Truncation error and modified Romberg extrapolation

for the Filon-trapezoidal rule

by

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Abstract

Two asymptotic expansions are derived for the error in the Filon-trapezoidal rule (a numerical integration method for integrals of Fourier type

$$\int_0^1 f(x)e^{i\omega x}dx \quad \text{with} \quad |\omega| >> 1.$$ 

Classical Romberg extrapolation can be based on the first expansion. However, because of the dependence on $\omega$ of the coefficients it is successful only in the very limit $\theta := \omega h \to 0$. With the second expansion a modification of Romberg extrapolation is introduced that coincides with classical Romberg for $\theta \to 0$, but gives improved results already for $\theta \sim \pi$. 
Introduction

Of all methods of Filon type for numerical integration of

\[ I_\omega(f) := \int_0^1 f(x)e^{i\omega x}dx, \quad |\omega| > 1, \]

the Filon-trapezoidal rule (Tuck, 1967, [1])

(1) \[ FT_{\omega,h}(f) := h \sum_{j=0}^{N-1} \left( \frac{1 + i\theta - e^{i\theta}}{\theta^2} f_j e^{i\omega x_j} + \frac{1 - i\theta - e^{-i\theta}}{\theta^2} f_{j+1} e^{i\omega x_{j+1}} \right) \]

\[ = \frac{\sin(|\theta|)}{|\theta|} h \sum_{j=0}^{N} f_j e^{i\omega x_j} + (1 - \frac{\sin(|\theta|)}{|\theta|}) h \frac{\sin(|\omega|)}{|\omega|} \]

\( h := 1/N, x_j := jh, \theta := \omega h, \sum' = \sum_1^N \) with coefficient \( \frac{1}{2} \) for \( j = 0, N \)

is the most simple, but it is a method of only low order accuracy:

\[ R := I_\omega(f) - FT_{\omega,h}(f) = O(h^2), \quad |\theta| \leq 2\pi, f \in C^2[0,1]. \]

To obtain more accurate results from a sequence \( FT_{\omega,h_m}(f) \) for decreasing \( h_m \rightarrow 0 \), we show the possibility of \textit{Romberg extrapolation} based on the asymptotic expansion

(2) \[ R = \frac{[1 + (m-1)/2]}{k=1} \alpha_k(\omega,f) h^{2k} + O(h^m), \quad h \rightarrow 0, f \in C^m[0,1] \text{ (Th. 1)}, \]

and of a modification of \textit{Romberg extrapolation} based on

(3) \[ R = \sum_{k=2}^{m-1} \beta_k(\theta)I_\omega(D^k f) h^k + O(h^m), \quad |\theta| < 2\pi, f \in C^m[0,1] \text{ (Th. 2)}. \]
For the sequence $h_m := h_{m-1}/2$ the modified Romberg extrapolation is defined by

$$\begin{align*}
\text{FT}_{\omega, h}^{(1)}(f) & := \text{FT}_{\omega, h}(f) \\
\text{FT}_{\omega, h}^{(j)}(f) & := \text{FT}_{\omega, h}^{(j-1)}(f) + \gamma_j(\theta)(\text{FT}_{\omega, h}^{(j-1)}(f) - \text{FT}_{\omega, h}^{(j-1)}(f)), j > 1,
\end{align*}$$

where the first two coefficients $\gamma$ are

$$\begin{align*}
\gamma_2(\theta) & := -(\frac{\theta}{2} \cotan(\frac{\theta}{2}))((\frac{\theta}{2} \cotan(\frac{\theta}{2}) - 1)/(\frac{\theta}{2})^2) \\
\gamma_3(\theta) & := \frac{\theta \cotan(\theta)\{((\theta \cotan(\theta) - 1 + \frac{1}{3} \theta^2)/\theta^4\}}{((\theta \cotan(\theta) - 1)/\theta^2).
\end{align*}$$

Finally we demonstrate the usefulness of the modification by a closer analysis of (2) and (3), estimating the order of magnitude of the coefficients $\beta_k$ in (3) (Th. 3).

