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Numerical simulation of water-hammer in tapered pipes using an implicit least-squares approach

Mohsen Lashkarbolok^{a,*}, Arris S. Tijsseling^b

^a Department of Civil Engineering, Faculty of Engineering, Golestan University, Aliabad Katoul, Iran

^b Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, 5600 MB, the Netherlands

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ABSTRACT

A technique using radial-basis functions and least-squares optimisation is applied for the numerical solution of one-dimensional equations governing transient pipe flow. The method can deal with geometrically non-uniform pipes by employing an arbitrary distribution of scattering nodes in the space domain. The formulation is implicit in time with a sparse and symmetric matrix equation to be solved at each time step. One tapered-pipe problem with available analytical solution is used for verification and one straight-pipe problem with available experimental data is used for validation. The effect of gradually clogged and swollen pipe sections on pressure wave propagation is investigated. The latter is of importance for transient-based fault-detection techniques.

1. Introduction

Any rapid change in the momentum of a fluid flow in a pipeline system produces a pressure wave that propagates through the pipes. This phenomenon is called water-hammer. One of the important parameters that affects the frequency and shape of the pressure wave is the variation of the cross section of the pipe. Diffuser junctions in pipelines, the presence of locally clogged or swollen sections, and blood flow in tapered vessels, are some examples of cross-sectional area variation. In these kinds of problems, the conventional governing equations of water-hammer, which can be found in Ref. [1], and its standard numerical solution, Method of Characteristics (MOC) [1], should be rewritten. Streeter et al. [2] were among the first researchers who dealt with wave propagation in tapering pipes. They derived the governing equations for pulsating pressure in tapered distensible vessels and provided a modified formulation of the MOC to solve them numerically. They evaluated their numerical simulation with experimental results for a tapering vessel. The comparison was not satisfactory and the authors presented a useful discussion about the sources of discrepancies between numerical and experimental results. Rouleau and Young [3,4] studied the distortion of short pulses in tapered tube pulse transformers analytically, using Laplace transforms, for both inviscid and viscous liquids. They concluded that tapering geometry along with viscous shear can severely distort pulses that propagate through a tapered transmission line. Tarantine and Rouleau [5] calculated the transient behavior of a tapered

tube using a step-line impedance technique. They continued their research by performing laboratory experiments to show the validity of their theoretical research. They showed that the pressure surge resulting from quick closing-reopening or opening-reclosing of a valve can be effectively reduced by a tapered section at the end of a long uniform line [6]. Yoshizawa and Ando [7] mentioned that the step-line impedance technique, that was used by Tarantine and Rouleau [5], is inefficient computationally. They presented an analytical solution based on the Laplace transform and produced solutions for two shapes of tapered pipes (linear and exponential). Washio et al. [8] carried out a frequency response test on a linearly tapered line filled with oil. They also compared the results of Zielke's method of characteristics [9] and conventional step-line approach with their experimental results and concluded that the step-line approach is not efficient for the solution of tapering pipes due to its complexity. Tanahashi et al. [10] studied travelling pressure pulses in tapered pneumatic pipelines with an end reservoir to investigate the effect of tapering on the shape and peak of the pressure history, both experimentally and analytically. They concluded that the characteristics of the pressure waves are largely influenced by the tapering geometry. Wu and Ferng [11] reviewed the fundamental equations for the study of water-hammer in tapered pipes and solved the governing equations numerically using the MOC method. They investigated numerically the effect of a mild expansion or contraction in the pipe on the pressure history at the valve. There have been more researchers who used modified versions of the MOC to solve

* Corresponding author.

E-mail addresses: m.lbolok@gu.ac.ir (M. Lashkarbolok), a.s.tijsseling@tue.nl (A.S. Tijsseling).

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Fig. 1. A sketch of a longitudinal section of a tapered pipe (not to scale).

the water-hammer equations in tapering pipes. They can be found for instance in Ref. [12–14]. The effect of tapering on a pressure wave travelling through a distensible tube is an important topic in hemodynamics. Evans studied the effects of tapering geometry on the pulsating flow in the artery, analytically [15]. Belardinelli and Cavalcanti [16] developed a nonlinear two-dimensional model of blood flow in tapered and elastic vessels. They concluded, numerically, that in the tapered tube the axial velocity profile becomes flatter, the flow decreases and consequently the flow resistance increases considerably. Stergiopoulos et al. [17] solved one-dimensional blood flow equations numerically and investigated the effect of arterial and aortic stenoses on the pulsating flow. They showed that the presence of severe arterial stenoses significantly affects the flow pulses. Numerical simulation in this field recently is going toward practical medical usage. For instance, Sazonov et al. [18] presented a method for non-invasively detecting the severity and location of aortic aneurysms using numerical simulation of pulsating flow passing through tapered vessels. Similarly, the effects of blockage in water pipelines on a travelling pressure wave studied by Duan et al. [19] and Che et al. [20]. They provided a promising tool for blockage detection in pipes.

