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Global Solutions to the Complete Vehicle Energy Management Problem via Forward-Backward Operator Splitting

G.P. Padilla, G. Belgioioso, M.C.F. Donkers

Abstract—Complete Vehicle energy Management (CVEM) aims to minimize the energy consumption of all subsystems in a vehicle. We consider the case where the subsystems consist of energy buffers with linear dynamics and/or energy converters with quadratic power losses. In this paper, we show the existence of only global solutions for the CVEM optimal control problem and propose a reformulation of this problem so that it can be solved using a Forward-Backward splitting algorithm for nonconvex optimization problems. The regularization properties inherent to this algorithm allow us to solve CVEM cases that were difficult using other approaches as dual decomposition.

I. INTRODUCTION

Improving energy efficiency of vehicles is an important topic of research for the automotive industry. While for traditional and hybrid vehicles, a better energy efficiency leads to lower emissions, the main motivation to improve efficiency of electric vehicles is that it will extend the range of the vehicle and thereby mitigating range anxiety. Range anxiety is the concern experienced by the user to be unable to reach a final destination with the current energy [1]. An energy management strategy (EMS) aims at reducing the energy consumption by an optimal distribution of power among the powertrain components (to provide the required power to the wheels) and the auxiliary systems. In essence, the EMS is the solution to an optimal control problem and optimization techniques for energy control on hybrid vehicles have been discussed in [2], [3]. A recent trend in this research area is to consider all the energy consumers inside the vehicle and this concept is known as Complete Vehicle Energy Management (CVEM) [4]. CVEM requires the optimal control problem to be scalable, as it requires a larger number of subsystems to be connected to the power network compared to earlier solutions for EMS.

Static optimization techniques have emerged as tractable approaches to tackle the CVEM problem, see, e.g., [5]–[8]. In particular, [6] has incorporated the control of auxiliary systems into the EMS and [5] has used convex relaxations to guarantee optimality of the solution. However, due to the use of centralized optimization methods, those approaches are not flexible in the sense that subsystems cannot be easily added or removed. The research presented in [7] uses a game-theoretic approach to solve the CVEM problems in

a decentralized manner, where all the subsystems share a limited amount of information and are able to take some decisions autonomously. Global optimality of the centralized optimal control problem is not guaranteed.

A static distributed optimization approach for CVEM is presented in [8]. The method proposed in these papers are scalable and flexible. The main idea is to use dual decomposition to split the original optimal control problem into several simple problems related to interconnected subsystems. As a result, the computational time is drastically reduced and adding and removing subsystems becomes easy. However, several open questions exist related to numerical aspects of the algorithms proposed to solve the distributed optimization problem. For instance, [8] proposes a second-order dual update for which the convergence of the algorithm has not been formally proven. Additionally, the approach requires components to be described with linear dynamics and quadratic energy conversion models. In case the energy conversion is almost linear, the distributed optimization approach of [8] becomes ill conditioned, causing the algorithm to have difficulties to converge.

In this paper, we design an efficient algorithm with convergence guarantee for the CVEM problem, based on the Forward-Backward (FB) operator splitting method [9, §5.1]. First, we show that the non-convex CVEM problem only has global optimal solutions. Then, by taking advantage of the globality of the solutions, we massage the FB splitting method onto the CVEM problem to obtain a parallelizable static optimization algorithm. The convergence analysis of the proposed algorithm is based on a recent result on projected-gradient dynamics for non-convex optimization problems [9, §5.3]. Remarkably, the use of this projection method introduces regularization that prevents ill-conditioning of the optimization problem. This means that convergence is possible even if linear models are used to describe the power consumption of the vehicle subsystems.

Nomenclature: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, and $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ the set of extended real numbers. For $a, b \in \mathbb{R}$, a significant strict inequality that is denoted by $a \ll b$ implies that a is much less than b . The matrices $\mathbf{0}$ and $\mathbf{1}$ denote matrices with all elements equal to 0 and 1, respectively, and the matrix $\mathbf{I}$ denotes the identity matrix. To improve clarity, we sometimes add the dimension of these matrices as subscript. Furthermore, $\text{diag}(A_1, \dots, A_N)$ denotes a block-diagonal matrix with $A_1, \dots, A_N$ as diagonal blocks. Given N vectors $x_1, \dots, x_N \in \mathbb{R}^n$, we denote $\text{col}(\{x_i\}_{i \in \{1, \dots, N\}}) = [x_1^\top, \dots, x_N^\top]^\top$. The short hand notation $\{x_{p,q}\}$ is used to

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The authors are with the Dep. of Electrical Eng. of Eindhoven University of Technology, Netherlands. E-mail: {g.p.padilla.cazar, g.belgioioso, m.c.f.donkers}@tue.nl

denote $\{x_{p,q}\}_{p \in \mathcal{P}, q \in \mathcal{Q}}$.

