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Profit Division in Newsvendor Situations with Delivery Restrictions

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Abstract

This study considers a supply chain that consists of \( n \) retailers, each of them facing a newsvendor problem, and a supplier. Groups of retailers might increase their expected joint profit by joint ordering and inventory centralization, which means that they give a joint order and allocate this quantity among themselves to maximize the total profit after the demands are realized. Furthermore, we assume that the retailers pose some restrictions on the number of items that should be delivered to them. In this situation, we show that the associated cooperative game has a non-empty core. Afterwards, we concentrate on a dynamic situation, where the retailers change their delivery restrictions. We investigate how the profit division might be affected by these changes. We define four new monotonicity properties, which we think are interesting in general, and we derive necessary and sufficient conditions for pairs of totally balanced TU-games to satisfy these properties. We also show that pairs of cooperative games associated with newsvendor situations do not necessarily satisfy these properties in general. Finally, we define a class of games with retailers having a normally distributed demand where one of the monotonicity properties holds.

Keywords: supply chain management, game theory, newsvendor, balancedness and core-monotonicity.

1 Introduction

Consider a distribution system of a single item with \( n \) independent outlets. Each outlet faces a stochastic demand and has to order in advance as the supplier lead time is long. For example, fashion goods outlets in Europe, which are ordering seasonal products from a supplier in Asia, are subject to such long lead times. In such an environment, it is known that these outlets might benefit from inventory pooling and might increase their
total expected profit if they can coordinate their individual orders. Additionally, more will be gained if the outlets can coordinate the allocation of the ordered units according to the realized demand instead of managing their inventories and demands separately.

In the literature (see Hartman et al (2000), Müller et al (2002), Slikker et al (2001,2005) and Özen et al (2004)), there are usually no limitations in the extent of cooperation. However, such cooperation might be subject to some restrictions. Several motivating scenarios can be given for the existence of these restrictions. Companies could have market or industry related constraints to survive in the market (e.g., service level constraints), which might contradict with profit maximization objectives. Companies have to satisfy these constraints not only when they are working alone but when they are a member of a cooperation as well. Furthermore, the companies may pose some conditions, which might be trust related, in order to join the cooperation.

It is usually hard to obtain and sustain a stable cooperation. There are several criteria for stability of cooperation. Consider a group of retailers forming a coalition to benefit from inventory pooling. A natural criterion is that, being in the coalition, each retailer expects to see the advantage of cooperation when compared to acting alone. So, they should expect to profit from the coalition at least as much as they can obtain if they were alone. Besides, this argument can be extended to groups of retailers. Cooperative game theory offers the core concept as a solution to this problem. Basically, the core consists of all stable distributions of total benefit such that no group of retailers would like to split from the biggest coalition and form a smaller coalition. In this research, we work with the core concept.

Another issue for the retailers in such coalitions would be of importance when a change in the environment or a situation that affects the outcome of the cooperation occurs. The retailer should feel that they are not discriminated or deceived under such a situation. One example of such a case might occur when a new retailer joins the coalition. Each retailer would expect to be at least as profitable as they were before expanding. In other words, a new distribution of the expected total profit that does not discriminate any of the retailers should be determined. Another example might be that profit of the coalition may change as a consequence of changing conditions of the retailers to be in the coalition. In general, monotonicity notions from cooperative game theory can be used to address this issue. In this research, we focus on the changes in the latter example and we use monotonicity properties that are instrumental in determining ”long term” stability of the coalition under such changes.

We consider a distribution system that consists of a supplier and \( n \) retailers, each facing a stochastic demand. Each retailer solves a single period problem (newsvendor problem). At the start of the period, every retailer determines his order quantity that maximizes his expected profit. After demands are realized, the products are delivered to the retailers to satisfy demands. As an alternative, we consider cooperation with transferable utility. By cooperating, the retailers give a coordinated order and allocate the received units after demand realizations to maximize the total coalition profit. However, we consider limitations (constraints) on how this allocation can be performed. Specif-
ically, we assume that each retailer poses a minimum quantity (related to the demand realization) that should be delivered. In this way, they guarantee to satisfy at least some amount of their realized demand. There could be several possible reasons for the retailers to pose such constraints. These are:

- **Ensure income from operations**: To survive in the market and preserve strategic strength, a retailer may want to stay active in the industry. A constraint on the delivered quantity guarantees the retailer to receive an income from the business and continue to be in the market.

- **Ensure some customer service level**: Another important factor in surviving in the market is customer satisfaction. By this constraint, the retailer can ensure a reasonable customer service level.

In non-technical terms, constraints allow us to incorporate long term targets of a retailer in a single period modeling framework. Of course, we expect that the limitations imposed by the constraint to change from one such season to another, simply representing an updated long term target.

We analyze cooperative games arising from newsvendor situations above. We first focus on the existence of stable profit distributions, which is shown by proving that these games have non-empty cores. Afterwards, we investigate the cases, where the retailers change their conditions of cooperation, i.e., the retailers increase or decrease their minimum delivery quantities, which affect the outcome of the coalition. We identify two types of changes. In the first one, a group of retailers either decrease or increase the delivery amounts, and in the latter a single retailer changes his minimum amount. We focus on the issue of whether we can find a core distribution of total profit for the new situation, which does not discriminate any of the retailers. This issue is not captured by the standard, known monotonicity properties in game theory. Hence, we formulate four new monotonicity properties associated with these two types of changes. We conceive that these properties are off interest in general settings, as well. We derive several necessary and sufficient conditions for the monotonicity properties to be satisfied by a general class of games. We also show that cooperative games associated with newsvendor situations do not necessarily satisfy these properties in general. We give an example of a newsvendor setting where such a property is satisfied.

The outline of the paper is as follows: In Section 2, we give a brief literature review. In Section 3, we introduce preliminaries on cooperative game theory. Section 4 introduces the newsvendor situations with delivery constraints and the associated cooperative games. Moreover, we show that these games have non-empty cores. In Section 5, we consider changes in the minimum delivery amounts of the retailers, which change the maximum profit of the coalitions. We then define four monotonicity properties associated with changes in minimum amounts. In Section 6, we analyze necessary and sufficient conditions for arbitrary pairs of totally balanced cooperative games to satisfy the defined monotonicity properties. We specify conditions for two-player games, three-player games.
and games with arbitrary number of players, separately. We conclude Section 6 by analyzing the monotonicity properties for pairs of games associated with newsvendor situations. After providing examples that none of the properties are guaranteed to hold for cooperative games associated with newsvendor situations, we focus on a special class of newsvendor games and show that one of the monotonicity properties holds. We conclude our paper in Section 7 with final remarks.

2 Literature review

It is known in the literature that inventory centralization leads to cost savings and profit increase. Eppen (1979), Eppen and Schrage (1981), Chin and Lin (1989), Chang and Lin (1991), and Cherikh (2000) showed this effect in different inventory settings. Several tools have been developed for the problem of sharing benefits in game theory literature. Especially, the core concept in cooperative game theory has received special interest by several papers. Hartman et al. (2000) studied the situation, where multiple retailers cooperate to benefit from inventory centralization. They focused on core divisions and showed that such divisions exist by proving that associated newsvendor games have non-empty cores under some restricted assumptions on demand distributions. Then, Müller et al. (2002) and Slikker et al. (2001) independently came with a more powerful result showing that newsvendor games have non-empty cores regardless of the demand distribution. Slikker et al. (2002) and Özen et al. (2004) consider several extensions on simple newsvendor situations. Slikker et al. (2005) introduced non-identical selling and purchasing prices for the retailers, and transshipment costs. They showed that newsvendor games with transshipments have non-empty cores. Similarly, Özen et al. (2004) showed that newsvendor games associated with newsvendor situations with warehouses have non-empty cores.

All studies above assume complete consolidation of the demands under profit maximization objective. A different type of inventory centralization is considered by Anupindi et al. (2001). They considered a distribution system where the retailers may transship excess inventory in one location to satisfy excess demand in another location. They followed a hybrid approach, which utilizes cooperative and non-cooperative game theory, to analyze the situation where the ordering decisions of the retailers take place competitively before demand realization and transshipments occur cooperatively after demand realization. They call the cooperative games that arise from the cooperative transshipment decisions Snapshot Allocation Games (SAG). They derived a profit sharing mechanism based on dual prices of the optimal shipping problem for SAG, which is a core solution and leads to joint optimal orders being an equilibrium. Granot and Sošić (2003) extended the model of Anupindi et al. (2001) by considering an intermediate stage where the retailers decide how much excess inventory/demand they want to share before playing SAG. After showing that the mechanisms based on dual prices might not induce the retailers to reveal their exact excess inventory/demand at all times, they analyzed mechanisms that are value preserving, i.e., mechanisms that induce all the retailers to share their excess
inventory/demand that do not create a decrease in the total additional profit. They first showed that any value preserving mechanism has to satisfy a specific monotonicity property and then they proved that both the Shapley value and the fractional rule have this monotonicity property, hence they are value preserving. Sošić (2004) analyzed the model of Granot and Sošić (2003) from a farsighted point of view and showed that Shapley value divisions, which are not necessarily stable in the myopic sense, are stable in the farsighted sense. In this study, we consider a system where the retailers consolidate their demands to maximize the total system profit. However, this consolidation is limited by the retailers’ constraints on their delivery quantities.

Reaching a certain service level is as important as maximizing the profit for some companies. Bartholdi and Kemahlıoğlu (2005) considered a system of n retailers, who have certain service level requirements, and a common supplier, who keeps separate stock for the retailers and bears all the inventory risk. They investigated the cooperative game in which the players can form inventory-pooling coalitions instead of keeping separate stocks and, hence increase the total profit. After showing that mechanisms based on Shapley value are guaranteed to coordinate the supply chain, they investigated the effect of demand variance on profit division. Finally, they focused on the retailer collusion against the supplier, which appeared to be not always profitable for the retailers as intuition suggests, and they provided conditions under which such collusion benefits the retailers.

Monotonicity notions in game theory deal with the problem of fair profit division when the underlying situation of a game changes. Sprumont (1990) considered a dynamic number of players and investigated benefit sharing mechanisms which assign increasing payoffs to each player as the coalition to which he belongs grows larger. These mechanisms are called population monotonic allocation schemes. In this study, however, we focus on situations with a fixed number of players. On the other hand, we concentrate on cases in which the output of the associated game changes because of changing the delivery constraints of the retailers. Megiddo (1974) studied aggregate monotonicity, which states that if the value of the grand coalition increases, while the value of other coalitions remains fixed, then none of the players should get less than before. Afterwards, Young (1985) studied coalitional monotonicity. Coalitional monotonicity considers a value increase of a particular coalition, while the values of other coalitions remain constant and states than none of the members of the changing coalition should get less than before. Megiddo (1974) and Young (1985) investigated several single valued sharing mechanisms and checked whether they have these properties or not. Sasaki (1995) and Nunez et. al. (2002) analyzed assignment games, where an increase in the worth of a pair of players may increase the value of several coalitions containing the pair. Nunez and Rafels (2002) showed that the τ-value satisfies pairwise monotonicity, i.e., if the value of a couple increases then each member of the couple should receive more. Sasaki (1995) considered a weaker version of pairwise monotonicity for a set valued mechanism (Core) and used it for axiomatic characterization of the core of assignment games. None of these monotonicity properties, however, covers the cases that we analyze in this paper. We discuss this issue
in more detail when we introduce four new monotonicity properties in section 5.

3 Preliminaries

In this section, we give a brief introduction to cooperative game theory and introduce some notation. Let $N$ be a finite set of players, $N = \{1, ..., n\}$. A subset of $N$ is called a coalition. A function $v$, assigning a value $v(S)$ to every coalition $S \subseteq N$ with $v(\emptyset) = 0$, is called a characteristic function. The value $v(S)$ is interpreted as the maximum total profit that coalition $S$ can obtain through cooperation. Assuming that the benefit of a coalition $S$ can be transferred between the players of $S$, a pair $(N, v)$ is called a cooperative game with transferable utility (TU-game). For a game $(N, v)$, $S \subset N$ and $S \neq \emptyset$, the subgame $(S, v|_S)$ is defined by $v|_S(T) = v(T)$ for each coalition $T \subseteq S$.

