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ANALYTICITY SPACES OF SELF-ADJOINT OPERATORS SUBJECTED TO PERTURBATIONS WITH APPLICATIONS TO HANKEL INVARIANT DISTRIBUTION SPACES*

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Abstract. A new theory of generalized functions has been developed by one of the authors (de Graaf). In this theory the analyticity domain of each positive self-adjoint unbounded operator $\mathcal{A}$ in a Hilbert space X is regarded as a test space denoted by $\mathcal{S}_{X,\mathcal{A}}$. In the first part of this paper, we consider perturbations $\mathcal{P}$ on $\mathcal{A}$ for which there exists a Hilbert space Y such that $\mathcal{A} + \mathcal{P}$ is a positive self-adjoint operator in Y . In particular, we investigate for which perturbations $\mathcal{P}$ and for which $\nu > 0$, $\mathcal{S}_{X,\mathcal{A}^\nu} \subset \mathcal{S}_{Y,(\mathcal{A}+\mathcal{P})^\nu}$. The second part is devoted to applications. We construct Hankel invariant distribution spaces. The corresponding test spaces are described in terms of the $\mathcal{S}_\alpha^\beta$ -spaces introduced by Gel'fand and Shilov. It turns out that the modified Laguerre polynomials establish an uncountable number of bases for the space of even entire functions in $\mathcal{S}_\mu^\mu$ ($\frac{1}{2} \leq \mu \leq 1$). For an even entire function φ we give necessary and sufficient conditions on the coefficients in the Fourier expansion with respect to each basis such that $\varphi \in \mathcal{S}_\mu^\mu$.

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Introduction. Let X be a separable infinitely dimensional Hilbert space and let $\mathcal{L}$ be a linear operator in X . Then $\mathfrak{D}^\omega(\mathcal{L})$, the analyticity domain of $\mathcal{L}$, consists of all vectors $v \in \bigcap_{n=1}^\infty \mathfrak{D}(\mathcal{L}^n)$ satisfying

$$\exists_{a>0} \exists_{b>0} \forall_{n \in \mathbb{N}} : \|\mathcal{L}^n v\| \leq n! a^n b.$$

For a positive self-adjoint operator $\mathcal{A}$ in X , Nelson [13] proved that $\mathfrak{D}^\omega(\mathcal{A})$ can also be described as

$$\mathfrak{D}^\omega(\mathcal{A}) = \bigcup_{t>0} e^{-t\mathcal{A}}(X) = \{e^{-t\mathcal{A}}w \mid w \in X, t>0\}.$$

Instead of $\mathfrak{D}^\omega(\mathcal{A})$ we use the notation $\mathcal{S}_{X,\mathcal{A}}$ introduced by de Graaf. The spaces of type $\mathcal{S}_{X,\mathcal{A}}$ are called analyticity spaces. They are nonstrict inductive limits of Hilbert spaces. Together with their strong duals $\mathcal{T}_{X,\mathcal{A}}$ they establish the functional analytic description of the distribution theory in [7].

For each positive constant ν the operator $\mathcal{A}^\nu$ is well defined, positive and self-adjoint in X . So it makes sense to write $\mathcal{S}_{X,\mathcal{A}^\nu}$. The question arises for which perturbations $\mathcal{P}$ on $\mathcal{A}$ there can be found a Hilbert space Y such that $\mathcal{A} + \mathcal{P}$ is a positive self-adjoint operator in Y and $\mathcal{S}_{X,\mathcal{A}^\nu} \subset \mathcal{S}_{Y,(\mathcal{A}+\mathcal{P})^\nu}$. In the paper [1] the case $\nu=1$ has been considered. Also some results concerning analytic dominancy can be found there.

In the second part of this paper we study a class of Hankel invariant test and distribution spaces, and also their relations to the $\mathcal{S}_\alpha^\beta$ -spaces of Gel'fand and Shilov [9]. With our papers [2] and [4] we have started this study. There we have shown that the space of even functions in $\mathcal{S}_{1/2}^{1/2}$ remains invariant under the modified Hankel transforms $\mathbf{H}_\alpha$, $\alpha > -1$, defined by

$$(\mathbf{H}_\alpha f)(x) = \int_0^\infty (xy)^{-\alpha} J_\alpha(xy) f(y) y^{2\alpha+1} dy.$$

Moreover, for each $\alpha > -1$ the space of even functions in $\mathcal{S}_{1/2}^{1/2}$ equals the analyticity space $\mathcal{S}_{X_\alpha, \mathcal{A}_\alpha}$ where $X_\alpha = \mathfrak{L}_2((0, \infty), x^{2\alpha+1} dx)$ and $\mathcal{A}_\alpha = -d^2/dx^2 + x^2 - (2\alpha+1)xd/dx$.

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The operator $\mathcal{A}_\alpha$ has an orthonormal basis of eigenvectors $(L_n^{(\alpha)})_{n=0}^\infty$ with eigenvalues $4n + 2\alpha + 2$. So for each even $f \in \mathcal{S}_{1/2}^{1/2}$ there exists an l_2 -sequence $(\omega_n)_{n=0}^\infty$ and $t > 0$ such that $f = \sum_{n=0}^\infty \exp(-(4n + 2\alpha + 2)t) \omega_n L_n^{(\alpha)}$. Here we prove similar results for the spaces $\mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu}$ with $\nu \geq \frac{1}{2}$ and $\alpha > -1$. It will follow that for all $\alpha, \beta > -1$ and all $\nu \geq \frac{1}{2}$

$$\mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu} = \mathcal{S}_{X_\beta, (\mathcal{A}_\beta)^\nu}.$$

For $\nu \in [\frac{1}{2}, 1]$ the analyticity space $\mathcal{S}_{X_{-1/2}, (\mathcal{A}_{-1/2})^\nu}$ contains just the even functions in $\mathcal{S}_{1/2}^{1/2\nu}$.

