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A boundary-layer model for a plane free-burning high-pressure gas-discharge arc

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A boundary-layer model for a plane free-burning high-pressure gas-discharge arc is given. For the dependence on the temperature, a rule given by Elenbaas [*The High Pressure Mercury Vapour Discharge* (North-Holland, Amsterdam, 1951)] is used for the electrical conductivity and power-law rules for the dynamic viscosity and the thermal conductivity. The specific heat is assumed to be constant. It is shown that a similarity transformation can be applied. The upward velocity increases with the square root of the core temperature. Graphs are given which show how the temperature rise in the arc depends upon the applied current.

INTRODUCTION

Various authors¹⁻⁹ have recognized that convection can play an important role in the performance of high-pressure discharge lamps. This is in sharp contrast with low-pressure discharge lamps where convection does not play any role at all. Earlier research on this subject dates back to the 1930s. Among the research workers of those early days, two names are particularly prominent: Elenbaas¹ and Kenty.² The latter measured convection speeds by introducing small particles into the tube containing the arc. Kenty refers to what he calls "a splendid series of studies" by Elenbaas on the various theoretical aspects of convection in high-pressure discharges. These and other early papers are all mentioned in Elenbaas's equally splendid and still very readable book¹ on the high-pressure mercury discharge.

In those early days convection effects were modeled with the so-called channel approximation. The tubes being elongated, there was thought to be an extended central section in which the temperature and flow characteristics were independent of the axial coordinate. These models are inherently one dimensional. Of course, a prerequisite for the validity of these models is that the tubes are in a vertical position, i.e., aligned with the gravitational field. In horizontal or otherwise slanting tubes, the flow fields are at least two dimensional and far more complicated models are needed to account for these.

Until the early 1970s progress in this particular field, i.e., the modeling of convection effects, seems to have been rather slow. When more and more computing power became available, some researchers recognized the possibility of studying convection effects on a level far more sophisticated than Elenbaas's. Lowke⁴ wrote down the full Navier-Stokes and energy equations for this system, entering energy input and radiation output terms where necessary. He considered time-dependent systems, which allowed him to simulate the temporal evolution of the arc, that is to say, of his model for it, in a high-pressure system. Zollweg³ uses Lowke's numerical procedure to study various aspects of high-pressure arcs. Of particular interest are Zollweg's observations¹⁰ on the instability of thin concentrated arcs, i.e., the "snaking" effect. Other relevant studies concern highly nonlinear time-dependent flows which are dependent on the radial coordi-

nate only. These are more sophisticated versions of the classical channel models that were pioneered by Elenbaas. Recent examples are the papers by Dakin and Rautenberg¹¹ and Rutan and Matthews.¹²

If one compares the development of this field as described above with that of other fields in fluid dynamics and heat transfer, then one is struck by the almost complete absence of analytical or semianalytical approaches which are on a higher level of sophistication than the simple models of Elenbaas. A related field such as aerodynamics, for instance, is now dominated by the large-scale computerized approach. This later stage in its development, however, was preceded by a rich era in which boundary-layer models of all kinds were studied with great success. The knowledge gained from these (semi)analytical models is still indispensable for a complete understanding of the field of aerodynamics. And, indeed, many research workers are still active in the further development of this particular aspect of aerodynamics. The same is true for other fields, from creeping flows right through to turbulence.

Boundary-layer studies seem to be completely absent in discharge-lamp research. The extreme complexity of the field is no doubt partly to blame for this. Nevertheless, concentrated temperature and velocity fields have been observed in this particular area. The "snaking" of a concentrated arc,^{1,2,10} which is an instability effect, can probably be fully understood only with the help of a stability analysis on the basis of a boundary-layer study. As to the occurrence of boundary layers in high-pressure environments, Fig. 10 of Lowke⁴ is quite illustrative. Lowke shows two sets of isotherms for arcs in mercury vapor. The first of these concerns a pressure of 1 atm. The isotherm structure is like that of the channel flows studied by Elenbaas. At a pressure of 10 atm, on the other hand, a typical boundary-layer profile is seen to develop in the lower section of the tube, occupying almost two-thirds of its total length. It would seem, therefore, that there is a strong case for a separate line of research, within discharge-lamp research as a whole, which is specially devoted to the study of boundary-layer effects.

In this paper we shall present a boundary-layer analysis of a problem which is relevant for high-pressure discharge lamps. Our aim will be to obtain insight into the structure of

the thermal and velocity field of a free-burning arc. This is an arc which is influenced relatively little by outer boundaries. Owing to the extremely strong, namely, exponential dependence of the electrical conductivity on the temperature, the structure of the arc is characterized by two layers. One of these layers is a thin one located in the core of the arc. Almost all of the Joule heating occurs within it. It is surrounded by a thicker boundary layer in which the temperature drops from a high value in the core of the arc to a much lower ambient value. Because our aim is to acquire insight into this particular structure and not to present a realistic simulation of any practical situation yet, we shall analyze a plane, i.e., two-dimensional arc, here. Having studied the structure in this mathematically simpler geometry, we will be better prepared to do a similar analysis, but now for the more realistic cylindrical case. We shall also assume, as have other researchers before us, e.g., Lowke,⁴ that the heat produced within the arc is carried off solely by conduction and convection. Of course, within a system as hot as a high-pressure gas discharge, this is a gross simplification. Again, a step-by-step approach should be favored here. Having studied simple systems first, albeit fairly unrealistic ones, models of ever-increasing complexity can be devised by adding new effects one by one.

