequality for the last integral, we obtain the bound

\[ W_2^{(1)}(t) \leq \frac{\left\| w(\cdot, 1) \right\|^2_2}{\sqrt{\rho'(a_s)}} \sum_{k = 0}^\infty e^{a^{(k)}(t-1)} \left( \int_0^\infty e_k(h)^2 \right)^{\frac{1}{2}} dh, \quad t \geq 1. \]  

In the same way, we find that

\[ W_2^{(2)}(t) \leq \frac{\left\| w(\cdot, 1) \right\|^2_2}{\sqrt{\rho'(a_s)}} \sum_{k = 0}^\infty e^{a^{(k)}(t-1)} \left( \int_0^\infty h e_k(h)^2 \right)^{\frac{1}{2}} dh, \quad t \geq 1. \]  

Substitute (2.23–2.24) into (3.16–3.17) and use that \( a^{(k)} \leq a^{(0)} = -a^{**} \), to estimate

\[ W_2^{(1)}(t) \vee W_2^{(2)}(t) \leq c e^{-a^{**}(t-2)} \sum_{k = 0}^\infty e^{a^{(k)} k^3}, \quad t \geq 2. \]  

By (2.22), the sum in the right-hand side converges. Since \( a_s < a^{**}, s \geq 0, \) is bounded away from \( a^{**} \), it is now obvious that the maps \( t \mapsto e^{a_s t} W_2^{(1)}(t) \) and \( t \mapsto e^{a_s t} W_2^{(2)}(t) \) are integrable at infinity uniformly in \( s \).

\[ \square \]  

4. Proof of Theorems 1.2–1.3

In Sections 4.2–4.3 we give the proof of Theorems 1.2–1.3 with the help of Propositions 3.1–3.2. In Section 4.1 we derive a technical proposition that is needed along the way.

4.1 Growth rate of a restricted moment generating function

Abbreviate \( B_{[0,T]} = \{ B_t : t \in [0, T] \} \) for the range of the path up to time \( T \). For \( T > 0 \) and \( \delta, C \in (0, \infty) \), define events

\[ \mathcal{E}(\delta; T) = \{ B_{[0,T]} \subset [-\delta, B_T + \delta] \}, \]

\[ \mathcal{E}^\leq(\delta, C; T) = \left\{ \max_{x \in [-\delta, \delta]} L(T, x) \leq C, \quad \max_{x \in [B_T - \delta, B_T + \delta]} L(T, x) \leq C \right\}. \]

In words, on \( \mathcal{E}(\delta; T) \) the path does not visit more than the \( \delta \)-neighborhood of the interval between its starting point 0 and its endpoint \( B_T \), while on \( \mathcal{E}^\leq(\delta, C; T) \) its local times in the \( \delta \)-neighborhoods of these two points are bounded by \( C \). Note that both \( \mathcal{E}(\infty; T) \) and \( \mathcal{E}^\leq(\delta, \infty; T) \) are the full space.
Proposition 4.1. Fix $\mu > -\rho(a^*)$. Then, for any $\delta, C \in (0, \infty]$ there exists a constant $K_1(\delta, C) \in (0, \infty)$ such that, for any $\mu_T \to \mu$ as $T \to \infty$,

$$e^{\rho^{-1}(-\mu_T) T} \mathbb{E} \left( e^{-H_T} e^{\alpha_T B_T} \mathbb{1}_{E(\delta, T)} \mathbb{1}_{E(\delta, C, T)} \mathbb{1}_{\{B_T \geq 0\}} \right) = K_1(\delta, C) + o(1). \quad (4.3)$$

Moreover, if $\mu = \mu_b$ solves $I(b) = \mu b - \Lambda^+(\mu)$, then the same is true when $\mathbb{1}_{\{B_T \geq 0\}}$ is replaced by $\mathbb{1}_{\{B_T \approx bT\}}$.

Proof. We may assume that $\mu_T > -\rho(a^*)$ for all $T$. Fix $\delta, C \in (0, \infty]$ and choose $a_T$ such that $\mu_T + \rho(a_T) = 0$, i.e., $a_T = \rho^{-1}(-\mu_T) < a^*$. Clearly, $\lim_{T \to \infty} a_T = \rho^{-1}(-\mu) < a^*$. Since, on $E(\delta; T) \cap \{B_T \leq 2\delta\}$, we can estimate

$$H_T = 4\delta \int_{-\delta}^{\delta} \frac{dx}{4\delta} L(T, x)^2 \geq 4\delta \left( \int_{-\delta}^{\delta} \frac{dx}{4\delta} L(T, x) \right)^2 = \frac{T^2}{4\delta}, \quad (4.4)$$

we may insert the indicator of $\{B_T \geq 2\delta\}$ in the expectation on the left-hand side of (4.3), paying only a factor $1 + o(1)$ as $T \to \infty$.

1. Introduce the following subsets of $C^+_0$, respectively, $C^+$ (see below (3.1)):

$$G^{\leq}_{\delta, C} = \{ g \in C^+_0 : g(\delta) = 0, \max g \leq C \}, \quad (4.5)$$

$$F^{\leq}_{\delta, C} = \{(y, f) \in C^+ : y \geq 2\delta, \max f \leq C, \max \left[ f_{[y-\delta, y]} \right] \leq C \}. \quad (4.6)$$

Note that

$$E(\delta; T) \cap E(\delta, C, T) \cap \{B_T \geq 2\delta\} = \{ L(T, B_T + \cdot) \in G^{\leq}_{\delta, C} \} \cap \{ L(T, - \cdot) \in G^{\leq}_{\delta, C} \} \cap \{(B_T, L(T, B_T - \cdot)) \in [0, B_T] \} \subset F^{\leq}_{\delta, C}. \quad (4.7)$$

Apply Proposition 3.1 for $a = a_T$ with $F = F^{\leq}_{\delta, C}$ and $C^+ = G^-$, $G^- = G^{\leq}_{\delta, C}$, to get

l.h.s. of (4.3) $= (1 + o(1)) \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \mathbb{1}_{\{t_1 + t_2 \leq T\}} e^{\alpha_T(t_1 + t_2)}$

$$\times \mathbb{E}_{\alpha_T} \left( \frac{w_{G^{\leq}_{\delta, C}}(X_0, t_1)}{x_{\alpha_T}(X_0)} \mathbb{1}_{\{A^{-1}(T-t_1-t_2) \geq 2\delta\}} \mathbb{1}_{\{\max_{0, \delta} X \leq C\}}, \right) \mathbb{1}_{\{\max_{A^{-1}(T-t_1-t_2) - \delta, A^{-1}(T-t_1-t_2)} X \leq C\}} \frac{w_{G^{\leq}_{\delta, C}}(Y_{T-t_1-t_2}, t_2)}{x_{\alpha_T}(Y_{T-t_1-t_2})}. \quad (4.8)$$

2. In the case $C = \infty$, the last two indicators vanish and we can identify the limit of the integrand as $T \to \infty$ with the help of Lemma 2.1. Indeed, apply Lemma 2.1 for $f(\cdot) = w_{G_\delta}(\cdot, t_1)$ and $g(\cdot) = w_{G_\delta}(\cdot, t_2)$, where we put $G_\delta = G^{\leq}_{\delta, \infty} = \{ g \in C^+_0 : g(\delta) = 0 \}$. Then we obtain that the integrand converges to

$$e^{a(t_1 + t_2)} \langle w_{G_\delta}(\cdot, t_1), x_\alpha \rangle \frac{1}{\rho'(a)} \langle w_{G_\delta}(\cdot, t_2), x_\alpha \rangle, \quad (4.9)$$

where we also use that $A^{-1}(\infty) = \infty$ because $X$ never hits 0 (recall (2.7)). According to Proposition 3.2, we are allowed to interchange the limit $T \to \infty$ with the two integrals over $t_1$ and $t_2$. This implies that (4.3) holds with $K_1(\delta, \infty)$ identified as

$$K_1(\delta, \infty) = \langle y^{(i)}(x_\alpha), x_\alpha \rangle \frac{1}{\rho'(a)} \langle y^{(i)}(x_\alpha), x_\alpha \rangle, \quad (4.10)$$

where $y^{(i)}(h)$ is defined as (recall (3.1))

$$y^{(i)}(h) = \int_0^{\infty} dt e^{a(t_1 + t_2)} w_{G_\delta}(h, t) = \mathbb{E}_h^{\ast} e^{\int_0^\infty [a X_t^* - (X_t^*)^2] dt} \mathbb{1}_{\{X_t^* = 0\}}. \quad (4.11)$$
Trivially, $K_1(\delta, \infty) > 0$. Since $y^{(t)}_a \leq y_a$, it follows from (2.2) and (2.16) that $K_1(\delta, \infty) < \infty$.