Remark. $\text{FT}_{\omega, h}^{(2)}(f)$ in (4) is the classical Filon (-Simpson) approximation to $I_{\omega}(f)$ with step $h$.

§ 1. The asymptotic expansions (2) and (3)

By the method of Peano (Davis, Rabinowitz, 1967, [2]) we derive an expression for the truncation error:

$$R = h^3 \sum_{j=0}^{N-1} \int_0^1 \{D^2 f(x_j + th)e^{i\omega(x_j+th)}\}K_2(\theta, t)dt$$

with

$$K_2(\theta, t) := \{te^{-i\theta(t-1)} - (t - 1)e^{-i\theta t} - 1\}/\theta^2,$$

or

$$R = h^3 \sum_{j=0}^{N-1} e^{i\omega x_j} \int_0^1 \{D^2 f(x_j + th)\}L_2(\theta, t)dt$$

with

$$L_2(\theta, t) := e^{i\theta t}K_2(\theta, t) = \{1 - e^{i\theta t} + t(e^{i\theta} - 1)\}/\theta^2.$$
From (6) we prove:

**Theorem 1 (Asymptotic expansion of R in even powers of h with \( \omega \)-dependent coefficients)**

Let the coefficients \( \lambda_{2k,p} \) \((k \geq 1, 0 \leq p \leq 2k-2)\) be defined by

\[
\lambda_{2k,p} := -\frac{2(2k+1-p)}{(2k+2)!} \sum_{q=0}^{p} \frac{(2k+2)_q B_q}{(2k+1)_q} + \frac{2}{(2k+1)_q} \sum_{q=p+2}^{2k} \frac{(2k+2)_q B_q}{(2k+1)_q}
\]

\( B_q \) are the Bernoulli numbers) and \( \alpha_{2k}(\omega,f) \) \((k \geq 1)\) by

\[
\alpha_{2k}(\omega,f) := (-1)^k \omega^{2k-2} \sum_{p=0}^{k-2} \lambda_{2k,p} \int_{0}^{1} \left( \frac{D}{\omega} \right)^p D^2 f(x) e^{i\omega x} dx
\]

Then for \( f \in C^m[0,1] \)

\[
R = \sum_{k=1}^{\lfloor (m-1)/2 \rfloor} \alpha_{2k}(\omega,f) h^{2k} + O(h^m), \ h \to 0.
\]

**Proof.** In (6) we write

\[
K_2(\theta, t) = \sum_{n=2}^{\infty} g_n(t)(-i\theta)^{n-2}
\]

introducing \( g_n(t) := \{(t-1)t^n - (t-1)^n\}/n! \). The expansion of the polynomial \( g_n(t) \) as a sum of Bernoulli polynomials

\[
g(t) = \int_{0}^{1} g(t)dt + B_1(t)(g(1) - g(0)) + \frac{B_2(t)}{2!} (g'(1) - g'(0)) \text{ etc.}
\]

is

\[
g_n(t) = -\frac{(1+(-1)^n)}{(n+2)!} + \sum_{q=2}^{n} \frac{B_q(t)}{q!} (q-1) \frac{(1+(-1)^{n-q})}{(n-q+2)!}
\]
Integrating (6) by parts \((m-2)\) times using the expansions (11),

\[
\frac{d}{dt} \frac{B_{q+1}(t)}{q+1} = \frac{B_q(t)}{q!} \text{ and } B(1) = B(0) = B_q \quad (q \geq 2),
\]
we derive for \(f \in C^m[0,1]\)

\[
R = - \sum_{n=2}^{\infty} (-i\theta)^n (-1)^{n-2} \sum_{p=0}^{\min\{n-2\}} (-1)^p \lambda_{n+p,p} \int_0^1 D^p D^2 f(x)e^{i\omega x} dx +
\]

