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Discrete analogues of self-decomposability and stability

by

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Abstract

Analogues are proposed for the concepts of self-decomposability and stability for distributions on the nonnegative integers. It turns out that these "discrete self-decomposable" and "discrete stable" distributions have properties that are quite similar to those of their continuous counterparts.

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Discrete analogues of self-decomposability and stability

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1. Introduction and preliminaries

A probability distribution on \( \mathbb{R} \) is said to be self-decomposable (or, of class \( L \)) if its characteristic function (c.f.) satisfies (cf. [5], p. 161)

\[
\varphi(t) = \varphi(at) \varphi_a(t) \quad (t \in \mathbb{R}; a \in (0, 1]),
\]

with \( \varphi_a \) a c.f. For the corresponding random variables (r.v.'s) this means that (in distribution)

\[
X = aX' + X_a \quad (a \in (0, 1]),
\]

where \( X' \) and \( X_a \) are independent and \( X' \) is distributed as \( X \). Clearly, apart from \( X = 0 \), no lattice r.v. can satisfy (1.2); in fact, all nondegenerate self-decomposable (self-dec) distributions are known to be absolutely continuous (see e.g. [3]).

In this note we propose analogues of self-decomposability and stability for distributions on \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \). It turns out that the discrete self-dec distributions and the discrete stable distributions share the basic properties with their continuous counterparts. The discrete self-dec distributions, for instance, are unimodal, and the discrete stable distributions are very similar to their continuous analogues on \( (0, \infty) \).

We shall need the following two lemmas for probability generating functions (p.g.f.'s), and on infinite divisibility (inf div). For a proof of the second lemma we refer to [1] and [6]. The generating function of sequences \( (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, \) etc. will be denoted by \( A, B, \) etc.

**Lemma 1.1.** If \( P \) is a p.g.f., then

\[
\lim_{x \uparrow 1} (1 - x) P'(x) = 0.
\]

**Proof.** For \( x \in [0, 1) \) we have \( 1 - P(x) = (1 - x) P'(\xi) \) with \( \xi \in (x, 1) \). As \( P' \) is nondecreasing, we have \( (1 - x) P'(x) \leq (1 - x) P'(\xi) = 1 - P(x) \to 0 \) as \( x \uparrow 1. \)
Lemma 1.2. A p.g.f. \( P \) with \( 0 < p_0 < 1 \) is inf div iff \( P \) has the form

\[
P(z) = \exp(\lambda(G(z) - 1)),
\]

where \( \lambda > 0 \) and \( G \) is a (unique) p.g.f with \( G(0) = 0 \). Equivalently, \( P \) is inf div iff

\[
P(z) = \exp\left(- \int_0^1 R(u) du\right),
\]

where \( R(u) = \sum_{n=0}^{\infty} r_n u^n \), with \( r_n \geq 0 \) and, necessarily, \( \sum_{n=0}^{\infty} r_n (n+1)^{-1} < \infty \), i.e. iff the \( p_n \) satisfy

\[
(n + 1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \quad (n \in \mathbb{N}_0),
\]

with \( r_n \geq 0 \).

2. Discrete self-decomposability

We start with analogues of (1.1) and (1.2) that operate within the set of distributions on \( \mathbb{N}_0 \). For definiteness we shall assume \( 0 < p_0 < 1 \).

Definition 2.1. A distribution on \( \mathbb{N}_0 \) with p.g.f. \( P \) is called discrete self-decomposable if

\[
P(z) = P(1 - \alpha + \alpha z)P_\alpha(z) \quad (|z| \leq 1; \alpha \in (0,1)),
\]

with \( P_\alpha \) a p.g.f.