3. Next we return to (4.8) and consider the case $C \in (0, \infty)$. Note that the integrals over $t_1$ and $t_2$ can both be restricted to $[0, C\delta]$, since $w_{G_{\delta,C}}(h, t) = 0$ for $t > C\delta$ as is seen from (3.1) and (4.5).

Let us abbreviate $s = T - t_1 - t_2$. We first apply the Markov property for the process $X$ at time $\delta$ and integrate over all values $z = A(\delta)$. Because of the appearance of the indicator of $\{\max_{[0, \delta]} X \leq C\}$, we may restrict to $z \in [0, C\delta]$ (recall (2.7)). We note that the additive functional of the process $(X_{\delta+t})_{t \geq 0}$ given that $A(\delta) = z$, denoted by $\tilde{A} = (\tilde{A}(t))_{t \geq 0}$, is given by $\tilde{A}(t) = A(t+\delta) - z$. Making the change of variables $s = \tilde{A}(t) + z$, we see that $A^{-1}(s) = A^{-1}(s-z) + \delta$ for any $s \geq 0$. Defining $f_{s,T}^{t_1} : (0, \infty)^2 \to [0, \infty)$ by

$$
\int_0^s \bar{f}_{s,T}^{t_1}(h, z) d\mu h d\mu z = x_{\alpha T}(h) \bar{\mathbb{E}}^\mathbb{F} \left( \frac{w_{G_{\delta,C}}(X_0, t_1)}{x_{\alpha T}(X_0)} \mathbb{I}_{\{\max_{[0, \delta]} X \leq C\}} \mathbb{I}_{\{A^{-1}(s) \geq 2\delta\}} \mathbb{I}_{\{\max_{[A^{-1}((s-\delta), A^{-1}(s)]} X \leq C\}} \frac{w_{G_{\delta,C}}(Y, t_2)}{x_{\alpha T}(Y)} \right), \tag{4.12}
$$

we thus obtain that the expectation under the integral in (4.8) can be written as

$$
\mathbb{E}^\mathbb{F} \left( \frac{w_{G_{\delta,C}}(X_0, t_1)}{x_{\alpha T}(X_0)} \mathbb{I}_{\{\max_{[0, \delta]} X \leq C\}} \mathbb{I}_{\{A^{-1}(s) \geq 2\delta\}} \mathbb{I}_{\{\max_{[A^{-1}((s-\delta), A^{-1}(s)]} X \leq C\}} \frac{w_{G_{\delta,C}}(Y, t_2)}{x_{\alpha T}(Y)} \right)
= \int_0^{C\delta} dz \mathbb{E}^\mathbb{F} \left( \frac{f_{s,T}^{t_1}(X_0, z)}{x_{\alpha T}(X_0)} \mathbb{I}_{\{\max_{[A^{-1}((s-\delta), A^{-1}(s)]} X \leq C\}} \frac{w_{G_{\delta,C}}(X_{A^{-1}((s-\delta), A^{-1}(s)]+\delta)}{x_{\alpha T}(X_{A^{-1}((s-\delta), A^{-1}(s)]+\delta})} \right). \tag{4.13}
$$

(The tilde can be removed afterwards.) We next apply the Markov property for the process $Y$ at time $s-z$ (respectively, the strong Markov property for the process $X$ at time $A^{-1}(s-z)$), to write

$$
r.h.s. \text{ of (4.13)} = \int_0^{C\delta} dz \mathbb{E}^\mathbb{F} \left( \frac{f_{s,T}^{t_1}(X_0, z)}{x_{\alpha T}(X_0)} \frac{g_{T}^{t_2}(Y_{s-z})}{x_{\alpha T}(Y_{s-z})} \right), \tag{4.14}
$$

where $g_{T}^{t_2}$ is defined by

$$
g_{T}^{t_2}(h) = x_{\alpha T}(h) \bar{\mathbb{E}}^\mathbb{F} \left( \frac{w_{G_{\delta,C}}(X_{s}, t_2)}{x_{\alpha T}(X_{s})} \mathbb{I}_{\{\max_{[0, \delta]} X \leq C\}} \right). \tag{4.15}
$$

4. We want to take the limit $s \to \infty$ in (4.14) (recall that $s = T - t_1 - t_2$) and use Proposition 2.1. Therefore we need dominated convergence. To establish this, we note that

$$
\sup_{h \in [0, C]} \sup_{t \in [0, C]} \sup_{T \geq 1} \frac{w(h, t)}{x_{\alpha T}(h)} = K < \infty \tag{4.16}
$$

(see (2.17–2.19) and recall that $x_a$ is bounded away from zero on $[0, C]$ and continuous in $a$). By (4.15–4.16), the last quotient in the right-hand side of (4.14) is bounded above by $K$. Substituting (4.12) into (4.14) and using that $w_{G_{\delta,C}} \leq w_{C_0} = w$, we therefore obtain

$$
\text{integrand of r.h.s. of (4.14)} \leq K \mathbb{E}^\mathbb{F} \left( \frac{f_{s,T}^{t_1}(X_0, z)}{x_{\alpha T}(X_0)} \right)
\leq K \mathbb{E}^\mathbb{F} \left( \frac{w(X_0, t_1)}{x_{\alpha T}(X_0)} \mathbb{I}_{\{\max_{[0, \delta]} X \leq C\}} \frac{w_{G_{\delta,C}}(Y, t_2)}{x_{\alpha T}(Y)} \right)
\leq K^2 \frac{\mathbb{E}^\mathbb{F} (A(\delta) \in dz)}{dz}. \tag{4.17}
$$

It is easy to see from (2.9) that the right-hand side of (4.17) is bounded uniformly in $T \geq 1$ and $z \in [0, C\delta]$. Therefore we have an integrable majorant for (4.14), which allows us to interchange the limit $s \to \infty$ with the integral over $z$. 
5. In order to identify the limit as \( s \to \infty \) of the integrand on the right-hand side of (4.14), we apply Lemma 2.1 to see that this integrand converges to \( \langle f^{t_1}(\cdot, z), x_a(\cdot) \rangle_{\rho(a)}^{-1} \langle g^{t_2}, x_a \rangle_\circ \), with \( f^{t_1} \) and \( g^{t_2} \) the pointwise limit of \( f^{t_{R,T}}_s \) and \( g^{t_{R,T}}_s \), respectively:

\[
f^{t_1}(h, z) \, dh \, dz = x_a(h) \left( \frac{w_{G_{t_1,C}}(X_0, t_1)}{x_a(X_0)} \mathbb{I}_{\{ \max_{X \leq C} \delta(X) \leq C \}} \mathbb{I}_{\{ X_i \in dh \}} \mathbb{I}_{\{ A(\delta) \in dz \}} \right)
\]

\[
g^{t_2}(h) \, dh = x_a(h) \left( \frac{w_{G_{t_2,C}}(X_0, t_2)}{x_a(X_0)} \mathbb{I}_{\{ \max_{X \leq C} \delta(X) \leq C \}} \right),
\]

Using this in (4.14) and interchanging the integral over \( z \) with the limit \( s \to \infty \), we obtain that

\[
\lim_{s \to \infty} \text{lns. of (4.13)} = \langle f^{t_1}, x_a \rangle_{\rho(a)}^{-1} \langle g^{t_2}, x_a \rangle_\circ
\]

with \( f^{t_1}(h) = \int_0^{G_s} dz \, f^{t_1}(h, z) \).

