Solving parabolic problems using local defect correction in combination with the finite volume method
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We present a method for solving Partial Differential Equations characterized by highly localized properties in which the Local Defect Correction (LDC) algorithm for time-dependent problems is combined with a finite volume discretization. At each time step, LDC computes a numerical solution on a composite grid, union of a global uniform coarse grid and a local uniform fine grid. The main feature of the method is that the discrete conservation property, typical of the finite volume approach, is preserved on the composite grid.

1 Introduction

Time-dependent Partial Differential Equations (PDEs) are often characterized by solutions with a high activity which is mainly concentrated in a limited part of the spatial domain. By high activity we mean relatively high values of the spatial and time derivatives. In order to efficiently solve this kind of problems numerically, it is important to make use of adaptive grid techniques. In adaptive grid methods, a fine grid spacing is adopted only when the large variations occur, so that the computational effort and the memory requirements are minimized.

An adaptive grid technique of particular interest is the Local Defect Correction (LDC) method. LDC was first introduced in [6] for solving elliptic boundary value problems on composite grids. LDC is analysed in combination with a finite differences discretization in [3, 4], and in combination with finite elements in [9]. In [8, 5] the method is studied with different grid types. In [7] the LDC technique is extended to time-dependent PDEs. In a time-dependent setting LDC works as follows: first a time step is performed on a global uniform coarse grid. The global solution at the new time level provides artificial boundary conditions on a local uniform fine grid. A solution is then computed locally, possibly with a smaller time step than the one adopted on the global grid. At this point the local approximation provides an estimate of the coarse grid discretization error or defect. The defect, added to the right hand side...
of the coarse grid problem, leads to determining a more accurate (both in space and time) global approximation of the solution. This latter can now be used to update the boundary conditions locally and the entire procedure can be repeated until convergence. Like shown in [1] for elliptic problems, the convergence of the LDC method is generally very fast. The main advantage of LDC is that one may always work with uniform structured grids and uniform grid solvers, dealing thus with rather simple data structures.

In this paper we apply the time-dependent LDC technique in combination with standard finite volume discretizations on the global and local grid. Unlike in [7], where the local grid is adaptively placed at each time step where the high activity occurs, we assume that the high activity of the solution is always located, at each time level, in the same (limited) part of the global domain. Already for this simple situation, in the LDC method as presented in [7] the discrete conservation law, which is one of the main attractive features of the finite volume method, does not hold for the solution on the composite grid. Here, we introduce a finite volume adapted LDC method for parabolic problems for which the conservation property is preserved.

In [2] a finite volume adapted LDC algorithm which is conservative on the composite grid is presented for elliptic problems. The main idea there is that the defect correction should be such that the integrated fluxes across the interface between the coarse and fine grid are balanced. In the time-dependent case we consider here, we extend that idea and write the defect term is such a way that the balance across the interface holds at every time level. In doing so we deal with the complication that the time integration might be performed with different time steps on the global and local grid.

This paper is structured as follows: in Section 2 we formulate a two-dimensional time-dependent convection-diffusion problem and we initialize the LDC algorithm by computing a numerical approximation of the solution on a composite grid at a certain time level. In Section 3 we focus on the expression of the defect term. In Section 4 we discuss the properties of the finite volume adapted LDC technique; in particular, we prove that a discrete conservation property holds on the composite grid. In Section 5 we present the results of some numerical experiments.