1. General theory. Let $\mathcal{A}$ be a positive self-adjoint operator in a Hilbert space X and let $\nu > 0$. It makes sense to write $\mathcal{A}^\nu$ and the operator $\mathcal{A}^\nu$ is positive and self-adjoint in X . So the space $\mathcal{S}_{X, \mathcal{A}^\nu}$ is well defined. Its elements are characterized by

LEMMA 1.1. *For each $f \in \mathcal{D}(\mathcal{A}^\infty) \subset X$ the following statements are equivalent:*

- (i) $\exists_{a>0} \exists_{b>0} \forall_{k \in \mathbb{N}} : \|\mathcal{A}^k f\| \leq (k!)^{1/\nu} a^k b$.
- (ii) $f \in \mathcal{S}_{X, \mathcal{A}^\nu}$.

Proof. (i) $\Rightarrow$ (ii). Let $N \in \mathbb{N}$ and let $\tau > 0$. Consider the following estimation

$$(*) \quad \sum_{k=0}^N \frac{\tau^k}{k!} \|\mathcal{A}^k f\| \leq \sum_{k=0}^N \frac{\tau^k}{k!} \|\mathcal{A}^{-1+\nu k - [\nu k]}\| \|\mathcal{A}^{[\nu k]+1} f\|$$

$$\leq b_1 \sum_{k=0}^N \frac{\tau^k}{k!} (([\nu k] + 1)!)^{\nu k} a^k$$

where $b_1 = b \sup_{k \in \mathbb{N} \cup \{0\}} (\|\mathcal{A}^{-1+\nu k - [\nu k]}\|)$. The following inequalities are valid:

$$([\nu k] + 1)! \leq ([\nu k] + 1)([\nu k] + 1)^{[\nu k]} \leq e([\nu k] + 1)(\nu k)^{\nu k}.$$

So $([\nu k] + 1)! \leq (e([\nu k] + 1))^{1/\nu} (\nu e)^k k!$, and for $\tau < (\nu e a)^{-1}$ the series $(*)$ converges. It implies that $f \in \exp(-\tau \mathcal{A}^\nu)(X)$.

(ii) $\Rightarrow$ (i). Suppose $g \in \mathcal{S}_{X, \mathcal{A}^\nu}$. Then there exists $s > 0$ and $w \in X$ such that $g = \exp(-s \mathcal{A}^\nu)w$. Let $k \in \mathbb{N}$. Then we estimate as follows

$$\|\mathcal{A}^k f\| \leq \|\mathcal{A}^k \exp(-s \mathcal{A}^\nu)\| \|w\| = \|w\| \left(\frac{k}{\nu s} \right)^{k/\nu} e^{-k/\nu}$$

$$\leq \|w\| (1/\nu s)^{k/\nu} \cdot (k!)^{1/\nu}.$$

With $a = (\nu s)^{-1/\nu}$ and $b = \|w\|$ the implication (ii) $\Rightarrow$ (i) has been proved. $\square$

Let $\mathcal{L}$ be an unbounded linear operator in X . Then the operators $\mathcal{L}^2, \mathcal{L}^3, \dots$ are well defined. As a corollary of the previous theorem we get the following.

COROLLARY 1.2. *Let $n \in \mathbb{N}$ and let $f \in \mathcal{D}^\omega(\mathcal{L})$. The following statements are equivalent.*

- (i) $\exists_{a>0} \exists_{b>0} \forall_{k \in \mathbb{N}} : \|\mathcal{L}^k f\| \leq (k!)^{1/n} a^k b$.
- (ii) $f \in \mathcal{D}^\omega(\mathcal{L}^n)$.

As mentioned in the introduction we investigate perturbations $\mathcal{P}$ on $\mathcal{A}$ such that $\mathcal{D}^\omega((\mathcal{A} + \mathcal{P})^\nu) \supset \mathcal{S}_{X, \mathcal{A}^\nu}$. For $\nu = 1$ the following result has been proved in [1]. Here we consider general $\nu > 0$.

THEOREM 1.3. *Let $\mathcal{P}$ be a linear operator in X with $\mathfrak{D}(\mathcal{P}) \supset \mathcal{S}_{X, \mathcal{A}^\nu}$. Suppose the following conditions are satisfied.*

- (i) *There exists a Hilbert space Y such that $\exp(-t\mathcal{A}^\nu)$ maps X into Y for all $t > 0$.*
- (ii) *In addition, $\mathcal{A} + \mathcal{P}$ defined on $\mathcal{S}_{X, \mathcal{A}^\nu}$ is positive and essentially self-adjoint in Y .*
- (iii) *There exists an everywhere defined, monotone nonincreasing function φ on $(0, 1)$ such that*

$$\forall r: 0 < r < 1 : \|\exp(r\mathcal{A}^\nu)\mathcal{P}\mathcal{A}^{-1}\exp(-r\mathcal{A}^\nu)\|_X \leq \varphi(r).$$

Then $\mathcal{S}_{X, \mathcal{A}^\nu} \subset \mathcal{S}_{Y, (\mathcal{A} + \mathcal{P})^\nu}$.

