Solution to Problem 75-15: An eigenvalue problem

Citation for published version (APA):

Lossers, O. P. (1976). Solution to Problem 75-15: An eigenvalue problem. *SIAM Review*, 18(3), 502-505.
<https://doi.org/10.1137/1018092>

DOI:

[10.1137/1018092](https://doi.org/10.1137/1018092)

Document status and date:

Published: 01/01/1976

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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Now we assume that A and B are distinct. The convergence proof is simplified by sending A and B to ± 1 by the linear transformation

$$(4) \quad x_1 = \frac{1}{2}(A - B)Z + \frac{1}{2}(A + B).$$

The sequence of approximations satisfies

$$(5) \quad Z_{n+1} = \frac{1}{2}(Z_n + Z_n^{-1}), \quad n = 1, 2, \dots,$$

and thus the limit points satisfy a quadratic equation with roots ± 1 . [*Editorial note*: Subsequently, the author shows that $\operatorname{Re} Z_1 > 0 \Rightarrow \text{limit} = +1$, $\operatorname{Re} Z_1 < 0 \Rightarrow \text{limit} = -1$ and $\operatorname{Re} Z_1 = 0 \Rightarrow \text{no limit}$. A simpler proof follows easily from the explicit solution of (5), i.e.,

$$Z_{n+1} = \frac{(Z_1 + 1)^{2^n} + (Z_1 - 1)^{2^n}}{(Z_1 + 1)^{2^n} - (Z_1 - 1)^{2^n}}.]$$

Thus the imaginary axis represents the set of Z that does not converge and (4) maps this onto the right bisector of A and B in the original coordinates.

The original trial roots did not necessarily satisfy (2). The set of such roots, that give x'_1 and x'_2 on the right bisector of A and B , is found by taking

$$x'_1 = iC(A - B) + \frac{1}{2}(A + B)$$

for C real. Then using the definition of x'_1 and eliminating A and B by means of

$$P(x_i) = (x_i - A)(x_i - B), \quad i = 1, 2,$$

we obtain

$$\frac{1}{4C^2} = \frac{4P(x_1)P(x_2)}{[P(x_1) + P(x_2) - (x_1 - x_2)^2]^2} - 1,$$

contingent upon $x_1 \neq x_2$. Thus trial roots making the expression on the right positive real form a set of measure zero that does not give convergence. Of course, $x_1 = x_2$ must join this set as (1) is not defined.

Also solved partially F. CARTY (North Canton, Ohio), T. O. ESPELID (Universitet I Bergen, Bergen, Norway) and the proposers.

Problem 75-15, An Eigenvalue Problem, by E. WASSERSTROM (Israel Institute of Technology, Haifa, Israel).

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

where d_1, d_2 and d_3 are positive and $d_3 \leq d_1$. Show that if $d_3 < d_1/3$, then there are two other positive diagonal matrices D_1 and D_2 such that D, D_1 and D_2 are distinct but DT, D_1T and D_2T have the same eigenvalues. Show also that if $d_3 > d_1/3$ and D_1 is a positive diagonal matrix distinct from D , then DT and D_1T must have different sets of eigenvalues.

Illustration. For the three matrices $D = \text{diag}(5.5596, 1.4147, 1.5257)$, $D_1 = \text{diag}(5.1030, 2.4288, 0.9682)$ and $D_2 = \text{diag}(2.9782, 4.6565, 0.8653)$, correct to the given figures, the eigenvalues of DT , D_1T and D_2T are the same, i.e., $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 12$. On the other hand, with $D = \text{diag}(3.4530, 1.4584, 1.5887)$, the eigenvalues of DT are $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = 8$, and there is no other positive diagonal matrix D_1 such that the eigenvalues of D_1T are the same.

Remark. This problem arises from the discretization of the inverse eigenvalue problem $d^2y/dx^2 = \lambda\rho(x)y$, $y(0) = y(1) = 0$. For a given spectrum, λ , one is then required to find the density function $\rho(x)$. (See B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by means of two spectra*, Uspehi Mat. Nauk., 19 (1964), pp. 3–63.)

Editorial note. The proposer's solution is essentially a numerical one. It would be desirable to give an analytic solution.

Solution by O. P. LOSSERS (Technical University, Eindhoven, the Netherlands).

A. The eigenvalues of DT are the roots of the equation in λ :

$$(1) \quad \lambda^3 - 2a\lambda^2 + b\lambda - 4c = 0,$$

where

$$a = d_1 + d_2 + d_3, \quad b = 3d_2(d_1 + d_3) + 4d_1d_3, \quad c = d_1d_2d_3.$$

Eliminating d_1 and d_3 between the equalities (1), we find that d_2 is a root of the equation

$$(2) \quad 3x^3 - 3ax^2 + bx - 4c = 0$$

in x . Apart from d_2 , this equation has two roots satisfying

$$(3) \quad f(x) = 3x^2 - 3(d_1 + d_3)x + 4d_1d_3 = 0.$$

The function $f(x)$ has $m = -\frac{1}{4}(3d_1 - d_3)(d_1 - 3d_3)$ as its absolute minimum. We assume that throughout $d_3 \leq d_1$. The roots of (3) are imaginary if $d_3 > \frac{1}{3}d_1$. It follows that there does not exist in this case a 3×3 diagonal matrix M with real elements such that MT has the same eigenvalues as DT . We suppose, further $d_3 < \frac{1}{3}d_1$. Then $m < 0$. Since $f(d_1) = f(d_3) = d_1d_3 > 0$, we infer that (3) has two different real roots between d_3 and d_1 . Let δ_2 be one of these roots; then we have $d_3 < \delta_2 < d_1$.

