Solution to Problem 75-15: An eigenvalue problem

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Published in:
SIAM Review

DOI:
10.1137/1018092

Published: 01/01/1976

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Now we assume that $A$ and $B$ are distinct. The convergence proof is simplified by sending $A$ and $B$ to $\pm 1$ by the linear transformation (4)

\[ x_1 = \frac{1}{2}(A - B)Z + \frac{1}{2}(A + B). \]

The sequence of approximations satisfies

(5) \[ Z_{n+1} = \frac{1}{2}(Z_n + Z_n^{-1}), \quad n = 1, 2, \ldots, \]

and thus the limit points satisfy a quadratic equation with roots $\pm 1$. [Editorial note: Subsequently, the author shows that $\text{Re } Z_1 > 0 \Rightarrow \text{limit } = +1$, $\text{Re } Z_1 < 0 \Rightarrow \text{limit } = -1$ and $\text{Re } Z_1 = 0 \Rightarrow \text{no limit}$. A simpler proof follows easily from the explicit solution of (5), i.e.,

\[ Z_{n+1} = \frac{(Z_1 + 1)^{2^n} + (Z_1 - 1)^{2^n}}{(Z_1 + 1)^{2^n} - (Z_1 - 1)^{2^n}}. \]

Thus the imaginary axis represents the set of $Z$ that does not converge and (4) maps this onto the right bisector of $A$ and $B$ in the original coordinates.

The original trial roots did not necessarily satisfy (2). The set of such roots, that give $x_1$ and $x_2$ on the right bisector of $A$ and $B$, is found by taking

\[ x_1' = iC(A - B) + \frac{1}{2}(A + B) \]

for $C$ real. Then using the definition of $x_1'$ and eliminating $A$ and $B$ by means of

\[ P(x_i) = (x_i - A)(x_i - B), \quad i = 1, 2, \]

we obtain

\[ \frac{1}{4C^2} = \frac{4P(x_1)P(x_2)}{[P(x_1) + P(x_2) - (x_1 - x_2)^2]^2} - 1, \]

contingent upon $x_1 \neq x_2$. Thus trial roots making the expression on the right positive real form a set of measure zero that does not give convergence. Of course, $x_1 = x_2$ must join this set as (1) is not defined.

Also solved partially F. CARTY (North Canton, Ohio), T. O. ESPELID (Universitet I Bergen, Bergen, Norway) and the proposers.

**Problem 75-15, An Eigenvalue Problem**, by E. WASSERSTROM (Israel Institute of Technology, Haifa, Israel).

Let

\[ D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \]

where $d_1$, $d_2$ and $d_3$ are positive and $d_3 \leq d_1$. Show that if $d_3 < d_1/3$, then there are two other positive diagonal matrices $D_1$ and $D_2$ such that $D$, $D_1$ and $D_2$ are distinct but $DT$, $D_1T$ and $D_2T$ have the same eigenvalues. Show also that if $d_3 > d_1/3$ and $D_1$ is a positive diagonal matrix distinct from $D$, then $DT$ and $D_1T$ must have different sets of eigenvalues.
Illustration. For the three matrices $D = \text{diag}(5.5596, 1.4147, 1.5257)$, $D_1 = \text{diag}(5.1030, 2.4288, 0.9682)$ and $D_2 = \text{diag}(2.9782, 4.6565, 0.8653)$, correct to the given figures, the eigenvalues of $DT$, $D_1T$ and $D_2T$ are the same, i.e., $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 12$. On the other hand, with $D = \text{diag}(3.4530, 1.4584, 1.5887)$, the eigenvalues of $DT$ are $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = 8$, and there is no other positive diagonal matrix $D_1$ such that the eigenvalues of $D_1T$ are the same.