Proof. We note first that $\mathcal{S}_{X, \mathcal{A}^\nu} = \bigcup_{0 < t < 1} \exp(-t\mathcal{A}^\nu)(X)$. So let $0 < t < 1$, and let $0 < \tau < t$. Put $s = t - \tau$. We want to estimate the norm of the operator $\exp(\tau\mathcal{A}^\nu) \cdot (\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)$ for each $k \in \mathbb{N}$. Therefore we factor as follows

$$\begin{aligned} & \exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu) \\ &= \prod_{j=0}^{k-1} \left\{ \exp\left(\left(\tau + \frac{j}{k}s\right)\mathcal{A}^\nu\right)(\mathcal{A} + \mathcal{P}) \exp\left(-\left(\tau + \frac{j}{k}s\right)\mathcal{A}^\nu\right) \right\}. \end{aligned}$$

This factoring yields the estimate

$$\begin{aligned} & \|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)\| \\ & \leq \left\| \mathcal{A} \exp\left(-\frac{s}{k}\mathcal{A}^\nu\right) \right\| \prod_{j=0}^{k-1} \left\| \exp\left(\left(\tau + \frac{j}{k}s\right)\mathcal{A}^\nu\right)(\mathcal{A} + \mathcal{P}) \exp\left(-\left(\tau + \frac{j}{k}s\right)\mathcal{A}^\nu\right) \right\| \\ & \leq (k!)^{1/\nu} \left(\frac{1}{\nu s}\right)^{k/\nu} \prod_{j=0}^{k-1} \left(1 + \varphi\left(\tau + \frac{j}{k}s\right)\right). \end{aligned}$$

Since $\varphi(\tau + js/k) \leq \varphi(\tau)$ for all $j = 0, 1, \dots, k-1$, we get

$$\prod_{j=0}^{k-1} \left(1 + \varphi\left(\tau + \frac{j}{k}s\right)\right) \leq (1 + \varphi(\tau))^k.$$

Thus we have proved that

$$\forall_{t>0} \forall_{\tau, 0 < \tau < t} \exists_{a>0} \forall_{k \in \mathbb{N} \cup \{0\}} : \|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)\| \leq (k!)^{1/\nu} a^k.$$

Let $t > 0$ and let $w \in X$. Set $f = \exp(-t\mathcal{A}^\nu)w$. Then for $0 < \tau < t$ fixed there exists $a > 0$ such that

$$\begin{aligned} & \|(\mathcal{A} + \mathcal{P})^k f\|_Y \leq \|\exp(-\tau\mathcal{A}^\nu)\|_{X \rightarrow Y} \|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k f\|_X \\ & \leq \|\exp(-\tau\mathcal{A}^\nu)\|_{X \rightarrow Y} \|w\|_X a^k (k!)^{1/\nu}. \end{aligned}$$

From Lemma 1.1 it follows that $f \in \mathcal{S}_{Y, (\mathcal{A} + \mathcal{P})^\nu}$. $\square$

Remark. Suppose there exists $k \in \mathbb{N}$ such that the operator $\mathcal{A}^{-k}$ maps X continuously into Y . Then Condition (iii) of Theorem 1.3 is fulfilled because

$$\|\exp(-t\mathcal{A}^\nu)\|_{X \rightarrow Y} \leq \|\mathcal{A}^{-k}\|_{X \rightarrow Y} \|\mathcal{A}^k \exp(-t\mathcal{A}^\nu)\|_X.$$

COROLLARY 1.4. *Let $\mathcal{P}$ be an operator in X and let $n \in \mathbb{N}$ with $\mathfrak{D}(\mathcal{P}) \supset \mathcal{S}_{X, \mathcal{A}^n}$. Suppose there exists an everywhere defined monotone nonincreasing function φ on $(0, 1)$ such that*

$$\forall_{0 < r < 1} : \|\exp(r\mathcal{A}^n)\mathcal{P}\mathcal{A}^{-1}\exp(-r\mathcal{A}^n)\| \leq \varphi(r).$$

Then $\mathcal{S}_{X, \mathcal{A}^n} \subset \mathfrak{D}^\omega((\mathcal{A} + \mathcal{P})^n)$.

Proof. As in the proof of the previous theorem: $\forall_{t>0} \forall_{\tau, 0 < \tau < t} \exists_{a>0} \forall_{k \in \mathbb{N}}$:

$$\|\exp(\tau\mathcal{A}^n)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^n)\| \leq (k!)^{1/n} a^k.$$

So for $f = \exp(-t\mathcal{A}^n)w$, $t > 0$, $w \in X$, we get

$$\|(\mathcal{A} + \mathcal{P})^k f\|_X \leq \|\exp(\tau\mathcal{A}^n)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^n)\| \|w\| \leq (k!)^{1/n} a^k \|w\|. \quad \square$$

Remark. If $\mathcal{P}$ satisfies the conditions in Corollary 1.4, then $\mathcal{A}^n$ analytically dominates $(\mathcal{A} + \mathcal{P})^n$. (For the terminology, see [6].)

In order to prove the converse statement of Theorem 1.3, i.e.,

$$\mathcal{S}_{Y, (\mathcal{A} + \mathcal{P})^n} \subset \mathcal{S}_{X, \mathcal{A}^n}$$

we have to interchange the roles of $\mathcal{A}$ and $\mathcal{A} + \mathcal{P}$. Put differently, if we write $\mathcal{B} = \mathcal{A} + \mathcal{P}$ and hence $\mathcal{A} = \mathcal{B} - \mathcal{P}$, then we have to check whether the pair $\mathcal{B}$, $\mathcal{P}$ satisfies the conditions required in Theorem 1.3.

2. Hankel invariant distribution spaces. In our papers [2], [4] on Hankel invariant distribution spaces the following results have been proved.