In reality, the players are not primarily interested in benefits of a coalition but in their individual benefits that they make out of that coalition. A division is a payoff vector $y = (y_i)_{i \in N} \in \mathbb{R}^N$, specifying for each player $i \in N$ the benefit $y_i$. A division $y$ is called efficient if $\sum_{i \in N} y_i = v(N)$ and individually rational if $y_i \geq v(\{i\})$ for all $i \in N$. Individual rationality means that every player gets at least as much as what he could obtain by staying alone. The set of all individually rational and efficient divisions constitutes the imputation set:

$$I(v) = \{ y \in \mathbb{R}^N | \sum_{i \in N} y_i = v(N) \text{ and } y_i \geq v(\{i\}) \text{ for each } i \in N \}.$$  

If these rationality requirements are extended to all coalitions, we obtain the core:

$$Core(v) = \{ y \in \mathbb{R}^N | \sum_{i \in N} y_i = v(N) \text{ and } \sum_{i \in S} y_i \geq v(S) \text{ for each } S \subseteq N \}.$$  

Thus, the core consists of all imputations in which no group of players has an incentive to split off from the grand coalition $N$ and form a smaller coalition, because they collectively receive at least as much as what they can obtain by cooperating on their own. Note that the core of a game can be empty.

Bondareva (1963) and Shapley (1967) independently made a general characterization of games with a non-empty core by the notion of balancedness. Let us define the vector $e^S$ for all $S \subseteq N$ by $e^S_i = 1$ for all $i \in S$ and $e^S_i = 0$ for all $i \in N/S$. A map $\kappa : 2^N/\{\emptyset\} \to [0, 1]$ is called a balanced map if $\sum_{S \in 2^N/\{\emptyset\}} \kappa(S)e^S = e^N$. Further, a game $(N, v)$ is called balanced if for every balanced map $\kappa : 2^N/\{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N/\{\emptyset\}} \kappa(S)v(S) \leq v(N)$. The following theorem is due to Bondareva (1963) and Shapley (1967).

**Theorem 1** Let $(N, v)$ be a TU-game. Then $Core(v) \neq \emptyset$ if and only if $(N, v)$ is balanced.

A TU-game $(N, v)$ is called totally balanced if it is balanced and each of its subgames is balanced as well.
The anticore of a TU-game \((N, v)\) is defined by

\[
\text{Anticore}(v) = \{ y \in \mathbb{R}^N \mid \sum_{i \in N} y_i = v(N) \text{ and } \sum_{i \in S} y_i \leq v(S) \text{ for each } S \subseteq N \}.
\]

Two interesting properties that a game might satisfy are superadditivity and convexity. A game (or its characteristic function) is called superadditive if for all \(T, S \subseteq N\) with \(T \cap S = \emptyset\) it holds that

\[v(T \cup S) \geq v(T) + v(S).\]

Therefore, if a game is superadditive, two disjoint coalitions can increase their total worth by joining and forming a big coalition. We remark that superadditive games do not necessarily have non-empty cores. A game \((N, v)\) is called convex if for all \(i \in N\) and all \(S, T \subseteq N \setminus \{i\} \) with \(S \subset T\),

\[v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)\]

Hence, for convex games, the marginal contribution of a player to any coalition is greater than his marginal contribution to a smaller coalition. A game is strictly convex if all inequalities are strict. We remark that convex games have non-empty cores.

Before introducing marginal vectors, we first introduce orders. For any \(T \subseteq N\), an order \(\sigma\) of \(T\) is a bijection \(\sigma : \{1, \ldots, |T|\} \to T\), where \(|T|\) is the cardinality of \(T\) and \(\sigma(i) = j\) means that with respect to \(\sigma\) player \(j\) is in \(i^{th}\) position. The set of all orders is denoted by \(\Pi(T)\).

Let \((N, v)\) be a TU-game, then the marginal vector \(m^\sigma(v) \in \mathbb{R}^N\), for any \(\sigma \in \Pi(N)\), is defined by

\[m^\sigma_{\sigma(1)}(v) := v(\{\sigma(1)\});\]

\[m^\sigma_{\sigma(k)}(v) := v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}) \text{ for all } k \in \{2, \ldots, |N|\}.
\]

The following theorem is due to Shapley (1971) and Ichiischi (1981). Here, \(\text{Conv}\{A\}\) denotes the convex hull of the set of points \(A\).

**Theorem 2** Let \((N, v)\) be a TU-game. Then \((N, v)\) is convex if and only if \(\text{Core}(v) = \text{Conv}\{m^\sigma(v) | \sigma \in \Pi(N)\}\).

Finally, we introduce some notation and terminology. A normal distribution with mean \(\mu\) and variance \(\sigma^2\) is denoted by \(N(\mu, \sigma^2)\). \(\Theta\) denotes the distribution function of the standard normal distribution \(N(0, 1)\).

### 4 Model

In this section, we introduce newsvendor situations with delivery restrictions and define the associated cooperative games. Then, we show that these games have non-empty cores. The proofs of all theorems and lemmas in this section are presented in the appendix.
Consider a set \( N = \{1, ..., n\} \) of retailers selling the same product. Each retailer \( i \in N \) experiences a stochastic demand \( X_i \) with finite expectation and has to give an order to the same supplier before the demand realization\(^2\). Moreover, each retailer \( i \in N \) has a unit cost \( c_i \), which includes purchasing and transportation costs, and a selling price \( p_i \). Throughout the study, we assume that \( p_i \) and \( c_i \) are positive and \( p_i \geq c_i \) for all \( i \in N \). Besides, different from the standard newsvendor model, each retailer would like to satisfy his own demand up to a certain amount for sure. This amount is denoted by \( \varepsilon_i \) for retailer \( i \in N \). If realized demand is less than this amount, then the whole demand is critical and the retailer would like to satisfy it all. We call \( \varepsilon_i \) the critical demand level of retailer \( i \). In single newsvendor setting, the critical demand level can be seen as the minimum order quantity for a retailer to meet a certain service level. A tuple \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N})\) with \( N, X_i, c_i, p_i \) and \( \varepsilon_i \) as above is called a newsvendor situation with delivery restrictions. For convenience, we will, in the rest of the paper, refer to newsvendor situations with delivery restrictions simply as newsvendor situations.

Consider a newsvendor situation and a collection of retailers \( S \). If these retailers come together and form coalition \( S \), they might increase their total profit by giving a joint order and splitting it after demand realization. Such a joint order \( q_S \) should not create any infeasibility for coalition \( S \) with respect to the critical demand levels of the retailers in \( S \), i.e., \( q_S \geq \sum_{i \in S} \varepsilon_i \). The collection of possible orders of coalition \( S \) is given by
\[
Q^S := \{ q \in \mathbb{R} | q \geq \sum_{i \in S} \varepsilon_i \}.
\]

Let \((x_i)_{i \in S}\) be a realization of demand vector \( X^S = (X_i)_{i \in S} \). For notational convenience, we will denote this realization as the vector \( x^S \in \mathbb{R}^N \) where \( x^S_i = 0 \) for all \( i \in N \setminus S \) and \( x^S_i = x_i \) for all \( i \in S \). Suppose coalition \( S \) has ordered \( q^S \in Q^S \) and demands are realized as \( x^S \). Then the retailers in \( S \) can allocate the joint order among themselves to satisfy the demands. An allocation of \( q^S \) is a vector \( a^S \in \mathbb{R}_+^N \) with
\[
\begin{align*}
a^S_i &= 0 \text{ if } i \in N \setminus S; \\
\sum_{i \in S} a^S_i &= q^S; \\
a^S_i &\geq \min \{x_i, \varepsilon_i\} \text{ for all } i \in S.
\end{align*}
\]

Here, \( a^S_i \) denotes the amount of product that will be sent from the supplier to retailer \( i \). The set of all possible allocations for coalition \( S \) of an order vector \( q^S \in Q^S \) is denoted by \( M^S(q^S) \). Note that it is not allowed to ship goods to retailers that are not in coalition \( S \). Moreover, an allocation should satisfy \( a^S_i \geq \min \{x_i, \varepsilon_i\} \) for each retailer \( i \in S \), which is the delivery restriction associated with \( \varepsilon_i \). Finally, we assume that at the end of the period all ordered units should be transferred to the retailers. This assumption can be

\(^2\)In most practical applications \( X_i \) can not take negative values. However, in this work we allow \( X_i \) to take negative values with very low probabilities to cover some well known distributions (e.g., normal distribution). Besides, negative demand can be interpreted as returns from customers.
interpreted in several ways. First, the only opportunity to salvage the leftover products is at the retailers. Second, there is no opportunity to keep stock at the location where the allocation of the joint order takes place.

For a fixed order quantity \( q^S \in Q^S \), demand realization \( x^S \) of \( X^S \), and allocation vector \( a^S \in M^S(q^S) \) the profit of coalition \( S \) can be expressed as

\[
P^S(a^S, q^S, x^S) = -\sum_{i \in S} a^S_i c_i + \sum_{i \in S} p_i \min\{a^S_i, x^S_i\}.
\]

Note that in the profit function, we do not consider any extra cost for the allocation of the joint order. This is natural for the cases, where the individual orders of the retailers follow the same route up to a point. So, if the demand realization occurs before the orders reach this point, the allocation of the joint order of a coalition can take place without any additional cost.

The following lemma shows that an optimal allocation exists for a given coalition, order quantity and demand realization.

**Lemma 1** Let \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\epsilon_i)_{i \in N})\) be a newsvendor situation, let \( S \subseteq N \), let \( q^S \in Q^S \), and let \( x^S \) be a demand realization vector. There exists an allocation \( a^{S,*} \in M^S(q^S) \) that maximizes the profit \( P^S(\cdot, q^S, x^S) \) of coalition \( S \).

From now on, we refer to \( P^S(a^{S,*}, q^S, x^S) \) as \( r^S(q^S, x^S) \). The expected profit function of coalition \( S \) is defined by

\[
\pi^S(q^S) = E_{X^S}[r^S(q^S, \cdot)].
\]

The following theorem shows that for any coalition an optimal order quantity, which maximizes expected total profit of this coalition, exists.

**Theorem 3** Let \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\epsilon_i)_{i \in N})\) be a newsvendor situation and let \( S \subseteq N \). There exists an order quantity \( q^{S,*} \) that maximizes the expected profit function \( \pi^S(\cdot) \) of coalition \( S \).

The determination of an optimal order quantity requires solving a two stage stochastic program, where in the first stage the order quantity is determined and in the second stage, given the order quantity and demand realizations, an allocation decision is made. There exists a solution algorithm, which utilizes backward induction process. We refer to Anupindi et al. (2001) for a discussion about how such an algorithm can be applied.

Let \( \Gamma \) be a newsvendor situation. The associated *cooperative game* \((N, v^\Gamma)\) is defined by

\[
v^\Gamma(S) = \max_{q^S \in Q^S} \pi^S(q^S) \text{ for all } S \subseteq N.
\]
The value of a coalition is given by the optimal value of the profit maximization problem of the coalition. Recall that the optimal order quantity that maximizes the expected profit function of a coalition \( S \subseteq N \) is denoted by \( q^{S, *} \). We use the following lemma to prove that newsvendor games with delivery constraints are balanced.

**Lemma 2** Let \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N})\) be a news-vendor situation with delivery constraints. Let \( \kappa \) be an associated balanced map. Then

\[
\pi^N \left( \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)q^{S, *} \right) \geq \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)\pi^S \left( q^{S, *} \right). \tag{1}
\]

The following theorem shows that cooperative games associated with newsvendor situations are balanced.

**Theorem 4** Let \( \Gamma = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N}) \) be a newsvendor situation. The associated cooperative game \((N, v^\Gamma)\) is balanced.

From Theorem 1, the following corollary follows immediately.

**Corollary 1** Let \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N})\) be a newsvendor situation. The associated cooperative game has a non-empty core.

Since every subgame of a cooperative game associated with a newsvendor situation is a cooperative game associated with a newsvendor situation itself, the following corollary follows immediately from Theorem 4.

**Corollary 2** Let \((N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N})\) be a newsvendor situation. The associated cooperative game is totally balanced.

This concludes the proof that every newsvendor situation described has a nonempty core. This result is important as it extends the previous works and the core concept of stability is still valid even if we have restrictions on the quantities delivered.

### 5 Dynamic delivery restrictions and monotonicity properties

In the previous section, we showed that every cooperative game associated with a newsvendor situation has a nonempty core, which provides a stable profit division for the grand coalition under fixed critical demand levels of the retailers. Another important issue to consider is how the payoffs of the retailers are affected by a change in the critical demand levels of the retailers.