6. Finally, recall that \( s = T - t_1 - t_2 \) and that \( e^{a(t_1 + t_2)} \) times the left-hand side of (4.13) is equal to the integrand on the right-hand side of (4.8). According to Proposition 3.2, we are allowed to interchange the limit \( T \to \infty \) with the two integrals over \( t_1 \) and \( t_2 \). Hence we obtain that (4.3) holds with \( K_1(\delta, C) \) identified as the integral over \( t_1, t_2 \) of the right-hand side of (4.20), which is a strictly positive finite number. This proves the statement with the indicator on \( \mathbb{I}_{\{ B_{\gamma T} \geq 0 \}} \).

7. To prove the statement with \( \mathbb{I}_{\{ B_{\gamma T} \geq 0 \}} \) replaced by \( \mathbb{I}_{\{ B_{\alpha T} > 0 \}} \), we let \( \mu = \mu_b \) solve \( I(b) = \mu b - \Lambda^+ (\mu) \). The statement follows when we show that for every \( a \in \mathbb{R} \), we have that

\[
e^{-\gamma T} \int_{\gamma T} e^{a(\cdot - \mu)} \mathbb{I}_{\{ \xi \leq \gamma, C; T \}} \mathbb{I}_{\{ B_{\gamma T} \geq 0 \}} = e^{-\gamma T} \text{ some } \sigma^2 \in (0, \infty). \]

Indeed, (4.21) shows that \( \mathbb{I}_{\{ B_{\alpha T} > \gamma, B_{\gamma T} \geq 0 \}} \) is asymptotically negligible for any \( \gamma T \) such that \( \gamma T / \sqrt{T} \to \infty \).

In order to prove (4.21), we rewrite the left-hand side as

\[
e^{-\gamma T} \int_{\gamma T} e^{a(\cdot - \mu)} \mathbb{I}_{\{ \xi \leq \gamma, C; T \}} \mathbb{I}_{\{ B_{\gamma T} \geq 0 \}} = e^{a(\cdot - \mu)} \mathbb{I}_{\{ B_{\gamma T} > \gamma, B_{\gamma T} \geq 0 \}}.
\]

where \( \mu_{\alpha T} = \mu + \frac{a}{\sqrt{T}} \). Clearly, \( \mu_{\alpha T} \to \mu \), so that the second factor converges to \( K_1(\delta, C) \). We are therefore left to compute the exponential. We note that since \( \mu = \mu_b \) solves \( I(b) = \mu b - \Lambda^+ (\mu) \), we have that \( \rho'(-\mu_b) = 1/b \). Therefore,

\[
\rho^{-1}(\mu) = \rho^{-1}(\mu) - \frac{a}{\sqrt{T}} \rho'(-\mu) = \frac{a^2}{2T} \int \frac{d^2}{d\mu^2} \rho^{-1}(\mu) + o(T^{-1}).
\]

Therefore,

\[
e^{-\gamma T} \int_{-\gamma T} e^{a(\cdot - \mu)} \mathbb{I}_{\{ B_{\gamma T} > \gamma, B_{\gamma T} \geq 0 \}} = e^{-\gamma T} \int \frac{d^2}{d\mu^2} \rho^{-1}(\mu) + o(T^{-1}),
\]

which completes the proof with \( \sigma^2 = -\frac{d^2}{d\mu^2} \rho^{-1}(-\mu_b) \).

\[\square\]

4.2 Proof of Theorem 1.3(i–iii)

\textbf{Step 1.} For any \( \mu > -\rho(a^{**}) \), the limit in (1.7) exists and equals \( \Lambda^+(\mu) = -\rho^{-1}(-\mu) \). On \( (-\rho(a^{**}), \infty) \), the function \( \Lambda^+ \) is real-analytic and strictly convex, and satisfies \( \lim_{\mu \to -\rho(a^{**})} \Lambda^+(\mu) = b^{**} \).
Proof. Fix \( \mu > -\rho(a**) \), apply Proposition 4.1 with \( \delta = C = \infty \), and use the continuity of \( \rho \), to obtain that the limit in the definition of \( \Lambda^+(\mu) \) in (1.7) exists and equals \( -\rho^{-1}(-\mu) \). This proves the first assertion. The remaining assertions follow from (2.3–2.5).

In the following step, we consider paths that never go below \( -\delta \), have local times that are bounded by \( C \) in the \( \delta \)-neighborhood of the starting point 0, and have the endpoint \( B_T \) close to 0. Recall that \( \gamma_T \) is a function that satisfies \( \gamma_T/T \to 0 \) and \( \gamma_T/\sqrt{T} \to \infty \) as \( T \to \infty \).

**STEP 2.** For any \( \delta \in (0, \infty) \) and \( C \in (0, \infty) \),

\[
E\left(e^{-H_T} \mathbb{1}_{\{B_T \in [0, \gamma_T]\}} \mathbb{1}_{\left\{ \min_{[0,T]} B \geq -\delta \right\}} \mathbb{1}_{\left\{ \max_{[0,T]} L(T) \leq C \right\}} \right) \geq e^{-\rho(a**)T+o(T)}, \quad T \to \infty. \tag{4.25}
\]

Proof. Pick \( a = a** \) and apply Proposition 3.1 for

\[
F = F_{\delta,C} = \{(y, f) \in C^+: y \leq \delta, \max_{[y-\delta y]} f \leq C\}, \quad G^+ = G_{\delta,C}^+, \quad G^- = G_{\delta,C}^- \tag{4.26}
\]

(recall (4.5)). Note that the event under the expectation on the left-hand side of (4.25) contains the event

\[
\{L(T, B_T + \cdot) \in C_0^+ \} \cap \{(B_T, L(T, B_T - \cdot)) \in F_{\delta,C}\} \cap \{L(T, \cdot -) \in G_{\delta,C}^-\}. \tag{4.27}
\]

Also note that \( e^{-\rho(a**)B_T} \leq 1 \) when \( B_T \geq 0 \) because \( \rho(a**) > 0 \). Therefore we find

**l.h.s. of (4.25)**

\[
\int_0^\infty \int_0^\infty \int_0^\infty dt_1 dt_2 dt \frac{\mathbb{1}_{\{t_1 + t_2 + T \}} e^{-\rho(a**) t}}{x_{a**}} w(x_0, t_1) \mathbb{1}_{\{A^{-1}(s) \leq \delta\}} \mathbb{1}_{\{\max_{A^{-1}(s) - \delta, A^{-1}(s)} X \leq C\}} \frac{w_{g_{\delta,C}^+}(Y_s, t_2)}{x_{a**}(Y_s)}, \tag{4.28}
\]

where we again abbreviate \( s = T - t_1 - t_2 \). Next we interchange the two integrals, restrict the \( t_2 \)-integral to \([0, \delta]\) and the \( t_1 \)-integral to \([T - t_2 - \delta, T - t_2] \), estimate \( A^{-1}(s) \leq A^{-1}(\delta) \) for \( s \leq \delta \), and integrate over \( s = T - t_1 - t_2 \), to get

**l.h.s. of (4.25)**

\[
\int_0^\delta \int_0^\delta \int_0^\delta ds \frac{w(x_0, t_2 - s)}{x_{a**}(x_0)} \mathbb{1}_{\{A^{-1}(s) \leq \delta\}} \mathbb{1}_{\{\max_{[0,T]} L(T) \leq C\}} \frac{w_{g_{\delta,C}^+}(Y_s, t_2)}{x_{a**}(Y_s)} ds \tag{4.29}
\]

Now we use Proposition 2.2(i) to estimate \( w(x_0, T - s - t_2) \geq e^{-\rho(a**)T+o(T)} \), uniformly on the domain of integration. The remaining expectation on the right-hand side no longer depends on \( T \) and is strictly positive for any \( \delta \in (0, \infty) \) and \( C \in (0, \infty) \).