Let $\mathcal{A}_\gamma$ denote the differential operator $-d^2/dx^2 + x^2 - (2\gamma + 1)/x d/dx$ and let X_γ denote the Hilbert space $\mathfrak{L}_2((0, \infty), x^{2\gamma+1} dx)$ where we take $\gamma > -1$. Then for every $\alpha, \beta > -1$ we have shown that

$$\mathcal{S}_{X_\alpha, \mathcal{A}_\alpha} = \mathcal{S}_{X_\beta, \mathcal{A}_\beta}.$$

Moreover, $f \in \mathcal{S}_{X_\gamma, \mathcal{A}_\gamma}$ if and only if f is extendible to an even function in $\mathcal{S}_{1/2}^{1/2}$. Also, it has been proved that the space $\mathcal{S}_{X_\gamma, \mathcal{A}_\gamma}$ remains invariant under the modified Hankel transform $\mathbf{H}_\gamma$ defined by

$$(\mathbf{H}_\gamma f)(x) = \int_0^\infty f(y)(xy)^{-\gamma} J_\gamma(xy) y^{2\gamma+1} dy.$$

Here J_γ denotes the Bessel function of the first kind and of order γ . The Hankel transform $\mathbf{H}_\gamma$ extends to a unitary operator on X_γ and $\mathbf{H}_\gamma \mathcal{A}_\gamma = \mathcal{A}_\gamma \mathbf{H}_\gamma$. It follows that for all $\alpha, \beta > -1$, $\mathbf{H}_\alpha$ maps the space $\mathcal{S}_{X_\beta, \mathcal{A}_\beta}$ onto itself. By duality, each $\mathbf{H}_\alpha$ leaves invariant each space of generalized functions $\mathcal{T}_{X_\beta, \mathcal{A}_\beta}$ corresponding to $\mathcal{S}_{X_\beta, \mathcal{A}_\beta}$. The functions $\mathbf{L}_n^{(\gamma)}$ defined by

$$\mathbf{L}_n^{(\gamma)}(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\gamma+1)} \right)^{1/2} e^{-x^2/2} \mathcal{L}_n^{(\gamma)}(x^2), \quad n \in \mathbb{N} \cup \{0\}, \quad x > 0$$

establish an orthonormal basis in X_γ and they are the eigenfunctions of the self-adjoint operator $\mathcal{A}_\gamma$ with respective eigenvalues $4n + 2\gamma + 2$. Here $\mathcal{L}_n^{(\gamma)}$ denotes the n th generalized Laguerre polynomial of order γ . We note that $\mathbf{H}_\gamma \mathbf{L}_n^{(\gamma)} = (-1)^n \mathbf{L}_n^{(\gamma)}$. We recall that for each $\alpha, \beta > -1$ the functions $f \in \mathcal{S}_{X_\alpha, \mathcal{A}_\alpha}$ can be written as $f = \sum_{n=0}^\infty \omega_n \mathbf{L}_n^{(\beta)}$ where $\omega_n = O(e^{-nt})$ for some $t > 0$.

With the aid of the theory presented in the first part of this paper we extend the mentioned results and prove that

$$\mathcal{S}_{X_\alpha(\mathcal{A}_\alpha)^\nu} = \mathcal{S}_{X_\beta(\mathcal{A}_\beta)^\nu}$$

for all $\nu \geq \frac{1}{2}$ and all $\alpha, \beta > -1$. In addition, we show that for each $\nu \in [\frac{1}{2}, 1]$ and all $\alpha > -1$ the space $\mathcal{S}_{X_\alpha(\mathcal{A}_\alpha)^\nu}$ contains just the even functions of the Gel'fand–Shilov space $\mathcal{S}_{1/2\nu}^{1/2\nu}$. So each even function $f \in \mathcal{S}_{1/2\nu}^{1/2\nu}$ admits Fourier expansions $f = \sum_{n=0}^{\infty} \rho_n^{(\alpha)} \mathbf{L}_n^{(\alpha)}$ with $\rho_n^{(\alpha)} = O(\exp(-n^\nu t))$.

Let $\alpha, \beta > -1$. Then $\mathcal{A}_\alpha$ can be written as

$$\mathcal{A}_\alpha = \mathcal{A}_\beta + 2(\alpha - \beta)\mathcal{R}$$

where we put $\mathcal{R} = (1/x)d/dx$. Obviously, $\mathcal{A}_\alpha$ can be obtained from $\mathcal{A}_\beta$ by means of the “perturbation” $2(\alpha - \beta)\mathcal{R}$, and $\mathcal{A}_\beta$ from $\mathcal{A}_\alpha$ by means of $2(\beta - \alpha)\mathcal{R}$. In order to show that $\mathcal{R}$ and hence $c\mathcal{R}$, $c \in \mathbb{C}$, is a perturbation in the sense of Theorem 1.3 we compute the matrix of $\mathcal{R}$ with respect to the orthonormal basis $(\mathbf{L}_n^{(\gamma)})_{n=0}^{\infty}$. To this end, we mention that

$$\mathcal{R}\mathbf{L}_n^{(\gamma)} = -\mathbf{L}_n^{(\gamma+1)} - 2\mathbf{L}_{n-1}^{(\gamma+1)}$$

where the relation $d\mathcal{L}_n^{(\gamma)}/dx = -\mathcal{L}_{n-1}^{(\gamma+1)}$ is used.

Now $\mathcal{L}_k^{(\gamma+1)} = \sum_{j=0}^k \mathcal{L}_j^{(\gamma)}$ and hence

$$\mathcal{R}\mathbf{L}_n^{(\gamma)} = -\left(\frac{2\Gamma(n+1)}{\Gamma(n+\gamma+1)}\right)^{1/2} \left[\left(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\right)^{1/2} \mathbf{L}_n^{(\gamma)} + 2 \sum_{m=0}^{n-1} \left(\frac{\Gamma(m+\gamma+1)}{2\Gamma(m+1)}\right)^{1/2} \mathbf{L}_m^{(\gamma)}\right].$$