B. We propose to prove that there exist really two positive numbers δ_1 and δ_3 with $\delta_3 < \frac{1}{3}\delta_1$ such that

$$(4) \quad \delta_1 + \delta_2 + \delta_3 = a, \quad 3\delta_2(\delta_1 + \delta_3) + 4\delta_1\delta_3 = b, \quad \delta_1\delta_1\delta_3 = c.$$

We observe in the first place that the conditions

$$(5) \quad \delta_1 + \delta_2 + \delta_3 = a \wedge \delta_1\delta_2\delta_3 = c$$

together with the fact that δ_2 is a solution of (2) imply the second of the equalities (4). For we obtain, under the said conditions:

$$3\delta_2(\delta_1 + \delta_3) + 4\delta_1\delta_3 = \delta_2^{-1}\{3\delta_2^2(a - \delta_2) + 4c\} = \delta_2^{-1}b\delta_2 = b.$$

There exist different positive numbers δ_1 and δ_3 satisfying (5) iff

$$(\delta_1 - \delta_3)^2 = (a - \delta_2)^2 - 4\delta_2^{-1}c > 0,$$

that is, iff

$$\delta_2(a - \delta_2)^2 - 4c > 0.$$

Introducing the function $\varphi(x) = x(a - x)^2 - 4c$, we therefore have to prove $\varphi(\delta_2) > 0$. To this end, we observe:

$$\begin{aligned} \varphi(0) = \varphi(a) = -4c < 0; & \quad \varphi(d_3) = d_3(d_1 - d_3)^2 > 0; \\ \varphi(d_1) = d_1(d_2 - d_3) \geq 0; & \quad \varphi(\infty) > 0. \end{aligned}$$

It follows that the equation $\varphi(x) = 0$ has three different real roots $x_1 < x_2 < x_3$ satisfying:

$$0 < x_1 < d_3 < d_1 \leq x_2 < a < x_3.$$

Therefore $d_3 < x < d_1$ implies $\varphi(x) > 0$. Since we know from A that $d_3 < \delta_2 < d_1$ we obtain $\varphi(\delta_2) > 0$. The existence of δ_1 and δ_3 thus being ascertained we chose our notation such that $0 < \delta_3 < \delta_1$.

C. It remains to prove that $\delta_3 < \frac{1}{3}\delta_1$ and that the matrix $\text{diag}(\delta_1, \delta_2, \delta_3)$ is not equal to D . *It is not possible to derive this from the data provided by the proposer of the problem.* It will be necessary to make the additional assumption that (2) has no multiple root. In this case, d_2 is not a solution of (3) and therefore $d_2 \neq \delta_2$. Hence $\text{diag}(\delta_1, \delta_2, \delta_3) \neq D$. Moreover, the equation $3x^2 - 3(\delta_1 + \delta_3)x + 4\delta_1\delta_3 = 0$ has two different real roots. This means that

$$0 < 9(\delta_1 + \delta_3)^2 - 48\delta_1\delta_3 = 3(3\delta_1 - \delta_3)(\delta_1 - 3\delta_3),$$

whence in view of $\delta_3 < \delta_1$ the missing inequality: $\delta_3 < \frac{1}{3}\delta_1$. Now we have completely proved that there exist two different positive diagonal matrices D_1 and D_2 other than D and of precisely the same nature as D , such that DT , D_1T and D_2T have the same eigenvalues. We have to show that our additional assumption is in fact a necessary one. If we drop it, d_2 may happen to be a multiple root of (2). Then

$$(6) \quad 3d_2^2 - 3d_2(d_1 + d_3) + 4d_1d_3 = 0$$

and (5) with $\delta_2 = d_2$ leads to $\delta_1 = d_1$, $\delta_3 = d_3$. Therefore $D_1 = \text{diag}(\delta_1, \delta_2, \delta_3) = D$. Apart from d_2 , the equation (3) has $d_1 + d_3 - d_2$ as a root. Since it is easy to verify that $d_1 + d_3 - d_2 \neq d_2$, we may expect that this root provides us with a diagonal matrix $D_2 = \text{diag}(\delta_1, d_1 + d_3 - d_2, \delta_3) \neq D$ and of the same nature as D . To find δ_1 and δ_3 , we have the equations: $\delta_1 + \delta_3 = 2d_2$, $(d_1 + d_3 - d_2)\delta_1\delta_3 = c$. This means that δ_1 and δ_3 are the roots of the quadratic equation in z :

$$(d_1 + d_3 - d_2)z^2 - 2d_2(d_1 + d_3 - d_2)z + d_1d_2d_3 = 0.$$

This leads, in view of (6), to

$$(7) \quad \delta_1 = \frac{2d_1d_3}{d_1 + d_3 - d_2}, \quad \delta_3 = \frac{2d_1d_3}{3(d_1 + d_3 - d_2)}.$$

Here we have, instead of the required *inequality* $\delta_3 < \frac{1}{3}\delta_1$, the *equality* $\delta_3 = \delta_1/3$. The theorem is therefore in the redaction given by the proposer not unconditionally true.

Numerical example in which (2) has a multiple root.

$$D = \text{diag}(d_1, d_2, d_3) = \text{diag}(9, 3, 2) \Rightarrow a = 14, \quad b = 171, \quad c = 54.$$

Equation (2) becomes: $3(x^3 - 4x^2 + 57x - 72) = 0 \Rightarrow (x-3)^2(x-8) = 0$. Starting from $D_2 = (\delta_1, 8, \delta_3)$ we find $\delta_1 + \delta_3 = 6$, $\delta_1\delta_3 = 27/4$. Hence $\delta_1 = \frac{9}{2}$, $\delta_3 = \frac{3}{2}$; $\delta_3 = \frac{1}{3}\delta_1$.

Also solved by A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands), who also pointed out that the problem is not quite correct, and the proposer.