Minimizing the sum of squared residuals using least-squares techniques for the solution of PDEs has been used broadly in the finite element method (for instances [21–23]). The same technique was applied to minimize the squared residuals which came from moving least-squares interpolation method [24,25] and radial point interpolation methods [26,27]. In the present paper, we used radial point interpolation and least-squares techniques for the numerical simulation of the water-hammer problem in pipes with variable cross-sectional area. In this method, the space domain is discretized uniformly, randomly or purposefully using scatter nodes. The point-wise character of this method enables it to deal with non-uniform parameters (speed of the sound in the fluid, diameter, modulus of elasticity and thickness of the pipe wall) without any extra complication in the solution procedure. Here, the governing equations of one-dimensional water-hammer are derived by averaging the continuity and momentum equations over the pipe's cross-section. Frequency-dependent laminar viscous losses are included via Schohl's method [28]. The implicit Euler method is used in the time stepping to give the method stability. We presented a vectorized formulation for the numerical method so that all of the solution procedure consists of some sparse matrix algebra which makes it efficient in computation. After verification of the scheme, analytically and experimentally, the effect of clogged and swollen sections in a pipeline is investigated numerically.

2. Governing equations

Consider slightly compressible fluid flow (here, in z -direction) in an axisymmetric and non-uniform horizontal pipe of circular cross-section (tapered pipe) with radius $R(z)$ as shown in Fig. (1). We assume that the variation of pipe radius is so mild, $\frac{dR(z)}{dz} \ll 1$, that we can neglect the radial component of the velocity and consider the flow as one-dimensional.

The continuity equation for the flow of a fluid in cylindrical coordinates (r, θ and z) is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho u_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z} = 0 \quad (1)$$

where, ρ , t , u_r , u_θ and u_z are density of the fluid, time, radial, angular and longitudinal component of the velocity, respectively, and r is the distance from the centerline of the pipe which is in the range of: $0 \leq r \leq R(z)$. For axisymmetric, one-directional and slightly compressible flow (with uniform density in the cross section and excluding thermodynamic effects), $u_\theta = 0$ and $u_r = 0$, the continuity equation simplifies to:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_z}{\partial z} + u_z \frac{\partial \rho}{\partial z} = 0 \quad (2)$$

where u_z is a function of r . The effective bulk modulus, K' , in a slightly compressible fluid in an elastic tube is defined by:

$$K' = \rho \frac{\partial P}{\partial \rho} \quad (3)$$

where P is the pressure of the fluid. The speed of sound in the fluid is:

$$c = \sqrt{\frac{K'}{\rho}} \quad (4)$$

Substituting Eqs. (3) and (4) into Eq. (2), ignoring the advection term, because $c \gg u_z$, and writing the pressure in terms of head, $h = \frac{P}{\rho g}$, the continuity equation turns into:

$$\frac{\partial h}{\partial t} + \frac{c^2}{g} \frac{\partial u_z}{\partial z} = 0 \quad (5)$$

The effective bulk modulus in a fluid-pipe system, which the pipe is restrained from axial movement,¹ is calculated as:

$$K' = \frac{K}{1 + 2(1 - \nu^2) \frac{K R}{e E}} \quad (6)$$

where K , e , E and ν are bulk compression modulus of the fluid, wall thickness, modulus of elasticity and Poisson's ratio of the pipe, respectively. Since R , e , E and ν may vary with z , K' and consequently c may be a function of z . In order to obtain a one-dimensional equation for the continuity, we need to integrate Eq. (5) over the cross section of the pipe:

$$\int_0^{R(z)} \frac{\partial h}{\partial t} dA + \int_0^{R(z)} \frac{c^2}{g} \frac{\partial u_z}{\partial z} dA = 0 \quad (7)$$

where A is the cross-sectional area which is a function of z , but not of t . The cross-sectional averages of the head, H , and discharge, Q , are:

$$H = \frac{1}{A} \int_0^{R(z)} h dA \quad (8)$$

and

$$Q = \int_0^{R(z)} u_z dA \quad (9)$$

Now, the integrated Eq. (5) is:

$$\frac{\partial H}{\partial t} A + \frac{c^2}{g} \int_0^{R(z)} \frac{\partial u_z}{\partial z} 2\pi r dr = 0 \quad (10)$$

Using the Leibniz rule, Eq. (10) turns into:

¹ The formula for speed of sound in a fluid flow with other kinds of supports is provided in [1: pp. 5–6].

Fig. 2. Spatial domain, supporting domain and nodes.

Fig. 3. The effect of the parameter q on RBF, Φ .

$$\frac{\partial H}{\partial t} A + \frac{c^2}{g} \left(\frac{\partial}{\partial z} \int_0^{R(z)} u_z 2\pi r dr - u_z|_{r=R(z)} 2\pi R(z) \frac{dR(z)}{dz} \right) = 0 \quad (11)$$

Knowing that u_z is equal to zero at the wall (i.e. $u_z|_{r=R(z)} = 0$) and using Eq. (9), Eq. (11) becomes:

$$\frac{\partial H}{\partial t} + \frac{c^2}{gA} \frac{\partial Q}{\partial z} = 0 \quad (12)$$