\[
+ \sum_{j=0}^{N-1} \int_0^1 D^j D^2 f(x) e^{i\omega x} \big|_{x_j} + th K_m(\theta, t) dt,
\]

with the definitions

\[
\lambda_{m,p}(t) := \sum_{q=2}^{m} \frac{B_{q+p}(t)}{(q + p)!} \left(\frac{q - 1}{(m - p - q + 2)!}\right) - \sum_{q=p+2}^{m} \frac{B_q(t)}{q!} \left(\frac{p + 1 - q}{(m - q + 2)!}\right),
\]

and

\[
K_m(\theta, t) := (-1)^m \sum_{n=2}^{\infty} (-i\theta)^{n-2} \left(\lambda_{n+m-2,m-2}(t) - \lambda_{n+m-2,m-2}\right).
\]

Interchange of summation and integration in the derivation of (12) is allowed as

\[
\sum_{n=2}^{\infty} (-i\theta)^{n-2} \lambda_{n+p,p}(t) \quad (p \geq 0)
\]
is absolutely convergent for all \(\theta\), uniformly in \(t \in [0,1]\). This can be seen as follows:

since \(\lambda_{n+p,p}(t)\) can be written as
\[
\lambda_{n+p,p}(t) = \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} (q-1) \left( 1 + (-1)^{n+p-q} \right) \frac{(n+p-q)!}{(n+p-q+2)!} - \]

\[
- p \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} \left( 1 + (-1)^{n+p-q} \right) \frac{(n+p-q)!}{(n+p-q+2)!} + \sum_{q=2}^{p+1} (p+1-q) \frac{B_q(t)}{q!} \left( 1 + (-1)^{n+p-q} \right) \frac{(n+p-q)!}{(n+p-q+2)!},
\]

or, with (11) and similar expansions of \( \tilde{a}_n(t) := \frac{(t^n - (t-1)^n)}{(n+1)!} \), as

\[
\lambda_{n+p,p}(t) = \tilde{a}_{n+p}(t) - \sum_{q=2}^{p+1} (p+1-q) \frac{B_q(t)}{q!} \left( 1 + (-1)^{n+p-q} \right) \frac{(n+p-q)!}{(n+p-q+2)!} + \]

\[
+ pB_p(t) \left( 1 + (-1)^{n+p-1} \right) + \sum_{q=2}^{p+1} (p+1-q) \frac{B_q(t)}{q!} \left( 1 + (-1)^{n+p-q} \right) \frac{(n+p-q)!}{(n+p-q+2)!},
\]

obviously \( |\lambda_{n+p,p}(t)| \leq \text{const}(p)/n! \), a sufficient condition for uniform absolute convergence of

\[
\sum_{n=2}^{\infty} (-i\theta)^{n-2} \lambda_{n+p,p}(t).
\]

If in (12) we estimate the remainder with (14) it follows

\[
R = - \sum_{n=2}^{\infty} (-i\theta)^{n-2} \sum_{p=0}^{n+2} \lambda_{n+p,p} \int_0^1 p^p \{D^2 f(x)e^{iwx}\} dx + o(h^m), h \to 0.
\]

Writing \( \theta = \omega h \), interchanging summation indices and estimating terms of higher order in \( h \) than \( (m-1) \) we obtain our final result

\[
R = - \sum_{k=2}^{m-1} \sum_{k=2}^{k-2} \lambda_{k,p} \int_0^1 \left( \frac{D}{i\omega} \right)^p \{D^2 f(x)e^{iwx}\} dx + o(h^m), h \to 0, \text{ for } f \in C^m[0,1],
\]

an expansion in even powers of \( h \) because \( \lambda_{\text{odd},p} = 0 \).