2 Problem formulation and initialization of the LDC algorithm

In order to present the LDC method for parabolic problems in combination with a finite volume discretization, we consider the two-dimensional convection-diffusion equation for a quantity \( \varphi = \varphi(\mathbf{x}, t) \). The equation can be expressed in integral formulation as follows

\[
\frac{\partial}{\partial t} \int_{V} \varphi \, d\Omega + \int_{\partial V} (\varphi \mathbf{u} - \lambda \nabla \varphi) \cdot \mathbf{n} \, d\gamma = \int_{V} s \, d\Omega \quad \text{in} \, \Theta, \quad \text{for all} \, V \subset \Omega, \tag{2.1}
\]

where \( \mathbf{u} = \mathbf{u}(\mathbf{x}, t) \) is a given velocity field, \( \lambda > 0 \) is a diffusion coefficient and \( s = s(\mathbf{x}, t) \) is a known source term. Furthermore \( \Theta \) is the time interval \([0, t_{\text{end}}]\), while \( V \) is a generic volume contained in the spatial domain \( \Omega := (0, 1) \times (0, 1) \subset \mathbb{R}^2 \). With \( \partial V \) we indicate the boundary of \( V \) and with \( \mathbf{n} \) the outward unit vector perpendicular to \( \partial V \). We also introduce \( \partial \Omega \), the boundary of \( \Omega \), and we define \( \bar{\Omega} := \Omega \cup \partial \Omega \). We close problem (2.1) by prescribing the Dirichlet boundary condition

\[
\varphi = \psi, \quad \text{on} \, \partial \Omega \times \Theta, \tag{2.2}
\]
and the initial condition
\[ \varphi(\mathbf{x}, 0) = \eta, \quad \text{in } \tilde{\Omega}. \] (2.3)

In (2.2) and (2.3), \( \psi = \psi(\mathbf{x}, t) \) and \( \eta = \eta(\mathbf{x}) \) are given functions. If all the variables in (2.1) are sufficiently smooth and the velocity field is divergence free (\( \nabla \cdot \mathbf{u} = 0 \)), the integral formulation (2.1) is equivalent to the differential formulation
\[ \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi - \lambda \nabla^2 \varphi = s \quad \text{in } \Omega \times \Theta. \] (2.4)

We introduce the flux vector \( \mathbf{f} \) by
\[ \mathbf{f} = \mathbf{f}(\varphi) = \left( \begin{array}{c} f_x \\ f_y \end{array} \right) := \varphi \mathbf{u} - \lambda \nabla \varphi, \] (2.5)
so that equation (2.1) can be rewritten as
\[ \frac{\partial}{\partial t} \int_V \varphi \, d\Omega + \int_{\partial V} \mathbf{f} \cdot \mathbf{n} \, dy = \int_V s \, d\Omega \quad \text{in } \Theta, \quad \text{for all } V \subset \Omega. \] (2.6)

We assume that \( \varphi \), the continuous solution of (2.6) that satisfies (2.2) and (2.3), at each time in \( \Theta \) presents a region of high activity that covers a (small) part of \( \Omega \).

Problem (2.6) is first discretized in space on a global uniform coarse grid using the finite volume method. We consider, in particular, a standard vertex-centered approach. This is just for notational convenience and the usage of other approaches, e.g. cell-centered discretization, would guarantee the same properties in the method we are describing. We introduce the grid size parameter \( H = 1/N_x \), where \( N_x \) is a positive integer, and the grid points \((x_i, y_j) := (iH, jH)\). We also consider the space of grid functions that operate on \( \tilde{\Omega} \) as follows:
\[ \Omega_H := \{(x_i, y_j)\} \cap \Omega, \quad \partial \Omega_H := \Omega_H \cap \partial \Omega, \quad \Omega_H := \tilde{\Omega} \setminus \partial \Omega_H. \] (2.7)

We want to compute a discrete approximation of \( \varphi \) at the points of the computational grid \( \Omega_H \). Each point of \( \Omega_H \) is the center of a control volume
\[ V_{i,j} := (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}). \] (2.8)

The midpoints of the interfaces of volumes \( V_{i,j} \) form a dual grid
\[ V_H := \left( \{(x_{i+1/2}, y_j)\} \cup \{(x_i, y_{j+1/2})\} \right) \cap \Omega, \] (2.9)
on which we will define discrete fluxes. Figure 1 represents the global coarse grid we have introduced in \( \Omega \). The figure is drawn for \( N_x = 8 \); grid points \( \Omega_H \) are marked with a circle, while rhombi and small squares identify points of \( \partial \Omega_H \) and \( V_H \) respectively.