Given a set $\mathcal{S} \subseteq \mathbb{R}^n$, the mapping $\iota_{\mathcal{S}} : \mathbb{R}^n \rightarrow \{0, +\infty\}$ denotes the indicator function satisfying $\iota_{\mathcal{S}}(x) = 0$ if $x \in \mathcal{S}$ and $\iota_{\mathcal{S}}(x) = \infty$ if $x \notin \mathcal{S}$, and set-valued mapping $N_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator satisfying $N_{\mathcal{S}}(x) = \{v \in \mathbb{R}^n \mid \sup_{z \in \mathcal{S}} v^\top(z - x) \leq 0\}$ if $x \in \mathcal{S}$ and $N_{\mathcal{S}}(x) = \emptyset$ if $x \notin \mathcal{S}$. The mapping $\text{proj}_{\mathcal{S}} : \mathbb{R}^n \rightarrow \mathcal{S}$ for a closed set $\mathcal{S} \subseteq \mathbb{R}^n$ denotes the projection onto $\mathcal{S}$, i.e., $\text{proj}_{\mathcal{S}}(x) = \arg\min_{y \in \mathcal{S}} \|y - x\|^2$.

For a function $\psi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ with $\text{dom}(\psi) := \{x \in \mathbb{R}^n \mid \psi(x) < \infty\}$, the subdifferential set-valued mapping $\partial\psi : \text{dom}(\psi) \rightrightarrows \mathbb{R}^n$ is defined as $\partial\psi(x) := \{v \in \mathbb{R}^n \mid \psi(z) \geq \psi(x) + v^\top(z - x) \text{ for all } z \in \text{dom}(\psi)\}$. In case ψ is continuously differentiable, the subgradient is equal to its gradient, i.e., $\partial\psi(x) = \nabla\psi(x)$.

II. CVEM AS OPTIMAL CONTROL PROBLEM

In this section, we first discuss a mathematical description of a power network model for CVEM. Later, we will formulate the discrete optimal control problem for CVEM, which will turn out to be non-convex, and we will show that all the solutions to this non-convex optimal control problem are global optimal solutions. Finally, we will propose a convenient equivalent formulation of the problem that will be exploited in Section III to find an optimization algorithm based on operator splitting techniques.

A. Power Network Model for CVEM

The CVEM problem aims to minimize the energy consumption for a network of subsystems $m \in M := \{1, \dots, M\}$, where M is the total number of subsystems. These subsystems are composed of an energy converter possibly combined with an energy buffer. For instance, considering a battery, the capacity of storing energy is modeled as an energy buffer, while the power losses produced during the transformation of chemical energy into electrical energy in the battery are represented by an energy converter.

A power network for CVEM is schematically depicted in Fig. 1. It can be noted that the energy buffers are always connected to energy converters by the power input $u_{m,k}$. On the other hand, the converters are connected to each other according to a specific topology via the network nodes $j \in \mathcal{J} := \{1, \dots, J\}$, where J is the total number of nodes in the network. In this case, power outputs $y_{m,k}$ and inputs $u_{m,k}$ of the converters could be directly connected to a network node. Every node $j \in \mathcal{J}$ can have a known exogenous load signal $w_{j,k}$ given for each time instant k . The following assumption on the network topology is considered in this paper.

Assumption 1: The power network has a tree structure topology, i.e., every subsystem is connected to only one node, and two consecutive nodes are always bridged by an individual power converter. Moreover, only converters can be connected directly to a network node. $\square$

Note that, albeit of its simplicity, the network presented in Fig. 1 contains the essential features described in Assumption 1, which can be found in more complicated power networks.

Fig. 1: Power network for CVEM.

B. Optimal Control Problem

The CVEM problem is an optimal control problem that aims to minimize the total aggregated energy consumption of the components over a time horizon $\mathcal{K} := \{0, 1, \dots, K-1\}$, while considering the interaction of all the components interconnected in the power network. Thus, the optimal control problem is given by

$$\min_{\{y_{m,k}, u_{m,k}, x_{m,k}\}} \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} a_m y_{m,k} + b_m u_{m,k}, \quad (1a)$$

where $x_{m,k} \in \mathbb{R}^{n_m}$ are the states, $u_{m,k} \in \mathbb{R}$ are the (scalar) inputs and $y_{m,k} \in \mathbb{R}$ are the (scalar) outputs of the converter in subsystem $m \in \mathcal{M}$, while $a_m \in \mathbb{R}_+$ and $b_m \in \mathbb{R}$ are coefficients to define the objective function. A typical objective in CVEM is to minimize the fuel consumption, in which case assume the subsystem $m = 1$ corresponds to the combustion engine and $y_{1,k}$ denotes the chemical fuel power flow at time $k \in \mathcal{K}$. In this case, only $a_1 = 1$, while all other a_m and b_m are zero.