The momentum equation for a viscous fluid flow in cylindrical coordinates, using the one-directional motion assumption and neglecting again the advection term, $u_z \frac{\partial u_z}{\partial z}$, is written as:

$$\frac{\partial u_z}{\partial t} + g \frac{\partial h}{\partial z} + gh_f = 0 \quad (13)$$

where, h_f is the head-loss per unit length due to viscous forces. Now, we take the integral from Eq. (13) over the cross section of the pipe to achieve a one-dimensional momentum equation as follows:

$$\int_0^{R(z)} \frac{\partial u_z}{\partial t} dA + \int_0^{R(z)} g \frac{\partial h}{\partial z} dA + \int_0^{R(z)} gh_f dA = 0 \quad (14)$$

The cross-section average of the head loss per unit length is defined by:

$$H_f = \frac{1}{A} \int_0^{R(z)} h_f dA \quad (15)$$

Using Eqs. (8), (9), (14) and (15) and applying Leibniz rule, the one-dimensional momentum equation becomes:

$$\frac{\partial Q}{\partial t} + gA \frac{\partial H}{\partial z} + gAH_f = 0 \quad (16)$$

Zielke [9] presented a method for relating H_f to the mean velocity, $V = \frac{Q}{A}$, in laminar transient flow. Head loss calculation for transient turbulence flow, using a similar weighting function model, was presented by Vardy and Brown [29]. Zielke's method was not efficient computationally and many researchers have suggested alternatives to

modify the original method [30–33]. Here, the formulation by Schohl [28] is applied for the calculation of H_f . It is calculated numerically as:

$$H_f = \frac{32\nu}{gD^2} V + \frac{16\nu}{gD^2} (y_1 + y_2 + y_3 + y_4 + y_5) \quad (17)$$

with:

$$y_i(t + \Delta t) = y_i(t) e^{-W_i \left(\frac{4\nu}{D^2} \right) \Delta t} + M_i (V(t + \Delta t) - V(t)), \quad i = 1, 2, 3, 4, 5 \quad (18)$$

where Δt , ν and D are the time step size in the numerical calculation, kinematic viscosity of the fluid and the pipe diameter, respectively. The coefficients W_i and M_i determined by Schohl [28] are:

$$W_i = \{26.65, 100, 669.6, 6497, 57990\}$$

$$M_i = \{1.051, 2.358, 9.021, 29.47, 79.55\}$$

Starting with zero initial condition for $y_1, y_2, \dots, y_5$, H_f is calculated for the next time steps using Eqs. (17) and (18).

3. The least-squares formulation

Here, a matrix formulation is presented for the numerical solution of the governing equations. All matrices and vectors are shown by **bold** fonts and scalars have normal or *italic* font. Assume a one-dimensional spatial domain in the range of $[z_a, z_b]$ discretized by n arbitrarily distributed nodes (Fig. (2)). The value of an unknown function $u(z)$ at the i th node, with coordinate z_i , is denoted by u_i . The value of the unknown function at a point z_k , which is not a nodal point necessarily (call it a collocation point), is interpolated by:

$$u_k \cong \tilde{u}_k = \beta_0 + \beta_1 z_k + \beta_2 z_k^2 + \sum_{i=1}^M \alpha_i \Phi(r_{ik}) \quad (19)$$

where r_{ik} is the relative distance between points z_i and z_k :

$$r_{ik} = \frac{z_i - z_k}{l_k} \quad (20)$$

Here, it is assumed that only the M nearest nodes to z_k have an effect on $\tilde{u}_k$ and define its supporting domain. Their indices are I_k^i for $i = 1, 2, \dots, M$. The distance between the k th collocation point and the farthest node in its supporting domain is denoted by l_k , see Fig. (2).

The function Φ in Eq. (19) is a Radial Basis Function (RBF). There are different RBFs used for curve fitting [34], p. 73]. Here, we choose the function Φ as:

$$\Phi(d) = \frac{1}{(1 + d^2)^q} \quad \text{with } q > 0 \text{ and } 0 \leq d \leq 1 \quad (21)$$

where q is the shape parameter, the effect of which is shown in Fig. (3). A useful discussion on the range of the shape parameters is provided in [34, p. 74]. According to our numerical experiments, results did not show sensitivity to the value of q in the range of 5–15. We choose $q = 5$ for all simulations in this paper.

The polynomials in the interpolation function, Eq. (19), ensure the reproduction of the parabolic field (C^2 consistency). It is reported, as we concluded so, that adding polynomials can also improve the accuracy of the results and reduce the sensitivity of the shape parameters in RBFs on the solution accuracy [34, p. 74]. The coefficients $\beta_0, \beta_1, \beta_2$ and $\alpha_1, \alpha_2, \dots, \alpha_M$ (for the k th collocation) are obtained by solving the following set of equations [34, pp. 75–78]:

$$\beta_0 + \beta_1 z_k + \beta_2 z_k^2 + \sum_{i=1}^M \alpha_i \Phi(r_{ik}) = u_{I_k^i}, \quad i = 1, 2, \dots, M \quad (22)$$

$$\sum_{i=1}^M \alpha_i = 0, \quad \sum_{i=1}^M \alpha_i z_i^k = 0, \quad \sum_{i=1}^M \alpha_i z_i^2 = 0 \quad (23)$$