We denote by \( G(\Omega_H) \), \( G(\tilde{\Omega}_H) \) and \( G(V_H) \), the space of grid functions that operate on \( \Omega_H \), \( \tilde{\Omega}_H \) and \( V_H \) respectively. We introduce the following notation: for a certain \( T^H \in G(\Omega_H) \), we write \( T^H = \{T^H_{i,j}\}, i,j = 1,2, \ldots, N_x - 1 \), with \( T^H_{i,j} := T^H(x_i, y_j) \). Similarly it is done for elements in \( G(\tilde{\Omega}_H) \) and \( G(V_H) \). Given a certain \( F^H \in G(V_H) \), we introduce the central difference operator \( \nabla^H_\Sigma : G(V_H) \to G(\Omega_H) \) by
\[ (\nabla^H_\Sigma F^H)_{i,j} := F^H_{i,j+1/2} - F^H_{i,j-1/2} + F^H_{i+1/2,j} - F^H_{i-1/2,j}. \] (2.10)

We also define \( T(\varphi) \in G(\Omega_H) \), \( S \in G(\Omega_H) \) and \( F(\varphi) \in G(V_H) \) as follows:
\[ T_{i,j}(\varphi) := \int_{V_{i,j}} \varphi \, d\Omega, \] (2.11)
Figure 1: Global coarse grid.

\[ S_{i,j} := \int_{V_{i,j}} s \, d\Omega, \quad (2.12) \]

\[ F_{i+1/2,j}(\varphi) := \int_{y_{j-1/2}}^{y_{j+1/2}} f_x(x_{i+1/2}, \eta, t) \, d\eta, \quad (2.13) \]

\[ F_{i,j+1/2}(\varphi) := \int_{x_{i-1/2}}^{x_{i+1/2}} f_y(\xi, y_{j+1/2}, t) \, d\xi. \quad (2.14) \]

\( F \) is called the integrated flux and in its expression the flux vector that we defined in (2.5) occurs. Finally we use the operators and definitions that we have introduced so far to write the conservation law in (2.6) for \( V = V_{i,j} \):

\[ \frac{\partial}{\partial t} T_{i,j}(\varphi) + \left( \nabla_T^H F(\varphi) \right)_{i,j} = S_{i,j}, \quad \text{in } \Theta. \quad (2.15) \]

In a finite volume approach the continuous fluxes that appear in (2.13) and (2.14) are approximated by finite differences; furthermore a quadrature rule has to be used in order to approximate all the integrals (2.11)-(2.14). Here we will not be specific on the particular schemes to be employed. We denote the spatial discretization of \( T, F \) and \( S \) by \( T^H \in G(\Omega^H), F^H \in G(V^H) \) and \( S^H \in G(\Omega^H) \) respectively and we call \( \varphi^H = \varphi^H(t) \in G(\bar{\Omega}^H) \) the spatial approximation of \( \varphi = \varphi(x,t) \). The finite volume discretization applied to (2.15) leads to a set of \((N_x - 1)^2\) Ordinary Differential Equations that have to be satisfied by \( \varphi^H \):

\[ \frac{d}{dt} T^H(\varphi^H) + \nabla_T^H F^H(\varphi^H) = S^H, \quad \text{in } \Theta. \quad (2.16) \]

System (2.16) has still to be discretized in time in order to be solved numerically. For this we divide the time interval \( \Theta \) into \( N_t \geq 1 \) subintervals such that \( t_{end}/N_t =: \Delta t \). We also introduce \( t_n := n\Delta t \), with \( n = 0, 1, 2, \ldots, N_t \) and by \( \varphi^n_H \) we denote an approximation of \( \varphi(x, t_n) \) on \( \bar{\Omega}_H \).