Equation (2.1) can be written in terms of r.v.'s as follows:

\[
X = \alpha \circ X' + X_\alpha,
\]

where \( \alpha \circ X' \) and \( X_\alpha \) are independent, and \( X' \) is distributed as \( X \). Here \( \alpha \circ X \) is defined (in distribution) by its p.g.f. \( P(1 - \alpha + \alpha z) \), or by

\[
\alpha \circ X = \sum_{j=1}^{X} N_j,
\]

where \( P(N_j = 1) = 1 - P(N_j = 0) = \alpha \), all r.v.'s being independent. It then follows that \( \alpha \circ X \in \mathbb{N}_0 \), with \( 1 \circ X = X \), \( 0 \circ X = 0 \), and \( E \alpha \circ X = \alpha E X \), as in scalar multiplication; an empty sum is zero.

We first establish the canonical form of the discrete self-decomposable p.g.f.'s.
Theorem 2.2. A p.g.f. \( P \) is discrete self-dec iff it has the form

\[
P(z) = \exp\{-\lambda \int_0^1 \frac{1-G(u)}{1-u} \, du\},
\]

where \( \lambda > 0 \) and \( G \) is a (unique) p.g.f. with \( G(0) = 0 \). Equivalently, \( P \) is discrete self-dec iff it is inf div and has a canonical measure \( r_n \) (cf. lemma 1.2) that is nonincreasing.

Proof. Let \( P \) be s.d., i.e. satisfy (2.1). Then for \( r > 0 \) and \( r(1 - \alpha_n)^{-1} \in \mathbb{N} \),

\[
Q_{r,n}(z) := \{ P_{\alpha_n} (z) \}^{r/(1-\alpha_n)}
\]

is a p.g.f. As \( P(1 - \alpha + \alpha z) = P(z) + (1 - \alpha)(1 - z)P'(z) + o(1 - \alpha) \)
as \( \alpha \to 1 \), by (2.1) and (2.5), with \( \alpha_n \) such that \( \alpha_n \to 1 \) as \( n \to \infty \),

\[
Q_r(z) := \lim_{n \to \infty} Q_{r,n}(z) = \exp\{-r(1 - z)P'(z)/P(z)\}.
\]

As (cf. lemma 1.1) \( Q_r(z) \to 1 \) as \( z \to 1 \), by the continuity theorem for p.g.f.'s (cf. [11], p.280), \( Q_r \) is a p.g.f. for every \( r > 0 \). It follows that \( Q := Q_1 \) is infinitely divisible, and therefore by (2.6), and (1.3) applied to \( Q \), that

\[
R(z) := \frac{P'(z)}{P(z)} = \frac{-\log Q(z)}{1-z} = \lambda \frac{1-G(z)}{1-z},
\]
equivalent to (2.4). Comparing (2.4) and (1.4) we see that \( P \) is inf div, with

\[
r_n = \lambda(1 - \sum_{j=1}^{n} g_j) = \lambda \sum_{j=n+1}^{\infty} g_j,
\]

which is nonincreasing. Conversely, let \( P \) satisfy (2.4); this is easily seen to be the case if \( P \) is inf div with nonincreasing \( r_n \), i.e. satisfying (2.8). Then \( P \) satisfies (2.1) with

\[
P_{\alpha} (z) = \exp\{- \int_0^1 R(u) \, du\},
\]
i.e. with \( R(z) := P'(z)/P(z) = R(z) - \alpha R(1-\alpha(1-z)) \), with coefficients

\[
\begin{align*}
\frac{r_n}{a^{n+1}} - \sigma \sum_{k=0}^{\infty} \binom{k}{n} a^n (1-a)^{k-n} r_k &\geq \frac{r_n}{a^{n+1}} \left(1 - a^{n+1}\right) \sum_{j=0}^{\infty} \binom{n+j}{j} (1-a)^j = 0,
\end{align*}
\]

where we have used the fact that \( r_n \) is nonincreasing. It follows that \( P_n \) is a (infinitely divisible) p.g.f.

The unimodality of discrete self-decomposable distributions is a corollary to the following theorem.

