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Abstract

Two asymptotic expansions are derived for the error in the Filon-trapezoidal rule (a numerical integration method for integrals of Fourier type

\[ \int_0^1 f(x) e^{i\omega x} dx \text{ with } |\omega| >> 1 \].

Classical Romberg extrapolation can be based on the first expansion. However, because of the dependence on \( \omega \) of the coefficients it is successful only in the very limit \( \theta := \omega h \to 0 \). With the second expansion a modification of Romberg extrapolation is introduced that coincides with classical Romberg for \( \theta \to 0 \), but gives improved results already for \( \theta \approx \pi \).
Introduction

Of all methods of Filon type for numerical integration of

\[ I_\omega(f) := \int_0^1 f(x)e^{i\omega x}dx, \quad |\omega| >> 1, \]

the Filon-trapezoidal rule (Tuck, 1967, [1])

\[ FT_{\omega,h}(f) := \]

\[ = \left(\begin{array}{l l l l}
\sin(\frac{j\theta}{2})^2 h & \sum_{j=0}^{N-1} (\frac{1 + i\theta - e^{-i\theta}}{\theta^2})f_j e^{i\omega x_j} & + \left(\frac{1 - i\theta - e^{i\theta}}{\theta^2}\right)f_{j+1} e^{i\omega x_{j+1}}
\end{array}\right) \]

\[ = \left(\begin{array}{l l l l}
\sin(\frac{j\theta}{2})^2 h & \sum_{j=0}^{N-1} f_j e^{i\omega x_j} & + \left(1 - \frac{\sin(\theta)}{\theta}f(x) e^{i\omega x}\right)_j^0
\end{array}\right) \]

\[ (h := 1/N, x_j := jh, \theta := \omega h, \sum_{j=0}^N = \sum_{j=0}^N \text{with coefficient } \frac{1}{2} \text{ for } j = 0, N) \]

is the most simple, but it is a method of only low order accuracy:

\[ R := I_\omega(f) - FT_{\omega,h}(f) = 0(h^2), \quad |\theta| \leq 2\pi, f \in C^2[0,1]. \]

To obtain more accurate results from a sequence \( FT_{\omega,h}(f) \) for decreasing \( h_m \to 0 \), we show the possibility of Romberg extrapolation based on the asymptotic expansion

\[ R = \left[ \frac{(m-1)/2}{k=1} \right] a_{2k}(\omega,f)h^{2k} + 0(h^m), \quad h \to 0, f \in C^m[0,1] \quad (\text{Th. 1}), \]

and of a modification of Romberg extrapolation based on

\[ R = \left[ \sum_{k=2}^{m-1} \beta_k(\theta)I_\omega(D^k f)h^k + 0(h^m), \quad |\theta| < 2\pi, f \in C^m[0,1] \quad (\text{Th. 2}). \]
For the sequence \( h_m := h_{m-1}/2 \) the modified Romberg extrapolation is defined by

\[
\begin{align*}
\text{FT}^{(1)}_{\omega,h}(f) & := \text{FT}_{\omega,h}(f) \\
\text{FT}^{(j)}_{\omega,h}(f) & := \text{FT}^{(j-1)}_{\omega,h}(f) + \gamma_j(\theta) \left( \text{FT}^{(j-1)}_{\omega,2h}(f) - \text{FT}^{(j-1)}_{\omega,h}(f) \right), \quad j > 1,
\end{align*}
\]

where the first two coefficients \( \gamma \) are

\[
\begin{align*}
\gamma_2(\theta) & := -\left( \frac{\theta}{2} \cotan\left( \frac{\theta}{2} \right) \right) \left( \frac{\theta}{2} \cotan\left( \frac{\theta}{2} \right) - 1 \right) / \left( \frac{\theta}{2} \right)^2 \\
\gamma_3(\theta) & := \frac{\theta \cotan(\theta)}{((\theta \cotan(\theta) - 1 + \frac{1}{3} \theta^2) / \theta^2)}.
\end{align*}
\]

Finally we demonstrate the usefulness of the modification by a closer analysis of (2) and (3), estimating the order of magnitude of the coefficients \( b_k \) in (3) (Th. 3).