Because of the high activity of the solution, a coarse grid approximation computed at time \( t_n \) using a time step \( \Delta t \) might be not accurate enough to adequately represent \( \varphi^n := \varphi(t_n) \). Therefore we want to find a more accurate (both in space and time) local
approximation of $\varphi^n$ and eventually use it to correct and improve the global coarse grid solution. For this purpose we choose $\Omega_1$, an open subset of $\Omega$ such that the local high activity of $\varphi$ is entirely contained in $\Omega_1$, for all $t \in \Theta$. Using the same notation as for the global domain, $\partial \Omega_1$ indicates the boundary of $\Omega_1$ and $\bar{\Omega}_1 := \Omega_1 \cup \partial \Omega_1$. For convenience, the local region $\Omega_1$ is chosen in such a way that

$$\begin{align*}
(x_i, y_j) \in \Omega_H \cap \Omega_1 & \implies W_{i,j} \subset \Omega_1 \\
W_{i,j} & := (x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1}).
\end{align*}$$

holds with

This condition means that if a coarse grid point $(x_i, y_j)$ lies in $\Omega_1$, then its left, right, top and bottom neighbors lie in $\bar{\Omega}_1$. Note that $V_{i,j} \subset W_{i,j}$, so that $W_{i,j} \subset \Omega_1$ implies $V_{i,j} \subset \Omega_1$. Also, $\Omega_1$ is not a union of control volumes $V_{i,j}$. In $\Omega_1$, like in $\Omega$, we apply a vertex-centered finite volume method. Following the same procedure adopted for the global domain, in $\bar{\Omega}_1$ we introduce a local fine grid (size $h < H$), which we denote by $\Omega_{1,h}$. We also define, cf. (2.7),

$$\begin{align*}
\partial \Omega_{1,h} & := \bar{\Omega}_{1,h} \cap \partial \Omega_1, \\
\Omega_{1,h} & := \bar{\Omega}_{1,h} \setminus \partial \Omega_{1,h}.
\end{align*}$$

It is of practical convenience that points of $\Omega_{1,h}$ that lie in the area of refinement belong to $\Omega_{1,h}$ too, and that boundaries of control volumes in the local fine grid coincide with boundaries of control volumes in the global coarse grid. For that reason, we assume the factor of grid refinement

$$\sigma := \frac{H}{h}$$

(2.20)

to be an odd integer. The union of coarse and fine grid defines the composite grid $\Omega_{H,h} := \Omega_H \cup \Omega_{1,h}$. Figure 2 shows an example of a composite grid. In the figure the grid points $\Omega_{1,h}$ have been marked with an empty circle, while the empty rhombi denote points belonging to $\partial \Omega_H$. The region with gray background is $\Omega_1$; in that region the small circles are used to mark the fine grid points $\Omega_{1,h}$, while the small rhombi indicate $\partial \Omega_{1,h}$. From the figure one can see that coarse grid points lying in the area of refinement $\Omega_1$ belong to the fine grid too. In this example the factor of grid refinement $\sigma$ is 3.

Figure 2: Example of a composite grid.
As for the LDC method described in [7], together with a fine grid spacing we also introduce a fine time step. The time interval \((t_{n-1}, t_n]\), with \(n \geq 1\), is divided into \(n_t \geq 1\) subintervals such that \((t_n - t_{n-1})/n_t = \Delta t\). In this way the factor of time refinement

\[ \tau := \frac{\Delta t}{\Delta t} \]  

(2.21)

turns out to be a positive integer. We also introduce \(t_{n-1+k/\tau} := t_{n-1} + k\Delta t\), with \(k = 0, 1, 2, \ldots, n_t\) and by \(\varphi_{1,h}^{n-1+k/\tau}\) we denote an approximation of \(\varphi\) on the local fine grid \(\Omega_{1,h}\) at time \(t_{n-1+k/\tau}\).