The objective function (1a) is to be solved subject to a quadratic equality constraint that describes the input-output behavior of each converter, i.e.,

$$y_{m,k} = \frac{1}{2} q_{2,m} u_{m,k}^2 + q_{1,m} u_{m,k} + q_{0,m}, \quad (1b)$$

where $q_{0,m} \in \mathbb{R}$, $q_{1,m} \in \mathbb{R}$ and $q_{2,m} \in \mathbb{R}_+$ are efficiency coefficients of the converter $m \in \mathcal{M}$, and subject to the linear system dynamics of the energy buffer

$$x_{m,k+1} = A_m x_{m,k} + B_m u_{m,k}, \quad (1c)$$

for all $k \in \mathcal{K}$ with $A_m \in \mathbb{R}^{n_m \times n_m}$ and $B_m \in \mathbb{R}^{n_m \times 1}$. The initial state $x_{m,0}$ of the storage device is assumed to be given and the inputs and states are subject to

$$\underline{x}_m \leq x_{m,k} \leq \bar{x}_m \quad \text{and} \quad \underline{u}_m \leq u_{m,k} \leq \bar{u}_m, \quad (1d)$$

where for all $m \in \mathcal{M}$ the given state and input bounds are respectively $\underline{x}_m, \bar{x}_m \in \mathbb{R}^{n_m}$ and $\underline{u}_m, \bar{u}_m \in \mathbb{R}$. In [8], it has been shown that quadratic static models for energy converters and linear dynamical models for buffers are an adequate approximation to describe typical components in CVEM.

The interaction between the subsystems in the power network is given by the power balance at each node $j \in \mathcal{J}$. We distinguish between energy conserving nodes, given by

$$\sum_{m \in \mathcal{M}} c_{j,m} y_{m,k} + d_{j,m} u_{m,k} + w_{j,k} = 0, \quad (1e)$$

for $j \in \mathcal{J}_c = \{1, 2, \dots, J_c\}$ and energy dissipating nodes

$$\sum_{m \in \mathcal{M}} c_{j,m} y_{m,k} + d_{j,m} u_{m,k} + w_{j,k} \leq 0, \quad (1f)$$

for $j \in \mathcal{J}_d = \{J_c + 1, J_c + 2, \dots, J\}$. Observe that $\mathcal{J}_c \cap \mathcal{J}_d = \emptyset$ and $\mathcal{J}_c \cup \mathcal{J}_d = \mathcal{J}$. In these expressions, $c_{j,m}$ is 1 if the correspondent power signal $y_{m,k}$ is connected to the node j and 0 otherwise. The constant $d_{j,m}$ is -1 if the respective power signal $u_{m,k}$ flows into node j , it is 1 if the power flows out of node j , and 0 if the respective signal is not connected to the node. It should be noted that dissipating nodes (1f) exist, e.g., because mechanical braking can be modeled as a power dissipation in a node. Finally, due to Assumption 1, we have that for every $m \in \mathcal{M}$, $c_{j,m} = 1$ for only a single $j \in \mathcal{J}$.

C. Global Solutions to the Nonconvex CVEM Problem

The optimal control problem (1) is non-convex due to (1b). This might cause solvers to get stuck in a local minimum. In this section, we show that all (local) solutions to (1) are global solutions under very mild conditions.

To show that (1) has only global solutions, we relax the equality constraint (1b) to an inequality constraint, i.e.,

$$\frac{1}{2}q_{2,m}u_{m,k}^2 + q_{1,m}u_{m,k} + q_{0,m} - y_{m,k} \leq 0 \quad (1b')$$

for $m \in \mathcal{M}$ and $k \in \mathcal{K}$, allowing us to define a relaxed optimal control problem

$$\begin{aligned} \min_{\{y_{m,k}, u_{m,k}, x_{m,k}\}} \quad & \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} a_m y_{m,k} + b_m u_{m,k} \\ \text{s.t.} \quad & (1b'), (1c) - (1f). \end{aligned} \quad (2)$$

The discrete optimal control problem (2) is convex, thus has only global solutions. The optimal value of the original problem (1) and of the convex relaxation (2) satisfy

$$p^{\text{CR}} \leq p^{\text{NC}}, \quad (3)$$

where p^{CR} and p^{NC} denote the optimal value of the convex relaxation (2) and the non-convex optimal control problem (1), respectively, because the convex relaxation has a larger feasible set due to (1b').

The next Theorem states that for this particular optimal control problem and its relaxation, it holds that, $p^{\text{CR}} = p^{\text{NC}}$ and that (2) always has a solution that satisfies (1b') with equality, thus it satisfies (1b) and solves (1).

Theorem 1: Assume that the bounds $\{u_m\}_{m \in \mathcal{M}}$ and $\{\bar{u}_m\}_{m \in \mathcal{M}}$ are finite and that there exists a feasible point $\{u_{m,k}, y_{m,k}, x_{m,k}\}_{m \in \mathcal{M}, k \in \mathcal{K}}$ for (2) with strict inequalities (1b') and (1d). Then, an optimal solution to (2) exists that satisfies (1b') with equality, $p^{\text{CR}} = p^{\text{NC}}$, and (1) only has global optimal solutions.