**STEP 3.** \( \Lambda^+ \) equals \( -a** \) on \( (-\infty, -\rho(a**) \). 

Proof. For \( \mu \leq -\rho(a**) \), define \( \Lambda^*(\mu) \) and \( \Lambda^+ \) as in (1.7) with \( \lim \) replaced by \( \lim \inf \) and \( \lim \sup \), respectively. Since \( \Lambda^+ \) is obviously non-decreasing, we have \( \Lambda^+(\mu) \leq \Lambda^+(-\rho(a**)) + \varepsilon \) for \( \mu \leq -\rho(a**) \) and any \( \varepsilon > 0 \). Using Step 1 and the continuity of \( \rho \), we see that \( \lim_{\varepsilon \downarrow 0} \Lambda^+(-\rho(a**)) + \varepsilon = -\rho^{-1}(\rho(a**)) = -a** \), which shows that \( \Lambda^+(\mu) \leq -a** \). In order to get the reversed inequality for \( \Lambda^+(\mu) \), bound

\[
E\left(e^{-H_T e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}} \right) \geq E\left(e^{-H_T e^{\mu B_T} \mathbb{1}_{\{B_T \in [0, \gamma_T]\}}} \right) \geq e^{\mu T} E\left(e^{-H_T \mathbb{1}_{\{B_T \in [0, \gamma_T]\}}} \right), \tag{4.30}
\]

take logs, divide by \( T \), let \( T \to \infty \) and use Step 2, to obtain that \( \Lambda^+(\mu) \geq -a** \). Since \( \Lambda^+ \leq \Lambda^+ \), this implies the assertion.

**STEP 4.** \( \Lambda^+(\mu) = \frac{1}{2} \mu^2 + O(\mu^{-1}) \) as \( \mu \to \infty \).
Proof. According to Step 1, we have \( \Lambda^+(\mu) = -\rho^{-1}(-\mu) \) for \( \mu > -\rho(a^*) \). Hence, in order to obtain the asymptotics for \( \Lambda^+(\mu) \) as \( \mu \to \infty \), we need to obtain the asymptotics for \( \rho(a) \) as \( a \to -\infty \). In the following we consider \( a < 0 \).

We use Rayleigh’s Principle (see [G81, Proposition 10.10]) to write (recall (2.1))

\[
\rho(a) = \sup_{x \in L^2 \cap C^2: \|x\|_2 = 1} \langle K^a x, x \rangle = \sup_{x \in L^2 \cap C^2: \|x\|_2 = 1} \int_0^\infty \left[ -2h a' x(h)^2 + (ah - h^2) x(h)^2 \right] \, dh. \tag{4.31}
\]

Substituting \( x(h) = (-a)^{1/2} y((-a)^{1/2} h) \), we get

\[
\rho(a) = (-a)^{1/2} \sup_{y \in L^2 \cap C^2: \|y\|_2 = 1} \int_0^\infty \left[ -2h y'(h)^2 - (h + h^2(-a)^{-2}) y(h)^2 \right] \, dh. \tag{4.32}
\]

Hence, we have the upper bound \( \rho(a) \leq V(-a)^{1/2} \) with

\[
V = \sup_{y \in L^2 \cap C^2: \|y\|_2 = 1} \int_0^\infty \left[ -2h y'(h)^2 - h y(h)^2 \right] \, dh. \tag{4.33}
\]

By completing the square under the integral and partially integrating the cross term, we easily see that \( y^*(h) = \frac{1}{\sqrt{2}} e^{-h/\sqrt{2}} \) is the maximizer of (4.33) and \( V = -\sqrt{2} \). Substituting \( y^* \) into (4.32), we can also bound \( \rho(a) \) from below:

\[
\rho(a) \geq -\sqrt{2} (-a)^{1/2} - (-a)^{-1} \int_0^\infty h^2 y^*(h)^2 \, dh. \tag{4.34}
\]

Therefore,

\[
\rho(a) = -\sqrt{2} (-a)^{1/2} + \mathcal{O}(|a|^{-1}), \quad a \to -\infty. \tag{4.35}
\]

Consequently,

\[
\Lambda^+(\mu) = -\rho^{-1}(-\mu) = \frac{1}{2} \mu^2 + \mathcal{O}(\mu^{-1}), \quad \mu \to \infty. \tag{4.36}
\]

Steps 1, 3 and 4 complete the proof of Theorem 1.3(i–iii).

4.3 Proof of Theorem 1.2 and 1.3(iv)

For \( b \in \mathbb{R} \), define \( I_-(b) \) and \( I_+(b) \) as in (1.4) with \( \lim \) replaced by \( \lim \sup \) and \( \lim \inf \), respectively.

**STEP 5.** For any \( b > b^* \), the limit in (1.4) exists and (1.8) holds.

**Proof.** Fix \( b > b^* \).

1. To derive ‘\( \geq \)’ in (1.8) for \( I_- \) instead of \( I \), bound, for any \( \mu \in \mathbb{R} \),

\[
E(e^{-HT} \mathbb{1}_{\{T \leq \gamma_T \}}) \leq e^{-\mu b^* + |b| T} E(e^{-HT} e^{\mu BT} \mathbb{1}_{\{T \leq \gamma_T \}}) \leq e^{-\mu b^* + |b| T} E(e^{-HT} e^{\mu BT} \mathbb{1}_{\{T \geq 0 \}}), \tag{4.37}
\]

where the last inequality holds for any \( T \) sufficiently large because \( \gamma_T/T \to 0 \) as \( T \to \infty \). Take logs, divide by \( T \), let \( T \to \infty \), use (1.7) and minimize over \( \mu \in \mathbb{R} \), to obtain

\[
-I_-(b) \leq \min_{\mu \in \mathbb{R}} [-\mu b + \Lambda^+(\mu)]. \tag{4.38}
\]

This shows that ‘\( \geq \)’ holds in (1.8) for \( I \) replaced by \( I_- \).
2. To derive \( \leq \) in (1.8), bound, for any \( \mu \in \mathbb{R} \),

\[
E(e^{-H_T} \mathbb{1}_{\{B_T - bT \leq \gamma_T\}}) \\
\geq E(e^{-H_T} \mathbb{1}_{\mathcal{E}(\delta,T)} \mathbb{1}_{\{B_T - bT \leq \gamma_T\}} \mathbb{1}_{\{B_T \geq 0\}}) \\
\geq e^{-\mu bT - \|\mathbb{P}^\delta\| \gamma_T} P^{\mu,\delta,T}(B_T - bT \leq \gamma_T) E(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta,T)} \mathbb{1}_{\{B_T \geq 0\}}),
\]

(4.39)

where \( P^{\mu,\delta,T} \) denotes the probability law whose density with respect to \( P \) is proportional to \( e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta,T)} \mathbb{1}_{\{B_T \geq 0\}} \).