Remark. \( \text{FT}^{(2)}_{\omega,h}(f) \) in (4) is the classical Filon (-Simpson) approximation to \( I_\omega(f) \) with step \( h \).

\section{The asymptotic expansions (2) and (3)}

By the method of Peano (Davis, Rabinowitz, 1967, [2]) we derive an expression for the truncation error:

\[
R = h^3 \sum_{j=0}^{N-1} \left\{ D^2 f(x_j + th) e^{i\omega(x_j + th)} K_2(\theta,t) \right\} dt
\]

with

\[
K_2(\theta,t) := (te^{-i\theta(t-1)} - (t - 1)e^{-i\theta} - 1)/\theta^2,
\]

or

\[
R = h^3 \sum_{j=0}^{N-1} e^{i\omega x_j} \int_0^1 D^2 f(x_j + th) L_2(\theta,t) dt
\]

with

\[
L_2(\theta,t) := e^{i\theta t} K_2(\theta,t) = \left( 1 - e^{i\theta} + t(e^{i\theta} - 1) \right)/\theta^2.
\]
From (6) we prove:

**Theorem 1 (Asymptotic expansion of R in even powers of h with \( \omega \)-dependent coefficients)**

Let the coefficients \( \lambda_{2k,p} (k \geq 1, 0 \leq p \leq 2k-2) \) be defined by

\[
\lambda_{2k,p} := \frac{2(2k+1-p)}{(2k+2)^2} \sum_{q=0}^{p} \binom{2k+2}{q} B_q + \frac{2}{(2k+1)^2} \sum_{q=p+2}^{2k} \binom{2k+1}{q} B_q = \frac{2(2k+1-p)}{(2k+2)^2} \sum_{q=p+2}^{2k} \binom{2k+1}{q} B_q
\]

(8)

(\( B_q \) are the Bernoulli numbers) and \( a_{2k}(\omega,f) (k \geq 1) \) by

\[
a_{2k}(\omega,f) := (-1)^k \omega^{2k-2} \sum_{p=0}^{2k-2} \lambda_{2k,p} \int_{0}^{1} \frac{D^p D^2 f(x) e^{i\omega x}}{\omega^p} dx.
\]

(9)

Then for \( f \in C^m[0,1] \)

\[
R = \sum_{k=1}^{[m-1]/2} a_{2k}(\omega,f) h^{2k} + O(h^m), h \to 0.
\]

(10)= (2)

**Proof.** In (6) we write

\[
K_2(\theta,t) = \sum_{n=2}^{\infty} g_n(t)(-i\theta)^{n-2}
\]

introducing \( g_n(t) := ((t-1)t^n - (t-1)^n t)/n! \). The expansion of the polynomial \( g_n(t) \) as a sum of Bernoulli polynomials

\[
g(t) = \int_{0}^{1} g(t) dt + B_1(t)(g(1) - g(0)) + \frac{B_2(t)}{2!} (g'(1) - g'(0)) etc.
\]

is

\[
g_n(t) = -\frac{(1 + (-1)^n)}{(n+2)!} + \sum_{q=2}^{n} \frac{B_q(t)}{q!} (q-1) \frac{(1 + (-1)^{n-q})}{(n - q + 2)!}.
\]