D. An Equivalent Formulation of the CVEM Problem

In this subsection, we will reformulate the optimal control problem (1) into an equivalent form that will later be recast as a static optimization problem that is suitable to be solved with the operator splitting technique presented in Section III. The reformulation of (1) considers three main steps that are described and justified as follows:

- 1) Substitution of the quadratic equality constraint (1b) into (1a), (1e), (1f). This substitution aims to reduce the number of decision variables by the elimination of $\{y_{m,k}\}_{m \in \mathcal{M}, k \in \mathcal{K}}$ from the optimal control problem.

- 2) Conversion of the inequality constraint (1f) into an equality constraint using slack variables $\{z_{j,k}\}_{j \in \mathcal{J}, k \in \mathcal{K}}$, of which some are constrained to $z_{j,k} = 0$ for all $j \in \mathcal{J}_c$ and $k \in \mathcal{K}$. This yields that the Lagrangian of the optimal control problem (4) is a mapping from $\mathbb{R}$ to $\mathbb{R}$, which is necessary for the algorithm that will be proposed Section III, see [9, §5].

- 3) Introduction of quadratic penalty terms associated to the equality constraints (1e) and (1f). This is done without removing these constraints. The resulting Lagrangian function of the optimal control problem can be seen as an augmented Lagrangian function. The advantage of this formulation is that it introduces regularization to the optimization procedure, thereby improving the convergence properties of the algorithm [10, §3.2.1, §4.2].

The equivalent optimal control problem obtained as a result of the above steps is given by

$$\begin{aligned} \min_{\{u_{m,k}, x_{m,k}, z_{j,k}\}} \quad & \sum_{k \in \mathcal{K}} \left(\sum_{m \in \mathcal{M}} p_m(u_{m,k}) + \sum_{j \in \mathcal{J}} \frac{\sigma_j}{2} \left(\sum_{m \in \mathcal{M}} r_{j,m}(u_{m,k}) + \frac{1}{2} z_{j,k}^2 \right)^2 \right) \end{aligned} \quad (4a)$$

s.t. (1c), (1d)

$$\sum_{m \in \mathcal{M}} r_{j,m}(u_{m,k}) + \frac{1}{2} z_{j,k}^2 = 0 \text{ for all } j \in \mathcal{J}, \quad (4b)$$

$$z_{j,k} = 0 \text{ for all } j \in \mathcal{J}_c, \quad (4c)$$

where $\sigma_j \in \mathbb{R}_+$ are coefficients that weight the penalty terms in (4a). Furthermore,

$$p_m(u_{m,k}) = a_m \left(\frac{1}{2} q_{2,m} u_{m,k}^2 + q_{1,m} u_{m,k} + q_{0,m} \right) + b_m u_{m,k}, \quad (5)$$

$$r_{j,m}(u_{m,k}) = c_{j,m} \left(\frac{1}{2} q_{2,m} u_{m,k}^2 + q_{1,m} u_{m,k} + q_{0,m} \right) + d_{j,m} u_{m,k} + w_j. \quad (6)$$

It should be noted that constraint (4c) acts only on the non dissipating nodes $\mathcal{J}_c$, which means that the equality constraint (1e) is embedded in (4b) for this formulation.

The equivalence of the discrete time optimal control problems (1) and (4) is the main result of this section. In order to present this result, we will indicate a constraint qualification (CQ) that (4) satisfies. For a complete survey of CQ for non-linear programming see [11]. In the following lemma, we present a condition to guarantee that (4) satisfies a linear independence CQ (LICQ).

Lemma 1: The feasible set of the discrete optimal control problem (4) satisfies LICQ, if for all $m \in \mathcal{M}$ the bounds on $u_{m,k}$ in (1d) satisfy

$$\underline{u}_m > -\frac{q_{1,m}}{q_{2,m}} \quad \text{or} \quad \bar{u}_m < -\frac{q_{1,m}}{q_{2,m}}. \quad (7)$$

The condition to guarantee LICQ of discrete optimal control problem (4) presented in this lemma can be tested a priori via a simple inspection of the power bounds for all the subsystems. For realistic applications, it typically holds that

$|q_{2,m}| \ll |q_{1,m}|$, which suggests that (7) is a mild condition. The satisfaction of LICQ by the optimal control problem (4) indicates that its critical points are regular. This will be exploited in the following Theorem to show the equivalence between the problem formulations (1) and (4) in terms of the Karush-Kuhn-Tucker (KKT) conditions for optimality.

Theorem 2: The optimization problems (1) and (4) have the same necessary conditions for optimality, and same global minimizers, if the conditions in Lemma 1 are satisfied.

III. A FORWARD-BACKWARD ALGORITHM FOR CVEM

In the previous section, we showed that the CVEM problem (1) only has global solutions, thus solvers cannot get stuck in local minima. Moreover, it was demonstrated that (4) is an equivalent formulation to (1). In this section, we will take advantage of the previous results to propose a method to solve the CVEM problem (1) using an operator splitting approach. To achieve this we will recast the equivalent optimal control problem (4) into a non-convex static optimization problem. Then, we will split the cost function of this static optimization problem as the sum of a non-convex function and a convex set-valued function. This allows the forward-backward splitting method for non-convex optimization problems to be applied.