**Theorem 2.3.** Let \((P_n)_0^\infty\) and \((r_n)_0^\infty\) be sequences of real numbers with \( P_n \geq 0 \), \( P_0 > 0 \), and \( r_n \) nonincreasing. Furthermore let \( P_n \) and \( r_n \) be related by

\[
(n+1) P_{n+1} = \sum_{k=0}^{n} P_k r_{n-k} \quad (n \in \mathbb{N}_0).
\]

Then \((P_n)_0^\infty\) is unimodal, i.e. \( P_n - P_{n-1} \) changes sign at most once \( (p_{-1} = 0) \); \( P_n \) is nonincreasing iff \( r_0 \leq 1 \).

**Proof.** The proof is very similar to that in [7] for self-decomposable densities on \((0,\infty)\). Putting \( d_n = P_n - P_{n-1} \) and \( \lambda_n = r_n - r_{n+1} \), from (2.9) we obtain by subtraction

\[
(n+1)d_{n+1} = (r_0 - 1)P_n - \sum_{j=0}^{n-1} \lambda_j P_{n-j-1} \quad (n \in \mathbb{N}_0).
\]

Clearly, \( d_n \leq 0 \) for \( n \in \mathbb{N} \) iff \( r_0 \leq 1 \). Now let \( r_0 > 1 \), and suppose that

\[
d_1 > 0, d_2 \geq 0, \ldots, d_{n_1} \geq 0, d_{n_1+1} < 0, \ldots, d_{n_1+m} =: d_{n_2} \leq 0, d_{n_2+1} > 0.
\]

Then we have, putting \( p_{n-j} = 0 \) if \( j > n \),

\[
\begin{align*}
&\quad p_{n_1-j} \leq p_{n_2-j} \quad (j = m+1, m+2, \ldots) \\
&\quad p_{n_1-j} \leq p_{n_1} \quad (j = 1, 2, \ldots, m).
\end{align*}
\]

From (2.10) and (2.11) we have

\[
(n_1 + 1)d_{n_1+1} = (r_0 - 1)P_{n_1} - \sum_{j=0}^{n_1-1} \lambda_j P_{n_1-j-1} < 0,
\]

\[
(n_1 + 1)\alpha = (r_0 - 1)\alpha - \sum_{j=0}^{n_1-1} \lambda_j \alpha_j < 0.
\]
As \( \sum_{j=0}^{m-1} \lambda_j p_{n_2-j} \leq \sum_{j=0}^{m-1} \lambda_j p_{n_2-j-1} \), from (2.14) it follows that \( (r_0 - 1)p_{n_2} - \sum_{j=0}^{n_2-1} \lambda_j p_{n_2-j-1} > 0. \)

(2.15) \( (r_m - 1)p_{n_2} > \sum_{j=m}^{n_2-1} \lambda_j p_{n_2-j-1}. \)

But, from (2.12) and (2.15) we obtain

\[
\sum_{j=0}^{n_1-1} \lambda_j p_{n_1-j} \leq \sum_{j=0}^{m-1} \lambda_j p_{n_1} + \sum_{j=m}^{n_1-1} \lambda_j p_{n_2-j-1} < p_{n_1}(r_0 - r_m) + p_{n_2}(r_m - 1) < p_{n_1}(r_0 - 1),
\]

which contradicts (2.13). It follows that (2.11) is impossible.

Corollary 2.4. A discrete self-dec distribution \( (p_n)_0^\infty \) is unimodal; it is nonincreasing iff \( r_0 = p_1/p_0 \leq 1. \) Equivalently, an \( \text{inf div} \) distribution on \( \mathbb{N}_0 \) (with \( p_0 > 0 \)) is unimodal if \( r_n \) (cf. (1.5)) is nonincreasing; it is nonincreasing iff in addition \( r_0 \leq 1. \)

Remark 1. In theorem 2.3 the \( r_n \) are not supposed to be all nonnegative, i.e. we seem to find a sufficient condition for unimodality of more general sequences than \( \text{inf div} \) distributions. For nonnegative \( p_n \), however, \( r_n \) nonincreasing implies \( r_n \geq 0 \) \( (n \in \mathbb{N}_0) \).