(11)
Integrating (6) by parts \((m-2)\) times using the expansions (11),
\[
\frac{d}{dt} \frac{B_{q+1}(t)}{(q+1)!} = \frac{q}{q!} \quad \text{and} \quad B_q(1) = B_0(0) = B_q \quad (q \geq 2),
\]
we derive for \(f \in C^m[0,1]\)
\[
R = - \sum_{n=2}^{\infty} (-i\theta)^{n-2} \sum_{p=0}^{m-3} (-1)^p p^{p+2} \lambda_{n+p,p} \int_0^1 D^p \{D^2 f(x)e^{i\omega x}\} dx +
\]
\[
+ \ h^m \sum_{j=0}^{N-1} \int_0^1 D^{m-2} \{D^2 f(x)e^{i\omega x}\} |_{x_j} + t \ K_m(\theta, t) dt,
\]
with the definitions
\[
\lambda_{m,p}(t) := \sum_{q=2}^{m-p} \frac{\frac{B_{q+p}(t)}{(q+p)!}}{q!} (q - 1) \frac{(1 + (-1)^{m-p-q})}{(m-p-q+2)!},
\]
\[
\lambda_{m,p} := \lambda_{m,p}(0)
\]
and
\[
K_m(\theta, t) := (-1)^m \sum_{n=2}^{\infty} (-i\theta)^{n-2} (\lambda_{n+m-2,m-2}(t) - \lambda_{n+m-2,m-2}).
\]
Interchange of summation and integration in the derivation of (12) is allowed as
\[
\sum_{n=2}^{\infty} (-i\theta)^{n-2} \lambda_{n+p,p}(t) \quad (p \geq 0)
\]
is absolutely convergent for all \(\theta\), uniformly in \(t \in [0,1]\). This can be seen as follows:
since \(\lambda_{n+p,p}(t)\) can be written as
\[ \lambda_{n+p,p}(t) = \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} (q-1) \frac{(1+(-1)^{n+p-q})}{(n+p-q+2)!} - \]
\[ - p \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} (1+(-1)^{n+p-q}) \frac{(p+1-q)}{(n+p-q+2)!} - \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} \frac{B_q(t)}{(n+p-q+2)!} (1+(-1)^{n+p-q}) , \]

or, with (11) and similar expansions of \( \tilde{g}_{n,p}(t) := (t^{n+1} - (t-1)^{n+1})/(n+1)! \), as

\[ \lambda_{n+p,p}(t) = g_{n+p}(t) - \tilde{g}_{n+p}(t) + (p+1) \frac{(1+(-1)^{n+p})}{(n+p+2)!} + \]
\[ + pB_q(t) \frac{(1+(-1)^{n+p-1})}{(n+p+2)!} + \sum_{q=2}^{n+p} \frac{B_q(t)}{q!} (p+1-q) \frac{(1+(-1)^{n+p-q})}{(n+p-q+2)!} , \]

obviously \( |\lambda_{n+p,p}(t)| \leq \text{const}(p)/n! \), a sufficient condition for uniform absolute convergence of

\[ \sum_{n=2}^{\infty} (-i\theta)^{n-2} \lambda_{n+p,p}(t) . \]

If in (12) we estimate the remainder with (14) it follows

\[ R = - \sum_{n=2}^{\infty} (-i\theta)^{n-2} \sum_{p=0}^{m-3} (-1)^p \lambda_{n+p,p} \int_0^1 p^p \{ D^2 f(x) e^{i\omega x} \} dx + o(h^m), h \to 0 . \]

Writing \( \theta = \omega h \), interchanging summation indices and estimating terms of higher order in \( h \) than \( (m-1) \) we obtain our final result

\[ R = - \sum_{k=2}^{m-1} h^k (-i\omega)^{k-2} \sum_{p=0}^{k-2} \lambda_{k,p} \int_0^1 \{ D^k f(x) e^{i\omega x} \} dx + o(h^m), h \to 0, \text{ for } f \in C^m[0,1] , \]

an expansion in even powers of \( h \) because \( \lambda_{\text{odd},p} = 0 \).

From (13) one easily derives the second expression for \( \lambda_{2k,p} \) in (8) and with \( \sum_{q=0}^{k-1} (k) \frac{B_q}{q!} = 0 \) \( (k \geq 2) \) the first expression, completing the proof. \( \square \)

Remark. \( \lambda_{2k,2k-2} = B_{2k}/(2k)! \). So for \( \omega = 0 \), when the Filon trapezoidal rule reduces to the trapezoidal rule for \( f \), indeed (10) reduces to the Euler-Maclaurin expansion:

\[ R = - \sum_{k=1}^{[(m-1)/2]} B_{2k} (2k)! \{ \int_0^1 D^{2k} f(x) dx \} h^k + o(h^m), h \to 0, \text{ for } f \in C^m[0,1] . \]
From (7) we prove:

**Theorem 2 (Asymptotic expansion of R with \( \theta \)-dependent coefficients)**

Let the coefficients \( \delta_p(\theta) \) \( (p \geq 2) \) be defined as

\[
\delta_p(\theta) := \left( \frac{\theta}{2} \cot(\frac{\theta}{2}) - 1 \right) - \frac{p-1}{q} B(q) \frac{\theta^q}{(q^2 - \theta^2)}/(i\theta)^p, \quad |\theta| \neq 2k\pi, \quad k = 1,2 \text{ etc.}
\]

and the coefficients \( \beta_k(\theta) \) \( (k \geq 2) \) by the recurrence relation

\[
\beta_k(\theta) := -\left( \sum_{p=2}^{k-1} (-1)^p i\theta \delta_p(\theta) \delta_{k-p+1}(\theta) + (-1)^k \delta_k(\theta) \right).
\]

Then, if \( f \in C^m[0,1] \),

\[
R = \sum_{k=2}^{m-1} \beta_k(\theta) I_n(D^k f) h^k + O(h^m),
\]

for \( |\theta| \) bounded and bounded away from \( 2k\pi \), \( k = 1,2 \) etc.

If \( |\theta| < 2\pi \) we have the estimate for the order term in (17)

\[
|O(h^m)| \leq \text{const.}\left\{ \frac{13/3}{2\pi(1 - (\theta/2\pi)^2)} \right\}^{m-2} \max_{x \in [0,1]} |D^m f(x)| h^m.
\]

**Proof.**

The first part of the proof is analogous to that of Th.2 but now we start from (7) instead of (6). We write \( L_2(\theta,t) = \sum_{n=2}^{\infty} g_n(t) \theta^{n-2} \) introducing \( g_n^*(t) := (t^n - t)/n! \) with the Bernoulli expansion

\[
g_n^*(t) = \frac{(1-n)}{2(n+1)!} + \sum_{q=2}^{n} \frac{B_q(t)}{q!(n-q+1)!}.
\]

Integrating (7) by parts \( (m-2) \) times we have for \( f \in C^m[0,1] \)

\[
R = h \sum_{j=0}^{N-1} e^{j\omega x} \left[ \sum_{n=2}^{\infty} (i\theta)^{n-2} \left( \sum_{p=0}^{m-3} (-1)^p h^{p+1} \mu_{n+p,p} D^{p+1} f(x) \right) x_j^{x+1} 
+ h^m \int_0^1 D^m f(x) \left| x_j + \text{th} L_{m}(\theta,t) \right| dt \right].
\]
with the definitions

\[
\mu_{m,p}(t) := - \sum_{q=2}^{m-p} \frac{B_{q+p}(t)}{(q+p)!(m-p-q+1)!} = \sum_{q=0}^{p+1} \frac{B_q}{q!(m-q+1)!},
\]

\[
\mu_{m,p} := \mu_{m,p}(0)
\]

and

\[
L_m(\theta,t) := (-1)^m \sum_{n=2}^{\infty} \frac{(i\theta)^{n-2}}{n-2} \{ \mu_{n+m-2,m-2}(t) - \mu_{n+m-2,m-2} \}
\]

Interchange of summation and integration in the derivation of (20) is allowed because of uniform convergence of \( \sum_{n=2}^{\infty} \frac{(i\theta)^{n-2}}{n-2} \mu_{n+p,p}(t) \).