From (13) one easily derives the second expression for \( \lambda_{2k,p} \) in (8) and with

\[
\sum_{q=0}^{k-1} \binom{k}{q} B_q = 0 \quad (k \geq 2)
\]

the first expression, completing the proof. \( \square \)

Remark. \( \lambda_{2k,2k-2} = B_{2k}/(2k)! \). So for \( \omega = 0 \), when the Filon trapezoidal rule reduces to the trapezoidal rule for \( f \), indeed (10) reduces to the Euler-Maclaurin expansion:

\[
R = - \sum_{k=1}^{[(m-1)/2]} \frac{B_{2k}}{(2k)!} \int_0^1 D^{2k} f(x) dx h^k + o(h^m), h \to 0, \text{ for } f \in C^m[0,1].
\]
From (7) we prove:

**Theorem 2 (Asymptotic expansion of R with \( \theta \)-dependent coefficients)**

Let the coefficients \( \delta_p(\theta) \) \((p \geq 2)\) be defined as

\[
\delta_p(\theta) := \left( \frac{\theta}{2} \cot(\frac{\theta}{2}) - 1 - \sum_{q=2}^{p-1} \frac{B(q)}{q^p} \right) / (i\theta)^p, \quad |\theta| \neq 2k\pi, \quad k = 1, 2 \text{ etc.}
\]

\[
= \sum_{q=p}^{\infty} \frac{B(q)}{q^p} (i\theta)^{-p} \quad \text{for} \quad |\theta| < 2\pi,
\]

and the coefficients \( \beta_k(\theta) \) \((k \geq 2)\) by the recurrence relation

\[
\beta_k(\theta) := - \left( \sum_{p=2}^{k-1} (-1)^p i\theta \delta_p(\theta) \delta_{k-p+1}(\theta) + (-1)^k \delta_k(\theta) \right).
\]

Then, if \( f \in C^m[0,1] \),

\[
R = \sum_{k=2}^{m-1} \beta_k(\theta) I_\omega (D^k f) h^k + O(h^m),
\]

for \(|\theta|\) bounded and bounded away from \(2k\pi, k = 1, 2 \text{ etc.}\).

If \(|\theta| < 2\pi\) we have the estimate for the order term in (17)

\[
|O(h^m)| \leq \text{const.} \left\{ \frac{13/3}{2\pi(1 - (\theta/2\pi)^2)} \right\}^{m-2} \left\{ \max_{x \in [0,1]} |D^m f(x)| \right\} h^m.
\]

**Proof.**

The first part of the proof is analogous to that of Th. 2 but now we start from (7) instead of (6). We write \( L_2(\theta,t) = \sum_{n=2}^{\infty} g_n(t)(i\theta)^{n-2} \) introducing \( g_n(t) := (t^n-t)/n! \) with the Bernoulli expansion

\[
g_n(t) = \frac{(1-n)}{2(n+1)!} + \sum_{q=2}^{n} \frac{B(q)}{q!(n-q+1)!} t^{n-q}.
\]

Integrating (7) by parts \((m-2)\) times we have for \( f \in C^m[0,1] \)

\[
R = h \sum_{j=0}^{N-1} e^{i\omega x_j} \left[ \sum_{n=2}^{m-3} (-1)^{p+1} \mu_{n+p+1} D^{p+1} f(x) \right] x_{j+1} + h^m \int_0^1 D^m f(x) \left[ x_j + \text{th} L_m(\theta,t) dt \right],
\]
with the definitions

\[ u_{m,p}(t) := - \sum_{q=2}^{m-p} \frac{B_{q+p}(t)}{(q+p)!(m-p-q+1)!} = \sum_{q=0}^{p+1} \frac{B_q}{q!(m-q+1)!} \]

\[ \mu_{m,p} := u_{m,p}(0) \]

and

\[ L_m(\theta, t) := (-1)^m \sum_{n=2}^{\infty} (i\theta)^n - \{ \mu_{n+m-2,m-2}(t) - \mu_{n+m-2,m-2} \} \]

Interchange of summation and integration in the derivation of (20) is

allowed because of uniform convergence of \( \sum_{n=2}^{\infty} (i\theta)^n \mu_{n+p,p}(t) \).