MATHEMATICAL MODEL

Let us consider the geometry of Fig. 1. An electric high-pressure gas-discharge arc exists between A and B, where the cathode and the anode are located, respectively. The horizontal lines denoted by A and B are a distance l apart and are in a vertical position relative to each other. The x coordinate points from A to B. In this investigation the conditions are assumed to be such that l is large in comparison with the thickness of the arc, so that throughout the arc the electric field is more or less parallel to the x axis. Moreover, the field is assumed to be independent of the coordinate y . These as-

FIG. 1. Geometrical configuration.

sumptions are in line with most studies on convection in lamps. As a result, if $\mathbf{i}$ is the electric-current density vector, we have

$$i_x = \sigma(t)E_x. \quad (1)$$

Here, $\sigma(t)$, the electric conductivity of the gas, is assumed to be a function of the local temperature t only.

Integrating (1) across the arc, invoking the usual boundary-layer approximations, we have

$$I = 2E_x \int_0^\infty \sigma(t)dy, \quad (2)$$

where I denotes the total current, which is independent of x . The factor 2 appears for reasons of symmetry.

The energy dissipation per unit volume, which is given by $q = i_x E_x$, can now be written as follows:

$$q = \frac{1}{4} I^2 \sigma(t) \left(\int_0^\infty \sigma(t)dy \right)^{-2}. \quad (3)$$

This is fully analogous to the presentation of Lowke.⁴

If u and v are the velocity components in the x and y directions, respectively, then the boundary-layer equations governing the flow and thermal field within the arc are the following:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (4)$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \mu(t) \frac{\partial u}{\partial y} + (\rho_\infty - \rho)g, \quad (5)$$

$$\rho c_p \left(u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} \right) = \frac{\partial}{\partial y} \lambda(t) \frac{\partial t}{\partial y} + q, \quad (6)$$

which are the continuity, momentum, and energy equations, respectively. In (4)–(6) ρ is the density, μ is the dynamic viscosity, c_p is the specific heat at constant pressure, and λ is the thermal conductivity of the gas. Furthermore, ρ_∞ is the density at the outer edge of the arc. Here, we shall assume that ρ_∞ is constant. Following Lowke,⁴ we only consider those cases for which the radiation losses are small in comparison with those attributable to conduction and convection.

The buoyancy force is represented by the last term on the right-hand side of (5). It can be derived in the manner explained by Gebhart *et al.*¹³ (Sec. 8.3), where ideal-gas conditions are assumed. Since the variations in the pressure owing to buoyancy effects are of the order of 1 N m^{-2} or less ($\rho g l$), the pressure can be regarded as being practically constant in the equation of state, i.e., for high-pressure ($p \approx 10^5 \text{ N m}^{-2}$ or more) arcs. Therefore,

$$\rho = \frac{\rho_\infty t_\infty}{t}, \quad (7)$$

where t_∞ is the (constant) temperature at the outer edge of the arc.

It is well known that $\sigma(t)$ is a very rapidly varying function of t ; that is to say, it increases strongly with temperature. For the high-pressure mercury discharge, Elenbaas¹ (p. 48) gives the formula

$$\sigma(t) = \gamma t^{3/4} \exp(-t_i/t), \quad (8)$$

where γ is a constant that is inversely proportional to the

square root of the gas pressure and the collision cross section of the mercury atoms. It is understood that the temperatures of the atoms and the electrons are the same. Elenbaas¹ emphasizes that this is a reasonable assumption for high-pressure discharges. Further, in (8), t_i denotes the ionization temperature

$$t_i = eV_i/2k, \quad (9)$$

where e is the charge of an electron, V_i is the ionization potential, and k is Boltzmann's constant.

As was emphasized before, the thermal properties λ and μ are functions of local temperature only. For many gases a good approximation¹³ is

$$\mu(t) = \mu_r(t/t_r)^{3/4}, \quad (10)$$

where μ_r is the dynamic viscosity at some reference temperature t_r . If we assume that λ contains no component other than that due to conduction, i.e., there is no additive effect representing excitation and ionization, we can write, similarly,

$$\lambda = \lambda_r(t/t_r)^{3/4}, \quad (11)$$

where λ_r is the thermal conductivity at t_r . The specific heat c_p is assumed to be constant. We shall show later (see Concluding Remarks) that the functions (8), (10), and (11) are in good agreement with the property values normally used by other authors investigating high-pressure gas-discharge arcs. The value of λ_r is closely related to that of μ_r . Indeed, for an ideal gas the Prandtl number is equal to $\frac{2}{3}$. Therefore, we have

$$\text{Pr} = \frac{\nu}{\kappa} = \frac{\nu_r}{\kappa_r} = \frac{\mu_r/\rho_r}{\lambda_r/\rho_r c_p}, \quad (12)$$

so that

$$\lambda_r = \text{Pr}^{-1} \mu_r c_p = \frac{2}{3} \mu_r c_p, \quad (13)$$

where κ is the thermal diffusivity.