With \( \sum_{q=0}^{\infty} (\frac{\xi+1}{q})B_q = \frac{1}{\xi}, \xi > 0 \) we transform

\[
\sum_{n=2}^{\infty} \frac{(i\theta)^{n-2}}{n-2} \mu_{n+p,p}
\]

\[
= \frac{e^{i\theta}-1}{i\theta} \sum_{q=0}^{p+1} \frac{B_q(i\theta)^q}{q!} - \frac{1}{(i\theta)^p+3} \sum_{q=0}^{p+1} \frac{(i\theta)^{q+1}}{(q+1)!} \sum_{q=0}^{\frac{p+1}{\xi}} \frac{(\xi+1)}{q} B_q
\]

\[
= - \frac{e^{i\theta}-1}{i\theta} \delta_{p+2}(\theta).
\]

Here we have introduced

\[(21) = (15) \delta_p(\theta) := \]

\[
= \frac{\theta}{2} \cotan\left(\frac{\theta}{2}\right) - \sum_{q=2}^{p-1} \frac{B_q(i\theta)^q}{q!}/(i\theta)^p, |\theta| \neq 2k\pi, k = 1,2 \text{ etc.}
\]

\[(21^*) = \sum_{q=p}^{\infty} \frac{B_q(i\theta)^q}{q!} \quad \text{if } |\theta| < 2\pi \text{ (Abramowitz, [3], 4.3.70).}
\]

Similarly we transform
(-1)^m L_m(\theta, t)

= \sum_{n=2}^{\infty} \frac{(i\theta)^{n-2}}{(n+2)!} \left( \frac{t^{n+m-2}}{(n+m-2)!} - \sum_{q=0}^{m-1} \frac{B_q(t) - B_q}{q!(n+m-1-q)!} \right)

= \frac{1}{(i\theta)^m} \left( e^{i\theta t} \left( \frac{e^{i\theta} - 1}{i\theta} \right) \sum_{q=0}^{m-1} \frac{B_q(t) - B_q}{q!} - \frac{B_q(t) - B_q}{q!} \right)

\quad - \frac{1}{(i\theta)^m} \sum_{q=0}^{m-1} \frac{B_q(t) - B_q}{q!} \right)

and since (because of (19)) in the last sum

\frac{t^\ell}{\ell!} - \frac{t^\ell}{\ell!} \left( \frac{q}{q!} \right)^{\ell+1-q} = 0, \ell > 0.

(22) \quad = \frac{1}{(i\theta)^m} \left( e^{i\theta t} \left( \frac{e^{i\theta} - 1}{i\theta} \right) \sum_{q=0}^{m-1} \frac{B_q(t)(i\theta)^q}{q!} - \frac{e^{i\theta} - 1}{i\theta} \delta_m(\theta) \right)

or (Abramowitz, [3], 23.1.1)

(22*) \quad = \frac{e^{i\theta} - 1}{i\theta} \sum_{q=m}^{\infty} \frac{B_q(t)(i\theta)^q}{q!} \quad \text{if } |\theta| < 2\pi.

If |\theta| < 2\pi from (21*) and (22*) we have the elementary estimates (Abramowitz, [3], 23.2.13-15):

\begin{align*}
|\delta_{2p}(\theta)| & \leq \frac{10/3}{(2\pi)^{2p}(1-c)} \quad (c := (\theta/2\pi)^2)
|\delta_{2p+1}(\theta)| & \leq \frac{10/3}{(2\pi)^{2p+1}(1-c)} \quad \left| \frac{\theta}{2\pi} \right|
|L_m(\theta, t)| & \leq 2 \left| \frac{e^{i\theta} - 1}{i\theta} \right| \frac{10/3}{(2\pi)^m(1-|\theta|/(2\pi))} \leq \text{const.} \quad (2\pi)^m
\end{align*}

Substitution of (21) and (22) in (20), summation over the points of integration x_j and estimation of the remainder as O(h^m) with (22) and (22*) for |\theta| bounded results in
from (22*) and (23) we have the estimate for the order term in (24)

\[ |0(h^m)| \leq \frac{\text{const.} M h^m}{(2\pi)^m}, \text{ where } M := \max_{x \in [0,1]} |D^m f(x)|. \]