With \( \sum_{q=0}^{\infty} (\frac{\xi+1}{\theta}) B_q = \frac{1}{\theta} \) \( \xi > 0 \) we transform

\[ \sum_{n=2}^{\infty} (i\theta)^n \mu_{n+p,p} \]

\[ = \frac{(e^{i\theta} - 1)}{i\theta} \sum_{q=0}^{p+1} \frac{B_{q}(i\theta)^q}{q!} - \frac{1}{(i\theta)^{p+3}} \sum_{\xi=0}^{p+1} \frac{(i\theta)^{\xi+1}}{(\xi+1)!} \sum_{q=0}^{\xi} \frac{B_q}{q!} \]

\[ = - \frac{(e^{i\theta} - 1)}{i\theta} \delta_{p+2}(\theta). \]

Here we have introduced

\[ (21) = (15) \delta_p(\theta) := \]

\[ = \left( \frac{\theta}{2} \cotan\left( \frac{\theta}{2} \right) - 1 \right) \sum_{q=2}^{p+1} \frac{B_{q}(i\theta)^q}{q!} (i\theta)^p, |\theta| \neq 2k\pi, k = 1,2 \text{ etc.} \]

\[ (21^*) \]

\[ \sum_{q=p}^{\infty} \frac{B_{q}(i\theta)^{q-p}}{q!} \text{ if } |\theta| < 2\pi \text{ (Abramowitz, [3], 4.3.70).} \]

Similarly we transform
\[ (-1)^m \int_0^l (e^t)^{-m} \frac{B(q)^q}{q!} \frac{e^t - 1}{i^m} \sum_{q=0}^{\infty} \frac{B(q)^q}{q!} (i^m)^q \]

and since (because of (19)) in the last sum

\[ \sum_{q=0}^{\infty} \frac{B(q)^q}{q!} (i^m)^q \]

and the rest follows from (22) and (22*) in the last sum

\[ \sum_{q=0}^{\infty} \frac{B(q)^q}{q!} (i^m)^q \]

or (Abramowitz, [3], 23.1.1)

\[ (22^*) = \frac{e^{\theta} - 1}{i^m} \sum_{q=0}^{\infty} \frac{B(q)^q}{q!} (i^m)^q \]

If \(|\theta| < 2\pi\) from (21*) and (22*) we have the elementary estimates

(Abramowitz, [3], 23.2.13-15):

\[ \left| \delta_{2p+1}(\theta) \right| \leq \frac{10/3}{(2\pi)^{2p+1}(1-c)} \left( |\theta|^2 \right) \]

Substitution of (21) and (22) in (20), summation over the points of integration \(x_j\) and estimation of the remainder as \(O(h^m)\) with (22) and (22*) for \(|\theta|<2\pi\) bounded results in
\[(24) \quad R = \sum_{k=2}^{m-1} (-1)^k \delta_k^k \delta_k^k(\theta) \{i \omega F_{\omega, h}(D^{k-1}f) - D^{k-1}f(x) e^{i \omega x} |_{x=0} \} + O(h^m). \]

If \(|\theta| < 2\pi\), from (22) and (23) we have the estimate for the order term in (24)

\[(25) \quad |O(h^m)| \leq \frac{\text{const.} M h^m}{(2\pi)^m}, \quad \text{where} \quad M_\theta := \max_{x \in [0, 1]} |D^m f(x)|. \]