After substituting the coefficients into Eq. (19), the following relationship is derived:

$$\tilde{u}_k = \sum_{i=1}^M N_i^k(z_k) u_i^k \quad (24)$$

where $N_j(z_k)$ is the value of j th shape function at the k th collocation point. There are nc collocation points and n nodal points (they may be equal and they are taken equal herein). We define the $\tilde{\mathbf{U}}$ and $\mathbf{U}$ vectors as:

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{nc} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad (25)$$

where $\tilde{\mathbf{U}}$ and $\mathbf{U}$ are vectors containing the approximated values of the unknown function, u , at the collocation points and the values of the unknown function at the nodal points, respectively. Equation (24) is used to make a relation between $\tilde{\mathbf{U}}$ and $\mathbf{U}$ as:

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{nc} \end{bmatrix} = \begin{bmatrix} N_1(z_1) & N_2(z_1) & \cdots & N_n(z_1) \\ N_1(z_2) & N_2(z_2) & \cdots & N_n(z_2) \\ \vdots & \vdots & \cdots & \vdots \\ N_1(z_{nc}) & N_2(z_{nc}) & \cdots & N_n(z_{nc}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad (26)$$

Equation (26) is written in compact form as:

$$\tilde{\mathbf{U}} = [\mathbf{N}] \mathbf{U} \quad (27)$$

The $nc \times n$ matrix $[\mathbf{N}]$ is a highly sparse matrix since there are few nodes in the supporting domain of collocation points. For instance, if we have 10,000 nodes and collocation points ($nc = n = 10,000$), the matrix $[\mathbf{N}]$ has 10^8 elements. With 7 nodes in the supporting domain of each collocation point ($M = 7$), see Fig. (2), each row of $[\mathbf{N}]$ contains only 7 non-zero members. Hence, the matrix $[\mathbf{N}]$ will have 70,000 non-zero elements (that is 0.07% of the matrix). Working with sparse matrices is highly desirable in numerical simulations. The element-wise first derivative of the shape function matrix with respect to z ($\frac{d\mathbf{N}}{dz}$) is obtained via Eq. (19).

Now, we are ready to discretize the governing equations Eq. (12) and Eq. (16) with considering the required boundary conditions. On the boundaries we have:

$$H = H_b; \dots \text{in points where the head is prescribed (here, at reservoir)} \dots \quad (28)$$

$$Q = Q_b; \dots \text{in points where the discharge is prescribed (here, at valve)} \dots \quad (29)$$

where, H_b and Q_b are prescribed values of the pressure-head and discharge at the boundaries. In the proposed method, the governing equations, are solved simultaneously or in a coupled form. A close look at Eq. (12) reveals that the second coefficient ($\frac{c_s^2}{g}$) is much larger than one,² and this affects the condition number of the final coefficient matrix. Here, we fix this problem by making the governing equations dimensionless so that all terms become of the order of one. The dimensionless pressure head, H^* , discharge Q^* , time, t^* , density, ρ^* , coordinate, z^* , pressure-head loss per unit length, H_f , sound speed, c^* , cross-sectional area, A^* and gravity acceleration, g^* , are defined by:

where, A_0 , Q_0 , c_0 , L_0 , ρ_0 and g_0 are reference cross-section area, discharge, sound speed, length, density of fluid and gravity acceleration,

respectively. By substituting the dimensionless variables into the governing equations and boundary conditions, Eqs. ((12), (16), (28) and (29), the dimensionless form of the governing equations and boundary conditions becomes:

$$\frac{\partial H^*}{\partial t^*} + \frac{c_s^2}{g^* A^*} \frac{\partial Q^*}{\partial z^*} = 0 \quad (30)$$

$$\frac{\partial Q^*}{\partial t^*} + g^* A^* \frac{\partial H^*}{\partial z^*} + g^* A^* H_f = 0 \quad (31)$$

$$H^* = H_b^*; \text{ at points where the head is prescribed} \quad (32)$$

$$Q^* = Q_b^*; \text{ at points where the discharge is prescribed} \quad (33)$$

where, $H_b^* = H_b \frac{g_0 A_0}{Q_0 c_0}$, $Q_b^* = \frac{Q_b}{Q_0}$. Now, the equations are more balanced from the numerical point of view since all of the coefficients, variables and their derivatives are near to one. Here, the values of the unknowns at the next time step ($m+1$) are calculated from the previous time step (m) by solving a $2n \times 2n$ system of equations. In the temporal discretization, the first-order implicit Euler method is used:

$$\frac{H_s^{m+1} - H_s^m}{\Delta t^*} + \frac{c_s^2}{g^* A^*} \frac{\partial Q_s^{m+1}}{\partial z^*} = 0 \quad (34)$$

$$\frac{Q_s^{m+1} - Q_s^m}{\Delta t^*} + g^* A^* \frac{\partial H_s^{m+1}}{\partial z^*} + g^* A^* H_{fs}^m = 0 \quad (35)$$

$$H_s^{m+1} = H_b^*; \text{ at points where the head is prescribed} \quad (36)$$

$$Q_s^{m+1} = Q_b^*; \text{ at points where the discharge is prescribed} \quad (37)$$