In addition to the field equations, we need boundary conditions. There is symmetry with respect to the plane $y = 0$. Therefore,

$$\frac{\partial t}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad v = 0 \quad \text{at } y = 0. \quad (14)$$

At the outer edge of the arc, we have

$$t \rightarrow t_\infty, \quad u \rightarrow 0 \quad \text{when } y \rightarrow \infty. \quad (15)$$

In full agreement with the boundary-layer character of the present problem, we have also

$$t = t_\infty, \quad u = 0 \quad \text{at } x = 0. \quad (16)$$

These conditions characterize the parabolic character of the governing set of equations. The conditions (14)–(16) are all homogeneous. The nonhomogeneous element in this problem is provided by the given current I and the given voltage drop V from the anode to the cathode.

DIMENSIONLESS FORMULATION

Let us introduce scaled dimensionless variables (capital letters) as follows:

$$x = IX, \quad y = \delta_r IY, \quad t = t_r T, \quad u = u_r U, \quad v = \delta_r u_r V, \quad (17)$$

where δ_r and u_r are parameters which will be determined presently. Since the arc is assumed to be narrow, accordingly, the parameter δ_r is assumed to be much smaller than unity.

When (17) is substituted in (4)–(6), the various terms appearing in these equations are assigned weights which involve the parameters of the system. In a properly scaled system, these weights ought to be the same in each of the equations. This is achieved when

$$\delta_r = \left(\frac{\nu_r^2 \rho_r}{gl^3 \rho_\infty} \right)^{1/4}, \quad u_r = \left(\frac{\rho_\infty gl}{\rho_r} \right)^{1/2}, \quad (18)$$

where $\nu_r = \mu_r/\rho_r$. The resulting set of equations reads

$$\frac{\partial}{\partial X} \left(\frac{U}{T} \right) + \frac{\partial}{\partial Y} \left(\frac{V}{T} \right) = 0, \quad (19)$$

$$\frac{1}{T} \left(U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} \right) = \frac{\partial}{\partial Y} T^{3/4} \frac{\partial U}{\partial Y} + \left(1 - \frac{T_\infty}{T} \right), \quad (20)$$

$$\frac{\text{Pr}}{T} \left(U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} \right) = \frac{\partial}{\partial Y} T^{3/4} \frac{\partial T}{\partial Y} + S^2 \Phi(T, T_i), \quad (21)$$

where

$$T_{(\cdot)} = \frac{t_{(\cdot)}}{t_r}, \quad (22)$$

and

$$\Phi(T, T_i) = T^{3/4} T_i^{5/2} \exp\left(-\frac{T_i}{T}\right) \times \left[\int_0^\infty T^{3/4} \exp\left(-\frac{T_i}{T}\right) dY \right]^{-2}. \quad (23)$$

The boundary conditions are

$$Y = 0: \quad \frac{\partial T}{\partial Y} = 0, \quad \frac{\partial U}{\partial Y} = 0, \quad V = 0, \quad (24)$$

$$Y \rightarrow \infty: \quad T \rightarrow T_\infty, \quad U \rightarrow 0, \quad (25)$$

$$X = 0: \quad T = T_\infty, \quad U = 0. \quad (26)$$

This system contains the parameters T_∞ , T_i , Pr , and S , where the latter is given by

$$S = \frac{1}{2} I \gamma^{-1/2} \lambda_i^{-1/2} t_i^{-7/8}. \quad (27)$$

Since $\text{Pr} = \frac{2}{3}$, the actual number of parameters is only three.

Of course, as we are still free to choose the reference temperature t_r , the parameter set can be further reduced. We might, for instance, choose $t_r = t_\infty$. In that case the parameter T_∞ is equal to unity. However, it is doubtful whether this would be a suitable choice. Indeed, the characteristics of the arc are determined first and foremost by what happens near the plane of symmetry, where we expect the highest temperatures and the largest velocities. Alternatively, we could select $t_r = t_i$, so that the temperature is scaled with respect to the ionization temperature. This leads to difficulties of another kind. Elenbaas¹ gives a value of $t_i = 60\,500$ K for mercury, and this value is much higher than the expected maximum temperatures in the arc which are in the range 5000–10 000 K. Therefore, a balanced system is obtained only when t_r is chosen in this range. Only then will the scaling factors of (18) be truly characteristic of the conditions in the arc.