From (24) by induction with respect to \(m\) we obtain our final result

\[(26) = (17) \quad R = \sum_{k=2}^{m-1} \delta_k^k(\theta) L_{\omega, k}(D^k f) h^k + O(h^m), \quad f \in C^{m-1}[0, 1], \]

for values of \(|\theta|\) bounded and bounded away from \(2k\pi, k = 1, 2, \text{etc.}\) with the coefficients \(\delta_k^k(\theta)\) defined by the recurrence relation

\[(27) = (16) \quad \delta_k^k(\theta) := -\left\{ \sum_{p=2}^{k-1} (-1)^p i \theta \delta_p^p(\theta) \delta_k-p^k(\theta) + (-1)^k \delta_k^k(\theta) \right\}. \]

If \(|\theta| < 2\pi\) the estimate for the order term in (26)

\[|O(h^m)| \leq \frac{\text{const.}}{(2\pi)^2} \left(1 + \frac{10/3}{1-c} m^{-2} M h^m \right) \]

or (18) holds by induction from (23) and (25).

**Remarks.**

i) For \(\omega = 0\), so \(\theta = 0\), (17) indeed reduces to the Euler-Maclaurin expansion for the trapezoidal rule.

ii) The first coefficients \(\delta_k^k(\theta)\) are

\begin{align*}
\delta_2^2(\theta) &= -\delta_2^2(\theta) \\
\delta_3^3(\theta) &= i \theta \{ \delta_4^4(\theta) + \delta_2^2(\theta) \} \\
\delta_4^4(\theta) &= -\{ \delta_4^4(\theta) + (i \theta)^2 (\delta_2^2(\theta) + 2 \delta_2^2(\theta) \delta_2^4(\theta)) \} \\
\delta_5^5(\theta) &= i \theta \{ \delta_6^6(\theta) + 2 \delta_2^2(\theta) \delta_4^4(\theta) + (i \theta)^2 (\delta_2^4(\theta) + 3 \delta_2^4(\theta) \delta_2^4(\theta) + \delta_4^4(\theta)) \}.
\end{align*}

§ 2. Modified Romberg extrapolation (4)

Now we compare classical Romberg extrapolation based on (2) for successively halved intervals:
\( \tilde{F}_\omega(f) := \hat{F}_\omega(f) \)

\( \tilde{F}_\omega(f) := \frac{\tilde{F}_\omega(f) - \tilde{F}_\omega(2h(f))}{2^{j-1}(2^j-1)} \), \( j > 1 \),

and a modified version of Romberg extrapolation based on (3):

\[
\begin{align*}
\tilde{F}_\omega(f) &:= \tilde{F}_\omega(f) \\
(28) &= (4) \\
\tilde{F}_\omega(f) &:= \tilde{F}_\omega(f) + \gamma_j(\theta)(\tilde{F}_\omega(f) - \tilde{F}_\omega(2h(f))) \), \( j > 1 \),
\end{align*}
\]

with the coefficients

\[
\begin{align*}
\gamma_j(\theta) &:= \frac{1}{(2^j \beta_j(\theta)) (2^j \beta_j(\theta))} \), \( j > 1 \\
\beta_k(1)(\theta) &:= \beta_k(\theta) \\
\beta_k(j)(\theta) &:= \beta_k(j-1)(\theta) + \gamma_j(\theta)(\beta_k(j-1)(\theta) - 2 \beta_k(j-1)(\theta)) \), \( k > j \).
\end{align*}
\]

Firstly an obvious advantage of classical Romberg is that we don't need to evaluate the complicated \( \theta \)-dependent functions \( \gamma_j(\theta) \) (\( j > 1 \)) for every next halving of \( h \).

Secondly, we remark that both methods are asymptotically equivalent for \( \theta \to 0 \) since \( \gamma_j(\theta) \approx 1/(2^j \beta_j(\theta)) \) for \( \theta \to 0 \).

So, to motivate the extra trouble of evaluating the coefficients \( \gamma_j(\theta) \) in our modification, we have to prove a substantial gain in accuracy over classical Romberg for relatively large values of \( |\theta| \). Since \( |\theta| \) must be less than \( 2\pi \) anyhow if we want regular convergence (\( |R| \leq \text{const. } h^2 \)) of \( \hat{F}_\omega(f) \) to \( I_\omega(f) \), it is sufficient to prove this higher accuracy for \( \theta \approx \pi \).

