Solution to Problem 86-5*: Conjectured trigonometric identities

Also solved by the proposer who uses Whipple's quadratic ${}_3F_2$ transformation as a generating function and who draws attention to the case $c=v$ as well as $c=u$.

Conjectured Trigonometrical Identities

Problem 86-5*, *by* M. HENKEL (University of Bonn, W. Germany) *and* J. LACKI (University of Geneva, Switzerland).

In computing low-temperature series for a Z_n-symmetric Hamiltonian [1] (in statistical mechanics), we apparently discovered some new trigonometrical identities. These were verified up to $n=100$ on a computer. Prove or disprove the following conjectured identities for general n.

In what follows, we use the notation,

$$\omega=\exp\left(\frac{2\pi i}{n}\right), \qquad \theta(x)=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x\leqq 0,\end{cases}$$

and

$$\sum_r \text{ stands for } \sum_{k_r=1}^{n-1}, \qquad \sum_{r\neq 1} \text{ stands for } \sum_{k_r=1,k_r\neq k_1}^{n-1}.$$

(1) $$\sum_1\sum_{2\neq 1}[(1-\omega^{k_1})(1-\omega^{k_2-k_1})(1-\omega^{-k_2})]^{-1}=0.$$

(2) $$\sum_1\sum_{2\neq 1}\sum_{3\neq 2}[(1-\omega^{k_1})(1-\omega^{k_2-k_1})(1-\omega^{k_3-k_2})(1-\omega^{-k_3})]^{-1}=\frac{(n^2-1)(n^2-4)}{180}.$$

(3) $$\sum_1\sum_2\frac{1}{1+\theta(k_2-k_1)}\left[\sin^2\frac{\pi k_1}{n}\sin^2\frac{\pi k_2}{n}\right]^{-1}=\frac{(n^2-1)(4n^2-1)}{45}.$$

(4) $$\sum_1\sum_2\sum_3\frac{1}{1+\theta(k_2-k_1)+\theta(k_3-k_2)}\left[\sin^2\frac{\pi k_1}{n}\sin^2\frac{\pi k_2}{n}\sin^2\frac{\pi k_3}{n}\right]^{-1}=\frac{(n^2-1)(22n^4-13n^2+15)}{945}.$$

(5) $$\sum_{k=1}^{n-1}k\sin^{-2}\frac{\pi k}{n}=\frac{n(n^2-1)}{6}.$$

(6) $$\sum_1\sum_{2\neq 1}(k_1+k_2)[(1-\omega^{k_1})(1-\omega^{k_2-k_1})(1-\omega^{-k_2})]^{-1}=0.$$

(7) $$\sum_1\sum_2\frac{k_1+k_2}{1+\theta(k_1-k_2)}\left[\sin^2\frac{\pi k_1}{n}\sin^2\frac{\pi k_2}{n}\right]^{-1}=\frac{n(n^2-1)(4n^2-1)}{45}.$$

REFERENCE

[1] M. HENKEL AND J. LACKI, Bonn preprint, submitted to J. Phys. A.

Editorial note. The proposers also had eleven more similar conjectured identities with triple, quadruple and quintuple summations.

Composite solution by S. W. GRAHAM *and* O. RUEHR (Michigan Technological University), O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands), *and* M. RENARDY (University of Wisconsin).

All of the conjectured identities are true. Identities 1 and 6 follow from symmetry arguments alone. The remaining identities reduce to evaluation of the sums

$$S_p = \sum_{k=1}^{n-1} \csc^{2p} \frac{\pi k}{n}, \qquad p = 1, 2, 3.$$

The required sums,

$$S_1 = \frac{n^2-1}{3}, \qquad S_2 = \frac{(n^2-1)(n^2+11)}{45},$$

$$S_3 = \frac{(n^2-1)(2n^4+23n^2+191)}{945},$$

may be found in [1] or may be computed using any one of several methods. For example, S_p may be derived by considering the contour integral

$$\int_C \cot z \csc^{2p} \frac{z}{n}\, dz$$

where C consists of the vertical line $\operatorname{Re} z = -\delta$ and $\operatorname{Re} z = n\pi - \delta$, $0 < \delta < \pi$, traversed in opposite directions. Clearly, the integral vanishes, so by residue calculus one has

$$S_p = -\operatorname*{Res}_{z=0} \cot z \csc^{2p} \frac{z}{n}.$$

We now turn to the conjectured identities. Let $T_1, T_2, \cdots, T_7$ denote the given sums.

To prove the first identity, note that the summand changes sign if we reverse the roles of k_1 and k_2. Thus the terms cancel in pairs and the sum vanishes. The same argument proves the sixth identity as well.