Remark 2. Theorem 2.3 could be used to give a slightly simpler proof of the unimodality of continuous self-dec distributions on \( (0,\infty) \), as any such distribution is the limit of discrete self-dec distributions. This procedure amounts to a more drastic discretization than the one used in [7].

3. Discrete stability

The set of distributions on \( \mathbb{R} \) that are (strictly) stable with exponent \( \gamma \) is the subset of the set of self-decomposable distributions with r.v.'s \( \mathbf{X} \) satisfying (cf.[2], p. 171)

\[
(s + t)^{1/\gamma} \mathbf{X} = s^{1/\gamma} \mathbf{X}_1 + t^{1/\gamma} \mathbf{X}_2 \quad (s, t > 0),
\]

in distribution, where \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are independent and distributed as \( \mathbf{X} \). We rewrite (3.1) as

\[
X = \alpha \mathbf{X}_1 + (1 - \alpha) \mathbf{X}_2 \quad (0 < \alpha < 1).
\]
Now replacing $\alpha X_1$ by $\alpha X_1$ as defined in (2.3), and similarly for the other term, we obtain the discrete analogue of (3.2). In terms of p.g.f.'s we then have

\begin{equation}
P(z) = P(1 - \alpha(1-z)) P\left(1 - (1 - \alpha^\gamma)^{1/\gamma}(1 - z)\right) \quad (|z| \leq 1 ; \alpha \in (0,1)),
\end{equation}

and we give the following definition.

**Definition 3.1.** A p.g.f. $P$ (with $0 < P(0) < 1$) is called (strictly) discrete stable with exponent $\gamma > 0$ if it satisfies (3.3).

From (3.3) it follows that

\begin{equation}
1 - P(1 - (1 - \alpha^\gamma)^{1/\gamma}(1 - z)) = \frac{P(1 - \alpha(1-z)) - P(z)}{(1 - \alpha)(1 - z)P(1 - \alpha(1-z))} \Rightarrow \frac{P'(z)}{P(z)}
\end{equation}

as $\alpha \to 1$. Putting $(1 - \alpha^\gamma)^{1/\gamma} = u$, this means that

\begin{equation}
1 - \frac{P(z)}{u(1-z)^{\gamma}} \Rightarrow \gamma^{-1}(1 - z)^{1-\gamma} \frac{P'(z)}{P(z)} \quad (u > 0),
\end{equation}

and with $z = 0$,

\begin{equation}
1 - \frac{P(1 - u)}{u^{\gamma}} \Rightarrow \frac{P(1)}{P_0} \quad (u > 0).
\end{equation}

Combining (3.4) and (3.5) we conclude that

\begin{equation}
P'(z) = \frac{P(1)}{P_0} (1 - z)^{\gamma-1} \quad (z \in [0,1)).
\end{equation}

As $P'(1) > 0$ (possibly infinite) unless $P(0) = 1$, from (3.6) we see that $0 < \gamma \leq 1$.

Integrating (3.6) we obtain

\begin{equation}
P(z) = P_0^\lambda \left\{z \right\} := \exp\{-\lambda(1 - z)^{\gamma}\} \quad (|z| \leq 1 ; \lambda > 0),
\end{equation}

by analytic continuation. As any $P$ satisfying (3.7) satisfies (3.3), we have now proved

**Theorem 3.2.** Discrete stable p.g.f.'s (i.e. satisfying (3.3)) only exist for $\gamma \in (0,1]$, and all stable p.g.f.'s with exponent $\gamma$ are given by (3.7).