Thus we obtain the matrix of $\mathcal{R}$ with respect to the basis $(\mathbf{L}_n^{(\gamma)})_{n=0}^{\infty}$

$$(\mathcal{R}\mathbf{L}_k^{(\gamma)}, \mathbf{L}_l^{(\gamma)})_\gamma = \begin{cases} -1 & \text{if } l=k, \quad k \in \mathbb{N}, \\ 0 & \text{if } l>k, \quad k \in \mathbb{N} \cup \{0\}, \\ -2\left(\frac{\Gamma(k+1)}{\Gamma(k+\gamma+1)} \frac{\Gamma(l+\gamma+1)}{\Gamma(l+1)}\right)^{1/2} & \text{if } 0 \leq l < k, \quad k \in \mathbb{N}. \end{cases}$$

The inequality (cf. [11])

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s}, \quad 0 \leq s \leq 1, \quad n \in \mathbb{N}$$

yields

$$|(\mathcal{R}\mathbf{L}_k^{(\gamma)}, \mathbf{L}_l^{(\gamma)})| \leq \begin{cases} 2 & \text{if } \gamma \geq 0, 0 \leq l < k, k \in \mathbb{N} \cup \{0\}, \\ 2k^{-\gamma/2} & \text{if } -1 < \gamma < 0, 0 \leq l < k, k \in \mathbb{N} \cup \{0\}. \end{cases}$$

For each $\nu \geq \frac{1}{2}$, the operator $\exp(r(\mathcal{A}_\gamma)^\nu)\mathcal{R}(\mathcal{A}_\gamma)^{-1}\exp(-r(\mathcal{A}_\gamma)^\nu)$ has to satisfy Condition (iii) of Theorem (1.3). We define the weighted shift operators $\mathcal{W}_{\gamma,\nu}^{(n)}(r)$, $n \in \mathbb{N} \cup \{0\}$,

$$(\mathcal{W}_{\gamma,\nu}^{(n)}(r)\mathbf{L}_k^{(\gamma)}, \mathbf{L}_l^{(\gamma)})_\gamma = \begin{cases} 0 & \text{if } k \neq l+n, \\ (\mathcal{R}\mathbf{L}_{l+n}^{(\gamma)}, \mathbf{L}_l^{(\gamma)}) \frac{\exp(-r(4(l+n)+2\gamma+2))^\nu - (4l+2\gamma+2)^\nu}{4(l+n)+2\gamma+2} & \text{if } k = l+n. \end{cases}$$

with norms

(*)

$$\|\mathcal{W}_{\gamma, \nu}^{(n)}(r)\|_{X_\gamma} = \sup_{l \in \mathbf{N} \cup \{0\}} |(\mathcal{R}\mathbf{L}_{l+n}^{(\gamma)}, \mathbf{L}_l^{(\gamma)})| \frac{\exp(-r(4(l+n)+2\gamma+2))^\nu - (4l+2\gamma+2)^\nu}{4(l+n)+2\gamma+2}.$$

So $\|\mathcal{W}_{\gamma, \nu}^{(0)}(r)\| \leq 1/(2\gamma+2)$. Now let $n \in \mathbf{N}$. The inequality

$$(4(l+n)+2\gamma+2)^\nu - (4l+2\gamma+2)^\nu \geq (l+n)^{1/2} - l^{1/2}$$

is valid for all $l \in \mathbf{N} \cup \{0\}$ and all $\nu \geq \frac{1}{2}$. In addition, the matrix elements $|(\mathcal{R}\mathbf{L}_{l+n}^{(\gamma)}, \mathbf{L}_l^{(\gamma)})|$ are smaller than $2(l+n)^{-\gamma/2}$ for $-1 < \gamma < 0$ and smaller than 2 for $\gamma \geq 0$. If $-1 < \gamma \leq 0$ we therefore get

$$\begin{aligned} \|\mathcal{W}_{\gamma, \nu}^{(n)}(r)\| &\leq \sup_{l \in \mathbf{N} \cup \{0\}} \frac{2(l+n)^{-\gamma/2}}{4(l+n)+2\gamma+2} \exp(-r((l+n)^{1/2} - l^{1/2})) \\ &\leq \sup_{l \in \mathbf{N} \cup \{0\}} \left(\frac{1}{2} (l+n)^{-1/2\gamma-1} \exp\left(-\frac{1}{2} r n (l+n)^{-1/2}\right) \right) \\ &\leq \frac{1}{2} \left(1 + \frac{1}{2}\gamma\right)^{2+\gamma} \left(\frac{1}{r}\right)^{2+\gamma} \left(\frac{1}{n}\right)^{2+\gamma} \exp(2+\gamma) =: d_1 \left(\frac{1}{r}\right)^{2+\gamma} \left(\frac{1}{n}\right)^{2+\gamma}. \end{aligned}$$

Since

$$\exp(r(\mathcal{A}_\gamma)^\nu) \mathcal{R}(\mathcal{A}_\gamma)^{-1} \exp(-r(\mathcal{A}_\gamma)^\nu) = \sum_{n=0}^{\infty} \mathcal{W}_{\gamma, \nu}^{(n)}(r)$$

we can use the following straightforward estimate for all $r > 0$

$$\begin{aligned} \|\exp(r(\mathcal{A}_\gamma)^\nu) \mathcal{R}(\mathcal{A}_\gamma)^{-1} \exp(-r(\mathcal{A}_\gamma)^\nu)\| &\leq \sum_{n=0}^{\infty} \|\mathcal{W}_{\gamma, \nu}^{(n)}(r)\| \\ &\leq \frac{1}{2\gamma+2} + d_1 \left(\frac{1}{r}\right)^{2+\gamma} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{2+\gamma} \\ &\leq d_\gamma \left(\frac{1}{r}\right)^{2+\gamma} + \frac{1}{2\gamma+2} \end{aligned}$$

where $d_\gamma = d_1 \sum_{n=1}^{\infty} (1/n)^{2+\gamma}$. Summarized we have

LEMMA 2.1. *Let $\gamma > -1$. Then there exist constants $d_\gamma > 0$ and $p_\gamma > 0$ such that*

$$\forall_{r>0}: \|\exp(r(\mathcal{A}_\gamma)^\nu) \mathcal{R}\mathcal{A}_\gamma^{-1} \exp(-r(\mathcal{A}_\gamma)^\nu)\| \leq d_\gamma \left(\frac{1}{r}\right)^{p_\gamma} + \frac{1}{2\gamma+2}.$$