From (24) by induction with respect to \( m \) we obtain our final result

\[ (26) = (17) R = \sum_{k=2}^{m-1} \delta_k(\theta) I^k \omega h^k + O(h^m), f \in C^m[0,1], \]

for values of \( |\theta| \) bounded and bounded away from \( 2k\pi \), \( k = 1,2, \text{ etc. with the coefficients } \delta_k(\theta) \text{ defined by the recurrence relation} \]

\[ (27) = (16) \delta_k(\theta) := -\frac{1}{p=2} \sum_{k} (-1)^p i^p \delta_{k-p}(\theta) + (-1)^{k} \delta_{k}(\theta). \]

If \( |\theta| < 2\pi \) the estimate for the order term in (26)

\[ |0(h^m)| \leq \frac{\text{const.}}{(2\pi)^2} \left( 1 + \frac{10/3}{1-c} m^2 \right) M h^m \]

or (18) holds by induction from (23) and (25).

Remarks.

i) For \( \omega = 0 \), so \( \theta = 0 \), (17) indeed reduces to the Euler-Maclaurin expansion for the trapezoidal rule.

ii) The first coefficients \( \delta_k(\theta) \) are

\[ \begin{align*}
\delta_2(\theta) &= -\delta_2(\theta) \\
\delta_3(\theta) &= i\theta \{ 2\delta_4(\theta) + \delta_2^2(\theta) \} \\
\delta_4(\theta) &= -(\delta_4(\theta) + (i\theta)^2 (\delta_2^3(\theta) + 2\delta_2^2(\theta) \delta_4(\theta))) \\
\delta_5(\theta) &= i\theta \{ \delta_6(\theta) + 2\delta_2(\theta) \delta_4(\theta) + (i\theta)^2 (\delta_2^4(\theta) + 3\delta_2^3(\theta) \delta_4(\theta) + \delta_2^2(\theta) \delta_4(\theta)) \}
\end{align*} \]

§ 2. Modified Romberg extrapolation (4)

Now we compare classical Romberg extrapolation based on (2) for successively halved intervals:
\[ \tilde{\mathcal{F}}_{\omega,h}(f) := FT_{\omega,h}(f) \]

\[ \tilde{\mathcal{F}}_{\omega,h}(f) := \tilde{\mathcal{F}}_{\omega,h}(f) + \{ \tilde{\mathcal{F}}_{\omega,h}(f) - \tilde{\mathcal{F}}_{\omega,h}(f) \}/(2^{j-1}-1), \quad j > 1, \]

and a modified version of Romberg extrapolation based on (3):

\[ \left\{ \begin{array}{l}
\tilde{\mathcal{F}}_{\omega,h}(f) := FT_{\omega,h}(f) \\
\tilde{\mathcal{F}}_{\omega,h}(f) := \tilde{\mathcal{F}}_{\omega,h}(f) + \gamma_j(\theta)\{ \tilde{\mathcal{F}}_{\omega,h}(f) - \tilde{\mathcal{F}}_{\omega,h}(f) \}, \quad j > 1,
\end{array} \right. \]

with the coefficients

\[ \gamma_j(\theta) := 1/(2^j\beta_j(j-1)(2\theta)/\beta_j(j-1) - 1), \quad j > 1 \]

\[ \beta_k(1)(\theta) := \beta_k(\theta) \]

\[ \beta_k(j)(\theta) := \beta_k(j-1)(\theta) + \gamma_j(\theta)\{ \beta_k(j-1)(\theta) - 2\beta_k(j-1)(2\theta) \}, \quad k > j. \]

Firstly an obvious advantage of classical Romberg is that we don't need to evaluate the complicated \( \theta \)-dependent functions \( \gamma_j(\theta) \) for every next halving of \( h \).

Secondly, we remark that both methods are asymptotically equivalent for \( \theta \to 0 \) since \( \gamma_j(\theta) \to 1/(2^{j-1}-1) \) for \( \theta \to 0 \).

So, to motivate the extra trouble of evaluating the coefficients \( \gamma_j(\theta) \) in our modification, we have to prove a substantial gain in accuracy over classical Romberg for relatively large values of \( |\theta| \). Since \( |\theta| \) must be less than 2\( \pi \) anyhow if we want regular convergence \( |R| \leq \text{const. } h^2 \) of \( FT_{\omega,h}(f) \) to \( I_{\omega}(f) \), it is sufficient to prove this higher accuracy for \( \theta \simeq \pi \).