To investigate this situation, we focus on the cases, in which the critical demand levels of the retailers change in the same direction. The following proposition shows how the value of a coalition is affected by a change in the critical demand levels of its retailers. We skip the obvious proof.
Proposition 1. Let \( \Gamma^1 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, \epsilon^1_i)_{i \in N} \) and \( \Gamma^2 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, \epsilon^2_i)_{i \in N} \) be two newsvendor situations such that there is a \( T \subseteq N \) with \( \epsilon^1_i < \epsilon^2_i \) for all \( i \in T \) and \( \epsilon^1_i = \epsilon^2_i \) for all \( i \in N/T \). Then the following relation holds for their associated games \((N, v^{\Gamma^1})\) and \((N, v^{\Gamma^2})\):

\[
\begin{align*}
v^{\Gamma^1}(S) &\geq v^{\Gamma^2}(S) \quad \text{for all } S \text{ with } S \cap T \neq \emptyset; \\
v^{\Gamma^1}(S) &= v^{\Gamma^2}(S) \quad \text{for all } S \text{ with } S \cap T = \emptyset.
\end{align*}
\]

In other words, the value of a coalition increases if a group of retailers in the coalition decreases their critical demand levels. This result is quite intuitive, since the possibilities of what a coalition can do are enlarged with lower critical demand levels of the retailers, i.e., the solution space of the profit maximization problem is enlarged.

We focus on two types of changes in critical demand levels. In the first one, all retailers weakly change their critical demand levels simultaneously in the same direction, i.e., they all weakly increase or they all weakly decrease these levels. This might lead to a change in the value of the grand coalition as well as in the values of some other coalitions as stated in Proposition 1. Since it is hard to distinguish the effect of retailers on the change of these values, the simplest fair argument states that none of the retailers should get less (more) than before if the values of the coalitions increase (decrease). In other words, the payoffs of the retailers should be affected in the same direction as the game changes due to the change in the critical demand levels. Note that we want all profit divisions (before and after a change in critical demand levels) to be stable since we consider it as a necessity for a possible cooperation.

In the latter type of change, only a single retailer changes its critical demand level, which might lead to a change in the value of the grand coalition as well as in the values of some other coalitions all involving this player. Since it is known that the changes are caused by this specific retailer, other retailers do not want to be harmed by this change. A fair argument to follow would be that none of these non-changing retailers should get less than what they got before.

The previous specifications lead us to issues of monotonicity with respect to the division of profit in case a change in the delivery restrictions occur. In the literature, several forms of monotonicity have been investigated. However, none of them covers our situation. Aggregate monotonicity (Megiddo (1974)) and coalitional monotonicity (Young 1985) consider the cases, where the value of a single coalition is changed only. In our case, however, the value of several coalitions can change because of changing critical demand levels. Similar behavior can be seen in assignment games. An increase in the worth of a pair of players may increase the value of several coalitions containing the pair. Sasaki (1995) and Nunez et. al. (2002) use pairwise monotonicity to analyze solution concepts for assignment games. However, pairwise monotonicity is hard to apply to our situations since it is defined specifically for assignment games. Moreover, we are interested in how a change could affect not only the players causing the change but also all other players.
To analyze the two types of changes above, we define four monotonicity properties
where the first two consider the first type and the last two properties are related with
the second type. The fairness arguments stated above can also be made for a general
class of games, in which any change in the underlying data or conditions has a similar
effect as in Proposition 1. Therefore, in the following part of this section, we introduce
the monotonicity properties for a general class of games.

Consider two games \((N, \alpha)\) and \((N, \beta)\). The triple \((N, \alpha, \beta)\) is called a pair of games
if both games are totally balanced and

\[
\alpha(S) \geq \beta(S) \quad \text{for all } S \subseteq N.
\]

We call game \((N, \alpha)\), which has larger values, the larger game and game \((N, \beta)\), which
has smaller values, the smaller game.

The first fair argument states that if the values of the coalitions in a game increase
(decrease), it is possible that no player gets less (more) than before.

- **MP1**: The pair of games \((N, \alpha, \beta)\) has monotonicity property 1 (MP1) if

  for all \(y \in \text{Core}(\beta)\) there exist an \(x \in \text{Core}(\alpha)\) such that \(x \geq y\).

- **MP2**: The pair of games \((N, \alpha, \beta)\) has monotonicity property 2 (MP2) if

  for all \(x \in \text{Core}(\alpha)\) there exist a \(y \in \text{Core}(\beta)\) such that \(y \leq x\).

If a pair of games has MP1 (MP2), then for all core solutions of the smaller (larger)
game, there is a core solution of the larger (smaller) game such that no player gets less
(more) than before. We remark that we want the new payoff vectors to be in the core,
since those are stable. Moreover, we check the entire core of game \(\beta\) (\(\alpha\)) for MP1 (MP2),
since the grand coalition can have any division in the core of this game.

Consider two games \((N, \alpha)\) and \((N, \beta)\). The tuple \((N, \alpha, \beta, i)\) is called a single deviation pair of games
if both games are totally balanced and \(i\) is a player in \(N\) such that

\[
\alpha(S) \geq \beta(S) \quad \text{for all } S \text{ containing } i \text{ and } \alpha(S) = \beta(S) \quad \text{for all } S \text{ not containing } i.
\]

We call player \(i\) the deviating player. Regarding to cooperative games associated with newsvendor
situations, the deviating player represents the retailer, who changes his critical demand
level. The second fair argument states that if the value change of the coalitions is caused
by one player, the other players should not get less than before. Besides, the deviating
player should not get less either if its deviation improves these values.

- **MP3**: The single deviation pair of games \((N, \alpha, \beta, i)\) has monotonicity property 3 (MP3) if

  for all \(y \in \text{Core}(\beta)\) there exist an \(x \in \text{Core}(\alpha)\)
  such that \(x_i \geq y_i\) and \(x_j \geq y_j\) for all \(j \in N/\{i\}\).
• **MP4:** The single deviation pair of games \((N, \alpha, \beta, i)\) has monotonicity property 4 (MP4) if

\[
\text{for all } x \in \text{Core}(\alpha) \text{ there exist a } y \in \text{Core}(\beta) \\
such that \ y_i \leq x_i \text{ and } y_j \geq x_j \text{ for all } j \in N/\{i\}.
\]

Similar as for MP1 and MP2, the cores of the games play a significant role for MP3 and MP4 as well.

Note that monotonicity properties MP1 and MP3 differ from each other only because MP3 is defined for a special class of pairs of games, i.e., single deviation pair of games.

## 6 Necessary and sufficient conditions

In this section, we focus on necessary and sufficient conditions for pairs of TU-games to satisfy the monotonicity properties. In separate subsections, we investigate situations with two, three and an arbitrary number of players. In the final subsection, we discuss monotonicity properties for newsvendor games.

### 6.1 Two-player games

In this subsection, we focus on two-player games and derive necessary and sufficient conditions for pairs of these games to satisfy monotonicity properties.

Let \((N, v)\) be a balanced game with \(N = \{1, 2\}\). Then, the extreme points of \(\text{Core}(v)\) can be denoted by

\[
m^{12}(v) = (v(\{1\}), v(\{1, 2\}) - v(\{1\})) , \\
m^{21}(v) = (v(\{1, 2\}) - v(\{2\}), v(\{2\})).
\]

Since all balanced two-player games are convex, by Theorem 2, the core of \((N, v)\) can be described by

\[
\text{Core}(v) = \text{Conv}\{m^{12}(v), m^{21}(v)\}
\]

The following theorem considers MP1.

**Theorem 5** Let \((N, \alpha, \beta)\) be a pair of games such that \(N = \{1, 2\}\). Then \((N, \alpha, \beta)\) has MP1 if and only if \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\) for all \(S \subseteq N\).

**Proof:** First, we prove the if-part. Suppose that \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\) for all \(S \subseteq N\). In other words, \(\alpha(N) - \alpha(S) \geq \beta(N) - \beta(S)\) for all \(S \subseteq N\). Since \((N, \alpha)\) and \((N, \beta)\) are two-player games with non-empty cores, their cores are given by \(\text{Core}(\alpha) = \text{Conv}(m^{12}(\alpha), m^{21}(\alpha))\) and \(\text{Core}(\beta) = \text{Conv}(m^{12}(\beta), m^{21}(\beta))\), respectively.

Let \(y\) be in the core of the game \((N, \beta)\). We can express \(y\) as a convex combination of the extreme points of the core of the game \((N, \beta)\), i.e., \(y = \lambda_1 m^{12}(\beta) + \lambda_2 m^{21}(\beta)\) for
some $\lambda_1 \in [0,1]$ and $\lambda_2 = 1 - \lambda_1$. Consider the division $x = \lambda_1 m^{12}(\alpha) + \lambda_2 m^{21}(\alpha)$. Since $x$ is given by a convex combination of the extreme points of $Core(\alpha)$, $x \in Core(\alpha)$. Since $\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)$ and $\alpha(S) \geq \beta(S)$ for all $S \subseteq N$, $m^{12}(\alpha) \geq m^{12}(\beta)$ and $m^{21}(\alpha) \geq m^{21}(\beta)$. Hence, $x \geq y$. This completes the proof of the if-part.

Now, we prove the only-if-part. Suppose that the pair of games $(N, \alpha, \beta)$ has MP1. Consider one of the extreme points of the core of $(N, \beta)$, $\tilde{y} = m^{12}(\beta)$. MP1 implies that there is an $x \in Core(\alpha)$ such that $x \geq \tilde{y}$. If $x \in Core(\alpha)$, then $x_2 \leq \alpha\{1,2\} - \alpha\{1\}$. Furthermore, $x_2 \geq \beta\{1,2\} - \beta\{1\}$ since $x \geq \tilde{y}$. Hence, $\alpha\{1,2\} - \alpha\{1\} \geq \beta\{1,2\} - \beta\{1\}$. In other words, $\alpha\{1,2\} - \beta\{1,2\} \geq \alpha\{1\} - \beta\{1\}$. From the same line of reasoning for the other extreme point of $Core(\beta)$, that is $\tilde{y} = m^{21}(\beta)$, we derive that $\alpha\{1,2\} - \beta\{1,2\} \geq \alpha\{2\} - \beta\{2\}$. This completes the proof. □

This theorem states that if there is an increase in the values of the coalitions, a necessary and sufficient condition for both players to benefit from this change is that the value of the grand coalition should increase more than any other coalition. In the following sections, we will investigate to what extent this result can be generalized to situations with more players.

The following theorem considers MP2 for two-player games.

**Theorem 6** Let $(N, \alpha, \beta)$ be a pair of games such that $N = \{1,2\}$. Then the pair of games $(N, \alpha, \beta)$ has MP2.

**Proof:** Consider another game $(N, \alpha')$ such that

\[
\begin{align*}
\alpha'(\{1,2\}) &= \alpha(\{1,2\}) \\
\alpha'(\{1\}) &= \beta(\{1\}) \\
\alpha'(\{2\}) &= \beta(\{2\})
\end{align*}
\]

Since $\alpha'(S) = \beta(S) \leq \alpha(S)$ for all $S \subseteq N$ and $\alpha'(N) = \alpha(N)$, $Core(\alpha) \subseteq Core(\alpha')$. Let $x$ be in the core of $(N, \alpha)$. Then, $x \in Core(\alpha')$. Therefore, we can express $x$ as a convex combination of the extreme points of $Core(\alpha')$. Let $\lambda$ be such that $x = \lambda_1 m^{12}(\alpha') + \lambda_2 m^{21}(\alpha')$ with $\lambda_1 \in [0,1]$ and $\lambda_2 = 1 - \lambda_1$. So $x = \lambda_1 m^{12}(\alpha') + \lambda_2 m^{21}(\alpha') = \lambda_1 (\beta(\{1\}) - \beta(\{1\})) + \lambda_2 (\alpha(\{1,2\}) - \beta(\{2\}))$. Consider the division $y = \lambda_1 m^{12}(\beta) + \lambda_2 m^{21}(\beta)$. Since $y$ is given by a convex combination of the extreme points of $Core(\beta)$, $y \in Core(\beta)$. Furthermore, since $m^{12}(\beta) \leq (\beta(\{1\}), \alpha(\{1,2\}) - \beta(\{1\}))$ and $m^{21}(\beta) \leq (\alpha(\{1,2\}) - \beta(\{2\}))$, we derive that $y \leq x$. This completes the proof. □

This theorem states that if there is a decrease in the values of the coalitions, the players can find a division of joint profit such that none of them benefits by this change. Hence, none of them feels discriminated. Later, Example 6 will show that this argument does not hold for games with more than two players.

The following theorem considers MP3 for two-player games.
Theorem 7 Let \((N, \alpha, \beta, i)\) be a single deviation pair of games such that \(N = \{1, 2\}\) and \(i \in N\). Then \((N, \alpha, \beta, i)\) has MP3 if and only if \(\alpha(N) - \beta(N) \geq \alpha(\{i\}) - \beta(\{i\})\).