Proof. For $-1 < \gamma \leq 0$ the assertion has already been proved. For $\gamma > 0$ it follows from the matrix expressions for $\mathcal{R}$ that

$$\|\exp(r(\mathcal{A}_\gamma)^\nu) \mathcal{R}\mathcal{A}_\gamma^{-1} \exp(-r(\mathcal{A}_\gamma)^\nu)\| \leq d_0 \left(\frac{1}{r}\right)^{p_0} + \frac{1}{2\gamma+2}.$$

□

In addition, we show that given $r > 0$, $\gamma, \delta > -1$, the operator $\exp(-r(\mathcal{A}_\gamma)^\nu)$ maps X_γ into X_δ . In [2, p. 17], the following result has been proved

$$\forall_{s \in \mathbb{N}} \exists_{l \in \mathbb{N}} : \|\mathcal{Q}^{2s}(\mathcal{A}_\gamma)^{-l}\|_\gamma < \infty.$$

Here $\mathcal{Q}$ denotes the multiplication operator in X_γ given by

$$(\mathcal{Q}f)(x) = xf(x).$$

Now let $\delta > -1$ and let $f \in X_\gamma$. Put $s := [\max\{0, (\delta - \gamma)/2\}] + 1$. Then there exists $l_0 \in \mathbb{N}$ such that $\|\mathcal{Q}^{2s}\mathcal{A}_\gamma^{-l}\|_\gamma < \infty$ for all $l \geq l_0$. So we derive

$$\begin{aligned} (*) \quad \int_1^\infty \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right|^2 x^{2\delta+1} dx &= \int_1^\infty x^{2(\delta-\gamma)} \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right|^2 x^{2\gamma+1} dx \\ &\leq \int_1^\infty x^{4s} \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right|^2 x^{2\gamma+1} dx \\ &\leq \|\mathcal{Q}^{2s}(\mathcal{A}_\gamma)^{-l}\|_\gamma^2 \|f\|_\gamma^2. \end{aligned}$$

Following [12, p. 248], there exists $l_1 \in \mathbb{N}$ and $d > 0$ such that

$$\max_{x \in [0,1]} |\mathbf{L}_k^{(\gamma)}(x)| \leq d(k+1)^{l_1}.$$

For $l > l_1$ it yields

$$\begin{aligned} (**) \quad \int_0^1 \left| ((\mathcal{A}_\gamma)^{-2}f)(x) \right|^2 x^{2\delta+1} dx &= \int_0^1 \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right|^2 x^{2\delta+1} dx \\ &\leq \left(\max_{x \in [0,1]} \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right| \right)^2 \int_0^1 x^{2\delta+1} dx \\ &\leq \frac{1}{2\delta+2} \left(\sum_{k=0}^\infty (f, \mathbf{L}_k^{(\gamma)})_\gamma \left(\frac{1}{4k+2\gamma+2} \right)^l \max_{x \in [0,1]} |\mathbf{L}_k^{(\gamma)}(x)| \right)^2 \\ &\leq \frac{1}{2\delta+2} \left(d^2 \sum_{k=0}^\infty \frac{(k+1)^{2l_1}}{(4k+2\gamma+2)^{2l}} \right) \|f\|_\gamma^2. \end{aligned}$$

From (*) and (**) we get

$$\begin{aligned} \forall_{\gamma > -1} \forall_{\delta > -1} \exists_{l \in \mathbb{N}} \exists_{c > 0} \forall_{f \in X_\gamma} : \\ \left\| (\mathcal{A}_\gamma)^{-l}f \right\|_\delta^2 = \int_0^\infty \left| ((\mathcal{A}_\gamma)^{-l}f)(x) \right|^2 x^{2\delta+1} dx \leq c \|f\|_\gamma^2 \end{aligned}$$

i.e. $(\mathcal{A}_\gamma)^{-l}$ is a continuous linear operator from X_γ into X_δ .

LEMMA 2.2. Let $\gamma > -1$. Then for every $r > 0$, $\nu > 0$ and $\delta > -1$ the operator $\exp(-r(\mathcal{A}_\gamma)^\nu)$ is a continuous linear operator from X_γ into X_δ .

Proof. Let $r > 0$, $\nu > 0$ and let $\delta > -1$. Then there exists $l \in \mathbb{N}$ such that $(\mathcal{A}_\gamma)^{-l}$ is a continuous linear mapping from X_γ into X_δ . Hence $\exp(-r(\mathcal{A}_\gamma)^\nu) = (\mathcal{A}_\gamma)^{-l} \{ (\mathcal{A}_\gamma)^l \exp(-r(\mathcal{A}_\gamma)^\nu) \}$ is also a continuous linear mapping from X_γ into X_δ .

□

Lemmas 2.1 and 2.2 yield the following important result.