Now the \( k \)-th term \( \alpha_{2k}(\omega,f) \) of (2) is a sum from the \( k \)-th term of the power series expansion about \( \theta = 0 \) of \( \beta_2(\theta)I_{\omega}(D^2f) \) until the first term of the expansion of \( \beta_{2k}(\theta)I_{\omega}(D^{2k}f) \) in (3). Assuming that all \( I_{\omega}(D^k f), k = 2, 3 \) etc., are of the same order of magnitude \( M \), we conclude that after elimination of \( k \) terms in (2) because of \( |\omega| \gg 1 \) the remainder is of the order of \( M \theta^2 2^k \simeq \frac{M \theta^2}{(2\pi)^2} \frac{2k}{2^k} \) and in (3) the remainder is of the order of \( |\beta_{k+1}(\theta)| M \theta^{k+1} \).
Thus, from \((\theta/2\pi)^2 \gg h\) for \(|\theta| \gg 1\), we conclude that our modification of Romberg extrapolation is an improvement of classical Romberg, if only for \(|\theta| \leq \pi\) we can bound the coefficients \(\beta_k(\theta)\) uniformly for all \(k \geq 2\). This finally we prove in Theorem 3.

**Theorem 3.** The coefficients \(\beta_k(\theta)\) defined by (16) are bounded uniformly for all \(k \geq 2\) and \(|\theta| \leq \pi\).

**Proof.** By a straightforward proof by induction we derive the following bounds:

\[
\begin{align*}
|\beta_{2m}(\theta)| &\leq \frac{1}{2\pi} \left(\frac{10/3}{2\pi(1-c)}\right)^{2m-1} \varepsilon_{2m} \tag{30} \\
|\beta_{2m+1}(\theta)| &\leq \frac{1}{2\pi} \left(\frac{10/3}{2\pi(1-c)}\right)^{2m} \frac{\theta}{2\pi} \varepsilon_{2m+1}
\end{align*}
\]

for any sequence \(\varepsilon\) satisfying the relations

\[
\begin{align*}
\varepsilon_2 &= 1 \\
\varepsilon_{2m-1} &\geq \frac{10}{9} \varepsilon_{2m-2} + \left(\frac{1-c}{10/3}\right)^{2m-3} \\
\varepsilon_{2m} &\geq \varepsilon_{2m-1} + \left(\frac{1-c}{10/3}\right)^{2m-2}, \quad m > 1.
\end{align*}
\tag{31}
\]

(The induction hypothesis is: the inequalities

\[
\begin{align*}
\varepsilon_{2m} &\geq c \sum_{p=1}^{m-1} \left(\frac{1-c}{10/3}\right)^{2p-2} \varepsilon_{2m-2p+1} + \sum_{p=1}^{m-1} \left(\frac{1-c}{10/3}\right)^{2p-1} \varepsilon_{2m-2p} \\
\varepsilon_{2m+1} &\geq \sum_{p=1}^{m} \left(\frac{1-c}{10/3}\right)^{2p-2} \varepsilon_{2m-2p+2} + \sum_{p=1}^{m} \left(\frac{1-c}{10/3}\right)^{2p-1} \varepsilon_{2m-2p+1}
\end{align*}
\]

and (30) hold for \(m\).

The sequence \(\varepsilon_{2m} := \left(\frac{14}{9}\right)^{m-1}, \varepsilon_{2m+1} := \left(\frac{14}{9}\right)^{m-0.2} (m \geq 1)\) satisfies (31).

So, obviously the coefficients \(\beta_k\) are uniformly bounded if \(\left|\frac{10/3}{2\pi(1-c)}\sqrt{\frac{1-c}{9}}\right| \leq 1\) and certainly if \(c \leq \frac{1}{4}\) or \(|\theta| \leq \pi\). \(\square\)

**Remark.**

From (29) the first coefficients \(\gamma_2, \gamma_3\) are calculated with result (5).
References