A. Reformulation as a Static Optimization Problem

To reformulate the optimal control problem (4) as a static optimization problem, we define the following vectors

$$\mathbf{u}_m = \text{col}(\{u_{m,k}\}_{k \in \mathcal{K}}) \in \mathbb{R}^K, \quad (8a)$$

$$\mathbf{x}_m = \text{col}(\{x_{m,k}\}_{k \in \mathcal{K}}) \in \mathbb{R}^{K n_m}, \quad (8b)$$

$$\mathbf{z}_j = \text{col}(\{z_{j,k}\}_{k \in \mathcal{K}}) \in \mathbb{R}^K, \quad (8c)$$

$$\mathbf{w}_j = \text{col}(\{w_{j,k}\}_{k \in \mathcal{K}}) \in \mathbb{R}^K, \quad (8d)$$

for $m \in \mathcal{M}$ and $j \in \mathcal{J}$. This notation allows us to write the solutions to (1c) in the compact form

$$\mathbf{x}_m = \Phi_m x_{m,0} + \Gamma_m \mathbf{u}_m, \quad (9)$$

with matrices $\Phi_m \in \mathbb{R}^{K n_m \times n_m}$ and $\Gamma_m \in \mathbb{R}^{K n_m \times K}$, and write (1d) as

$$x_m \mathbf{1} \leq \mathbf{x}_m \leq \bar{x}_m \mathbf{1} \quad \text{and} \quad u_m \mathbf{1} \leq \mathbf{u}_m \leq \bar{u}_m \mathbf{1}. \quad (10)$$

Now, by exploiting (8) - (10), we rewrite the discrete optimal control problem (4) as the following static optimization problem:

$$\min_{\{\mathbf{u}_m, \mathbf{z}_j\}} \sum_{m \in \mathcal{M}} P_m(\mathbf{u}_m) + \sum_{j \in \mathcal{J}} \frac{\sigma_j}{2} \left\| \sum_{m \in \mathcal{M}} R_{m,j}(\mathbf{u}_m) + \frac{1}{2} \mathbf{z}_j^2 \right\|_2^2 \quad (11a)$$

$$\text{s.t.} \quad \sum_{m \in \mathcal{M}} R_{m,j}(\mathbf{u}_m) + \frac{1}{2} \mathbf{z}_j^2 = \mathbf{0}, \quad \text{for all } j \in \mathcal{J} \quad (11b)$$

$$\mathbf{z}_j = \mathbf{0}, \quad \text{for all } j \in \mathcal{J}_c \quad (11c)$$

$$\mathbf{u}_m \in \Omega_m, \quad \text{for all } m \in \mathcal{M}, \quad (11d)$$

where

$$P_m(\mathbf{u}_m) = \mathbf{1}_K^T (a_m \mathbf{y}_m(\mathbf{u}_m) + b_m \mathbf{u}_m), \quad (12a)$$

$$R_{m,j}(\mathbf{u}_m) = c_{j,m} \mathbf{y}_m(\mathbf{u}_m) + d_{j,m} \mathbf{u}_m + \mathbf{w}_j, \quad (12b)$$

$$\Omega_m = \{\mathbf{u}_m \in \mathbb{R}^K \mid \mathbf{1}_K x_m \leq \Phi_m x_{m,0} + \Gamma_m \mathbf{u}_m \leq \mathbf{1}_K \bar{x}_m, \mathbf{1}_K u_m \leq \mathbf{u}_m \leq \mathbf{1}_K \bar{u}_m\}, \quad (12c)$$

in which

$$\mathbf{y}_m(\mathbf{u}_m) = \text{col}(\{\frac{1}{2} q_{2,m} u_{m,k}^2 + q_{1,m} u_{m,k} + q_{0,m}\}_{k \in \mathcal{K}}). \quad (13)$$

This convenient reformulation shows a non-convex static optimization problem that will be used to design a parallelizable algorithm to solve the CVEM problem (1), based on the FB operator splitting method [9, §5.1].

B. KKT Conditions

Operator splitting methods [12, §26] and [13] can be used to find zeros of (set-valued) mappings. In constrained optimization theory, these splitting methods are used to find the points that satisfy the KKT conditions. In particular, constraints (11d) will be embedded in the optimization problem using indicator functions, which will make the optimization problem non-smooth. In doing so, the KKT conditions of the static optimization problem (11) can be represented as a set-valued mapping, which is shown in this subsection, so that (candidate) minimizers can be found using operator splitting methods.