Proof: The proof of this theorem follows similar to the proof of Theorem 5.

Finally, the following theorem considers MP4 for two-player games.

Theorem 8 Let \((N, \alpha, \beta, i)\) be a single deviation pair of games such that \(N = \{1, 2\}\) and \(i \in N\). Then \((N, \alpha, \beta, i)\) has MP4 if and only if \(\alpha(N) - \beta(N) \leq \alpha(\{i\}) - \beta(\{i\})\).

Proof: Without loss of generality, assume \(i = 1\). First we prove the if-part. So, suppose \(\alpha(N) - \beta(N) \leq \alpha(\{1\}) - \beta(\{1\})\). Let \(x \in \text{Core}(\alpha)\), and let \(K = \alpha(N) - \beta(N)\). Consider payoff vector \(y\) with \(y_2 = x_2\) and \(y_1 = x_1 - K\). Then \(y_2 = x_2 \geq \alpha(\{2\}) = \beta(\{2\})\) and \(y_1 = x_1 - K \geq \alpha(\{1\}) - K \geq \beta(\{1\})\). Hence, \(y \in \text{Core}(\beta)\). Furthermore, \(y_1 \leq x_1\) and \(y_2 \geq x_2\). Hence, \((N, \alpha, \beta, i)\) has MP4.

Now, we prove the only-if-part. Suppose \((N, \alpha, \beta, i)\) has MP4. Consider one of the extreme points of the core of \((N, \alpha)\), namely \(x = (\alpha(\{1\}), \alpha(N) - \alpha(\{1\}))\). MP4 implies that there is an \(y \in \text{Core}(\beta)\) such that \(y_2 \geq x_2\). Hence, \(y_1 = \beta(N) - y_2 \leq \beta(N) - x_2 = \beta(N) - \alpha(N) + \alpha(\{1\})\). Moreover, \(y_1 \geq \beta(\{1\})\). We conclude that \(\alpha(N) - \beta(N) \leq \alpha(\{1\}) - \beta(\{1\})\). This completes the proof.

6.2 Three-player games

In this subsection, we focus on three-player games and try to extend the results found for two-player games. The following example shows that the condition for MP1 and MP3 that states that the increase of the value of the grand coalition should be bigger than the increase in the value of other coalitions is not a necessary condition for pairs of three-player games and single deviation pairs of three-player games, respectively.

Example 1 Consider pair of games \((N, \alpha, \beta)\) and single deviation pair of games \((N, \alpha, \beta, 1)\) described in Table 1. It is easy to check that these games are totally balanced. Note that the value of coalition \(\{1\}\) increased more than the value of the grand coalition. If \(y \in \text{Core}(\beta)\), then \(\sum_{j \in S} y_j \geq \beta(S)\) for all \(|S| = 2\). Adding these relations we find that \(2 \sum_{j \in N} y_j \geq 6\). However, \(\sum_{j \in N} y_j = 3 = \beta(N)\), so all inequalities \(\sum_{j \in S} y_j \geq \beta(S)\) with \(|S| = 2\) must be equalities. These have a unique solution \(y = (1, 1, 1)\), which constitutes the unique element of \(\text{Core}(\beta)\). Consider the division \(x = (1.3, 1.3, 1.3, 1.3)\), it is easy to check that \(x \in \text{Core}(\alpha)\). Moreover, \(x_j \geq y_j\) for all \(j \in N\), which implies that \((N, \alpha, \beta)\) satisfies MP1 and \((N, \alpha, \beta, i)\) satisfies MP3.

Although the condition in Theorem 5 is not necessary for three-player games, it is still a sufficient condition for the pairs of three-player games. The following theorem captures this.
Theorem 9 Let \((N, \alpha, \beta)\) be a pair of games with \(N = \{1, 2, 3\}\). If \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\) for all \(S \subseteq N\), then \((N, \alpha, \beta)\) has MP1.

The proof of this theorem is presented in the appendix.

The results for MP2 derived for pairs of two-player games cannot be extended to pairs of games with three or more players. The following example illustrates a pair of three-player games, which does not satisfy MP2.

Example 2 Consider pair of games \((N, \alpha, \beta)\) described in Table 2. Let \(y \in \text{Core}(\beta)\).

\[
\begin{array}{|c|c|c|}
\hline
S & \alpha(S) & \beta(S) \\
\hline
\{1,2,3\} & 0.72 & 0.62 \\
\{1,2\} & 0.3 & 0.3 \\
\{1,3\} & 0.44 & 0.44 \\
\{2,3\} & 0.6 & 0.2 \\
\{1\} & 0 & 0 \\
\{2\} & 0 & 0 \\
\{3\} & 0.2 & 0.2 \\
\hline
\end{array}
\]

Table 2: The pair of games (Example 2)

Then

\[
y_1 + y_2 \geq 0.3; \quad (2)
\]

\[
y_1 + y_3 \geq 0.44; \quad (3)
\]

\[
\sum_{i \in N} y_i = 0.62. \quad (4)
\]

Adding (2) and (3), we find that \(2y_1 + y_2 + y_3 \geq 0.74\). Therefore, using (4), \(y_1 \geq 0.12\). Moreover, from (2) and (4), \(y_3 \leq 0.32\).
Consider payoff vector $x = (0.02, 0.28, 0.42)$. It is easy to check that $x \in \text{Core}(\alpha)$. Since $x_1 = 0.02$ and $y_1 \geq 0.12$ for all $y \in \text{Core}(\beta)$, there is no $y \in \text{Core}(\beta)$ such that $y \leq x$. Hence, the pair of games $(N, \alpha, \beta)$ does not satisfy MP2.

In the following example, we show that the condition, which states that the decrease of the value of the grand coalition should be less than the decrease of other coalitions having the deviating player, is not a necessary condition for pairs of three-player games satisfying MP4. However, in the next section, we will show that it is a sufficient condition (see Theorem 15).

**Example 3** Consider single deviation pair of games $(N, \alpha, \beta, 1)$ described in Table 3. It is easy to check that these games are totally balanced. Note that the value of coalition $\{1\}$ decreased less than the value of grand coalition. Similar to Example 1, we derive that $\text{Core}(\alpha) = \{(3, 1, 3)\}$. Consider the division $y = (2, 1, 3)$, it is easy to check that $y \in \text{Core}(\beta)$. Moreover, $y_1 \leq x_1$, $y_2 \geq x_2$ and $y_3 \geq x_3$, which implies that $(N, \alpha, \beta, i)$ satisfies MP4.

This example can be adjusted easily to cover pairs of games with more than three players.

**6.3 Games with an arbitrary number of players**

In this subsection, we consider games with an arbitrary number of players and present sufficient conditions for the monotonicity properties.

In Theorems 5 and 9, we showed a sufficient condition, which states that the increase of the value of the grand coalition should be bigger than the increase in the value of other coalitions for pairs of two and three-player games to have MP1. The following example shows that this condition is not sufficient for pairs of games with an arbitrary number of players.
Table 4: The pair of games (Example 4)

<table>
<thead>
<tr>
<th>S</th>
<th>α(S)</th>
<th>β(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,2,3,4}</td>
<td>4.5</td>
<td>4</td>
</tr>
<tr>
<td>{1,2,3}</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>{1,2,4}</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>{1,3,4}</td>
<td>3.5</td>
<td>3</td>
</tr>
<tr>
<td>{2,3,4}</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>{1,2}</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>{1,3}</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>{1,4}</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>{2,3}</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>{2,4}</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>{3,4}</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>{1}</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>{2}</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>{3}</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>{4}</td>
<td>0.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 4 Consider pair of games \((N, \alpha, \beta)\) described in Table 4. It is easy to check that these games are totally balanced. Note that since \(\alpha(S) = \beta(S) + 0.5\) for all \(S \subseteq N\), \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\). If \(x \in \text{Core}(\alpha)\), then \(\sum_{i \in S} x_i \geq \alpha(S)\) for all \(S \subset N\). Adding the relations associated with coalitions \(\{1,2\}\), \(\{2,3\}\), \(\{1,3,4\}\) and \(\{4\}\), we find that \(2 \sum_{i \in N} x_i \geq 9\). However, \(\sum_{i \in N} x_i = \alpha(N) = 4.5\), so all inequalities associated with these coalitions must be equalities. These have a unique solution \(x = (1.5, 1, 1.5, 0.5)\), which constitutes the unique element of \(\text{Core}(\alpha)\). Consider the division \(y = (1, 1, 1, 1)\), it is easy to check that \(y \in \text{Core}(\beta)\). Moreover, there is no \(x \in \text{Core}(\alpha)\) such that \(x_i \geq y_i\) for all \(i \in N\), since \(\text{Core}(\alpha) = \{(1.5, 1, 1.5, 0.5)\}\). Hence, pair of games \((N, \alpha, \beta)\) does not have MP1.

The following two theorems consider sufficient conditions for MP1.

Theorem 10 Let \((N, \alpha, \beta)\) be a pair of games. If \((N, \alpha)\) is a convex game and \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\) for all \(S \subseteq N\), then \((N, \alpha, \beta)\) has MP1.

Proof: Suppose that \((N, \alpha)\) is a convex game and \(\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)\) for all \(S \subseteq N\).

Let \(K = \alpha(N) - \beta(N)\) and define game \((N, \beta')\) as follows

\[
\beta'(N) = \alpha(N) - K;
\]

\[
\beta'(S) = \alpha(S) - K \quad \text{for all } S \subset N.
\]
Since $K = \alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)$ for all $S \subseteq N$,
\[
\begin{align*}
\beta'(N) &= \beta(N); \\
\beta'(S) &\leq \beta(S) \text{ for all } S \subset N.
\end{align*}
\]

Therefore $\text{Core}(\beta) \subseteq \text{Core}(\beta')$.

Consider $(N, \beta')$. Since we have diminished all values of coalitions of $(N, \alpha)$ by $K$ to create $(N, \beta')$, we can describe the marginal vectors of game $(N, \beta')$, using the marginal vectors of $(N, \alpha)$ as follows. For all orders $\sigma \in \Pi(N)$,
\[
\begin{align*}
m_{\sigma(i)}^\sigma(\beta') &= m_{\sigma(i)}^\sigma(\alpha) \text{ for all } i \in \{2, ..., |N|\}; \\
m_{\sigma(1)}^\sigma(\beta') &= m_{\sigma(1)}^\sigma(\alpha) - K.
\end{align*}
\]

Note that for all $\sigma \in \Pi(N)$, $m^\sigma(\alpha) \geq m^\sigma(\beta')$.

Consider $\sigma \in \Pi(N)$. Since $(N, \alpha)$ is convex, from Theorem 2, $m^\sigma(\alpha) \in \text{Core}(\alpha)$. Then
\[
\sum_{i \in S} m_i^\sigma(\alpha) \geq \alpha(S) \text{ for all } S \subseteq N,
\]
\[
\sum_{i \in S} m_i^\sigma(\beta') \geq \sum_{i \in S} m_i^\sigma(\alpha) - K \geq \alpha(S) - K = \beta'(S) \text{ for all } S \subseteq N.
\]

Therefore, $m^\sigma(\beta')$ is in the core of $(N, \beta')$. Moreover, this is true for every possible order $\sigma \in \Pi(N)$. Hence, from Theorem 2, we conclude that $(N, \beta')$ is a convex game as well.

In the following part of the proof, we show that for all $y \in \text{Core}(\beta)$, we can find an $x \in \text{Core}(\alpha)$ such that $x \geq y$.

Let $y$ be in the core of $(N, \beta)$. Then $y \in \text{Core}(\beta')$, since $\text{Core}(\beta) \subseteq \text{Core}(\beta')$. Since $(N, \beta')$ is convex, we can express $y$ as a convex combination of the extreme points of the core of the game $(N, \beta')$, which are the marginal vectors of $(N, \beta')$. Let $\lambda \in \mathbb{R}_+^k$ be such that $y = \sum_{i=1}^k \lambda_i m^{\sigma_i}(\beta')$ with $\sigma_i \in \Pi(N)$ for all $i \in \{1, ..., k\}$ and $\sum_{i=1}^k \lambda_i = 1$.

Consider $x = \sum_{i=1}^k \lambda_i m^{\sigma_i}(\alpha)$. Recall that $(N, \alpha)$ is convex. Therefore, from Theorem 2, $x \in \text{Core}(\alpha)$. Furthermore, $x \geq y$, since $m^\sigma(\alpha) \geq m^\sigma(\beta')$ for all $\sigma \in \Pi(N)$. This completes the proof. \hfill \Box

Another sufficient condition for MP1, which does not require convexity, is given in the following theorem.