THEOREM 2.3. *Let $\alpha, \beta > -1$. Then for every $\nu \geq \frac{1}{2}$*

$$\mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu} = \mathcal{S}_{X_\beta, (\mathcal{A}_\beta)^\nu}.$$

Proof. Let $\nu \geq \frac{1}{2}$. We have shown that:

- $\exp(-t(\mathcal{A}_\alpha)^\nu)$, $t > 0$, maps X_α continuously into X_β .
- $\mathfrak{D}(\mathcal{R}) \supset \mathcal{S}_{S_\alpha, (\mathcal{A}_\alpha)^\nu}$, and $\mathcal{A}_\beta = \mathcal{A}_\alpha + 2(\alpha - \beta)\mathcal{R}$ is positive and self-adjoint in X_β .
- There exist constants $d_\alpha, p_\alpha > 0$ such that for all $r > 0$

$$\|\exp(r(\mathcal{A}_\alpha)^\nu)\mathcal{R}(\mathcal{A}_\alpha)^{-1}\exp(-r(\mathcal{A}_\alpha)^\nu)\|_\alpha \leq d_\alpha \left(\frac{1}{r}\right)^{p_\alpha} + \frac{1}{2\alpha + 2}.$$

So by Theorem 1.3, $\mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu} \subset \mathcal{S}_{X_\beta, (\mathcal{A}_\beta)^\nu}$. Interchanging α and β we get the wanted result. $\square$

Let $\alpha > -1$. Since $\mathbf{H}_\alpha \mathcal{A}_\alpha = \mathcal{A}_\alpha \mathbf{H}_\alpha$, also $\mathbf{H}_\alpha (\mathcal{A}_\alpha)^\nu = (\mathcal{A}_\alpha)^\nu \mathbf{H}_\alpha$. So the Hankel transform $\mathbf{H}_\alpha$ is a continuous bijection on the space $\mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu}$, $\nu \geq \frac{1}{2}$, and hence on the spaces $\mathcal{S}_{X_\beta, (\mathcal{A}_\beta)^\nu}$, $\nu \geq \frac{1}{2}$, $\beta > -1$. By duality each transform $\mathbf{H}_\alpha$ leaves invariant the spaces of generalized functions $\mathcal{T}_{X_\beta, (\mathcal{A}_\beta)^\nu}$. For $\alpha = -\frac{1}{2}$ we get $X_{-1/2} = \mathfrak{L}_2((0, \infty))$ and $\mathcal{A}_{-1/2} = -(d^2/dx^2) + x^2$. The functions $\mathbf{L}_k^{(-1/2)}$ are the even Hermite functions. With the aid of the papers [8] and [10] the following characterization of the spaces $\mathcal{S}_{X_{-1/2}, (\mathcal{A}_{-1/2})^\nu}$, $\nu \in [\frac{1}{2}, 1]$, can be obtained,

$$f \in \mathcal{S}_{X_{-1/2}, (\mathcal{A}_{-1/2})^\nu} : \Leftrightarrow f \text{ is extendible to an even function in the space } \mathcal{S}_{1/2\nu}^{1/2\nu}.$$

The spaces $\mathcal{S}_p^q$, $p + q \geq 1$, $p, q \geq 0$, are introduced by Gel'fand and Shilov in [9]. In this connection we note that in our paper [5] we have proved that the spaces $\mathcal{S}_{1/k+1}^{k/k+1}$ are analyticity spaces; explicitly

$$\mathcal{S}_{1/k+1}^{k/k+1} = \mathcal{S}_{\mathfrak{L}_2(\mathbf{R}), \mathcal{B}_k} \quad \text{with } \mathcal{B}_k = \left(-\frac{d^2}{dx^2} + x^{2k} \right)^{(k+1)/2k}.$$

Relevant for the present paper are the spaces $\mathcal{S}_\mu^\mu$, $\frac{1}{2} \leq \mu \leq 1$. We have

$$\varphi \in \mathcal{S}_\mu^\mu, \quad \frac{1}{2} \leq \mu \leq 1 \quad \text{if and only if } \varphi \text{ is an entire function satisfying} \\ \exists_{A, B, C > 0} : |\varphi(x + iy)| \leq C \exp(-A|x|^{1/\mu} + B|y|^{1/1-\mu})$$

and

$$\varphi \in \mathcal{S}_1^1 \quad \text{if and only if } \varphi \text{ is analytic on a strip about the real axis say of} \\ \text{width } r > 0 \text{ and satisfying } \exists_{A, C > 0} : \sup_{|y| < r} |\varphi(x + iy)| \leq C \exp(-A|x|).$$

Now Theorem 2.3 leads to the following important results.

COROLLARY 2.4. *Let $\alpha > -1$ and let $\nu \in [\frac{1}{2}, 1]$. Then $f \in \mathcal{S}_{X_\alpha, (\mathcal{A}_\alpha)^\nu}$ if and only if f is extendible to an even function in the space $\mathcal{S}_{1/2\nu}^{1/2\nu}$.*

COROLLARY 2.5. *Let $f \in \mathcal{S}_{1/2\nu}^{1/2\nu}$ be even, with $\nu \in [\frac{1}{2}, 1]$. Then for each $\gamma > -1$, there exists an l_2 -sequence $(\omega_n^{(\gamma)})_{n=0}^\infty$ and $t > 0$ such that $f = \sum_{n=0}^\infty \exp(-n^\nu t) \omega_n^{(\gamma)} \mathbf{L}_n^{(\gamma)}$ where the series converges pointwise.*

Appendix. The set of so-called entire vectors for a positive self-adjoint operator $\mathcal{A}$ in a Hilbert space X is equal to

$$\mathfrak{D}^\infty(e^{\mathcal{A}}) = \bigcap_{t>0} e^{-t\mathcal{A}}(X).$$

In [3], van Eijndhoven has used the Fréchet space $\mathfrak{D}^\infty(e^{\mathcal{A}})$ as the test space in a theory of generalized functions which is a kind of reverse of the theory in [7]. The space $\mathfrak{D}^\infty(e^{\mathcal{A}})$ is denoted by $\tau(X, \mathcal{A})$ and it may be called the entireness space. To our opinion the well-known theory of tempered distributions is considerably generalized in [3]. (Put $\mathcal{A} = \log(-d^2/dx^2 + x^2 + 1)$. Then $\tau(\mathfrak{L}_2(\mathbf{R}), \mathcal{A})$ is the space $\mathcal{S}(\mathbf{R})$ of functions of rapid decrease.)