As we shall see in the next section, the problem defined by (19)–(26) leads to a constant temperature in the plane $Y = 0$. Although this temperature is unknown to begin with, it is ideally suited as a parameter to scale the temperature field with. In that case we add an artificial boundary condition

$$T = 1 \text{ at } Y = 0 \quad (28)$$

to the ones already given by (24). The system is now overdetermined so that we must leave one parameter free in order to be able to satisfy the extra boundary condition (28). Clearly, the parameter S is ideally suited for this purpose. We have, as it were, inverted the definition of the problem. We now ask ourselves: What energy input is needed to achieve a given temperature in the plane $Y = 0$? This procedure has the added advantage that we do not have to calculate the integral appearing in (23) during the solution stage, but only afterwards, since we can make it formally part of the unknown parameter. Accordingly, we define

$$H = S^2 e^{T_i T_i^{5/2}} \left\{ \int_0^\infty T^{3/4} \exp \left[T_i \left(1 - \frac{1}{T} \right) \right] dY \right\}^{-2}, \quad (29)$$

and write

$$S^2 \Phi(T, T_i) = HT^{3/4} \exp [T_i (1 - 1/T)] \quad (30)$$

in (21). H is now the free parameter. Moreover, the exponential term in (30) now attains a value equal to unity at $Y = 0$. However, it is sharply peaked when $T_i \gg 1$.

A SIMILARITY SOLUTION

The system of Eqs. (19)–(23) and the boundary conditions (24)–(26) and (28) can be reduced to a system involving ordinary differential equations by means of a suitable similarity transformation. In fact, similar systems have been considered before (Sparrow and Gregg,¹⁴ Gebhart *et al.*¹³), but then without the energy production term (30). As explained by Gebhart *et al.*¹³ (Sec. 8.3), one should proceed by introducing the stream function Ψ as follows:

$$U = T \frac{\partial \Psi}{\partial Y}, \quad V = -T \frac{\partial \Psi}{\partial X}, \quad (31)$$

and then put

$$T(X, Y) = \theta(\eta), \quad \Psi(X, Y) = X^{3/4} f(\eta), \quad (32)$$

where η is the similarity variable defined by

$$\eta = X^{-1/4} \int_0^Y \frac{dY}{T}. \quad (33)$$

After a lengthy calculation the following set of equations and boundary conditions results:

$$(\theta^{-1/4} f'')' + \frac{3}{4} f f'' - \frac{1}{2} (f')^2 + \theta - T_\infty = 0, \quad (34)$$

$$(\theta^{-1/4} \theta')' + \frac{3}{4} \text{Pr} f \theta' + \omega \theta^{7/4} \exp [T_i (1 - 1/\theta)] = 0, \quad (35)$$

$$\eta = 0: \quad \theta = 1, \quad \theta' = 0, \quad f = 0, \quad f'' = 0, \quad (36)$$

$$\eta \rightarrow \infty: \quad \theta \rightarrow T_\infty, \quad f' \rightarrow 0, \quad (37)$$

where a prime stands for differentiation with respect to the argument, and where

$$\omega(T_i) = S^2 T_i^{5/2} e^{T_i} \left\{ \int_0^\infty \theta^{7/4} \exp \left[T_i \left(1 - \frac{1}{\theta} \right) \right] d\eta \right\}^{-2}. \quad (38)$$

Strictly speaking, the system (34)–(37) is not well posed. The energy-production term in (35) does not approach zero, as it should, when η tends to infinity. However, the term involves the factor $\exp(-T_i/T_\infty) = \exp(-t_i/t_\infty)$, which is practically equal to zero for real arcs (compare $t_i = 60\,500$ K and $t_\infty = 1000$ K).

In view of what was said in the previous paragraph, we might go as far as setting T_∞ equal to zero. This appears to be a realistic approximation for those cases where the core temperature exceeds the ambient temperature by an order of magnitude. At the same time this solves the problem of non-zero heat production outside the boundary layer. With the Prandtl number fixed and H being free, this leads to a one-parameter (T_i) problem definition for the plane arc. A new difficulty arises because a negative power of the temperature multiplies the highest derivatives in Eqs. (34) and (35). We show in the Appendix how this problem can be resolved.