3. Let \( \mu_b \) be the maximizer of the map \( \mu \mapsto \mu b - \Lambda^+(\mu) \). (Note that, by Step 1, the maximizer is unique and is characterized by \( (\Lambda^+)'(\mu_b) = b \).) Next we argue that

\[
\lim_{T \to \infty} P^{\mu,\delta,T}(B_T - bT \leq \gamma_T) = 1.
\]

(4.40)

Indeed, pick \( \varepsilon_T = \gamma_T/cT > 0 \) (with \( c > 0 \) to be specified later) and estimate

\[
\mathbb{1}_{\{B_T \leq bT + \gamma_T\}} \leq e^{\varepsilon_T [B_T - bT] - \varepsilon_T \gamma_T},
\]

(4.41)

This implies, with the help of Step 1 and Proposition 4.1 with \( \mu_T = \mu + \varepsilon_T, \ C = \infty \), that

\[
P^{\mu,\delta,T}(B_T \geq bT + \gamma_T) \leq e^{-\varepsilon_T [bT + \gamma_T] e^{[\Lambda^+(\mu_b + \varepsilon_T) - \Lambda^+(\mu_b)]T}} (1 + o(1)), \quad T \to \infty.
\]

(4.42)

A Taylor expansion of \( \Lambda^+ \) around \( \mu_b \), in combination with the observation that \( (\Lambda^+)'(\mu_b) = b \) and \( c = (\Lambda^+)'(\mu_b) > 0 \), yields that the right-hand side of (4.42) is equal to

\[
e^{\varepsilon_T [1 + o(\varepsilon_T)] - \varepsilon_T \gamma_T} = e^{-\frac{2}{\varepsilon_T} [1 + o(\varepsilon_T)]}, \quad T \to \infty.
\]

(4.43)

The right-hand side vanishes as \( T \to \infty \) because \( \varepsilon_T/T \to 0 \) and \( \gamma_T/\sqrt{T} \to \infty \). This shows that \( \lim_{T \to \infty} P^{\mu,\delta,T}(B_T \geq bT + \gamma_T) = 0 \). Analogously, replacing \( \varepsilon_T \) by \( -\varepsilon_T \), we can prove that \( \lim_{T \to \infty} P^{\mu,\delta,T}(B_T \leq bT - \gamma_T) = 0 \). Hence, (4.40) holds.

4. Use (4.40) in (4.39) for \( \mu = \mu_b \), take logs, divide by \( T \), let \( T \to \infty \), and use Step 1 and Proposition 4.1, to obtain

\[
-I^+(b) \geq -\mu_b b + \Lambda^+(\mu_b) = -\max_{\mu \in \mathbb{R}} [\mu b - \Lambda^+(\mu)].
\]

(4.44)

This shows that \( \leq \) holds in (1.8) for \( I \) replaced by \( I_+ \). Combine (3.38) and (4.44) to obtain that \( I_- = I = I_+ \) and that (1.8) holds on \( (b^*, \infty) \).

\[ \square \]

STEP 6. For any \( b \geq 0 \), \( I_-^+ \geq -bp(a^*) + a^{**} \).

**Proof.** Estimate

\[
\mathbb{1}_{\{B_T \leq bT \leq \gamma_T\}} \leq \mathbb{1}_{\{B_T \leq bT + \gamma_T\}} \leq e^{-p(a^{**}) [B_T - bT - \gamma_T]}.
\]

(4.45)

to obtain, for \( T \) sufficiently large,

\[
E\left(e^{-H_T} \mathbb{1}_{\{B_T \leq bT \leq \gamma_T\}}\right) \leq 2E\left(e^{-H_T} \mathbb{1}_{\{B_T \leq bT \leq \gamma_T\}} \mathbb{1}_{\{B_T \geq 0\}}\right) \\
\leq 2e^{bp(a^{**}) T + \gamma_T p(a^{**})} E\left(e^{-H_T} e^{-p(a^{**}) B_T} \mathbb{1}_{\{B_T \geq 0\}}\right).
\]

(4.46)

According to the definition of \( \Lambda^+ \) in (1.7), the expectation in the right-hand side is equal to \( e^{\Lambda^+(-p(a^{**})) T + o(T)} \). We therefore obtain that \( I(b) \geq -bp(a^{**}) - \Lambda^+(-p(a^{**})) \). Now Step 3 concludes the proof.

\[ \square \]

STEP 7. For any \( 0 \leq b \leq b^* \), \( I_+^+ \leq -bp(a^{**}) + a^{**} \).
Proof. Fix $0 \leq b \leq b^{**}$, pick $b' > b^{**}$ and put $a = b/b' \in [0, 1)$. We split the path $(B_s)_{s \in [0,T]}$ into two pieces: $s \in [0, \alpha T]$ and $s \in [\alpha T, T]$. First we bound from below by inserting several indicators:

$$E \left( e^{-H_T} \mathbb{1}_{\{B_{\alpha T} - a|T| \leq \gamma T \}} \right) \geq E \left( e^{-H_T} \mathbb{1}_{\{B_{\alpha T} - b|T| \leq \gamma T / 2 \}} \prod_{\max p, \alpha T} \left( B \leq B_{\alpha T} + \delta \right) \prod_{\max \left| \frac{B - B_{\alpha T}}{\delta} \right| - \delta} \left( L(\alpha T) \leq C \right) \right) \times \prod_{\min \left| \frac{B - B_{\alpha T}}{\delta} \right| - \delta} \left( L((1-\alpha)T) \leq C \right).$$

(4.47)

Here, $(\tilde{B}_s)_{s \in [0, (1-\alpha)|T|]}$ is the Brownian motion with $\tilde{B}_s = B_{\alpha T + s} - B_{\alpha T}$, and $\tilde{L}(\alpha T, x) = L(T, x) - L(\alpha T, x), x \in \mathbb{R}$, are its local times.

On the event under the expectation in the right-hand side, we may estimate

$$H_T = H_{\alpha T} + \tilde{H}_{(1-\alpha)T} + 2 \int_{B_{\alpha T} - \delta}^{B_{\alpha T} + \delta} L(\alpha T, x) \tilde{L}(\alpha T, x) \, dx \leq H_{\alpha T} + \tilde{H}_{(1-\alpha)T} + 4\delta C^2,$$

(4.48)

where $\tilde{H}_{(1-\alpha)T}$ denotes the intersection local time for the second piece. Using the Markov property at time $\alpha T$, we therefore obtain the estimate

$$E \left( e^{-H_T} \mathbb{1}_{\{B_{\alpha T} - b|T| \leq \gamma T \}} \right) \geq e^{-4\delta C^2} E \left( e^{-H_{\alpha T}} \mathbb{1}_{\{B_{\alpha T} - b\alpha T \leq \gamma T / 2 \}} \prod_{\max \left| \frac{B - B_{\alpha T}}{\delta} \right| - \delta} \left( L(\alpha T) \leq C \right) \right) \times \prod_{\min \left| \frac{B - B_{\alpha T}}{\delta} \right| - \delta} \left( L((1-\alpha)T) \leq C \right).$$

(4.49)

(The tilde can be removed afterwards.) Now use Proposition 4.1 (in combination with an argument like in parts 2-3 of the proof of Step 5) for the first term (with $T$ replaced by $\alpha T$) and use Step 2 for the second term (with $T$ replaced by $(1-\alpha)T$), to conclude that

$$I(b) \leq \alpha I(b') + (1-\alpha) a^{**} = \frac{b}{b'} \left( I(b') - a^{**} \right) + a^{**}.$$  

(4.50)

Let $b' \downarrow b^{**}$, use the continuity of $I$ in $b^{**}$, and note that $I(b^{**}) - a^{**} = -b^{**} \rho(a^{**})$ by Step 5, to conclude the proof.

\[\square\]

STEP 8. Theorems 1.2 and 1.3(iv) hold.