Similar to Theorem 1.3 we prove

THEOREM A.1. *Let $\mathcal{P}$ be a linear operator in X with $\mathfrak{D}(\mathcal{P}) \supset \exp(-\sigma\mathcal{A}^\nu)(X)$ for some $\sigma > 0$ sufficiently large. Suppose the following conditions are satisfied.*

- (i) *There exists a Hilbert space Y such that $\exp(-t\mathcal{A}^\nu)$ maps X into Y for all $t > 0$.*
- (ii) *Also, $\mathcal{A} + \mathcal{P}$ defined on $\exp(-\sigma\mathcal{A}^\nu)(X)$ is a positive essentially self-adjoint operator in Y .*
- (iii) *There exist positive constants $r_0 \geq 1$, $d > 0$ and $0 \leq q < 1/\nu$ such that for all $r > r_0$*

$$\|\exp(r\mathcal{A}^\nu)\mathcal{P}\mathcal{A}^{-1}\exp(-r\mathcal{A}^\nu)\|_X < dr^q.$$

Then $\tau(X, \mathcal{A}^\nu) \subset \tau(Y, (\mathcal{A} + \mathcal{P})^\nu)$.

Proof. Since $\tau(X, \mathcal{A}^\nu) = \bigcap_{t>r_0} \exp(-t\mathcal{A}^\nu)(X)$, we consider $t > r_0$ only. Let $0 < \tau < 1$ with $s = t - \tau > 1$. The factoring used in Theorem 1.3 yields the following estimate

$$\|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)\| \leq k! \left(\frac{1}{\nu s}\right)^{k/\nu} \prod_{j=0}^{k-1} (1 + d(\tau + js/k)^q).$$

Put $b_\tau = 1 + d\tau^q$. Then

$$\prod_{j=0}^{k-1} (1 + d(\tau + js/k)^q) \leq b_\tau \prod_{j=1}^{k-1} (1 + d) \left(\frac{k + js}{k}\right)^q \leq b_\tau (1 + d)^k 2^{qk} s^{qk}.$$

Set $a = (1 + d)2^{q(1/\nu)}$. Then

$$\|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)\|_X \leq (k!)^{1/\nu} \left(\frac{1}{s}\right)^{(-q+1/\nu)k} a^k b_\tau.$$

For $f \in \exp(-t\mathcal{A}^\nu)(X)$ it yields

$$\begin{aligned} \|(\mathcal{A} + \mathcal{P})^k f\|_Y &\leq \|\exp(-\tau\mathcal{A}^\nu)\|_{X \rightarrow Y} \|\exp(\tau\mathcal{A}^\nu)(\mathcal{A} + \mathcal{P})^k \exp(-t\mathcal{A}^\nu)\|_X \|\exp(t\mathcal{A}^\nu)f\| \\ &\leq (k!)^{1/\nu} \left(a \cdot \left(\frac{1}{s}\right)^{1/\nu - q}\right)^k b_\tau \|\exp(-\tau\mathcal{A}^\nu)\|_{X \rightarrow Y} \|\exp(t\mathcal{A}^\nu)f\|_X. \end{aligned}$$

Thus we find that $f \in \exp(-r(\mathcal{A} + \mathcal{P})^\nu)(Y)$ for all $r < (1/\nu ae)s^{-q+1/\nu}$. Now put $r(t) = (1/(\nu ae + 1))s^{-q+1/\nu}$ with $s = t + 1/t - 1$ for instance. Then we get

$$\begin{aligned} \tau(X, \mathcal{A}^\nu) &= \bigcap_{t>r_0} (\exp(-t\mathcal{A}^\nu)(X)) \subset \bigcap_{t>r_0} \left(\exp(-r(t)(\mathcal{A} + \mathcal{P})^\nu)(Y)\right) \\ &= \bigcap_{r>0} \left(\exp(-r(\mathcal{A} + \mathcal{P})^\nu)(Y)\right) = \tau(Y, (\mathcal{A} + \mathcal{P})^\nu). \end{aligned}$$

□

It is not hard to see that the spaces $\tau(X_\alpha, (\mathcal{A}_\alpha)^\nu)$, $\alpha > -1$, are Hankel invariant, and hence their strong duals $\sigma(X_\alpha, (\mathcal{A}_\alpha)^\nu)$. The previous theorem and the Lemmas 2.1 and 2.2 lead to the following classification.

THEOREM A.2. *Let $\alpha, \beta > -1$ and let $\nu \geq \frac{1}{2}$. Then*

$$\tau(X_\alpha, (\mathcal{A}_\alpha)^\nu) = \tau(X_\beta, (\mathcal{A}_\beta)^\nu).$$

By [2] and [8] we obtain the following characterizations

$$f \in \tau(X_{-1/2}, \mathcal{A}_{-1/2}) \text{ iff } f \text{ is extendible to an even entire function for which} \\ \forall_{0 < a < 1} \exists_{C > 0} \forall_{x+iy \in \mathbb{C}} : |f(x+iy)| \leq C \exp(-\tfrac{1}{2}ax^2 + \tfrac{1}{2a}y^2)$$

and

$$f \in \tau(X_{-1/2}, (\mathcal{A}_{-1/2})^{1/2}) \text{ iff } f \text{ is extendible to an even entire function for} \\ \text{which } \forall_{r > 0} : \sup_{|y| < r, -\infty < x < \infty} e^{r|x|} |f(x+iy)| < \infty.$$

Finally, Theorem A.2 gives the characterization in classical analytic terms of the elements in each $\tau(X_\alpha, \mathcal{A}_\alpha)$, respectively $\tau(X_\alpha, (\mathcal{A}_\alpha)^{1/2})$, $\alpha > -1$.

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