Let us first introduce vectors $\mathbf{u} = \text{col}(\{\mathbf{u}_m\}_{m \in \mathcal{M}})$, $\mathbf{z} = \text{col}(\{\mathbf{z}_j\}_{j \in \mathcal{J}})$, $\boldsymbol{\lambda} = \text{col}(\{\boldsymbol{\lambda}_j\}_{j \in \mathcal{J}})$, and the Lagrangian function of (11), which is given by

$$\mathbf{L}(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda}) = h(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda}) + \sum_{m \in \mathcal{M}} \iota_{\Omega_m}(\mathbf{u}_m) + \sum_{j \in \mathcal{J}_c} \iota_{\{\mathbf{0}\}}(\mathbf{z}_j), \quad (14)$$

where $\iota_{\{\mathbf{0}\}}(\mathbf{z}_j)$ and $\iota_{\Omega_m}(\mathbf{u}_m)$ are the indicator functions corresponding to (11c) and (11d), respectively. Moreover,

$$h(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda}) = \sum_{m \in \mathcal{M}} P_m(\mathbf{u}_m) + \sum_{j \in \mathcal{J}} \left(\frac{\sigma_j}{2} \left\| \sum_{m \in \mathcal{M}} R_{m,j}(\mathbf{u}_m) + \frac{1}{2} \mathbf{z}_j^2 \right\|_2^2 + \boldsymbol{\lambda}_j^T \left(\sum_{m \in \mathcal{M}} R_{m,j}(\mathbf{u}_m) + \frac{1}{2} \mathbf{z}_j^2 \right) \right), \quad (15)$$

with Lagrange multipliers $\boldsymbol{\lambda}_j \in \mathbb{R}^K$, $j \in \mathcal{J}$. The specific design of the Lagrangian function (14) is connected to the splitting algorithm that will be presented in Section III-C. Namely, the linear part of the feasible set is expressed as indicator functions while the non-linear and possibly non-convex part of the problem (11) is embedded in the continuously differentiable function h . The linear part of the feasible set is also separable per $m \in \mathcal{M}$, which will make the algorithm parallelizable.

The KKT conditions of the augmented optimization problem (11) can be expressed in terms of the subdifferential of the Lagrangian function (14) as $\mathbf{0} \in \partial \mathbf{L}(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda})$, with

$$\partial \mathbf{L}(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda}) = \underbrace{\begin{bmatrix} \text{col}(\{\mathbf{F}_m(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda})\}_{m \in \mathcal{M}}) \\ \text{col}(\{\mathbf{G}_j(\mathbf{u}, \mathbf{z}_j, \boldsymbol{\lambda}_j)\}_{j \in \mathcal{J}_c}) \\ \text{col}(\{\mathbf{G}_j(\mathbf{u}, \mathbf{z}_j, \boldsymbol{\lambda}_j)\}_{j \in \mathcal{J}_d}) \\ \text{col}(\{-\boldsymbol{\lambda}_j(\mathbf{u}, \mathbf{z}_j)\}_{j \in \mathcal{J}}) \end{bmatrix}}_{:= \nabla h} + \underbrace{\begin{bmatrix} \text{col}(\{\mathbf{N}_{\Omega_m}(\mathbf{u}_m)\}_{m \in \mathcal{M}}) \\ \text{col}(\{\mathbf{N}_{\{\mathbf{0}\}}(\mathbf{z}_j)\}_{j \in \mathcal{J}_c}) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}}_{:= \mathbf{N}_{\Omega \times \mathbb{R}^{(J_d + J)K}}}, \quad (16)$$

where $\Omega = (\prod_{m \in \mathcal{M}} \Omega_m) \times \{\mathbf{0}\}^{J_c}$ and

$$\mathbf{F}_m(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda}) = \nabla P_m(\mathbf{u}_m) + \sum_{j \in \mathcal{J}_c} \nabla R_m(\mathbf{u}_m)(\boldsymbol{\lambda}_j + \sigma_j \mathbf{A}_j(\mathbf{u}, \mathbf{z}_j)), \quad (17a)$$

$$\mathbf{G}_j(\mathbf{u}, \mathbf{z}_j, \boldsymbol{\lambda}_j) = \text{diag}(\mathbf{z}_j)(\boldsymbol{\lambda}_j + \sigma_j \mathbf{A}_j(\mathbf{u}, \mathbf{z}_j)), \quad (17b)$$

$$\mathbf{A}_j(\mathbf{u}, \mathbf{z}_j) = \sum_{m \in \mathcal{M}} R_m(\mathbf{u}_m) + \frac{1}{2} \mathbf{z}_j^2, \quad (17c)$$

in which

$$\nabla P_m(\mathbf{u}_m) = a_m \mathbf{1}_K \nabla \mathbf{y}_m(\mathbf{u}_m) + b_m \mathbf{1}_K, \quad (18a)$$

$$\nabla R_m(\mathbf{u}_m) = c_{j,m} \nabla \mathbf{y}_m(\mathbf{u}_m) + d_{j,m} \mathbf{I}_K, \quad (18b)$$

$$\nabla \mathbf{y}_m(\mathbf{u}_m) = \text{diag}(\{q_{2,m} u_{m,k} + q_{1,m}\}_{k \in \mathcal{K}}). \quad (18c)$$

In (16), we characterized the KKT conditions related to the static optimization problem (11). Under certain regularity conditions, the points $\{\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*\}$ satisfying $\mathbf{0} \in \partial \mathbf{L}(\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*)$ provide candidate minima of the optimization problem (11), see [10, §3.3.1] and [14] for a detailed discussion on this topic. In the theorem below, we formally state that $\mathbf{0} \in \partial \mathbf{L}(\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*)$ leads to global minimizers to the discrete time optimal control problem (1).