Δt^* is the non-dimensional time step size which is calculated from the dimensional time step size, Δt , by: $\Delta t^* = \Delta t \frac{c_0}{L_0}$. The initial values of H^* and Q^* at time zero are given. H_s^{m+1} and Q_s^{m+1} are unknown functions of t^* and z^* . In the space domain discretization, we use n number of nodes and nc number of arbitrarily distributed collocation points. The values of the unknowns at the collocation points are found by:

$$\tilde{\mathbf{H}}^* = [\mathbf{N}] \mathbf{H}^* \quad (38)$$

and

$$\tilde{\mathbf{Q}}^* = [\mathbf{N}] \mathbf{Q}^* \quad (39)$$

where $\tilde{\mathbf{H}}^*$, $\tilde{\mathbf{Q}}^*$, $\mathbf{H}^*$ and $\mathbf{Q}^*$ are vectors that contain the values of H^* and Q^* at all collocation and nodal points, respectively (take a look at Eq. (27)). Since $\tilde{\mathbf{H}}^*$ and $\tilde{\mathbf{Q}}^*$ are approximated values, substituting Eqs. (38) and (39) into Eqs. 34–37, residuals ($\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{R}_4$) remain. They are calculated by:

$$[\mathbf{N}] \mathbf{H}_s^{m+1} - \tilde{\mathbf{H}}_s^m + \frac{\Delta t^*}{g^*} \mathbf{c}_s^2 \bullet \left(\frac{d\mathbf{N}}{dz^*} \mathbf{Q}_s^{m+1} \right) = \mathbf{R}_1 \quad (40)$$

$$[\mathbf{N}] \mathbf{Q}_s^{m+1} - \tilde{\mathbf{Q}}_s^m + \Delta t^* g^* \mathbf{A}^* \bullet \left(\frac{d\mathbf{N}}{dz^*} \mathbf{H}_s^{m+1} \right) + \Delta t^* g^* \mathbf{A}^* \bullet \tilde{\mathbf{H}}_{fs}^m = \mathbf{R}_2 \quad (41)$$

$$[\mathbf{N}_{bQ}] \mathbf{Q}_s^{m+1} - \mathbf{Q}_{bs}^* = \mathbf{R}_3 \quad (42)$$

$$[\mathbf{N}_{bH}] \mathbf{H}_s^{m+1} - \mathbf{H}_{bs}^* = \mathbf{R}_4 \quad (43)$$

where $\mathbf{A}^*$, $\mathbf{H}_{fs}$ and $\mathbf{c}_s$ are vectors which store the values of the A^* , H_{fs} and c_s at each collocation point. The matrices $[\mathbf{N}_b]$ and $[\mathbf{N}_{bH}]$ are shape function matrices for the boundary nodes. To build them, all elements of the matrix $[\mathbf{N}]$, except those rows associated with boundaries, are set to zero. The vectors $\mathbf{Q}_{bs}^*$ and $\mathbf{H}_{bs}^*$ are vectors with nc rows which only have non-zero values at boundary points. Element by element multiplication of two vectors is denoted by the $\bullet$ symbol. The function J , which is the

² It is of the order of 10^5 m in water-filled steel pipes.

sum of the squared residuals, is written as:

$$J = \frac{1}{2} (\mathbf{R}_1 \mathbf{R}_1 + \mathbf{R}_2 \mathbf{R}_2 + \alpha \mathbf{R}_3 \mathbf{R}_3 + \alpha \mathbf{R}_4 \mathbf{R}_4) \quad (44)$$

In this paper, the residuals at the boundaries are scaled by a very big number (α) to enforce the boundary conditions. This is the same as the well-known penalty method in the finite element method [35], pp: 104–107]. Substituting Eqs. 40–43 into Eq. (44) yields:

Using Eqs. (48) and (49), the coefficient matrix ($\mathbf{K}$) and right-hand side vector ($\mathbf{F}$) are written as:

$$\mathbf{K}_{11} = \mathbf{N}^T \mathbf{N} + \left(\Delta t_* g_* \mathbf{A}_* \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \left(\Delta t_* g_* \mathbf{A}_* \otimes \frac{d\mathbf{N}}{dz_*} \right) + \alpha \mathbf{N}_{bH}^T \mathbf{N}_{bH} \quad (51)$$

$$\begin{aligned} J = \frac{1}{2} & \left[\mathbf{N} \mathbf{H}_*^{m+1} - \tilde{\mathbf{H}}_*^m + \frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{Q}_*^{m+1} \right) \right]^T \times \left[\mathbf{N} \mathbf{H}_*^{m+1} - \tilde{\mathbf{H}}_*^m + \frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{Q}_*^{m+1} \right) \right] \\ & + \frac{1}{2} \left[\mathbf{N} \mathbf{Q}_*^{m+1} - \tilde{\mathbf{Q}}_*^m + \Delta t_* g_* \mathbf{A}_* \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{H}_*^{m+1} \right) \right. \\ & \left. + \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m \right]^T \times \left[\mathbf{N} \mathbf{Q}_*^{m+1} - \tilde{\mathbf{Q}}_*^m + \Delta t_* g_* \mathbf{A}_* \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{H}_*^{m+1} \right) + \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m \right] \\ & + \frac{1}{2} \alpha (\mathbf{N}_{bQ} \mathbf{Q}_*^{m+1} - \mathbf{Q}_{b*})^T \left(\mathbf{N}_{bQ} \mathbf{Q}_*^{m+1} - \mathbf{Q}_{b*} \right) + \frac{1}{2} \alpha (\mathbf{N}_{bH} \mathbf{H}_*^{m+1} - \mathbf{H}_{b*})^T \left(\mathbf{N}_{bH} \mathbf{H}_*^{m+1} - \mathbf{H}_{b*} \right) \end{aligned} \quad (45)$$