**Theorem 11** Let $(N, \alpha, \beta)$ be a pair of games. If the game $(N, w)$ with $w(S) = \alpha(S) - \beta(S)$ for all $S \subseteq N$ has a non-empty core, then $(N, \alpha, \beta)$ has MP1.

**Proof:** Suppose that $(N, w)$ has a non-empty core. Let $y \in \text{Core}(\beta)$. Choose an arbitrary $z \in \text{Core}(w)$. Then $z_i \geq w(\{i\}) = \alpha(\{i\}) - \beta(\{i\}) \geq 0$ for all $i \in N$ since $\alpha(S) \geq \beta(S)$ for all $S \subseteq N$. Furthermore, $\sum_{i \in S} y_i \geq \beta(S)$ for all $S \subseteq N$, $\sum_{i \in N} y_i = \beta(N)$, $\sum_{i \in S} z_i \geq w(S)$ for all $S \subset N$ and $\sum_{i \in N} z_i = w(N)$. Consider the payoff vector $x$ such that $x_i = y_i + z_i$. Then $\sum_{i \in S} x_i = \sum_{i \in S} y_i + \sum_{i \in S} z_i \geq \beta(S) + w(S) = \alpha(S)$ for all
Then his payoff and the new payoff vector is in the core of the game \((N, \alpha, \beta)\) as follows:

\[
\begin{align*}
\sigma^*_{\alpha}(N) &= \sigma^*_{\alpha}(\alpha); \\
\sigma^*_{\beta}(S) &= \sigma^*_{\beta}(\beta) \quad \text{for all } S \subseteq N.
\end{align*}
\]

Since \(\beta(S) \leq \alpha(S)\) for all \(S \subseteq N\), \(Core(\alpha) \subseteq Core(\alpha')\).

Consider \((N, \alpha')\). Since \(\sigma^*_{\alpha}(N) \geq \sigma^*_{\beta}(N)\) and the subcoalition values of \((N, \alpha')\) and \((N, \beta)\) are the same, we can derive the marginal vectors of \((N, \alpha')\), using the marginal vectors of \((N, \beta)\). For any order \(\sigma \in \Pi(N)\),

\[
\begin{align*}
m^\sigma_{\sigma(i)}(\alpha') &= m^\sigma_{\sigma(i)}(\beta) \quad \text{for all } i = \{1, \ldots, |N| - 1\}; \\
m^\sigma_{\sigma(|N|)}(\alpha') &= m^\sigma_{\sigma(|N|)}(\beta) + K,
\end{align*}
\]

where \(K = \alpha(N) - \beta(N)\). Note that \(m^\sigma(\alpha') \geq m^\sigma(\beta)\).

Consider order \(\sigma \in \Pi(N)\). Since \((N, \beta)\) is convex, from Theorem 2, \(m^\sigma(\beta) \in Core(\beta)\). Then

\[
\begin{align*}
\sum_{i \in S} m^\sigma_i(\beta) &\geq \beta(S) \quad \text{for all } S \subseteq N; \\
\sum_{i \in S} m^\sigma_i(\alpha') &= \sum_{i \in S} m^\sigma_i(\beta) \geq \beta(S) = \sigma^*_{\alpha}(\alpha') \quad \text{for all } S \subseteq N \text{ with } \sigma(|N|) \notin S; \\
\sum_{i \in S} m^\sigma_i(\alpha') &= \sum_{i \in S} m^\sigma_i(\beta) + K \geq \beta(S) + K \geq \sigma^*_{\alpha}(\alpha') \quad \text{for all } S \subseteq N \text{ with } \sigma(|N|) \in S.
\end{align*}
\]

Therefore \(m^\sigma(\alpha')\) is also in the core of \((N, \alpha')\). Hence, from Theorem 2, we conclude that \((N, \alpha')\) is a convex game as well.

In the following part of the proof, we show that for all \(x \in Core(\alpha)\), we can find a \(y \in Core(\beta)\) such that \(y \leq x\).

Let \(x\) be in the core of the game \((N, \alpha)\). Then \(x \in Core(\alpha')\), since \(Core(\alpha) \subseteq Core(\alpha')\). Since \((N, \alpha')\) is convex, we can express \(x\) as a convex combination of the extreme points of the core of the game \((N, \alpha')\), which are the marginal vectors of \((N, \alpha')\). Let \(\lambda \in \mathbb{R}^k_+\) be such that \(x = \sum_{i=1}^k \lambda_i m^\sigma_i(\alpha')\) with \(\sigma_i \in \Pi(N)\) for all \(i = \{1, \ldots, k\}\) and
\( \sum_{i=1}^{k} \lambda_i = 1 \). Consider \( y = \sum_{i=1}^{k} \lambda_i m^\alpha(\beta) \). Recall that \((N, \beta)\) is convex. Therefore from Theorem 2, \( y \in \text{Core}(\beta) \). Furthermore, \( y \leq x \), since \( m^\sigma(\beta) \leq m^\sigma(\alpha') \) for all \( \sigma \in \Pi(N) \). This completes the proof. \( \square \)

Another sufficient condition for MP2, which does not require convexity, is given in the following theorem.

**Theorem 13** Let \((N, \alpha, \beta)\) be a pair of games. If the anticore of the game \((N, w)\) with \( w(S) = \alpha(S) - \beta(S) \) for all \( S \subseteq N \) has an element \( z \) with \( z_i \geq 0 \) for all \( i \in N \), then \((N, \alpha, \beta)\) has MP2.

**Proof:** Let \( x \) be in the core of game \((N, \alpha)\). Choose an arbitrary \( \bar{z} \in \text{Anticore}(N, w) \) such that \( \bar{z}_i \geq 0 \) for all \( i \in N \). Then \( \sum_{i \in S} x_i \geq \alpha(S) \) for all \( S \subseteq N \), \( \sum_{i \in N} x_i = \alpha(N) \), \( \sum_{i \in S} \bar{z}_i \leq w(S) \) for all \( S \subseteq N \) and \( \sum_{i \in N} \bar{z}_i = w(N) \). Consider the payoff vector \( y \) such that \( y_i = x_i - \bar{z}_i \). Then \( \sum_{i \in S} y_i = \sum_{i \in S} x_i - \sum_{i \in S} \bar{z}_i \geq \alpha(S) - w(S) = \beta(S) \) for all \( S \subseteq N \) and \( \sum_{i \in N} y_i = \sum_{i \in N} x_i - \sum_{i \in N} \bar{z}_i = \alpha(N) - w(N) = \beta(N) \). Hence, \( y \in \text{Core}(\alpha) \). Furthermore, \( y_i \leq x_i \) for all \( i \in N \), since \( \bar{z}_i \geq 0 \) for all \( i \in N \). This completes the proof. \( \square \)

Note that the sufficient conditions in Theorem 12 and Theorem 13 do not imply each other for pairs of games with three or more players. The proof of Theorem 13 implies that by choosing one proper payoff vector in the anticore of the difference game \((N, w)\), we can create a new division, which satisfies the requirements of MP2 (i.e., every player weakly decreases his payoff and the new payoff vector is in the core of the game \((N, \beta)\)), for every starting payoff vector.

The following theorem gives a sufficient condition for MP3.

**Theorem 14** Let \((N, \alpha, \beta, i)\) be a single deviation pair of games. If \( \alpha(N) - \beta(N) \geq \alpha(S) - \beta(S) \) for all \( S \) such that \( i \in S \) and \( S \subseteq N \), then \((N, \alpha, \beta, i)\) has MP3.

**Proof:** Suppose that \( \alpha(N) - \beta(N) \geq \alpha(S) - \beta(S) \) for all \( S \) such that \( i \in S \) and \( S \subseteq N \). Recall that \( \alpha(S) = \beta(S) \) for all \( S \subseteq N \setminus \{i\} \) since \((N, \alpha, \beta, i)\) is a singleton deviating pair of games. Let \( y \in \text{Core}(\beta) \), and let \( K = \alpha(N) - \beta(N) \). Consider the payoff vector \( x \) with \( x_j = y_j \) for all \( j \in N \setminus \{i\} \) and \( x_i = y_i + K \). Then \( \sum_{j \in S} x_j = \sum_{j \in S} y_j \geq \beta(S) = \alpha(S) \) for all \( S \subseteq N \setminus \{i\} \), \( \sum_{j \in S} x_j = \sum_{j \in S} y_j + K \geq \beta(S) + K \geq \alpha(S) \) for all \( S \subseteq N \) with \( i \in S \), and \( \sum_{j \in N} x_j = \sum_{j \in N} y_j + K = \beta(N) + K = \alpha(N) \). Hence, \( x \in \text{Core}(\alpha) \). Furthermore, \( x_i \geq y_i \) and \( x_j \geq y_j \) for all \( j \in N \setminus \{i\} \). This completes the proof. \( \square \)

The following theorem gives a sufficient condition for MP4.

**Theorem 15** Let \((N, \alpha, \beta, i)\) be a single deviation pair of games. If \( \alpha(N) - \beta(N) \leq \alpha(S) - \beta(S) \) for all \( S \) such that \( i \in S \) and \( S \subseteq N \), then \((N, \alpha, \beta, i)\) has MP4.

**Proof:** Suppose that \( \alpha(N) - \beta(N) \leq \alpha(S) - \beta(S) \) for all \( S \) such that \( i \in S \) and \( S \subseteq N \). Recall that \( \alpha(S) = \beta(S) \) for all \( S \subseteq N \setminus \{i\} \) since \((N, \alpha, \beta, i)\) is a singleton deviating pair
of games. Let $x \in \text{Core}(\alpha)$, and let $K = \alpha(N) - \beta(N)$. Consider the payoff vector $y$ with $y_j = x_j$ for all $j \in N \setminus \{i\}$ and $y_i = x_i - K$. Then $\sum_{j \in S} y_j = \sum_{j \in S} x_j \geq \alpha(S) = \beta(S)$ for all $S \subseteq N \setminus \{i\}$, $\sum_{j \in S} y_j = \sum_{j \in S} x_j - K \geq \alpha(S) - K \geq \beta(S)$ for all $S \subseteq N$ with $i \in S$, and $\sum_{j \in N} y_j = \sum_{j \in S} x_j - K = \alpha(N) - K \geq \beta(N)$. Hence, $y \in \text{Core}(\beta)$. Furthermore, $y_i \leq x_i$ and $y_j \geq x_j$ for all $j \in N \setminus \{i\}$. This completes the proof.

Tables 5 and 6 summarize our findings in the first three subsections. Note that the conditions are valid for the class of totally balanced games including the newsvendor games. We remark that the conditions for MP3 and MP4 are easier to check.

<table>
<thead>
<tr>
<th>Two-player games</th>
<th>MP1</th>
<th>MP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Necessary &amp; Sufficient conditions</td>
<td>$\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)$ for all $S \subseteq N$</td>
<td>no condition required</td>
</tr>
<tr>
<td>Sufficient conditions</td>
<td>$\alpha(N) - \beta(N) \geq \alpha(S) - \beta(S)$ for all $S \subseteq N$</td>
<td>see $n$-player games</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Three-player games</th>
<th>MP3</th>
<th>MP4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sufficient conditions</td>
<td>$\alpha(N) - \beta(N) \geq \alpha({i}) - \beta({i})$ for all $S \subseteq N$</td>
<td>$\alpha(N) - \beta(N) \leq \alpha({i}) - \beta({i})$ for all $S \subseteq N$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$-Player games</th>
<th>MP3</th>
<th>MP4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sufficient conditions</td>
<td>$\alpha(N) - \beta(N) \geq \alpha({i}) - \beta({i})$ for all $S \subseteq N$</td>
<td>$\alpha(N) - \beta(N) \leq \alpha({i}) - \beta({i})$ for all $S \subseteq N$</td>
</tr>
</tbody>
</table>

Table 5: Summary of the results for MP1 and MP2

Table 6: Summary of the results for MP3 and MP4

6.4 Newsvendor games

From Proposition 1, we know how the value of a coalition changes with a shift in the minimum delivery quantities of the retailers. However, knowing the direction (sign) of the change is not sufficient to ensure the games to satisfy the monotonicity properties. From the necessary and sufficient conditions presented in previous subsections, we have seen that magnitudes of changes in the values of coalitions play a critical role. In a newsvendor environment, the magnitude of the change in the profit of a coalition depends on the parameters of each retailer (e.g., critical demand levels, selling prices, demand distributions) in the coalition and, hence, can differ significantly from one coalition to another. Therefore, it is very much parameter dependent whether a pair or a single deviation pair of newsvendor games satisfies the monotonicity properties or not. In this subsection, we first give examples of a pair and a singleton pair of newsvendor games.
that do not satisfy the monotonicity properties. Then, we focus on a class of newsvendor situations with normal demand distributions and show that pairs of newsvendor games associated with this class satisfy MP1.