**Remark.** The discrete stable p.g.f.'s are quite similar to the Laplace transforms $\exp(-\lambda z^{\gamma})$ of the stable distributions on $(0,\infty)$ (cf.[2], p. 448). Rather curiously, the Poisson distribution replaces the degenerate one,
i.e. we have

**Corollary 3.3.** The Poisson distribution is discrete stable with exponent one.

Further, as in the continuous case, we have by (3.3) and (2.1)

**Corollary 3.4.** A discrete stable distribution is discrete self-decomposable, and hence unimodal.

**Remark.** If we define a p.g.f. \( P \) to be in the domain of (discrete) attraction of a stable p.g.f. \( P_y \) if there exist \( a_n \) such that

\[
\lim_{n \to \infty} \left( P(1 - \alpha + \frac{a_n z}{n}) \right)^n = P(y) ,
\]

then it follows that all distributions with finite first moment are attracted by the Poisson distribution: take \( \alpha = n^{-1} \). A general theory of attraction could easily be developed. However, as for \( y \in (0,1) \) we have

\[
P_y(1 - \tau) = \exp(-\tau y) ,
\]

and for every finite \( \tau \geq 0 \)

\[
P^n(1 - \frac{a_n \tau}{n}) = \left\{ E \exp(X \log(1 - \frac{a_n \tau}{n})) \right\}^n \sim \left\{ E \exp(-a_n \tau X) \right\}^n \quad (n \to \infty),
\]

\( X \in \mathbb{N}_0 \) is in the domain of discrete attraction of \( P^\lambda \) iff it is in the domain of attraction of \( \exp(-\lambda y) \) (cf. remark following theorem 3.2).

4. **Concluding remarks**

We were led to consider equation (2.1) by first considering a more formal analogue of (1.1), viz. (cf. [4])

\[
(4.1) \quad P(z) = \frac{P(az)}{P(a)} P_\alpha(z) \quad \quad (|z| \leq 1, \alpha \in (0,1)).
\]

This equation can be treated in the same way as (2.1), and it turns out that one has

**Theorem 4.1.** A p.g.f. \( P \), with \( P(0) > 0 \), satisfies (4.1) iff it is infinitely divisible, i.e. (cf. (1.3)) iff it is compound Poisson.

Defining \( \alpha \ast X \) (in distribution) by its p.g.f. \( 1 - \alpha + \alpha P(z) \), or by

\[
\alpha \ast X = \sum_{j=1}^{N} X_j ,
\]
with N as in (2.3), we may consider the equation \( X = \alpha \cdot X' + X_\alpha' \), or in terms of p.g.f.'s

\[
(4.2) \quad P(z) = \{1 - \alpha + \alpha P(z)\} P_\alpha(z) \quad (|z| \leq 1 ; \alpha \in (0,1)),
\]

to obtain

**Theorem 4.2.** A p.g.f. \( P \), with \( P(0) > 0 \), satisfies (4.2) iff it is compound geometric.

Equation (1.1) can be handled in a similar fashion, avoiding the use of triangular arrays, and one finds in exactly the same way: \( \psi \) satisfies (1.1) iff (this seems to be new)

\[
\psi(t) = \exp \int_0^t h(u)u^{-1}du \quad (t \in \mathbb{R}),
\]

where \( \exp(h(u)) \) is an inf div characteristic function. To prove this, however, one needs to know that \( \psi' \) exists in \( \mathbb{R} \setminus \{0\} \), and is such that \( t\psi'(t) \to 0 \) as \( t \to 0 \).

No such complication arises in the case of distributions on \([0,\infty)\) if one uses Laplace transforms instead of c.f.'s.

Corollary 3.3 seems to suggest that the distribution of a sum of i.i.d. random variables with only a first moment should be approximated by a discrete stable Poisson distribution rather than by a stable degenerate distribution. If higher moments exist, a normal approximation would, of course, be preferable.

It might be possible to develop a theory of discrete limiting distributions for maxima of i.i.d. random variables in \( \mathbb{R} \). This will be investigated later.