RESULTS

A numerical integration of the system (34)–(37) with $T_\infty = 0$ produced the results listed in Table I. The Prandtl number Pr was taken as $\frac{2}{3}$, which is the usual value for an ideal gas. In the first column values of t_r/t_i are given, where t_r is the temperature in the center of the boundary layer, i.e., in the middle of the arc. Clearly, most practical cases corre-

TABLE I. Characteristic values resulting from the similarity solution. [S defined by Eq. (27), ω defined by Eq. (38); for f' , see Eq. (39); for $f(\infty)$, see Eq. (41); and for $\int_0^\infty \theta d\eta$, see Eq. (43)].

t_{core}/t_i	S	ω	$f'(0)$	$f(\infty)$	$\int_0^\infty \theta d\eta$
0.05	0.315(−6)	3.409	0.943	2.59	1.704
0.06	0.227(−5)	2.965	0.946	2.59	1.713
0.07	0.964(−5)	2.646	0.949	2.59	1.721
0.08	0.294(−4)	2.406	0.952	2.60	1.728
0.09	0.715(−4)	2.218	0.955	2.60	1.735
0.10	0.148(−3)	2.066	0.957	2.60	1.742
0.12	0.460(−3)	1.837	0.961	2.60	1.755
0.14	0.107(−2)	1.672	0.965	2.61	1.766
0.16	0.208(−2)	1.547	0.968	2.61	1.777
0.18	0.355(−2)	1.448	0.971	2.62	1.786
0.20	0.554(−2)	1.368	0.974	2.62	1.795
0.25	0.130(−1)	1.222	0.980	2.63	1.815
0.30	0.241(−1)	1.123	0.985	2.63	1.832
0.35	0.389(−1)	1.050	0.989	2.64	1.846
0.40	0.570(−1)	0.994	0.992	2.64	1.860
0.45	0.784(−1)	0.950	0.996	2.65	1.871
0.50	0.103	0.915	0.998	2.65	1.882
0.55	0.130	0.885	1.001	2.65	1.891
0.60	0.159	0.860	1.003	2.66	1.900
0.65	0.191	0.838	1.005	2.66	1.908
0.70	0.226	0.819	1.007	2.66	1.916
0.75	0.262	0.803	1.008	2.67	1.923
0.80	0.300	0.788	1.010	2.67	1.929
0.85	0.340	0.775	1.011	2.67	1.935
0.90	0.382	0.763	1.012	2.67	1.941
0.95	0.426	0.753	1.014	2.68	1.947
1.00	0.471	0.743	1.015	2.68	1.952

spond to the first few lines of the table. In the second column we find the parameter S , which, according to (27), is directly proportional to the current I . Since we are discussing a plane arc here, it should be understood that I is the current per unit width of the arc, this width being measured in a direction perpendicular to the plane of Fig. 1. When integrating Eq. (27), we should realize that λ_i is not the actual thermal conductivity at t_i , but rather the projected value according to Eq. (11). This equation is assumed to be valid in the temperature range which applies to the arc, and this range is likely to be far below the temperature t_r . Therefore, the projected thermal conductivity $\lambda_i(t_i)$ is given by $\lambda_i = \lambda_r (t_i/t_r)^{3/4}$, where t_r is a temperature within the range where (11) applies.

The t_r/t_i -vs- S relationship is also shown graphically in Fig. 2. It is seen that the core temperature rises rapidly when the current is increased from zero before it varies more or less linearly with S . Again, it would seem that most practical cases are to be found within this range. In Fig. 3 we show a logarithmic plot which gives a more detailed picture of this range.

Two further columns of Table I give information about the flow field. The parameter $f'(0)$ is directly related to the upward velocity in the central plane. It is seen to vary only very slowly with S . This is due to the scalings we have chosen. If we retrace our steps through the various transformations and scalings of Eqs. (33), (32), (31), (18), (17), and (7), we find, for the maximum upward velocity,

$$u_{\max} = \left(gx \frac{t_r}{t_\infty} \right)^{1/2} f'(0). \quad (39)$$

Since $f'(0)$ is almost independent of t_r , we conclude that the

FIG. 2. Relationship between the core temperature and the dimensionless current.

FIG. 3. Relationship between the core temperature and the dimensionless current in the lower current range.

upward velocity is more or less proportional to the square root of the core temperature t_r . Had we used inadvertently the classical Boussinesq-induced free-convective rule $u_{\max} \propto [g\beta(t_r - t_\infty)x]^{1/2}$, where β is the coefficient of thermal expansion, then we would have concluded erroneously that $u_{\max}$ was independent of t_r , since $\beta \approx 1/t_r$.

The parameter $f(\infty)$ is related to the total mass flow through the arc

$$M = 2 \int_0^\infty \rho u dy. \quad (40)$$

Indeed, it can be shown that

$$M = 2\rho_\infty (gv_\infty^2 x^3)^{1/4} (t_r/t_\infty)^{1/8} f(\infty). \quad (41)$$

Since $f(\infty)$ is almost independent of t_r/t_∞ , which can be seen from Table I, we conclude that M is only very weakly dependent on the temperature rise in the core.

We also present Fig. 4, which shows the temperature and velocity profiles for $t_r/t_i = 0.1$. Graphs of this kind are very well known in the free-convective literature, as they are typical for plumes which have been studied intensively (see, for instance, Gebhart *et al.*,¹³ Fig. 3.7.1).