Proof. Steps 1 and 5 allow us to identify $I$ on $(b^{**}, \infty)$ as $I(b) = -b \rho(a_b) + a_b$, where $a_b$ solves $\rho'(a_b) = 1/b$ (the maximum in (1.8) is attained at $\mu = -\rho(a_b)$). From this and (2.3–2.5) it follows that

$$I'(b) = -\rho(a_b), \quad I''(b) = -\rho'(a_b) \frac{d}{db} a_b = \frac{[\rho(a_b)]^3}{\rho''(a_b)} > 0, \quad b > b^{**}.$$  

(4.51)

In particular, $I$ is real-analytic and strictly convex on $(b^{**}, \infty)$. Since $a_{b^{**}} = a^{**}$, it in turn follows that

$$\min_{b \geq 0} I(b) = \min_{0 < b < b^{**}} I(b) = I(b^*) = a^*,$$

(4.52)

where $a^*$ solves $\rho(a^*) = 0$ (the minimum is attained at $b^* = 1/\rho'(a^*)$). This, together with Steps 5–7, proves Theorem 1.2(i–iii).

Step 5 shows that (1.8) holds on $(b^{**}, \infty)$. To show that it also holds on $[0, b^{**}]$, use Step 3 to get

$$-b \rho(a^{**}) + a^{**} = \max_{\mu \in \mathbb{R}} \left( b \mu - \Lambda^+(\mu) \right), \quad 0 \leq b \leq b^{**},$$

(4.53)
since the maximum is attained at \( \mu = -\rho(a^{**}) \). Recall from Steps 6–7 that the left-hand side is equal to \( I(b) \). Thus we have proved Theorem 1.3(iv).

Finally, Theorem 1.2(iv) is an immediate consequence of Theorem 1.3(iii–iv). \(\square\)

5. Addendum 1: An Extension of Proposition 4.1

At this point we have completed the proof of the main results in Section 1. In Sections 5–6 we derive an extension of Proposition 4.1 that will be needed in a forthcoming paper [vdHdHK02]. In that paper we show that several one-dimensional polymers models in discrete space and time, such as the weakly self-avoiding walk, converge to the Edwards model, after appropriate scaling, in the limit of vanishing self-repellence. The proof is based on a coarse-graining argument, for which we need Proposition 5.1 below.

Recall the events in (4.1–4.2). For \( \delta \in (0, \infty), \alpha \in [0, \infty) \), define the event

\[
\mathcal{E}^{\geq}(\delta, \alpha; T) = \left\{ \max_{x \in [B_T-\delta, B_T+\delta]} L(T, x) \geq \alpha \delta^{-\frac{1}{2}} \right\}. \tag{5.1}
\]

Note that \( \mathcal{E}^{\geq}(\delta, 0; T) \) is the full space.

**Proposition 5.1.** Fix \( \mu > -\rho(a^{**}) \). Then:

(i) For any \( \delta \in (0, \infty) \) and \( \alpha \in [0, \infty) \) there exists a \( K_2(\delta, \alpha) \in (0, \infty) \) such that

\[
e^{-H_T\mu B_T} E \left[ e^{-H_T e^{\mu B_T} 1_{\mathcal{E}^{\geq}(\delta, \alpha; T)} 1_{\{B_T \geq 0\}}} \right] = K_2(\delta, \alpha) + o(1), \quad T \to \infty. \tag{5.2}
\]

(ii) For any \( \alpha \in (0, \infty) \),

\[
\lim_{\delta \downarrow 0} \frac{K_2(\delta, \alpha)}{K_1(\delta, \infty)} = 0, \tag{5.3}
\]

where \( K_1(\delta, \infty) \) is the constant in Proposition 4.1 (recall (4.10)).

**Proof.** (i) As in the proof of Proposition 4.1, we may insert the indicator on \( \{B_T \geq 2\delta\} \) in the expectation on the left-hand side of (5.2) and add a factor of \( 1 + o(1) \).

Introduce the following measurable subsets of \( C_0^+ \), respectively, \( C^+ \):

\[
G_{\delta, \alpha}^{\geq} = \{ g \in C_0^+ : g(\delta) = 0, \max_{\alpha} g \geq \alpha \delta^{-\frac{1}{2}} \}, \tag{5.4}
\]

\[
F_{\delta, \alpha}^{\geq} = \left\{ (y, f) \in C^+ : y \geq 2\delta, \max_{[0, \delta]} f \geq \alpha \delta^{-\frac{1}{2}} \right\}. \tag{5.5}
\]

Note from (4.1) and (5.1) that

\[
\mathcal{E}(\delta; T) \cap \mathcal{E}^{\geq}(\delta, \alpha; T) \cap \{ B_T \geq 2\delta \} = \{ L(T, - \cdot) \in G_\delta \} \cap
\]

\[
\left( \{ L(T, B_T + \cdot) \in G_{\delta, \alpha}^{\geq} \} \cup \{ \{ (B_T, L(T, B_T - \cdot)|_{[0, B_T]}) \in F_{\delta, \alpha}^{\geq} \} \cap \{ L(T, B_T + \cdot) \in G_\delta \} \right) \tag{5.6}
\]

with \( G_\delta = \{ g \in C_0^+ : g(\delta) = 0 \} \).

Pick \( a \in \mathbb{R} \) such that \( \mu + \rho(a) = 0 \), i.e., \( a = \rho^{-1}(-\mu) < a^{**} \). Apply Proposition 3.1 twice for \( G^- = G_\delta \) and the two choices: (1) \( F = F_{\delta, \alpha}^{\geq}, \quad G^+ = G_\delta \); (2) \( F = C^+, \quad G^+ = G_{\delta, \alpha}^{\geq} \). Sum the two
resulting equations, to obtain
\[
\text{l.h.s. of (5.2)} = (1 + o(1)) \int_0^\infty dt_1 \int_0^\infty dt_2 \mathbb{I}_{[t_1 + t_2 \leq T]} e^{\alpha(t_1 + t_2)} \\
\times \mathbb{P}_a \left( \left[ \frac{w_{G, a}(X_0, t_1)}{x_a(X_0)} \mathbb{I}_{\{\max_{p, q} X \geq \alpha \delta^{-\frac{1}{2}} \}} \right] \right) + \mathbb{I}_{\{A^{-1}(T - t_1 - t_2) \geq 2\delta \}} \mathbb{I}_{\{\max_{p, q} X \geq \alpha \delta^{-\frac{1}{2}} \}}
\]
\[
\times w_{G, a}(X_0, t_1) \mathbb{I}_{\{A^{-1}(T - t_1 - t_2) \geq 2\delta \}} + \frac{w_{G, a}(Y_{T - t_1 - t_2}, t_2)}{x_a(Y_{T - t_1 - t_2})}.
\]
\[
(5.7)
\]

In the same way as in the proof of Proposition 4.1, we obtain that (recall (4.10–4.11))
\[
\lim_{T \to \infty} (\text{r.h.s. of (5.7)}) = K_2(\delta, \alpha)
\]
\[
(5.8)
\]
with
\[
K_2(\delta, \alpha) = \mathbb{E}_h \left( \frac{y_\alpha^{(\delta, \alpha)}(X_0)}{x_a(X_0)} \mathbb{I}_{\{\max_{p, q} X \geq \alpha \delta^{-\frac{1}{2}} \}} \right) + \frac{\langle x_a, y_\alpha^{(\delta, \alpha)} \rangle}{\mathbb{E}_a \langle x_a, y_\alpha^{(\delta, \alpha)} \rangle},
\]
\[
(5.9)
\]
where \(y_\alpha^{(\delta)}\) is defined in (4.11) and \(y_\alpha^{(\delta, \alpha)}\) is defined as (recall (3.1))
\[
y_\alpha^{(\delta, \alpha)}(h) = \int_0^\infty dt e^{at} w_{G, a}(h, t) = \mathbb{P}_h \left( e^{\int_0^\infty [X(t) - (X_0)^2] dt} \mathbb{I}_{\{X_0 = 0 \}} \mathbb{I}_{\{\max_{p, q} X \geq \alpha \delta^{-\frac{1}{2}} \}} \right).
\]
\[
(5.10)
\]
The right-hand side of (5.9) is a strictly positive finite number.