Those values of unknowns ($\mathbf{H}_*^{m+1}$ and $\mathbf{Q}_*$), which minimize the function J are the desired solutions. They are found by:

$$\frac{\partial J}{\partial \mathbf{H}_*^{m+1}} = 0 \quad (46)$$

and

$$\frac{\partial J}{\partial \mathbf{Q}_*^{m+1}} = 0 \quad (47)$$

Equations arising from conditions (46) and (47) are:

$$\mathbf{K}_{12} = \mathbf{N}^T \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right) + \left(\Delta t_* g_* \mathbf{A}_* \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \mathbf{N} \quad (52)$$

$$\mathbf{K}_{21} = \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \mathbf{N} + \mathbf{N}^T \left(\Delta t_* g_* \mathbf{A}_* \bullet \left(\frac{d\mathbf{N}}{dz_*} \right) \right) \quad (53)$$

$$\mathbf{K}_{22} = \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right) + \mathbf{N}^T \mathbf{N} + \alpha \mathbf{N}_{bQ}^T \mathbf{N}_{bQ} \quad (54)$$

$$\begin{aligned} \mathbf{N}^T & \left[\mathbf{N} \mathbf{H}_*^{m+1} - \tilde{\mathbf{H}}_*^m + \frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{Q}_*^{m+1} \right) \right] + \left(\Delta t_* g_* \mathbf{A}_* \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \times \left[\mathbf{N} \mathbf{Q}_*^{m+1} - \tilde{\mathbf{Q}}_*^m + \Delta t_* g_* \mathbf{A}_* \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{H}_*^{m+1} \right) + \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m \right] \\ & + \alpha \mathbf{N}_{bH}^T (\mathbf{N}_{bH} \mathbf{H}_*^{m+1} - \mathbf{H}_{b*}) = 0 \end{aligned} \quad (48)$$

$$\begin{aligned} & \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \times \left[\mathbf{N} \mathbf{H}_*^{m+1} - \tilde{\mathbf{H}}_*^m + \frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{Q}_*^{m+1} \right) \right] \\ & + \mathbf{N}^T \left[\mathbf{N} \mathbf{Q}_*^{m+1} - \tilde{\mathbf{Q}}_*^m + \Delta t_* g_* \mathbf{A}_* \bullet \left(\frac{d\mathbf{N}}{dz_*} \mathbf{H}_*^{m+1} \right) + \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m \right] \\ & + \alpha \mathbf{N}_{bQ}^T (\mathbf{N}_{bQ} \mathbf{Q}_*^{m+1} - \mathbf{Q}_{b*}) = 0 \end{aligned} \quad (49)$$

The multiplication of a vector into a matrix is denoted by the $\otimes$ symbol which, here, is defined as:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \quad (55)$$

$$\mathbf{F}_1 = \mathbf{N}^T \tilde{\mathbf{H}}_*^m + \left(\Delta t_* g_* \mathbf{A}_* \otimes \frac{d\mathbf{N}}{dz_*} \right)^T (\tilde{\mathbf{Q}}_*^m - \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m) + \alpha \mathbf{N}_{bH}^T \mathbf{H}_{b*} \quad (56)$$

$$\mathbf{F}_2 = \left(\frac{\Delta t_*}{g_*} \mathbf{c}_*^2 \bullet \frac{1}{\mathbf{A}_*} \otimes \frac{d\mathbf{N}}{dz_*} \right)^T \tilde{\mathbf{H}}_*^m + \mathbf{N}^T (\tilde{\mathbf{Q}}_*^m - \Delta t_* g_* \mathbf{A}_* \bullet \tilde{\mathbf{H}}_{f*}^m) + \alpha \mathbf{N}_{bQ}^T \mathbf{Q}_{b*} \quad (57)$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{nc} \end{bmatrix} \otimes \begin{bmatrix} N_1(z_1) & N_2(z_1) & \cdots & N_n(z_1) \\ N_1(z_2) & N_2(z_2) & \cdots & N_n(z_2) \\ \vdots & \vdots & \cdots & \vdots \\ N_1(z_{nc}) & N_2(z_{nc}) & \cdots & N_n(z_{nc}) \end{bmatrix} = \begin{bmatrix} a_1 \cdot N_1(z_1) & a_1 \cdot N_2(z_1) & \cdots & a_1 \cdot N_n(z_1) \\ a_2 \cdot N_1(z_2) & a_2 \cdot N_2(z_2) & \cdots & a_2 \cdot N_n(z_2) \\ \vdots & \vdots & \cdots & \vdots \\ a_{nc} \cdot N_1(z_{nc}) & a_{nc} \cdot N_2(z_{nc}) & \cdots & a_{nc} \cdot N_n(z_{nc}) \end{bmatrix} \quad (50)$$

Fig. 4. Flow chart of the solution procedure.