Remark. This problem arises from the discretization of the inverse eigenvalue problem $\frac{d^2y}{dx^2} = \lambda \rho(x)y$, $y(0) = y(1) = 0$. For a given spectrum, $\lambda$, one is then required to find the density function $\rho(x)$. (See B. M. Levitan and M. G. Gasymov, Determination of a differential equation by means of two spectra, Uspehi Mat. Nauk., 19 (1964), pp. 3–63.)

Editorial note. The proposer’s solution is essentially a numerical one. It would be desirable to give an analytic solution.

Solution by O. P. Lossers (Technical University, Eindhoven, the Netherlands).

A. The eigenvalues of $DT$ are the roots of the equation in $\lambda$:

$$\lambda^3 - 2a\lambda^2 + b\lambda - 4c = 0,$$

where

$$a = d_1 + d_2 + d_3, \quad b = 3d_2(d_1 + d_3) + 4d_1d_3, \quad c = d_1d_2d_3.$$

Eliminating $d_1$ and $d_3$ between the equalities (1), we find that $d_2$ is a root of the equation

$$3x^3 - 3ax^2 + bx - 4c = 0$$

in $x$. Apart from $d_2$, this equation has two roots satisfying

$$f(x) = 3x^2 - 3(d_1 + d_3)x + 4d_1d_3 = 0.$$

The function $f(x)$ has $m = -\frac{1}{4}(3d_1 - d_3)(d_1 - d_3)$ as its absolute minimum. We assume that throughout $d_3 \leq d_1$. The roots of (3) are imaginary if $d_3 > \frac{1}{3}d_1$. It follows that there does not exist in this case a $3 \times 3$ diagonal matrix $M$ with real elements such that $MT$ has the same eigenvalues as $DT$. We suppose, further $d_3 < \frac{1}{3}d_1$. Then $m < 0$. Since $f(d_1) = f(d_3) = d_1d_3 > 0$, we infer that (3) has two different real roots between $d_3$ and $d_1$. Let $d_2$ be one of these roots; then we have $d_3 < d_2 < d_1$.

B. We propose to prove that there exist really two positive numbers $d_2$ and $d_3$ with $d_3 < \frac{1}{3}d_1$ such that

$$d_1 + d_2 + d_3 = a, \quad 3d_2(d_1 + d_3) + 4d_1d_3 = b, \quad \delta_1 \delta_2 \delta_3 = c.$$

We observe in the first place that the conditions

$$d_1 + d_2 + d_3 = a \land \delta_1 \delta_2 \delta_3 = c$$

together with the fact that $d_2$ is a solution of (2) imply the second of the equalities (4). For we obtain, under the said conditions:

$$3d_2(d_1 + d_3) + 4d_1d_3 = \delta_2^{-1}\{3\delta_2^2(a - \delta_2) + 4c\} = \delta_2^{-1}b\delta_2 = b.$$
There exist different positive numbers $\delta_1$ and $\delta_3$ satisfying (5) iff
\[(\delta_1 - \delta_3)^2 = (a - \delta_2)^2 - 4\delta_2^{-1}c > 0,
\]
that is, iff
\[\delta_2(a - \delta_2)^2 - 4c > 0.
\]
Introducing the function $\varphi(x) = x(a - x)^2 - 4c$, we therefore have to prove $\varphi(\delta_2) > 0$. To this end, we observe:
\[\varphi(0) = \varphi(a) = -4c < 0; \quad \varphi(\delta_3) = \delta_3(d_3 - d_3)^2 > 0; \quad \varphi(\delta_1) = d_1(d_2 - d_3) \geq 0; \quad \varphi(\infty) > 0.
\]
It follows that the equation $\varphi(x) = 0$ has three different real roots $x_1 < x_2 < x_3$ satisfying:
\[0 < x_1 < x_3 < x_3 < x_2 < a < x_3.
\]
Therefore $d_3 < x < d_1$ implies $\varphi(x) > 0$. Since we know from A that $d_3 < \delta_2 < d_1$ we obtain $\varphi(\delta_2) > 0$. The existence of $\delta_1$ and $\delta_3$ thus being ascertained we chose our notation such that $0 < \delta_3 < \delta_1$.