The thickness of the arc can be deduced from the scalings of Eqs. (17), (18), and (33). Inverting the latter equation, we find

$$Y_\infty = X^{1/4} \int_0^\infty \theta(\eta) d\eta, \quad (42)$$

revealing that the thickness is finite. Using (17) and (18), we find

$$y_\infty = \left(\frac{v_\infty^2 x}{g} \right)^{1/4} \left(\frac{t_r}{t_\infty} \right)^{5/8} \int_0^\infty \theta(\eta) d\eta. \quad (43)$$

Since the dimensionless temperature θ is only weakly depen-

FIG. 4. Dimensionless temperature (θ) and velocity (f') profiles as functions of the similarity variable η .

dent upon T_i or t_r , the integral appearing in (42) and (43) is nearly a constant number of order unity (see last column of Table I). Therefore, the thickness of the plane arc increases only slightly faster than the $\frac{3}{4}$ th power of t_r . Also, it grows in proportion to the one-fourth power of x , which is a classical free-convective result.

CONCLUDING REMARKS

In this paper we have analyzed a boundary-layer model for a free-burning gas-discharge arc. To minimize the complexity as much as possible, the plane arc was studied here. The advantage of this approach is that the structure of the arc can be studied within a simpler context. Having gained

insight through this simpler model, we will be better prepared to tackle the more complicated rotationally symmetric case. This will be the subject of future investigations.

At this stage we are not in a position to compare the results of our boundary-layer model with those of full-scale numerical simulations, since these were all carried out for a rotational symmetry. We can, however, check whether our assumptions about the material properties, such as the density (7), the electrical conductivity (8), the viscosity (10), and the thermal conductivity (11) are in line with those used by other authors. Zollweg³ presents such values for mercury arcs in three different pressure environments. His Table III refers to a 400-W, 2-A mercury arc in an environment of 5.32 atm. Zollweg's values of the electrical conductivity can be modeled fairly well with Eq. (8) when $\gamma = 11.4$ mho/cm/K^{-3/4} and $t_i = 55\,820$ K. His values of the density are such that ρt is almost constant in the temperature range considered by him. Also, the viscosity μ is modeled satisfactorily by the rule of Eq. (10). The thermal conductivity, on the other hand, follows (11) reasonably well only in the lower-temperature range. At higher temperatures it increases faster than predicted by (11). This may possibly be attributed to additional thermal exchange effects associated with excitation and ionization. The specific heat c_p , which we have assumed to be constant, is also seen to rise fairly rapidly in the range of higher temperatures. Clearly, more refined models will be needed to take these effects into account. A full comparison of the various data is given in Table II. It is interesting to note that the thermal conductivity and viscosity have the same behavior in the larger part of the temperature range shown in the table. When divided by the $\frac{3}{4}$ power of the temperature, both physical entities increase with t before decreasing again. This trend continues for the viscosity, but the thermal conductivity rises sharply in the higher-temperature range.

It would seem from Zollweg's tabulated values that the thermal behavior of λ/c_p follows a rule such as that of Eq. (10) or (11) more closely than λ itself. In view of this, it is worthy of note that μ and λ/c_p have the same dimension. In

TABLE II. Comparison of present property values with those used by Zollweg^a (Zollweg's results are from his Table III).

t (K)	Zollweg	Electrical conductivity $11.4t^{3/4} \exp\left(-\frac{55820}{t}\right)$	Viscosity $\mu_{\text{Zollweg}} t^{-3/4} \times 10^3$	Therm. cond. $\lambda_{\text{Zollweg}} t^{-3/4} \times 10^3$
1000	1.00×10^{-17}	1.16×10^{-21}	5.49	0.863
1500	1.58×10^{-13}	1.89×10^{-13}	5.89	0.922
2000	1.95×10^{-9}	2.58×10^{-9}	6.05	0.950
2500	6.88×10^{-7}	8.10×10^{-7}	6.10	0.958
3000	3.68×10^{-5}	3.84×10^{-5}	6.10	0.955
3500	6.45×10^{-4}	6.15×10^{-4}	6.08	0.947
4000	5.50×10^{-3}	4.99×10^{-3}	6.04	0.940
4500	2.87×10^{-2}	2.57×10^{-2}	5.99	0.936
5000	1.04×10^{-1}	0.96×10^{-1}	5.95	0.944
5500	2.92×10^1	2.85×10^{-1}	5.90	0.975
6000	7.07×10^{-1}	7.08×10^{-1}	5.86	1.062
6500	1.53	1.54	5.80	1.234
7000	3.00	3.00	5.72	1.525

^a See Ref 3.

any case, when λ increases with temperature far more rapidly than a rule such as (11) would predict, with μ still behaving in accordance with (10), and if this is also true for λ/c_p , then the effective Prandtl number will become much smaller than the value of $\frac{3}{2}$ used in this paper. This situation may arise in arcs in extremely-high-pressure surroundings, where thick radiative heat transfer, which may be simulated by an increased conduction, is present. For such cases special low-Prandtl-number studies, which have already been carried out for Boussinesq-type free-convective cases,¹⁵ can become important.