(ii) Fix \(\alpha \in (0, \infty)\). From (4.10) and (5.8) we see that \(K_2(\delta, \alpha)/K_1(\delta, \infty) = K^{(1)}(\delta, \alpha) + K^{(2)}(\delta, \alpha)\) with
\[
K^{(1)}(\delta, \alpha) = \frac{\int_0^\infty dh x_a(h) y_\alpha^{(\delta, \alpha)}(h) \mathbb{P}_h \left( \max_{p, q} X \geq \alpha \delta^{-\frac{1}{2}} \right)}{\mathbb{E}_a \langle x_a, y_\alpha^{(\delta, \alpha)} \rangle}, \quad K^{(2)}(\delta, \alpha) = \frac{\langle x_a, y_\alpha^{(\delta, \alpha)} \rangle}{\mathbb{E}_a \langle x_a, y_\alpha^{(\delta, \alpha)} \rangle}.
\]
\[
(5.11)
\]
To prove (5.3), we need the following lemma.

**Lemma 5.2.** Fix \(a < a^*\) and \(\alpha \in (0, \infty)\). Then:

(i) There exists \(d = d(\alpha) > 0\) such that, for any \(R > 0\) and any \(\delta > 0\) sufficiently small,
\[
\sup_{h \in [0, R]} \mathbb{P}_a \left( \max_{[0, \delta]} X \geq \alpha \delta^{-\frac{1}{2}} \right) \leq c e^{-\delta \frac{1}{2}} e^{c \sqrt{R}},
\]
\[
(5.12)
\]
\[
\sup_{h \in [0, R]} \frac{y_\alpha^{(\delta, \alpha)}(h)}{y_\alpha(h)} \leq c e^{-\delta \frac{1}{2}} e^{c \sqrt{R}}.
\]
\[
(5.13)
\]
(ii) For any \(\delta > 0\) sufficiently small,
\[
\inf_{h \in [0, \delta]} y_\alpha^{(\delta)}(h) \geq c.
\]
\[
(5.14)
\]
**Proof.** The proof is deferred to Section 6.

We use Lemma 5.2 to show that
\[
\lim_{\delta \to 0} K^{(1)}(\delta, \alpha) = \lim_{\delta \to 0} K^{(2)}(\delta, \alpha) = 0,
\]
\[
(5.15)
\]
which yields (5.3).

First note that, with the help of (5.14), the common denominator in (5.11) may be estimated from below by
\[
\langle x_a, y_\alpha^{(\delta)} \rangle \geq \int_0^\delta dh x_a(h) y_\alpha^{(\delta)}(h) \geq c \int_0^\delta dh x_a(h) \geq c \delta,
\]
\[
(5.16)
\]
where we use that $x_\alpha$ is bounded away from zero on $[0,\delta]$.

In order to estimate the numerator of $K^{(1)}(\delta, \alpha)$ from above, we split the integral in the numerator into two parts: $h \leq R$ and $h > R$. In the integral over $h \leq R$, estimate $y_\alpha^{(5)} \leq y_\alpha$ and use (5.12), to get the upper bound $ce^{-\frac{4}{3}\delta - \frac{1}{4}}e^{\sqrt{R}}$. In the integral over $h > R$, estimate $y_\alpha^{(\delta)} \leq y_\alpha$, estimate the probability against one and use (2.2) and (2.16), to get the upper bound $ce^{-c\delta R^{\frac{1}{2}}}$. Pick $R$ such that $c\sqrt{R} = \frac{4}{3}\delta - \frac{1}{4}$, to obtain that the numerator of $K^{(1)}(\delta, \alpha)$ is at most $ce^{-\frac{4}{3}\delta - \frac{1}{4}}$.

In the same way we show, with the help of (5.13), that the numerator of $K^{(2)}(\delta, \alpha)$ in (5.11) is at most $ce^{-\frac{4}{3}\delta - \frac{1}{4}}$. Now combine the two estimates with (5.16) to obtain (5.15).

\[ \square \]

6. Addendum 2: Proof of Lemma 5.2

We will need the following asymptotics for $x_\alpha$ and $y_\alpha$, which are refinements of (2.2) and (2.16), respectively.

**Step 1.** For $a < a^{**}$,

\[
\lim_{h \to \infty} \frac{1}{\sqrt{h}} \log \left[ e^{\frac{1}{2}h^2} x_\alpha(h) \right] = \lim_{h \to \infty} \frac{1}{\sqrt{h}} \log \left[ e^{\frac{1}{2}h^2} y_\alpha(h) \right] = \frac{a}{\sqrt{2}}. 
\]  

(6.1)

**Proof.** The statement for $y_\alpha$ is well-known, and follows from (2.15) together with the asymptotics of the Airy function given by (see [E56, p. 43])

\[
\text{Ai}(h) = \frac{1}{2\pi h^{\frac{3}{2}}} e^{-\frac{2}{3}h^{\frac{3}{2}}} [1 + o(1)], \quad h \to \infty. 
\]  

(6.2)

To prove the statement for $x_\alpha$, use [CL55, Theorem 2.1, pp. 143–144]. To this end, define

\[
\zeta_1(h) = x_\alpha(h^2), \quad \zeta_2(h) = h^{-2} \zeta'_1(h). 
\]  

(6.3)

Then the eigenvalue equation $K^\alpha x_\alpha = \rho(a)x_\alpha$ (recall (2.1)) can be written as (see also [CL55, equation (5.3)])

\[
\zeta'(h) = h^2 B(h) \zeta(h), 
\]  

(6.4)

with

\[
\zeta(h) = \begin{pmatrix} \zeta_1(h) \\ \zeta_2(h) \end{pmatrix}, \quad B(h) = \begin{pmatrix} 0 & 1 \\ 2 - 2a h^{-3} & \frac{1}{h^2} \end{pmatrix}. 
\]  

(6.5)

Note that $B(h) = \sum_{n=0}^\infty h^{-n} B^{(n)}$ ($B^{(0)} \neq 0$) is a convergent power series in $h^{-1}$, with $B^{(0)}$ having eigenvalues $\lambda_{1,2} = \pm \sqrt{2}$. Therefore (6.4) has formal solutions of the form

\[
Z(h) = P(h) h^R e^{Q(h)}, 
\]  

(6.6)

where the columns of the matrix $Z$ are the two linearly independent solutions to (6.4), $P(h) = \sum_{n=0}^\infty h^{-n} P^{(n)}$ (det($P^{(0)}$) $\neq 0$) is a formal power series in $h^{-1}$, $R$ is a complex diagonal matrix, and $Q(h) = \frac{1}{3} h^3 Q^{(0)} + \frac{1}{2} h^2 Q^{(1)} + h Q^{(2)}$ is a matrix polynomial with $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$ diagonal. In our case,

\[
Q^{(0)} = \text{diag}\{-\sqrt{2}, +\sqrt{2}\}, \quad Q^{(1)} = 0, \quad Q^{(2)} = \text{diag}\{\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\}. 
\]  

(6.7)

From the proof of [CL55, Theorem 2.1] it follows that $P(h), R, Q(h)$ can be chosen to be real, because $B(h) \lambda_1, \lambda_2$ are real. On [CL55, p. 151] there is the further remark that for every formal solution there exists an actual solution with the same asymptotics.
We need the solution that is in $L^2[0,\infty)$. By construction, we compute, for $R = \text{diag}\{r_1,r_2\}$ (with $r_1,r_2$ some functions of $a$),
\[ hR e^{\frac{1}{2}h^2Q(0) + \frac{1}{2}h^2Q^{(1)} + hQ^{(2)}} = \begin{pmatrix}
    h^{r_1}e^{-\frac{\sqrt{2}}{\sigma}h^3 + \frac{\sqrt{2}}{\sigma}h} & 0 \\
    0 & h^{r_2}e^{\frac{\sqrt{2}}{\sigma}h^3 - \frac{\sqrt{2}}{\sigma}h}
\end{pmatrix}. \tag{6.8}
\]

Therefore the solution in $L^2[0,\infty)$ must be
\[ \zeta(h) = h^{r_1}e^{-\frac{\sqrt{2}}{\sigma}h^3 + \frac{\sqrt{2}}{\sigma}h} \sum_{n=0}^{\infty} h^{-n} \left( \frac{P^{(n)}_{11}}{P^{(n)}_{21}} \right), \tag{6.9}
\]
where $P^{(n)}_{ij}$ denotes the element in the $i$-th row and the $j$-th column of the matrix $P^{(n)}$. Now return to (6.3) to read off the claim.