The following examples show that in general games associated with newsvendor situations are not guaranteed to have any of the monotonicity properties.

**Example 5** Consider two newsvendor situations $\Gamma^1 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon^1_i)_{i \in N})$ and $\Gamma^2 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon^2_i)_{i \in N})$ such that $N = \{1, 2, 3\}$, $c_i = 1$ for all $i \in N$, $p_i = 2$ for all $i \in N$, $\varepsilon^1 = (1, 0, 0)$ and $\varepsilon^2 = (1, 1, 0)$. The demand of the players are discrete, and independently distributed. The distribution functions are given in Table 7.

<table>
<thead>
<tr>
<th>Player</th>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 7: The demand distributions (Example 5)

The optimal order quantities of the coalitions for both situations are given in Table 8.

<table>
<thead>
<tr>
<th>$q^{S,*}$</th>
<th>{1,2,3}</th>
<th>{1,2}</th>
<th>{1,3}</th>
<th>{2,3}</th>
<th>{1}</th>
<th>{2}</th>
<th>{3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^1$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma^2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8: The optimal order quantities (Example 5)

Note that order quantity 1 for coalition \{1, 2\} in situation $\Gamma^2$ is not feasible because of the critical demand levels of players 1 and 2. Consider $\Gamma^1$ and coalition \{1, 2\}. Then the value of this coalition can be calculated as follows:

$$v^{\Gamma^1}(\{1, 2\}) = -1 \times 1 + 2 \times 0.91 = 0.82,$$

where 0.91 is the probability of having a positive total demand. Using similar calculations, we find the values of coalitions presented in Table 9.

Hence, $(N, v^{\Gamma^1})$ and $(N, v^{\Gamma^2})$ only differ in the value of coalition \{1, 2\}. Let $x \in Core(v^{\Gamma^1})$. Since $x_1 + x_2 + x_3 = 1.718$ and $x_1 + x_2 \geq 0.82$, we derive that $x_3 \leq 0.898$. 

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Table 9: The (single deviation) pair of cooperative games (Example 5)

Consider the payoff vector $y = (0.4, 0.4, 0.918) \in \text{Core}(v^{\Gamma_2})$. Then there is no $x \in \text{Core}(v^{\Gamma_1})$ such that $x_3 \geq y_3$, since $x_3 \leq 0.898$ for all $x \in \text{Core}(v^{\Gamma_1})$. So pair of games $(N, v^{\Gamma_1}, v^{\Gamma_2})$ (and single deviation pair of games $(N, v^{\Gamma_1}, v^{\Gamma_2}, 1)$) does not have MP1 (and MP3).

Example 6 Consider two newsvendor situations $\Gamma^1 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon^1_i)_{i \in N})$ and $\Gamma^2 = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon^2_i)_{i \in N})$ such that $N = \{1, 2, 3\}$, $c_i = 1$ for all $i \in N$, $p_i = 2$ for all $i \in N$, $\varepsilon^1 = (0, 0, 1)$ and $\varepsilon^2 = (0, 1, 1)$. Note that player 2 is the deviating player. The demand of the players are discrete and independently distributed. The distribution functions are given in Table 10.

Table 10: The demand distributions (Example 6)

The optimal order quantities of the coalitions for both situations are given in Table 11. Note that order quantity 1 for coalitions $\{1, 2, 3\}$ and $\{2, 3\}$ in situation $\Gamma^2$ is not feasible because of the critical demand levels of players 1, 2 and 3. Using similar calculations as in Example 5, we derive that $(N, v^{\Gamma_1})$ and $(N, v^{\Gamma_2})$ are equal to $(N, \alpha)$ and $(N, \beta)$ in Table 2, respectively.
Table 11: The optimal order quantities (Example 6)

<table>
<thead>
<tr>
<th>$q^S_i$</th>
<th>${1,2,3}$</th>
<th>${1,2}$</th>
<th>${1,3}$</th>
<th>${2,3}$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma^2$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

From Example 2, we know that the pair of games $(N, v^{\Gamma_1}, v^{\Gamma_2})$ does not satisfy MP2. Moreover, we know that for any payoff vector $y \in \text{Core}(v^{\Gamma_2})$, $y_3 \leq 0.32$. Consider payoff vector $x = (0.02, 0.28, 0.42) \in \text{Core}(v^{\Gamma_1})$. Since $x_3 \geq 0.32$ and $y_3 \leq 0.32$ for all $y \in \text{Core}(v^{\Gamma_2})$, we conclude that $x \notin \text{Core}(v^{\Gamma_2})$ and there is no $y \in \text{Core}(v^{\Gamma_2})$ such that $y_3 \geq x_3$. Note that player 3 is not the deviating player. Therefore, the single deviation pair of games $(N, v^{\Gamma_1}, v^{\Gamma_2}, 2)$ does not satisfy MP4 either.

In the remainder of this subsection, we focus on a class of newsvendor situations such that MP1 is satisfied by the associated games. Consider a newsvendor situation $\Gamma = (N, (X_i)_{i \in N}, (c_i)_{i \in N}, (p_i)_{i \in N}, (\varepsilon_i)_{i \in N})$ with independent normally distributed demands $X_i \sim N(\mu_i, \sigma^2)$ for all $i \in N$, $c_i = c$ for all $i \in N$, $p_i = p$ for all $i \in N$ and $\varepsilon_i = \mu_i + k\sigma$ with $k \in \mathbb{R}$ for all $i \in N$. Note that with this description, we consider retailers all having identical $c$ and $p$, non-identical mean demands, and identical variances. Moreover, we remark that for all $k \in \mathbb{R}$, $\varepsilon_i = \mu_i + k\sigma$ guarantees the same P1 type service level (probability of stockout) for each retailer $i \in N$. We call this special newsvendor situation a normal newsvendor situation, and we denote it by $\Pi = (N, (\mu_i)_{i \in N}, \sigma, c, p, k)$. In a normal newsvendor situation, there is no difference in satisfying demands for different retailers in terms of total profit of a coalition because of common purchasing cost and selling price. Therefore, the profit function of coalition $S$ can be written as follows

$$v(S) = \max_{q \geq \sum_{i \in S} \varepsilon_i} E_{X_S}[-c * q + p \min(q, \cdot)], \tag{5}$$

where $X_S = \sum_{i \in S} X_i$. Since $X_i$’s are independent and normally distributed, $X_S$ is normally distributed with parameters $\mu_S = \sum_{i \in S} \mu_i$ and $\sigma_S^2 = |S|\sigma^2$. In inventory literature, it is known that $E_{X_S}[-c * q + p \min(q, \cdot)]$ (the profit function of newsvendor problem without any delivery restrictions) is a concave function and the optimum $\bar{q}$ maximizing this function satisfies $F_S(\bar{q}) = 1 - c/p$, where $F_S$ is the cumulative distribution function of $X_S$. Since $X_S$ is normally distributed, $\bar{q} = \mu_S + k_{1-c/p}\sigma_S$, where $k_{1-c/p}$ is the unique number such that the standard normal distribution function $\Theta$ satisfies $\Theta(k_{1-c/p}) = 1 - c/p$. Since $E_{X_S}[-c * q + p \min(q, \cdot)]$ is concave, it follows immediately that the optimal order quantity of coalition $S$ is the maximum of the optimal order quantity of the newsvendor problem without any delivery restrictions and the sum of the critical demand levels of the retailers in $S$, i.e., $q^S = \max\{\mu_S + k_{1-c/p}\sigma_S, \mu_S + k|S|\sigma\}$. The following proposition provides some properties of newsvendor games associated with normal newsvendor
using (6), we can write that

\[ v^\Pi(S) = (p - c)q^S - p \sigma_S \int_{-\infty}^{\infty} \Theta(x) dx, \]

where \( q^S = \max\{\mu_S + k_1 - c/p \sigma_S, \mu_S + k|S|\sigma\} \), \( \mu_S = \sum_{i \in S} \mu_i \), and \( \sigma_S = \sigma |S| \); the second equality follows from Proposition 2(a), and (6). The last equality holds since \( k \).

Proposition 2 Let \( \Pi = (N, (\mu_i)_{i \in N}, \sigma, c, p, k) \) be a normal newsvendor situation. Let \( (N, v^\Pi) \) be the associated newsvendor game. Then,

(a) the profit of coalition \( S \) is given by

\[ v^\Pi(S) = (p - c)q^S - p \sigma_S \int_{-\infty}^{\infty} \Theta(x) dx, \]

where \( q^S = \max\{\mu_S + k_1 - c/p \sigma_S, \mu_S + k|S|\sigma\} \), \( \mu_S = \sum_{i \in S} \mu_i \), and \( \sigma_S = \sigma |S| \);

(b) \( (N, v^\Pi) \) is convex if \( k \leq k_1 - c/p |N| \).

The proof of this proposition is presented in the appendix.

The following lemma shows a relation for pairs of newsvendor games associated with normal newsvendor situations.

Lemma 3 Let \( \Pi^1 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k_1) \) and \( \Pi^2 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k_2) \) with \( c/p \leq 1/2 \) and \( k_1 \leq k_2 \) be two normal newsvendor situations. Let \( (N, v^{\Pi_1}) \) and \( (N, v^{\Pi_2}) \) be the newsvendor games associated with \( \Pi^1 \) and \( \Pi^2 \), respectively. Then,

\[ v^{\Pi_1}(T) - v^{\Pi_2}(T) \geq v^{\Pi_1}(S) - v^{\Pi_2}(S) \quad \text{for all } S \subset T \subseteq N. \]

Proof: Consider coalitions \( T \) and \( S \) such that \( S \subset T \subseteq N \). Let \( d(k) \) be the difference function

\[ d(k) = (p - c)(\bar{q}^T - \bar{q}^S) - \left( \frac{q^T - \mu_T}{\sigma_T} \int_{-\infty}^{\infty} \Theta(z) dz - \frac{q^S - \mu_S}{\sigma_S} \int_{-\infty}^{\infty} \Theta(z) dz \right), \]

where \( \bar{q}^T = \max\{\mu_T + k_1 - c/p \sigma_T, \mu_T + k \sum_{i \in T} \sigma_i\} \) and \( \bar{q}^S = \max\{\mu_S + k_1 - c/p \sigma_S, \mu_S + k \sum_{i \in S} \sigma_i\} \). We remark that \( d(k^1) = v^{\Pi_1}(T) - v^{\Pi_1}(S) \) and \( d(k^2) = v^{\Pi_2}(T) - v^{\Pi_2}(S) \). Using (6), we can write that

\[ v^{\Pi_1}(T) - v^{\Pi_2}(T) - v^{\Pi_1}(S) + v^{\Pi_2}(S) = v^{\Pi_1}(T) - v^{\Pi_1}(S) - v^{\Pi_2}(T) + v^{\Pi_2}(S) = d(k^1) - d(k^2) = - \int_{k^1}^{k^2} \dot{d}(k) dk. \]

The second equality follows from Proposition 2(a), and (6). The last equality holds since \( d(k) \) is continuous and differentiable in \( k \).