Owing to the occurrence of the exponential function in Eq. (35), the structure of the boundary layer is that of a two-layered system, and particularly so when T_i has a large value. The value of θ need drop only slightly below the core value of unity for the exponential function to reach extremely small values. Since the exponential represents Joule heating, this effect is restricted to a very narrow layer surrounding the plane of symmetry. In the remaining, much wider part of the boundary layer, the Joule-heating term can be left out of consideration. It is possible to study this two-layer structure by means of asymptotic techniques. Work is in progress to apply the so-called method of matched asymptotic expansions to derive a suitable asymptotic representation valid for $T_i \rightarrow \infty$.

There are some similarities between the mathematics presented in this paper and that which has been developed during the past 20 years for the description of combustion processes. This is caused by the fact that in both fields rapidly varying functions, namely, those representing the exponentially varying heat production, determine the temperature distributions. Good accounts of the mathematics relevant for combustion processes and flames can be found in books by Buckmaster and Ludford¹⁶ and Kapila.¹⁷

We shall conclude by summing up some of the main results of this investigation. For a plane free-burning arc in which conduction and convection are the primary modes of thermal transport, and where the material properties behave according to rules given elsewhere in this paper, a boundary-layer analysis is relevant when the thickness parameter δ_r as given by (18) is much smaller than unity. In that case, according to (17), the y scale is much smaller than the x scale. A typical value of ν_r is 10^{-4} m²/s. With $g \sim 10$ m/s², $l \sim 10^{-2}$ m, and $\rho_r/\rho_\infty \sim 10^{-1}$, we find $\delta_r \sim 10^{-1}$. The upward velocity is proportional to the square root of the core temperature. It was found that the current needed to produce a given temperature rise above the ambient increases rapidly with temperature at first. Later on, it increases only moderately fast.

APPENDIX

In an asymptotic solution of the system (34)–(37) with $T_\infty = 0$, which is valid for $\eta \rightarrow \infty$, the exponential term appearing in (35) does not play a role. Since $\theta \rightarrow 0$ as $\eta \rightarrow \infty$, this term decreases faster than any positive power of θ . In analogy with similar free-convective systems,¹³ we expect that $f \rightarrow c_0$ when $\eta \rightarrow \infty$, where c_0 is some positive constant. Therefore, the tail of the function θ will be governed approximately by

$$(\theta^{-1/4}\theta')' + \frac{3}{2}\text{Pr } c_0 \theta' = 0. \quad (\text{A1})$$

The general solution which tends to zero when $\eta \rightarrow \infty$ is

$$\theta \sim \tilde{\theta} = \left[\frac{3}{16} \text{Pr } c_0 (\eta + \eta_0) \right]^{-4}, \quad (\text{A2})$$

which shows that the behavior of the tail of θ is algebraic. Furthermore, η_0 is a free parameter.

We can now write

$$f \sim c_0 + \tilde{f}, \quad (\text{A3})$$

where $\tilde{f} \rightarrow 0$ when $\eta \rightarrow \infty$. Substituting this in (34) with $T_\infty = 0$ and dropping products of $\tilde{f}$ or its derivatives, we find that $\tilde{f}$ is governed by

$$(\tilde{\theta}^{-1/4}\tilde{f}'')' + \frac{3}{2}c_0\tilde{f}'' + \tilde{\theta} = 0. \quad (\text{A4})$$

A solution of the inhomogeneous equation is

$$\tilde{f}_1 = \frac{c_1}{(\eta + \eta_0)^2}, \quad (\text{A5})$$

where

$$c_1 = - \left(\frac{16}{3} \frac{1}{c_0 \text{Pr}} \right)^4 \frac{8}{27c_0} \frac{1}{\frac{3}{2} - \text{Pr}}, \quad (\text{A6})$$

which provides a negative correction of $f = c_0$ as required, but only for $\text{Pr} < \frac{3}{2}$.

Apart from the solution given by (A5), the homogeneous equation provides an additional suitable solution:

$$\tilde{f}_2 = \tilde{c}_2 (\eta + \eta_0)^{1-4/\text{Pr}}, \quad (\text{A7})$$

where $\tilde{c}_2$ is a free parameter. Looking at the asymptotic orders of the two independent solutions (A5) and (A7), we see that (A7) is the smaller of the two, i.e., as long as $\text{Pr} < \frac{3}{2}$. We have seen already that this condition must be satisfied anyway. Let us assume, for a moment, that we disregard (A7). Then, f behaves as $c_0 + c_1 (\eta + \eta_0)^{-2}$ when $\eta \rightarrow \infty$, where both c_0 and η_0 are free parameters. We can now return to Eq. (35) to find a correction on (A2). This leads to $\theta \sim \tilde{\theta} + O\{(\eta + \eta_0)^{-6}\}$ when $\eta \rightarrow \infty$. Continuing this process we find that the asymptotic expansions for f and θ will include all subsequent negative, but even, powers of $\eta + \eta_0$. There are two degrees of freedom expressed by the constants c_0 and η_0 .