Pick $a'$ such that $a < a' < a^{**}$ and define (recall (2.9))
\[ M_t = \frac{D_t^{(a')}}{D_t^{(a)}} = \frac{x_a(X_0)}{x_d'(X_t)} e^{-t|\rho(a') - \rho(a)|} e^{(a' - a) f_0 x_v dv}, \quad t \geq 0. \tag{6.10}
\]

**STEP 2.** For any $h \geq 0$, $(M_t)_{t \geq 0}$ is a martingale under $\hat{P}_h^a$.

**Proof.** Fix $0 < s < t$. If $\phi_s$ denotes the time-shift by $s \geq 0$ (i.e., $(\phi_s \circ f)((X_t)_{t \geq 0}) = f((X_{t+s})_{t \geq 0})$ for any bounded and measurable function $f$), then it is clear that $M_t = M_s(\phi_s \circ M_{t-s})$. Hence, using the Markov property at time $s$, we see that, for any $h \geq 0$,
\[ \hat{E}_h^a(M_t | M_s) = M_s \hat{E}_h^a(\phi_s \circ M_{t-s} | M_s) = M_s \hat{E}_h^a(\hat{E}_h^a(M_{t-s}))). \tag{6.11}
\]

Now use that, for any $x \geq 0$, according to the construction of the transformed process in (2.9-2.10),
\[ \hat{E}_x^a(M_{t-s}) = \hat{E}_x(D_{t-s}^{(a)}, M_{t-s}) = \hat{E}_x(D_{t-s}^{(a)}) = 1. \tag{6.12}
\]

**STEP 3. Proof of (5.12).**

**Proof.** Use Step 2, Doob’s martingale inequality and (6.12), to obtain
\[ \hat{E}_h^a \left( \max_{[0,\delta]} M \geq K \right) \leq \frac{1}{K} \max_{[0,\delta]} \hat{E}_h^a(M_t) = \frac{1}{K}, \quad h \geq 0, K > 0. \tag{6.13}
\]

Next note that by Step 1, for any $R > 0$,
\[ \inf_{[0,R]} \frac{x_a(X_t)}{x_a'(X_t)} \geq e^{-c\sqrt{R}}. \tag{6.14}
\]

Substitute this into (6.10), to get
\[ M_t \geq c\frac{x_d'(X_t)}{x_a(X_t)} e^{-c\sqrt{R}} \quad \hat{E}_h^a \text{-a.s., } 0 \leq t \leq 1, 0 \leq h \leq R. \tag{6.15}
\]

Pick $g_a: [0,\infty) \to (0,\infty)$ to be the largest increasing function not exceeding $x_{d'}/x_a$ anywhere on $[0,\infty)$. Then, by (6.15), $M_t \geq c g_a(X_t) e^{-c\sqrt{R}} \quad \hat{E}_h^a \text{-a.s., } 0 \leq t \leq 1, 0 \leq h \leq R$. Now use (6.13) to
estimate, for $0 \leq h \leq R$,

$$
\hat{\mathbb{P}}^a_h \left( \max_{t \leq 0} X \geq \alpha \delta^{-\frac{1}{2}} \right) = \mathbb{P}_h^a \left( \max_{t \leq 0} c g_a(X_t) \geq c g_a(\alpha \delta^{-\frac{1}{2}}) \right)
\leq \mathbb{P}_h^a \left( \max_{t \leq 0} M_t \geq c g_a(\alpha \delta^{-\frac{1}{2}}) e^{-c \sqrt{R}} \right)
\leq \frac{1}{c g_a(\alpha \delta^{-\frac{1}{2}})} e^{c \sqrt{R}}.
$$

(6.16)

By Step 1, it is possible to pick $g_a$ such that

$$
g_a(h) \geq e^{-c \sqrt{R}}, \quad h \to \infty.
$$

(6.17)

This implies the bound in (5.12) with $d(\alpha) = \sqrt{\alpha}$. □

**STEP 4. Proof of (5.13).**

**Proof.** Fix $a < a^*$. Define

$$
D_t^{(a,*)} = \frac{y_a(X_t^*)}{y_a(X_0^*)} e^{\int_0^t [a X_t^* - (X_t^*)^2] \, dt}, \quad t \geq 0.
$$

(6.18)

Then it is easy to check (see [RY94, Section VIII.3]) that $(D_t^{(a,*)})_{t \geq 0}$ is a martingale under $\mathbb{P}_h^a$ for any $h \geq 0$ (where $y_a$ is a strictly positive solution to the differential equation $2y''_a(h) = (h - a)y_a(h)$ on $[0, \infty$); recall (2.6) and (2.15)). Hence, analogously to (2.10), we may construct a transformed process via a Girsanov transformation by taking $D_t^{(a,*)}$ formally as a density with respect to $\mathbb{E}^{Q}_0$. Denote by $\mathbb{P}_h^{a,*}$ and $\mathbb{E}_h^{a,*}$ probability and expectation with respect to this transformed process starting at $h \geq 0$.

Recall that $y_a(0) = 1$. We have the following representation for the function $y_a^{(\delta,\alpha)}$ (recall (5.10)):

$$
y_a^{(\delta,\alpha)}(h) = y_a(h) \mathbb{P}_h^{a,*} \left( X_0^* = 0, \max X^* \geq \alpha \delta^{-\frac{1}{2}} \right).
$$

(6.19)

The proof of (5.13) is now analogous to Steps 2–3. Indeed, use (6.19), drop the restriction $X_0^* = 0$, and proceed analogously. Step 1 provides the necessary asymptotic bounds for $y_a$ and $y_a^*$, provided that $a < a' < a^*$. □

**STEP 5. Proof of (5.14).**

**Proof.** We return to the right-hand side of (4.11) and obtain a lower bound by inserting the indicator of the event $\{ \max X^* \leq 2\delta \}$. On this event, we may estimate the exponential from below by $c$. Hence, for $0 \leq h \leq \delta$,

$$
y_a^{(\delta)}(h) \geq c \mathbb{P}_h^{a} \left( \max X^* \leq 2\delta, X_0^* = 0 \right) = c \left[ \mathbb{P}_h^{a} (X_0^* = 0) - \mathbb{P}_h^{a} (\max X^* > 2\delta, X_0^* = 0) \right].
$$

(6.20)

Using the Markov property at the first time the BESQ$^0$ hits $2\delta$, we see that the latter probability is at most $\mathbb{P}_h^{a} (X_0^* = 0)$. Since the first probability is decreasing in $h$, we therefore have the bound

$$
y_a^{(\delta)}(h) \geq c (\mathbb{P}_h^{a} (X_0^* = 0) - \mathbb{P}_h^{a} (X_0^* = 0)).
$$

(6.21)

Now use that $\mathbb{P}_h^{a} (X_0^* = 0) = e^{-h/2\delta}$ for any $h, \delta \geq 0$ (see [RY94, Corollary XI(1.4)]) to complete the proof. □

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