In the remaining part, we will show that \( \dot{d}(k) \leq 0 \) for all \( k \in \mathbb{R} \). Since \( \sigma_i = \sigma \) for all \( i \in N \), it holds that \( \sigma_T = \sigma \sqrt{|T|} \), \( \sum_{i \in T} \sigma_i = |T| \sigma \), \( \sigma_S = \sigma \sqrt{|S|} \) and \( \sum_{i \in S} \sigma_i = |S| \sigma \). Hence,

\[ \bar{q}^T = \begin{cases} \mu_T + k|T|\sigma & \text{if } k \geq \frac{k_1 - c/p}{\sqrt{|T|}} ; \\ \mu_T + k_1 - c/p \sqrt{|T|} & \text{if } k \leq \frac{k_1 - c/p}{\sqrt{|T|}} , \end{cases} \]
and
\[ \hat{q}^S = \begin{cases} 
\mu_S + k|S|\sigma & \text{if } k \geq \frac{k_1-c/p}{\sqrt{|S|}}; \\
\mu_S + k_{1-c/p}\sigma\sqrt{|S|} & \text{if } k \leq \frac{k_1-c/p}{\sqrt{|S|}}.
\end{cases} \]

Moreover, \(0 \leq k_{1-c/p}/\sqrt{|T|} \leq k_{1-c/p}/\sqrt{|S|}\), since \(k_{1-c/p} \geq 0\). Taking the derivative of \(d(k)\), we obtain
\[
d'(k) = \begin{cases} 
|T|\sigma \left[ (p - c) - p\Theta(\frac{\hat{q}^T - \mu_T}{\sigma_T}) \right] & \text{if } k \geq \frac{k_{1-c/p}}{\sqrt{|S|}}; \\
-|S|\sigma \left[ (p - c) - p\Theta(\frac{\hat{q}^S - \mu_S}{\sigma_S}) \right] & \text{if } \frac{k_{1-c/p}}{\sqrt{|S|}} \geq k \geq \frac{k_{1-c/p}}{\sqrt{|T|}}; \\
|T|\sigma \left[ (p - c) - p\Theta(\frac{\hat{q}^S - \mu_S}{\sigma_S}) \right] & \text{if } \frac{k_{1-c/p}}{\sqrt{|T|}} \geq k.
\end{cases}
\]

We conclude that \(d'(k) \leq 0\) for all \(k \in \mathbb{R}\) since \([|p - c| - p\Theta((\hat{q}^T - \mu_T)/\sigma_T)] \leq [(p - c) - p\Theta((\hat{q}^S - \mu_S)/\sigma_S)] \leq 0\) for all \(k \in \mathbb{R}\). The inequalities hold since \(0 \leq k_{1-c/p}/\sqrt{|T|} \leq k_{1-c/p}/\sqrt{|S|}\), and \(\Theta((\hat{q}^S - \mu_S)/\sigma_S) \geq 1 - c/p\). This completes the proof. □

In other words, if the retailers in a normal newsvendor situation decrease their critical demand levels by the same amount, the contribution of these changes to a coalition is increasing in the size of the coalition.

The following theorem follows directly from Theorems 5 and 9, and Lemma 3.

**Theorem 16** Let \(\Pi^1 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k^1)\) and \(\Pi^2 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k^2)\) with \(|N| \in \{2, 3\}, c/p \leq 1/2\) and \(k^1 \leq k^2\) be two normal newsvendor situations. Let \((N, v^{\Pi^1})\) and \((N, v^{\Pi^2})\) be the newsvendor games associated with \(\Pi^1\) and \(\Pi^2\), respectively. Then pair of newsvendor games \((N, v^{\Pi^1}, v^{\Pi^2})\) has MP1.

Moreover, we have the following corollary, which follows from Lemma 3, Theorem 10 and Proposition 2(b).

**Theorem 17** Let \(\Pi^1 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k^1)\) and \(\Pi^2 = (N, (\mu_i)_{i \in N}, \sigma, c, p, k^2)\) with \(c/p \leq 1/2, k^1 \leq k_{1-c/p}/\sqrt{|N|}\) and \(k^1 \leq k^2\) be two normal newsvendor situations. Let \((N, v^{\Pi^1})\) and \((N, v^{\Pi^2})\) be the newsvendor games associated with \(\Pi^1\) and \(\Pi^2\), respectively. Then, pair of newsvendor games \((N, v^{\Pi^1}, v^{\Pi^2})\) has MP1.

Hence, we showed that MP1, one of the monotonicity properties specified in Section 5, holds for a fairly general class of cooperative newsvendor games.

## 7 Conclusion

In this study, we considered profit division problems arising from situations, where multiple retailers order jointly and allocate their order after demand realization to benefit from inventory centralization. We assumed transferable utility, however, we limited the cooperation by some restrictions. The retailers may pose restrictions on the allocation of
the available quantity after demand realization in the cooperation. These constraints can be motivated by minimum delivery constraints or service level constraints. The general setting that we considered is a newsvendor situation, where demand of each retailer is a random variable.

In this study, we first investigated the following question: "In the cooperative game setting with transferable utilities, does there exist a stable division of total profit under delivery restrictions?"

Existence of a stable division is needed for a possible cooperation in the newsvendor environment described. We would like to go beyond the single-period structure of the newsvendor situation and investigate the relation of the cores obtained by newsvendor situations with different delivery restrictions. We defined monotonicity properties for such pairs of games to summarize the desired relationship among the cores. This is what we consider as one possible approach to identify "stability" in the long-run. We then investigated the following: "Do the cores of pairs of cooperative games under changing delivery restrictions have the desired monotonicity properties?" This issue is important as it ensures the retailers that cooperation is not only beneficial in the short-run, but also in a dynamic environment where restrictions are expected to change.

We answered the first question positively by showing that cooperative games associated with newsvendor situations have non-empty cores. This result is important as it shows that delivery restrictions, as specified in this study, do not affect the existence of a stable solution.

We then concentrated on situations, where the retailers change their conditions to be a part of the coalition. These changes cause a change in the total profit of the cooperation. We considered two cases. In the first case, several retailers increase or decrease their minimum delivery quantity restriction and in the second case only one retailer changes his minimum delivery level. We focused on criteria for a new division of joint profit to be fair and we analyzed the existence of such divisions. These fairness criteria are captured by four monotonicity properties, where the first two of them are associated with the first case and the other two are related with the second case.

The properties defined apply to a more general class of cooperative games. Hence, we investigated conditions under which the properties will hold for general classes of cooperative games. Specifically, we derived sufficient conditions for two-player, three-player, and n-player games.

For the newsvendor situation described, we first showed by counterexamples that pairs of cooperative games do not necessarily satisfy these monotonicity properties in general. For the newsvendor situation, we defined a class of games with retailers having a normally distributed demand where one of the monotonicity properties holds.

We believe that results regarding the monotonicity properties are important for two different directions of future research. One direction is further refinement of the sufficient and necessary conditions required for monotonicity for the newsvendor situation described here, as well as situations within a more general class. A second direction is to use monotonicity properties to specify meaningful subsets of the core of the cooperative
games, and hence restrict the feasible set for the profit division problem. Additionally, to the same end, one can investigate whether a specific (given) profit division rule (such as the Shapley value) satisfies monotonicity conditions.

Appendix

This appendix contains the proofs of the Theorems and Lemmas in Section 4, and the proofs of Theorem 9 and Proposition 2 in section 6.

Proof of Lemma 1: Since $P^S(a^S, q^S, x^S) = -\sum_{i \in S} a^S_i c_i + \sum_{i \in S} p_i \min(a^S_i, x^S_i)$ is a continuous function of $a^S$ for given $q^S$ and $x^S$, and the domain of $P^S$ is the compact set $M^S(q^S)$, we conclude that $P^S(\cdot, q^S, x^S)$ attains its maximum.

Proof of Theorem 3: To prove this theorem, we will show that $\pi^S(\cdot)$ is a continuous function of the order and any order outside a specific compact set results in lower expected profits than the minimum possible order.

First, define $c_S = \max_{i \in S} c_i$, and $p_S = \max_{i \in S} p_i$. Let $q \in Q^S$ be an order quantity, and let $\rho > 0$. Define $\delta = \rho / (p_S + c_S)$. Let $q' \in Q^S$ be such that $|q - q'| < \delta$. Then

$$|\pi^S(q) - \pi^S(q')| = \left| E_{X^S} \left[ r^S(q, \cdot) - r^S(q', \cdot) \right] \right|$$

$$\leq E_{X^S} \left[ |r^S(q, \cdot) - r^S(q', \cdot)| \right]$$

$$\leq E_{X^S} [(p_S + c_S)|q - q'|]$$

$$< (p_S + c_S) \delta$$

$$= \rho.$$

The first inequality holds because of the triangle inequality. The second inequality holds since for each demand realization $x^S \in X^S$, the difference in order quantities $|q - q'|$ can not cause an extra cost (revenue) more than $c_S * |q - q'| (p_S * |q - q'|)$. The last two terms follow by definition of $\delta$ and $\rho$.

Now we will show that any order quantity outside a specific compact set results in lower expected profit than the expected profit of order quantity $\sum_{i \in S} \varepsilon_i$. Let

$$a^S = \frac{p_S}{\min_{i \in S} \{c_i\}} \left[ \sum_{i \in S} E[X_i] + \sum_{i \in S} E[Y_i^-] \right],$$

where $Y_i^- = \varepsilon_i + \max\{0, -X_i\}$. Since for all $i \in N$ we have $c_i > 0$ by definition, $a^S$ is well-defined. Then, for all $q \in Q^S$ with $q > a^S$ we have

$$\pi^S(q) \leq -q \min_{i \in S} \{c_i\} + p_S \sum_{i \in S} E[X_i]$$

$$< -p_S \sum_{i \in S} E[Y_i^-]$$
\[
= -p_S \sum_{i \in S} \varepsilon_i - p_S \sum_{i \in S} E[\max\{0, -X_i\}]
\leq -c_S \sum_{i \in S} \varepsilon_i - p_S \sum_{i \in S} E[\max\{0, -X_i\}]
\leq \pi^S(\sum_{i \in S} \varepsilon_i).
\]

The first inequality follows by taking the minimum possible cost and the maximum possible revenue into consideration. The second inequality follows by definition of \(a^S\). The third inequality holds since \(p_S \geq c_S\). The last inequality holds since if the coalition orders \(\sum_{i \in S} \varepsilon_i\), the purchasing and transportation cost will be lower and there is extra revenue from sales for positive demand realizations.

**Proof of Lemma 2:** Let \(x^N\) be a demand realization vector for the grand coalition and let \(x^S\) be the associated demand vector of coalition \(S \subset N\). As before, denote an allocation of \(q^{S,*}\) that maximizes the profit of coalition \(S\) for demand realization \(x^S\) by \(a^{S,*}\). Furthermore, let \(b^{N,*}\) be the optimal allocation for the grand coalition for order quantity \(\bar{q}^N = \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)q^{S,*}\) and demand realization \(x^N(= \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)x^S)\). Note that \(\bar{q}^N \in Q^N\). We define an allocation for the grand coalition by taking the corresponding weighted balanced sum of allocations of subcoalitions. Since \(a^S \in M^S(q^S)\) for all \(S \subseteq N\), \(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S)a^S \in M^N(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S)q^S)\). Define

\[
F(a, b) = -\sum_{i \in S} a_i c_i + \sum_{i \in S} p_i \min\{a_i, b\}, \text{ for all } a \in \mathbb{R}^N \text{ and } b \in \mathbb{R}.
\]

Let \(a, \bar{a} \in \mathbb{R}^N\), \(b, \bar{b} \in \mathbb{R}\) and \(\lambda_1, \lambda_2 > 0\) then

\[
F(\lambda_1 a + \lambda_2 \bar{a}, \lambda_1 b + \lambda_2 \bar{b}) \geq F(\lambda_1 a, \lambda_1 b) + F(\lambda_2 \bar{a}, \lambda_2 \bar{b})
= \lambda_1 F(a, b) + \lambda_2 F(\bar{a}, \bar{b})
\]

The inequality follows from \((\lambda_1 a + \lambda_2 \bar{a})c_i = \lambda_1 a c_i + \lambda_2 \bar{a} c_i\) and \(\min\{\lambda_1 a + \lambda_2 \bar{a}, \lambda_1 b + \lambda_2 \bar{b}\} \geq \min\{\lambda_1 a, \lambda_1 b\} + \min\{\lambda_1 a, \lambda_1 b\}\). The equality holds since \(F\) is homogeneous of degree one. Hence,

\[
\begin{align*}
F^N(\bar{q}^N, x^N) &= P^N \left( b^{N,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)x^S \right) \\
&\geq P^N \left( \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)a^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)x^S \right) \\
&= F \left( \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)a^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)x^S \right) \\
&\geq \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)F \left( a^{S,*}, x^S \right) \\
&= \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)D^S \left( a^{S,*}, q^{S,*}, x^S \right) \\
&= \sum_{S \subseteq N : S \neq \emptyset} \kappa(S)r^S \left( q^{S,*}, x^S \right)
\end{align*}
\]
The first, second and third equality hold by definition. The first inequality holds since \( b^{N,*} \) is an optimal allocation for the grand coalition for order quantity \( \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)q^{S,*} \) and demand realization vector \( \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)x^S \), while \( \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)a^{S,*} \) is a possible allocation in \( M^{N}(\sum_{S \subseteq N: S \neq \emptyset} \kappa(S)q^{S,*}) \). The second inequality holds by (8). The last equality holds since \( a^{S,*} \) is the optimal allocation vector for coalition \( S \), order quantity \( q^{S,*} \) and demand realization \( x^S \).