C. It remains to prove that $\delta_3 < \frac{1}{3}\delta_1$ and that the matrix $\text{diag} (\delta_1, \delta_2, \delta_3)$ is not equal to $D$. It is not possible to derive this from the data provided by the proposer. It will be necessary to make the additional assumption that (2) has no multiple root. In this case, $d_2$ is not a solution of (3) and therefore $d_2 \neq \delta_2$. Hence $\text{diag} (\delta_1, \delta_2, \delta_3) \neq D$. Moreover, the equation $3x^2 - 3(\delta_1 + \delta_3)x + 4\delta_1\delta_3 = 0$ has two different real roots. This means that
\[0 < 9(\delta_1 + \delta_3)^2 - 48\delta_1\delta_3 = 3(3\delta_1 - \delta_3)(\delta_1 - 3\delta_3),
\]
whence in view of $\delta_3 < \delta_1$ the missing inequality: $\delta_3 < \frac{1}{3}\delta_1$. Now we have completely proved that there exist two different positive diagonal matrices $D_1$ and $D_2$ other than $D$ and of precisely the same nature as $D$, such that $D_1T, D_1T$ and $D_2T$ have the same eigenvalues. We have to show that our additional assumption is in fact a necessary one. If we drop it, $d_2$ may happen to be a multiple root of (2). Then
\[(6) \quad 3d_2^2 - 3d_2(d_1 + d_3) + 4d_1d_3 = 0
\]
and (5) with $\delta_2 = d_2$ leads to $\delta_1 = d_1$, $\delta_3 = d_3$. Therefore $D_1 = \text{diag} (\delta_1, \delta_2, \delta_3) = D$. Apart from $d_2$, the equation (3) has $d_1 + d_3 - d_2$ as a root. Since it is easy to verify that $d_1 + d_3 - d_2 \neq d_2$, we may expect that this root provides us with a diagonal matrix $D_2 = \text{diag} (\delta_1, d_1 + d_3 - d_2, \delta_3) \neq D$ and of the same nature as $D$. To find $\delta_1$ and $\delta_3$, we have the equations: $\delta_1 + \delta_3 = 2d_2$, $(d_1 + d_3 - d_2)\delta_1\delta_3 = c$. This means that $\delta_1$ and $\delta_3$ are the roots of the quadratic equation in $z$:
\[(d_1 + d_3 - d_2)z^2 - 2d_2(d_1 + d_3 - d_2)z + d_1d_2d_3 = 0.
\]
This leads, in view of (6), to
\[(7) \quad \delta_1 = \frac{2d_1d_3}{d_1 + d_3 - d_2}, \quad \delta_3 = \frac{2d_1d_3}{3(d_1 + d_3 - d_2)}.
\]
Here we have, instead of the required inequality $\delta_3 < \frac{1}{3} \delta_1$, the equality $\delta_3 = \delta_1/3$. The theorem is therefore in the redaction given by the proposer not unconditionally true.

Numerical example in which (2) has a multiple root.

\[ D = \text{diag} (d_1, d_2, d_3) = \text{diag} (9, 3, 2) \Rightarrow a = 14, \quad b = 171, \quad c = 54. \]

Equation (2) becomes: \(3(x^3 - 4x^2 + 57x - 72) = 0\) \(\Rightarrow (x-3)^2(x-8) = 0\). Starting from \(D_2 = (\delta_1, 8, \delta_3)\) we find \(\delta_1 + \delta_3 = 6, \delta_1\delta_3 = 27/4\). Hence \(\delta_1 = \frac{9}{2}, \delta_3 = \frac{3}{2}, \delta_3 = \frac{1}{3} \delta_1\).

Also solved by A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands), who also pointed out that the problem is not quite correct, and the proposer.