Fig. 5. Sketch of reservoir-pipe-valve system [7].

Table 1
Pipe, fluid and tank properties.

L and L_0	30 m
g and g_0	9.81 m/s ²
c_0	1400 m/s
H_b	10 m
ν	0 m ² /s
A_0	3.14×10^{-4} m ²
ρ_0	1000 kg/m ³

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} \quad (58)$$

The values of the unknowns $\mathbf{Q}_*^{m+1}$ and $\mathbf{H}_*^{m+1}$ are finally calculated by solving the following sparse and symmetric linear system of equations:

$$\mathbf{K} \begin{bmatrix} \mathbf{H}_*^{m+1} \\ \mathbf{Q}_*^{m+1} \end{bmatrix} = \mathbf{F} \quad (59)$$

A flow chart of the solution procedure of the proposed scheme is shown in Fig. (4).

4. Test problems

The first test problem is a tapered tube for which an analytical solution is available [7]. The second problem is a well-known benchmark for frequency-dependent friction studies for which, herein, the effects of local tapering (through clogged and swollen pipe sections) are studied. In the numerical scheme two parameters (M and α) must be determined in advance. Larger numbers for M lead to better results but increase the computational cost. According to numerical experiments, using values larger than 7 for M will not improve the results significantly; therefore M is taken equal to 7. The value of α must be big enough to accurately enforce the boundary conditions. Here, for all problems, the boundary conditions are satisfied when $\alpha = 10^6$. Larger values of α will produce the same results. But much larger numbers (say 10^{20}) “deteriorate” the condition number of the coefficient matrix.

4.1. Tapered pipe

A large tank supplies $Q_0 = 0.0005$ m³/s of water via a tapered steel pipe with a valve at its end (as shown in Fig. (5)), which is closed instantaneously. The properties of the pipe, the fluid and the tank are given in Table 1. The diameter of the pipeline increases linearly from 0.02 m at the tank to 0.05 m at the valve. The kinematic viscosity of the fluid, ν , does not take part in this test problem. By assuming a constant

Fig. 6. Numerical and analytical [7] results for tapered pipe; dimensionless head at valve versus dimensionless time.

Fig. 7. Sketch of the experimental setup of Holmboe and Rouleau [36]; the concrete bedding is to prevent fluid-structure interaction (FSI).

Table 2
Pipe, fluid and tank properties.^a

H_b	0.927 m
L_0	36.1 m
Q_0	$6.49 \times 10^{-5} \text{ m}^3/\text{s}$
g and g_0	9.81 m/s^2
ρ_0	876 kg/m^3
ν	$3.97 \times 10^{-5} \text{ m}^2/\text{s}$
D_0	25.4 mm
c_0	1324 m/s
ν	0.2*
E	1* mm
K	2.39* GPa
E	107* GPa

^a The values of ν , e , K and E is chosen here so that the experimentally obtained c_0 is reproduced according to Eq. (6).

sound speed through the pipe, we can use the available analytical results in Ref. [7] to verify our presented scheme. So, here, $c_* = 1$.

A uniform distribution of nodes is used with $\Delta z_* = 0.001$. The problem is solved using different values of the Courant number, $C = c_* \frac{\Delta t_*}{\Delta z_*}$. The results are shown in Fig. (6). It shows that the method is stable even at high Courant numbers, although smaller time step sizes give better results. Figure (6) shows excellent agreement between the analytical and numerical results for $C = 0.1$.

4.2. Straight pipe with and without non-uniform section

The effect of viscous shear on transient flows in pipes has been investigated experimentally by Holmboe and Rouleau [36]. A sketch of their experimental setup is shown in Fig. (7). The pipe and fluid properties are given in Table 2. The problem is solved for three cases:

Case 1. The original problem as defined in [36]

Case 2. Considering a clogged part in the middle of the pipe

Case 3. Considering a swollen part in the middle of the pipe

4.2.1. Case 1. Straight pipe

Wave propagation due to sudden valve closure in a straight pipe is

Fig. 8. Dimensionless head history at the valve for the straight pipe.

Fig. 9. Geometry and location of clogged section.

investigated numerically. The speed of the sound was measured experimentally, $c_0 = 1324$ m/s. The reservoir head, $H_b = 0.927$, is the upstream and zero discharge is the downstream boundary condition. The space domain is discretized uniformly using $\Delta z^* = 0.001$. Numerical results for the dimensionless pressure head at different Courant numbers ($C = 10, 1, 0.1$) along with experimental results from Ref. [36] which was provided more clearly in Ref. [9] are shown in Fig. (8). The implicit formulation makes the method stable even at high Courant numbers, although smaller ones give better results. The differences between the

numerical and experimental results are due to the fact that the applied frequency-dependent friction model is not perfect.