Since this inequality holds for any realization \( x^N \) of \( X^N \), taking expectations proves that (1) holds. This completes the proof. \( \square \)

**Proof of Theorem 4:** Let \( \kappa : 2^N/\{\emptyset\} \rightarrow [0,1] \) be a balanced map. Let \( (q^{S,*})_{S \subseteq N: S \neq \emptyset} \) be optimal order quantities for the different coalitions. Then, \( z^N \) defined by \( z^N = \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)q^{S,*} \) denotes a possible order vector for coalition \( N \), while \( q^{N,*} \) is the optimal order vector. Hence, we know that \( \pi^N(q^{N,*}) \geq \pi^N(z^N) \). Then,

\[
v^F(N) = \pi^N(q^{N,*}) \\
\geq \pi^N(z^N) \\
= \pi^N\left( \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)q^{S,*} \right) \\
\geq \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)\pi^S(q^{S,*}) \\
= \sum_{S \subseteq N: S \neq \emptyset} \kappa(S)v^F(S). \tag{9}
\]

The first inequality follows from \( q^{N,*} \) being the optimal order quantity of the grand coalition. The second inequality holds because of Lemma 2. We conclude that the associated game is balanced. \( \square \)

**Proof of Theorem 9:** Suppose that \( \alpha(N) - \beta(N) \geq \alpha(S) - \beta(S) \) for all \( S \subset N \). Let \( K = \alpha(N) - \beta(N) \). Note that \( K \geq 0 \). Let us define the game \( (N, \bar{\alpha}) \) as follows:

\[
\bar{\alpha}(N) = \alpha(N) - K = \beta(N); \\
\bar{\alpha}(S) = \alpha(S) - K \quad \text{for all} \ S \subset N.
\]

Since \( \bar{\alpha}(S) = \alpha(S) - K \leq \beta(S) \) for all \( S \subset N \), we know that \( \text{Core}(\bar{\alpha}) \supseteq \text{Core}(\beta) \).

First, we show that for each possible extreme point of \( \text{Core}(\bar{\alpha}) \) there is an elementwise larger element of the core of \( (N, \alpha) \). Then, we extend this result to all elements of \( \text{Core}(\beta) \).

Since the game \( (N, \alpha) \) is totally balanced, we know that

\[
\alpha(\{i,j\}) \geq \alpha(\{i\}) + \alpha(\{j\}) \quad \text{for all} \ i, j \in \{1,2,3\} \text{ with } i \neq j; \\
\alpha(\{1,2,3\}) \geq \alpha(\{i,j\}) + \alpha(\{k\}) \quad \text{for all} \ i, j, k \in \{1,2,3\} \text{ with } i \neq j \neq k \neq i; \\
\alpha(\{1,2,3\}) \geq \frac{1}{2}\alpha(\{1,2\}) + \frac{1}{2}\alpha(\{1,3\}) + \frac{1}{2}\alpha(\{2,3\}).
\]
Since $\bar{\alpha}(S) = \alpha(S) - K$ for all $S \subseteq N$, we derive that

\[
\bar{\alpha}(\{i, j\}) \geq \bar{\alpha}(\{i\}) + \bar{\alpha}(\{j\}) + K \quad \text{for all } i, j \in \{1, 2, 3\} \text{ with } i \neq j;
\]

\[
\bar{\alpha}(\{1, 2, 3\}) \geq \bar{\alpha}(\{i, j\}) + \bar{\alpha}(\{k\}) + K \quad \text{for all } i, j, k \in \{1, 2, 3\}
\]

with $i \neq j \neq k \neq i$;

\[
\bar{\alpha}(\{1, 2, 3\}) \geq \frac{1}{2} \bar{\alpha}(\{1, 2\}) + \frac{1}{2} \bar{\alpha}(\{1, 3\}) + \frac{1}{2} \bar{\alpha}(\{2, 3\}) + 1/2K.
\]

Hence, $(N, \bar{\alpha})$ is totally balanced too.

The core of $(N, \bar{\alpha})$ is given by the convex set of points $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ satisfying the following constraints:

\[
\bar{x}_i \geq \bar{\alpha}(\{i\}) \quad \text{for all } i \in N;
\]

\[
\bar{x}_i + \bar{x}_j \geq \bar{\alpha}(\{i, j\}) \quad \text{for all } i, j \in N \text{ with } i \neq j;
\]

\[
\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{\alpha}(N).
\]

An extreme point of this convex set is given by three linearly independent binding constraints and we already know one of them, which is $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{\alpha}(N)$. Note that the binding constraints $\bar{x}_i = \bar{\alpha}(\{i\})$, $\bar{x}_j + \bar{x}_k = \bar{\alpha}(\{j, k\})$ and $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{\alpha}(N)$ with $i \neq j \neq k \neq i$ cannot define an extreme point since they are not linearly independent.

Totally balancedness of $(N, \bar{\alpha})$ implies that if $\bar{x}_i = \bar{\alpha}(\{i\})$, $\bar{x}_j = \bar{\alpha}(\{j\})$ with $i \neq j$ and $\bar{x} \in \text{Core}(\bar{\alpha})$, then $\bar{x}_i + \bar{x}_j = \bar{\alpha}(i, j)$. Moreover, balancedness of $(N, \bar{\alpha})$ implies that if $\bar{x}_1 + \bar{x}_2 = \bar{\alpha}(\{1, 2\})$, $\bar{x}_2 + \bar{x}_3 = \bar{\alpha}(\{2, 3\})$, $\bar{x}_1 + \bar{x}_3 = \bar{\alpha}(\{1, 3\})$ and $\bar{x} \in \text{Core}(\bar{\alpha})$, then $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{\alpha}(\{1, 2, 3\})$. Hence, the possible coalitions associated with binding constraints (tight coalitions) and their associated extreme points of $\text{Core}(\bar{\alpha})$ are given in Table 12.

<table>
<thead>
<tr>
<th>Type</th>
<th>Tight coalitions</th>
<th>Extreme point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$</th>
</tr>
</thead>
</table>
| 1    | $\{i\}$ and $\{i,j\}$ | $\bar{x}_i = \bar{\alpha}(\{i\})$  
$\bar{x}_j = \bar{\alpha}(\{i,j\}) - \bar{\alpha}(\{i\})$  
$\bar{x}_k = \bar{\alpha}(\{1, 2, 3\}) - \bar{\alpha}(\{i,j\})$ |
| 2    | $\{i,j\}$ and $\{i,k\}$ | $\bar{x}_i = \bar{\alpha}(\{i,j\}) + \bar{\alpha}(\{i,k\}) - \bar{\alpha}(\{1,2,3\})$  
$\bar{x}_j = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,k\})$  
$\bar{x}_k = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,j\})$ |

Table 12: Possible Binding Constraints and Extreme Points

Suppose that $\bar{x}$ is an extreme point of type 1. So, $\bar{x} \in \text{Core}(\bar{\alpha})$. Let $x = (\bar{x}_i + K, \bar{x}_j, \bar{x}_k)$ be its so-called corresponding point. Then $x \in \text{Core}(\alpha)$, since

\[
x_i = \bar{x}_i + K = \bar{\alpha}(\{i\}) + K = \alpha(\{i\});
\]

\[
x_j = \bar{x}_j = \bar{\alpha}(\{i,j\}) - \bar{\alpha}(\{i\}) \geq \bar{\alpha}(\{j\}) + K = \alpha(\{j\});
\]

\[
x_k = \bar{x}_k = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,j\}) \geq \bar{\alpha}(\{k\}) + K = \alpha(\{k\});
\]
Furthermore, 

\[ x_i + x_j = \bar{x}_i + \bar{x}_j + K \geq \bar{\alpha}(\{i,j\}) + K = \alpha(\{i,j\}); \]

\[ x_i + x_k = \bar{x}_i + \bar{x}_k + K \geq \bar{\alpha}(\{i,k\}) + K = \alpha(\{i,k\}); \]

\[ x_j + x_k = \bar{x}_j + \bar{x}_k = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i\}) \geq \bar{\alpha}(\{j,k\}) + K = \alpha(\{j,k\}); \]

\[ x_i + x_j + x_k = \bar{x}_i + \bar{x}_j + \bar{x}_k + K = \bar{\alpha}(\{1,2,3\}) + K = \alpha(\{1,2,3\}). \]

The inequality in the second line follows from (10). The inequality in the third line follows from (11). The inequalities in the fourth and fifth line, and the second equality in the last line hold since \( \bar{x} \in \text{Core}(\bar{\alpha}) \). The inequality in the sixth line follows from (11).

Suppose that \( \bar{x} \) is an extreme point of type 2. So, \( \bar{x} \in \text{Core}(\bar{\alpha}) \). Let \( x = (\bar{x}_i + K, \bar{x}_j, \bar{x}_k) \) be its so-called corresponding point. Then \( x \in \text{Core}(\alpha) \), since

\[ x_i = \bar{x}_i + K \geq \bar{\alpha}(\{i\}) + K = \alpha(\{i\}) \]

\[ x_j = \bar{x}_j = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,k\}) \geq \bar{\alpha}(\{j\}) + K = \alpha(\{j\}) \]

\[ x_k = \bar{x}_k = \bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,j\}) \geq \bar{\alpha}(\{k\}) + K = \alpha(\{k\}) \]

\[ x_i + x_j = \bar{x}_i + \bar{x}_j + K \geq \bar{\alpha}(\{i,j\}) + K = \alpha(\{i,j\}) \]

\[ x_i + x_k = \bar{x}_i + \bar{x}_k + K \geq \bar{\alpha}(\{i,k\}) + K = \alpha(\{i,k\}) \]

\[ x_j + x_k = \bar{x}_j + \bar{x}_k = 2\bar{\alpha}(\{1,2,3\}) - \bar{\alpha}(\{i,j\}) - \bar{\alpha}(\{i,k\}) \geq \bar{\alpha}(\{j,k\}) + K \]

\[ = \alpha(\{j,k\}) \]

\[ x_i + x_j + x_k = \bar{x}_i + \bar{x}_j + \bar{x}_k + K = \bar{\alpha}(\{1,2,3\}) + K = \alpha(\{1,2,3\}). \]

The inequalities in the second and third line follow from (11). The inequalities in the first, fourth and fifth line, and the second equality in the last line hold since \( \bar{x} \in \text{Core}(\bar{\alpha}) \). The inequality in the sixth line follows from (12).

Let \( (\bar{x}^1, ..., \bar{x}^k) \) be all extreme points that are in the core of \((N, \bar{\alpha})\) and \((x^1, ..., x^k)\) be their corresponding points. Note that all these corresponding points are in the core of \((N, \alpha)\). Consider \( y \in \text{Core}(\beta) \). Since \( \text{Core}(\bar{\alpha}) \supseteq \text{Core}(\beta), y \in \text{Core}(\bar{\alpha}) \). Hence, we can express \( y \) as a convex combination of the extreme points of the core of the game \((N, \bar{\alpha})\). Let \( \lambda \in \mathbf{R}_+^k \) be such that \( y = \sum_{i=1}^k \lambda_i \bar{x}^i \) with \( \sum_{i=1}^k \lambda_i = 1 \). Consider the division \( x = \sum_{i=1}^k \lambda_i x^i \). Since every corresponding point is in the core of \((N, \alpha)\), \( x \in \text{Core}(\alpha) \). Furthermore, \( x \geq y \). This completes the proof. \( \square \)

**Proof of Proposition 2:**

(a) From 5,

\[ v(S) = E[-c * q^S + p \min(q^S, X_S)] \]

\[ = -c * q^S + p \int_{-\infty}^{q^S} xf_S(x)dx + p \int_{q^S}^{\infty} q^S f_S(x)dx \]

\[ = -c * q^S + p \left( q^S F_S(q^S) - \int_{-\infty}^{q^S} F_S(x)dx \right) + p q^S (1 - F_S(q^S)) \]

33
\[
\begin{align*}
&= -c \cdot q^S - p \int_{-\infty}^{q^S} F_S(x) dx + pq^S \\
&= (p - c) \cdot q^S - p \int_{-\infty}^{q^S} F_S(x) dx \\
&= (p - c) \cdot q^S - p\sigma_S \int_{-\infty}^{q^S} \Theta(z) dz,
\end{align*}
\]
where \( q^S = \max\{\mu_S + k_1 - c/p\sigma_S, \sum_{i \in S} \mu_i + k \sum_{i \in S} \sigma_i\} \). The third equality holds by means of partial integration. The last equality follows from changing variables \( z = (x - \mu_S)/\sigma_S \).

(b) Since \( k \leq k_1 - c/p/\sqrt{|N|} \), it is easy to check that the optimal order quantity of coalition \( S \) is \( q^S = \mu_S + k_1 - c/p\sqrt{|S|} \). Hence, every coalition orders the optimal order quantity and reaches the optimal profit as if there are no delivery restrictions. From Theorem 2 of Özen et. al. (2005), \( v^\Pi \) is convex.

References


Holland, Amsterdam.