Theorem 3: Suppose conditions of Lemma 1 are satisfied, and sets Ω_m for all $m \in \mathcal{M}$ are compact. Then, global solutions $\{y_{m,k}^*, u_{m,k}^*, x_{m,k}^*\}$ to the discrete time optimal control problem (1) can be obtained from points $\{\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*\}$ satisfying $\mathbf{0} \in \partial \mathbf{L}(\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*)$.

C. Forward-Backward (FB) Splitting Algorithm

In this section, we apply the FB splitting method presented in [9, §5.1], which will be used to find points $\text{col}(\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*)$ satisfying $\mathbf{0} \in \partial \mathbf{L}(\mathbf{u}^*, \mathbf{z}^*, \boldsymbol{\lambda}^*)$, thereby finding global minimizers of the optimal control problem in (1). It should be noted that for convex optimization problems, FB splitting leads to the widely used proximal gradient method (see [12, §26.5]), while recent extensions of the FB splitting show application to solve non-convex problems [9], [15].

To see how FB splitting is applied here, note that the right-hand side of (16) consists of two terms: ∇h , which is the gradient of the continuously differentiable function (15), and $\mathbf{N}_{\Omega \times \mathbb{R}^{(J_d+J)K}}$, which is the operator that contains the normal cones of the linear feasible set defined by constraints (11c) and (11d). Now let Φ be a preconditioning matrix, given by

$$\Phi := \text{diag}(\{\alpha_m\}_{m \in \mathcal{M}}, \{\theta_j\}_{j \in \mathcal{J}}, \{\gamma_j\}_{j \in \mathcal{J}}) \quad (19)$$

where α_m, γ_j and θ_j are positive real constants that define the step sizes of the proposed algorithm. The FB operator

Fig. 2: Power network topology for a parallel-hybrid vehicle.

splitting method [9, §5.1], applied on the mapping in (16) with preconditioning Φ reads as

$$\boldsymbol{\omega}^{i+1} = \text{proj}_{\Omega \times \mathbb{R}^{(J_d+J)K}}(\boldsymbol{\omega}^i - \Phi \nabla h(\boldsymbol{\omega}^i)), \quad (20)$$

with $\boldsymbol{\omega} = \text{col}(\mathbf{u}, \mathbf{z}, \boldsymbol{\lambda})$ and superscript $i \in \{1, 2, \dots\}$ denoting the i -th iteration. It should be noted that both the function ∇h as well as the projection are highly structured. This allows for a parallelizable algorithm. Furthermore, the projections on Ω_m can be implemented efficiently as a constrained least squares problem. The final algorithm is presented below.

ALGORITHM 1: Forward-Backward Splitting

Initialization:

Select step-sizes $\alpha_m, m \in \mathcal{M}$, $\gamma_j, j \in \mathcal{J}$ and $\theta_j, j \in \mathcal{J}_d$; select regularization weights $\sigma_j, j \in \mathcal{J}$ and select initial values $\mathbf{u}_m^0 \in \Omega_m$ for all $m \in \mathcal{M}$, $\mathbf{z}_j^0 = \mathbf{0}_{K \times 1}$ for all $j \in \mathcal{J}_c$ and all $i \in \mathbb{N}$, $\mathbf{z}_j^0 \in \mathbb{R}^K$ for all $j \in \mathcal{J}_d$, and $\boldsymbol{\lambda}_j^0 \in \mathbb{R}^K$ for all $j \in \mathcal{J}$.

Iterate until convergence:

$$\begin{aligned} \mathbf{u}_m^{i+1} &= \text{proj}_{\Omega_m}(\mathbf{u}_m^i - \alpha_m \mathbf{F}_m(\mathbf{u}^i, \mathbf{z}^i, \boldsymbol{\lambda}^i)), & \text{for all } m \in \mathcal{M} \\ \mathbf{z}_j^{i+1} &= \mathbf{z}_j^i - \theta_j \mathbf{G}_j(\mathbf{u}^i, \mathbf{z}_j^i, \boldsymbol{\lambda}_j^i), & \text{for all } j \in \mathcal{J}_d \\ \boldsymbol{\lambda}_j^{i+1} &= \boldsymbol{\lambda}_j^i + \gamma_j \mathbf{A}_j(\mathbf{u}^i, \mathbf{z}_j^i), & \text{for all } j \in \mathcal{J} \end{aligned}$$

It remains to show how to choose the step sizes in Algorithm 1 so that the algorithm converges. In [9, Th. 5.3], it is shown that if α_m, γ_j and θ_j are chosen sufficiently small, the algorithm converges. For more details, the reader is referred to [9].