To prove the second identity, use the trigonometric formula $\csc x \csc y = \csc(x+y)[\cot x + \cot y]$ to write the summand as

$$\frac{1}{16} \csc \frac{\pi k_1}{n} \csc \frac{\pi k_3}{n} \csc \frac{\pi(k_2-k_1)}{n} \csc \frac{\pi(k_2-k_3)}{n}$$

$$= \frac{1}{16} \csc^2 \frac{\pi k_2}{n} \left(\cot \frac{\pi k_1}{n} + \cot \frac{\pi(k_2-k_1)}{n} \right) \left(\cot \frac{\pi k_3}{n} + \cot \frac{\pi(k_2-k_3)}{n} \right).$$

Since $\sum_{k=1}^{n-1} \cot \pi k/n = 0$, we have

$$\sum_{1\neq 2} \cot \frac{\pi k_1}{n} = \sum_{1\neq 2} \cot \frac{\pi(k_2-k_1)}{n} = -\cot \frac{\pi k_2}{n}$$

and

$$\sum_{3\neq 2} \cot \frac{\pi k_3}{n} = \sum_{3\neq 2} \cot \frac{\pi(k_2-k_3)}{n} = -\cot \frac{\pi k_2}{n}.$$

Thus

$$T_2 = \frac{1}{4} \sum_{k=1}^{n-1} \csc^2 \frac{\pi k}{n} \cot^2 \frac{\pi k}{n}$$

$$= \frac{1}{4}(S_2 - S_1) = \frac{(n^2-1)(n^2-4)}{180}.$$

To prove the third identity, write

$$T_3=\sum_1\sum_2 W(k_1,k_2)\csc^2\frac{\pi k_1}{n}\csc^2\frac{\pi k_2}{n}$$

where

$$W(k_1,k_2)=\frac{1}{2}\left\{\frac{1}{1+\theta(k_2-k_1)}+\frac{1}{1+\theta(k_1-k_2)}\right\}=\begin{cases}\frac{3}{4} & \text{if } k_1\neq k_2,\\ 1 & \text{if } k_1=k_2.\end{cases}$$

It follows that

$$T_3=\frac{3}{4}S_1^2+\frac{1}{4}S_2=\frac{(n^2-1)(4n^2-1)}{45}.$$

Identity 4 may be established similarly. Write

$$T_4=\sum_1\sum_2\sum_3 W(k_1,k_2,k_3)\csc^2\frac{\pi k_1}{n}\csc^2\frac{\pi k_2}{n}\csc^2\frac{\pi k_3}{n}$$

where

$$W(k_1,k_2,k_3)=\frac{1}{6}\sum\frac{1}{1+\theta(k_2-k_1)+\theta(k_3-k_2)},$$

the sum extending over all permutations of k_1, k_2, k_3. It is readily found that

$$W(k_1,k_2,k_3)=\begin{cases}\frac{5}{9} & \text{if } k_1,k_2,k_3 \text{ are distinct,}\\ \frac{2}{3} & \text{if exactly two of } k_1,k_2,k_3 \text{ are equal,}\\ 1 & \text{if } k_1=k_2=k_3.\end{cases}$$

We thus find

$$T_4=\frac{5}{9}S_1^3+\frac{1}{3}S_1S_2+\frac{1}{9}S_3=\frac{(n^2-1)(22n^4-13n^2+15)}{945}.$$

Finally, we make use of the fact that $\sin \pi k/n=\sin \pi(n-k)/n$ to reduce identities 5 and 7 to known sums. In particular, to evaluate T_5, write the sum a second time with k replaced by $n-k$ and add the two sums. We thus find $2T_5=nS_1$, so

$$T_5=\frac{n(n^2-1)}{6}.$$

Apply the same technique to T_7, replacing k_1 by $n-k_1$ and k_2 by $n-k_2$. Thus $2T_7=2nT_3$, so

$$T_7=\frac{n(n^2-1)(4n^2-1)}{45}.$$

REFERENCE

[1] M. E. FISHER, *Solution to Problem 69-14*, Sums of Inverse Powers of Cosines*, by L. A. GARDINER, JR., this Review, 13 (1971), pp. 116–119.

Also solved by C. GEORGHIOU (University of Patras, Greece), W. B. JORDAN (Scotia, New York), R. RICHBERG (Institut für Reine und Angewandte Mathematik der RWTH, Aachen, W. Germany) *and* W. VAN ASSCHE (Katholieke Universiteit Leuven, Belgium). *Solutions of all but the second identity were given* *by* D. FOULSER (Saxpy Computer Corporation, Sunnydale, California) *and* A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands). *Other partial solutions were provided by* J. W. S. CASSELS (Cambridge University, England), S. C. DUTTA ROY (Indian Institute of Technology, New Delhi, India) *and* U. EVERLING (Bonn, W. Germany).

Editorial note. A variety of methods was proposed for evaluating S_p. Different starting points included the Mittag–Leffler expansion of $\csc^2 z$ (A. A. Jagers), Chebyshev polynomials (W. van Assche), eigenvalues of the matrix

$$D=\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ & \ddots & \ddots & \\ & & & -1 \\ 0 & & -1 & 2 \end{bmatrix}$$

(D. Foulser) and the formula $\pi^2 \csc^2 \pi x = \psi'(1-x) + \psi'(x)$, $0 < x < 1$, where ψ denotes the digamma function (R. Richberg).

Late Solution

Problem 85-24: *by* S. LJ. DAMJANOVIĆ (TANJUG Telecommunicate Center, Belgrade, Yugoslavia).