Now the \( k \)-th term \( \alpha_{2k}^r(\omega,f) \) of (2) is a sum from the \( k \)-th term of the power series expansion about \( \theta = 0 \) of \( \beta_2(\theta)I_\omega(D^2f) \) until the first term of the expansion of \( \beta_2k(\theta)I_\omega(D^{2k}f) \) in (3). Assuming that all \( I_\omega(D^kf) \), \( k = 2,3, \text{etc.} \), are of the same order of magnitude \( M \), we conclude that after elimination of \( k \) terms in (2) because of \( |\omega| \gg 1 \) the remainder is of the order of \( \frac{\beta_k^2}{(2^k-1)!} \beta_k^2 \approx \frac{M^2}{(2\pi)^{2k}} \approx \frac{M^2}{2^k \pi^k} \) and in (3) the remainder is of the order of \( |\theta| \approx \pi \).
Thus, from \((\theta/2\pi)^2 \gg h\) for \(|\omega| \gg 1\), we conclude that our modification of Romberg extrapolation is an improvement of classical Romberg, if only for \(\theta \lesssim \pi\) we can bound the coefficients \(\beta_k(\theta)\) uniformly for all \(k \geq 2\). This finally we prove in Theorem 3.

**Theorem 3.** The coefficients \(\beta_k(\theta)\) defined by (16) are bounded uniformly for all \(k \geq 2\) and \(|\theta| \leq \pi\).

**Proof.** By a straightforward proof by induction we derive the following bounds:

\[
\begin{align*}
|\beta_{2m}(\theta)| & \leq \frac{1}{2\pi} \left(\frac{10/3}{2\pi(1-c)}\right)^{2m-1} |\varepsilon_m| \\
|\beta_{2m+1}(\theta)| & \leq \frac{1}{2\pi} \left(\frac{10/3}{2\pi(1-c)}\right)^{2m} \frac{\theta}{2\pi} |\varepsilon_{2m+1}|
\end{align*}
\]

(30)

for any sequence \(\varepsilon\) satisfying the relations

\[
\begin{align*}
|\varepsilon_2| & = 1 \\
|\varepsilon_{2m-1}| & \geq \frac{10}{9} |\varepsilon_{2m-2}| + \left(\frac{1-c}{10/3}\right)^{2m-3} \\
|\varepsilon_{2m}| & \geq |\varepsilon_{2m-1}| + \left(\frac{1-c}{10/3}\right)^{2m-2}, \quad m > 1.
\end{align*}
\]

(31)

(The induction hypothesis is: the inequalities

\[
|\varepsilon_{2m}| \geq c \sum_{p=1}^{m-1} \left(\frac{1-c}{10/3}\right)^{2p-2} |\varepsilon_{2m-2p+1}| + \sum_{p=1}^{m-1} \left(\frac{1-c}{10/3}\right)^{2p-1} |\varepsilon_{2m-2p}|
\]

and (30) hold for \(m\)).

The sequence \(\varepsilon_{2m} := \left(\frac{14}{9}\right)^{m-1}\), \(\varepsilon_{2m+1} := \left(\frac{14}{9}\right)^{m-0.2}\) (\(m \geq 1\)) satisfies (31).

So, obviously the coefficients \(\beta_k\) are uniformly bounded if \(\left|\frac{10/3}{2\pi(1-c)}\sqrt{\frac{1-c}{9}}\right| \leq 1\) and certainly if \(c \leq \frac{1}{4}\) or \(|\theta| \leq \pi\). \(\square\)

**Remark.**

From (29) the first coefficients \(\gamma_2, \gamma_3\) are calculated with result (5).
References