IV. NUMERICAL EXAMPLE

In this section, we will use the method presented in Section III-C to solve the CVEM optimal control problem for a parallel-hybrid electric vehicle, consisting of an internal combustion engine (ICE), electric machine (EM) and a high-voltage battery (HVB). For notational convenience, we use the set $\mathcal{M} = \{\text{ice}, \text{em}, \text{hvb}\}$, instead of $\mathcal{M} = \{1, 2, 3\}$, and the set $\mathcal{J} = \{a, b\}$, instead of $\mathcal{J} = \{1, 2\}$.

The power network for this study case is presented in Fig. 2. There exist two nodes $\mathcal{J}$ that describe the mechanical and electrical power balance in the vehicle. These nodes are bridged by the EG that acts as a single power converter to transform power between the electrical and mechanical domains. The power network also contains exogenous signals that represent the power consumed by auxiliary subsystems $w_a = 1.5[kw]$ and the traction power request $w_b = y_{req}$, which is presented in Fig. 3a. The presence of the mechanical braking power $y_{br} \leq 0$ in node b implies that it is a dissipative node, thus $\mathcal{J}_c = \{a\}$ and $\mathcal{J}_d = \{b\}$. The parameters that describe the subsystems are given in Table I and by selecting, $d_{b,ice} = d_{a,em} = -1$, $c_{a,hvb} = c_{b,em} = 1$ and all other coefficients as zero. Thus, the power network is described by

$$y_{hvb,k} + y_{em,k} + w_{a,k} = 0, \quad (21)$$

$$-u_{ice,k} - u_{em,k} + w_{b,k} = y_{br,k} \leq 0. \quad (22)$$

Fig. 3: Numerical Example

TABLE I: Subsystem Parameters

ICE		
$q_{ice,2} = 0$	$q_{ice,1} = 1$	$q_{ice,0} = 0$
$\bar{u}_{ice} = 310[kw]$	$\bar{u}_{ice} = 0[kw]$	
EM		
$q_{em,2} = 2 \times 10^{-5}$	$q_{em,1} = 2.53$	$q_{em,0} = 19$
$\bar{u}_{em} = 210[kw]$	$\bar{u}_{em} = -210[kw]$	
HVB		
$q_{hvb,2} = 0.001671$	$q_{hvb,1} = -1$	$q_{hvb,0} = 0$
$\bar{u}_{hvb} = 92.4[kw]$	$\bar{u}_{hvb} = -92.4[kw]$	
$\bar{x}_{hvb} = 210[kwh]$	$\bar{x}_{hvb} = -210[kwh]$	
$x_{hvb,0} = 11988[kJ]$	$x_{hvb,K} = 11988[kJ]$	
$A_{hvb} = 1$	$B_{hvb} = -5$	

For this case, the CVEM is formulated to minimize the energy consumed by the ICE, which implies that $a_{ice} = 1$ while the other a_m and all b_m are zero. In order to obtain results that reflect the energy consumption only related to fuel, constraints on the initial and final charge states of the battery are imposed, i.e., $x_{hvb,0} = x_{hvb,K}$. It is important to remark that this constraint on the final state can be easily included in the feasible set (12c) for $m = hvb$, without affecting the main results and the methodology presented in this paper. Finally, it is important to mention that the problem has been discretized using a sampling time $\tau = 5[s]$, which leads to $A_{hvb} = 1$ and $B_{hvb} = -5$.

Given the fact that condition (7) of Lemma 1 can be easily verified for this example, we have that Theorem 3 states that Algorithm 1 gives a global solution. For this example, two possible solutions are presented in Fig. 3b and Fig. 3c, which are obtained using the same initial conditions ($u_m^0 = 0$, $z_j^0 = 0$ and $\lambda_j^0 = 0$ for all $m \in \mathcal{M}$ and $j \in \mathcal{J}$) for Algorithm 1. but using different step sizes, see Table II. The energy consumed in both cases is $9.889551[l]/100[km]$, which shows that both solutions are indeed global minimizers of the optimal control problem. The convergence of Solution 1 was obtained after 365 iterations of the algorithm, while Solution 2 converged after 432 iterations. This is a remarkable result due to the fact that the use of the dual decomposition approach proposed in [8], does not converge for this numerical example. The observed improvement in convergence is related to the regularization properties introduced by the projections performed in Algorithm 1.

V. CONCLUSIONS

In this paper, we have proven that the non-convex CVEM problem only has global optimal solutions under mild condi-

TABLE II: Parameters used for Algorithm 1.

	Solution 1	Solution 2
α_{ice}	100.9	80.9
α_{em}	2.1	2.1
α_{hvb}	62.1	62.1
γ_a	0.001	0.001
γ_b	0.01	0.001
θ_a	0.04	0.1
σ_a	0.008	0.008
σ_b	0.01	0.01

tions. Moreover, we exploited this result in order to propose a reformulation of the CVEM problem that can be solved using a FB splitting algorithm. The guaranteed convergence and the regularization properties of the algorithm allowed us to solve a CVEM problem for a parallel hybrid vehicle that presents almost linear power losses, which was not possible with approaches like dual decomposition.

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