Let us now return to (A7). This term represents an additional degree of freedom. If $\frac{3}{2} < \text{Pr} < \frac{4}{3}$, it enters the f expansion between the terms of the order $(\eta + \eta_0)^{-2}$ and $(\eta + \eta_0)^{-4}$. The corresponding θ term enters the θ expansion between the terms which are of the order of $(\eta + \eta_0)^{-6}$ and $(\eta + \eta_0)^{-8}$. When $\frac{4}{3} < \text{Pr} < \frac{3}{2}$, the two terms enter one stage later, and so on. Once these terms have entered the expansion, product terms will arise, and the series become fairly complicated. When $\text{Pr} = 4/(2n + 1)$ with $n = 2, 3, 4$, etc., a further complication arises. For $\text{Pr} = \frac{4}{3}$, for instance, a term of the order of $(\eta + \eta_0)^{-4} \ln(\eta + \eta_0)$ will have to be included.

On the other hand, the additional term does not present any difficulty when $\text{Pr} = 2/(n + 1)$ with $n = 1, 2, 3$, etc., as it now gives rise to the inclusion of all negative odd powers of $\eta + \eta_0$ after a certain stage in the expansions. This simply means that after this particular stage in the expansions all negative integer powers have to be included, which is simple. It so happens that the important value $\text{Pr} = \frac{3}{2}$ belongs to this

selected category of Prandtl numbers. Henceforth, we shall carry out our analysis for this special value of Pr , which applies for ideal gases. The f expansion includes all negative integer powers of $\eta + \eta_0$ from $(\eta + \eta_0)^{-4}$ onwards. In the θ expansion the first such term is $(\eta + \eta_0)^{-6}$.

Proceeding in the manner described above, we can write down series expansions for f and θ that are valid for large enough values of η . This asymptotic solution has three degrees of freedom. Together with the free parameter ω appearing in Eq. (35), this provides sufficient freedom to satisfy the four conditions given by Eq. (36). We could do this by means of a shooting method in which the series provide the initial values and then integrate backwards down to $\eta = 0$. The only problem that now remains to be solved is to evaluate as many terms of the asymptotic series as are needed to calculate accurate initial values of f , θ , and their derivatives for each given set of c_0 , η_0 , and $\tilde{c}_2$. Obviously, this process should be computerized in such a way that numerical values of each of the terms are calculated recursively. In view of this, a term such as $\theta^{-1/4}$ presents a complication. Indeed, if θ is given as a series expansion, the evaluation of higher-order terms of the corresponding series expansion $\theta^{-1/4}$ becomes ever more cumbersome. Therefore, we introduce

$$K = \theta^{1/4}, \quad (\text{A8})$$

so that Eqs. (34) and (35) now read, i.e., for $T_\infty = 0$ and $Pr = \frac{2}{3}$,

$$Kf''' - K'f'' + K^2[\frac{3}{2}ff'' + \frac{1}{2}(f')^2 + K^4] = 0, \quad (\text{A9})$$

$$KK'' + 2(K')^2 + \frac{1}{2}K^2K' = 0, \quad (\text{A10})$$

where the exponential has been omitted for reasons that were explained before. Equations (A9) and (A10) now contain finite products of series expansions only, once the series expansions of f and K have been given. Redefining the notation of the coefficients, we write

$$f \sim \sum_{i=0}^{\infty} c_i (\eta + \eta_0)^{-i}, \quad (\text{A11})$$

$$K \sim (\eta + \eta_0)^{-1} \sum_{i=0}^{\infty} d_i (\eta + \eta_0)^{-i}, \quad (\text{A12})$$

where

$$c_1 = c_3 = d_1 = d_3 = 0. \quad (\text{A13})$$

The free parameters are c_0 , c_5 , and η_0 . Series expansions of

products of f and K with themselves or with their derivatives can easily be calculated and are readily programmable. For instance,

$$K^2 \sim (\eta + \eta_0)^{-2} \sum_{i=0}^{\infty} e_i (\eta + \eta_0)^{-i}, \quad (\text{A14})$$

where

$$e_i = \sum_{j=0}^i d_j d_{i-j}. \quad (\text{A15})$$

Then,

$$K^4 = (\eta + \eta_0)^{-4} \sum_{i=0}^{\infty} g_i (\eta + \eta_0)^{-i}, \quad (\text{A16})$$

with

$$g_i = \sum_{j=0}^i e_j e_{i-j}, \quad (\text{A17})$$

and so on. This process is easily implemented as a computer algorithm. We shall omit the details here. Once all the product series have been evaluated, which are only finite in number, Eqs. (A9) and (A10) each provide a single recurrence relation. If the coefficients $c_0, c_1, c_2, \dots, c_n$ are available, as are $d_0, d_1, d_2, \dots, d_{n-1}$, then d_n can be calculated from (A10). Subsequently, c_{n+1} can be found from (A9), and so on.

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