4.2.2. Case 2. Straight pipe with clogged section

Assume there is a small clogged part in the middle of the pipe. The geometry of this case is shown in Fig. (9). Here, area and speed of sound become a function of z . In the original experiment by Holmboe and Rouleau [36], speed of sound was determined experimentally. But here, we need the properties of the fluid and the pipe in order to calculate it

Fig. 10. The variation of the speed of sound in the clogged and swollen sections.

Fig. 11. Geometry and location of swollen section.

according to Eq. (6), thereby maintaining the measured value in the uniform sections. The chosen values of K , E , ν and e are indicated by asterisks in Table 2. The variation in speed of sound is shown in Fig. (10). The chosen space and time discretization are $\Delta z = 10^{-4}$ and $\Delta t = 10^{-4}$. Results for the first period of the head oscillation is shown in Fig. (12). The effect of the small clogged part is significant and clearly visible in the pressure-head history.

4.2.3. Case 3. Straight pipe with swollen section

Assume a swollen part as shown in Fig. (11). The required input data are provided in Table 2. The wave speed variation in the pipe is shown in Fig. (10). The pressure-head history at the valve after sudden closure of the valve is shown in Fig. (12). The swollen part has an opposite effect on the head history by introducing reverse bumps.

5. Conclusion

Water-hammer is simulated numerically using an implicit and coupled least-squares technique, which is formulated in a fully vectorized approach. The method produces sparse and symmetric matrices as a result of applying a compact-support technique for the radial basis functions and using the least-squares technique. In the proposed formulation, dealing with tapered geometry is the same as dealing with uniform geometry and there is no need for any extra considerations in the solution procedure. Blockage and leakage detection is one of the possible applications of the method. Here, it is applied to investigate the effect of localized clogged and swollen parts on pressure wave propagation due to water-hammer. It is shown that a clogged (swollen) part in a pipe produce bumps (reverse bumps) in the pressure history. The same

Fig. 12. Effect of clogged and swollen section on the pressure-head history at the valve.

approach can be used, after some modification in the governing equations, to simulate pulsating flow through tapered vessels including stenoses and aneurysms (same as clogged and swollen parts in this paper). The flexibility in the node distribution, gives the method a good potential for dealing with 2D and 3D problems where geometry dictates special considerations for space domain discretization.

CRediT authorship contribution statement

Mohsen Lashkarbolok: Methodology, Software, Writing - original draft. **Arris S. Tijsseling:** Conceptualization, Resources, Writing - review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

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List of symbols

A : cross-sectional of the pipe, function of z
 A_0 : reference cross-sectional area
 A^* : dimensionless cross-sectional area
 A^* : vectors of A^* at all collocations
 c : speed of sound in the fluid, function of z
 c_0 : reference sound speed
 c^* : dimensionless speed of sound in the fluid
 c^* : vectors of c^* at all collocations
 C : Courant number
 D : diameter of the pipe
 e : wall thickness of the pipe
 E : modulus of elasticity of the pipe
 F : right-hand side vector
 g : gravity acceleration
 g^* : dimensionless gravity acceleration
 g_0 : reference gravity acceleration
 h : pressure head of the fluid
 h_f : head-loss per unit length
 H : cross-sectional average of pressure head
 H_f : cross-sectional average of the head loss per unit length

H_b : value of the pressure head at related boundaries
 H^* : dimensionless pressure head
 H_b^* : dimensionless H_b
 H_f : dimensionless H_f
 $\bar{H}^*$: vectors of H^* at all collocations
 H^* : vectors of H^* at all nodes
 $\bar{H}_f$: vectors of H_f at all collocations
 H_b^* : vectorized H_b^*
 J : residual value
 K' : effective compression modulus of the fluid
 K : bulk compression modulus of the fluid
 $\mathbf{K}$: coefficient matrix
 L_0 : reference length
 M : number of nodes that support each collocation point
 n : number of nodes used in domain discretization
 nc : number of collocations used in domain discretization
 $N_j(z_k)$: value of j th shape function at the k th collocation point
 $\mathbf{N}$: shape function matrix
 $\mathbf{N}_b$: shape function matrix at discharge boundary nodes
 $\mathbf{N}_{HF}$: shape function matrix at pressure boundary nodes
 P : fluid pressure
 q : shape parameter of the radial basis function
 Q : discharge
 Q^* : dimensionless discharge
 Q_b : value of the discharge at related boundaries
 Q_b^* : dimensionless Q_b
 Q_0 : reference discharge
 Q^* : vectors of Q^* at all collocations
 Q^* : vectors of Q^* at all nodes
 Q_b^* : vectorized Q_b^*
 r : coordinate in the radial direction
 R : pipe radius, function of z
 t : time
 t^* : dimensionless time
 u_r : radial component of velocity
 u_θ : angular component of velocity
 u_z : longitudinal component of velocity
 V : cross-sectional mean velocity
 z : coordinate in the longitudinal direction
 z^* : dimensionless longitudinal coordinate
 α : penalty coefficient
 Δt : time step size
 Δt^* : dimensionless time step size
 θ : coordinate in the angular direction
 ν : Poisson’s ratio of the pipe
 ρ : density of the fluid
 ρ_0 : reference density of fluid
 ρ^* : dimensionless density
 ν : kinematic viscosity of the fluid
 Φ : radial basis function