In the remainder of this section we will first find an expression for \(\varphi_{1,h}^{n-1}\) and then we will define the local problem that leads to determining \(\varphi_{1,h}^{n}\). We assume that the LDC technique has been applied in the time interval \([t_{n-2}, t_{n-1}]\), with \(n > 1\). This means that a discrete problem has been solved on the global and on the local grid, and that the following approximation of \(\varphi^{n-1}\) is available on the composite grid \(\bar{\Omega}_{H,h}\):

\[ \varphi_{H,h}^{n-1} := \begin{cases} \varphi_{1,h}^{n-1}, & \text{in } \bar{\Omega}_{1,h}, \\ \varphi_{H}^{n-1}, & \text{in } \bar{\Omega}_{H,h} \setminus \bar{\Omega}_{1,h}. \end{cases} \]  

(2.22)

The approximation \(\varphi_{H,h}^{n-1}\) is called composite grid solution and its expression also includes values on the boundary of the global and local grid. For \(n = 1\), we simply have the initial condition (2.3)

\[ \varphi_{H,h}^{0} = \eta|_{\bar{\Omega}_{H,h}}. \]  

(2.23)

We indicate by \(\varphi_{H,h}^{n-1}|_{\Omega_{H}}\) the restriction of \(\varphi_{H,h}^{n-1}\) on \(\Omega_{H}\). Now, we can apply an implicit time discretization scheme to (2.16). We choose the Implicit Euler scheme; however, this is not restrictive and other implicit methods might be used as well. In [7] it is explained why time integration by an explicit method on the coarse grid is of minor interest in LDC. A coarse grid approximation \(\varphi_{H}^{n}\) can finally be computed solving

\[ T^H(\varphi_{H}^{n}) - T^H(\varphi_{H,h}^{n-1}|_{\Omega_{H}}) + \nabla^H \cdot F^H(\varphi_{H}^{n}) \Delta t = S_{H}^{n} \Delta t, \]  

(2.24)

with \(\varphi_{H}^{n} = \psi(t_n)\), on \(\partial \Omega_{H}\).

If we now want to solve a discrete analogue of (2.6) on \(\Omega_{1,h} \times (t_{n-1+k/\tau}, t_k, k = 1, \ldots, n_t)\), we have to provide conditions on the boundary of the local grid for each time \(t_{n-1+k/\tau}\), \(k = 1, \ldots, n_t\). For \(\partial \Omega \cap \partial \Omega_{1}\), i.e., the part of the local area's boundary in common with the global boundary, we can appropriately use the original condition (2.2). As for the rest of the local area's boundary, namely \(\Gamma := \partial \Omega_{1} \setminus (\partial \Omega \cap \partial \Omega_{1})\), we introduce the interpolation operator in space \(P_x\). With \(\Gamma_{H} := \Gamma \cup \Omega_{H}\) and \(\Gamma_h := \Gamma \cap \partial \Omega_{1,h}\), the operator \(P_x \in G(\Gamma_{H}) \rightarrow G(\Gamma_h)\) spatially interpolates \(\varphi_{H}^{n}|_{\Gamma_{H}}\) on \(\Gamma_h\). With \(P_x\) we are able to prescribe artificial Dirichlet boundary conditions on \(\Gamma_h\) at \(t_n\). Since we need boundary conditions not only at \(t_n\), but for all the \(t_{n-1+k/\tau}\), with \(k = 1, 2, \ldots, n_t\), we define another interpolation operator \(R_k^H \in G(\Gamma_{H}) \times G(\Gamma_h) \rightarrow G(\Gamma_h)\). The operator \(R_k^H\) performs time interpolation between the time levels \(t_{n-1}\) and \(t_n\); in particular, \(R_k^H\) interpolates between the restriction of \(\varphi_{H,h}^{n-1}\) on \(\Gamma_h\), see (2.22), and \(P_x(\varphi_{H}^{n})\). In this way, \(R_k^H\) enables us to specify artificial Dirichlet boundary conditions on \(\Gamma_h\) for every \(t_{n-1+k/\tau}\), with \(k = 1, 2, \ldots, n_t\). We can synthetically write the boundary condition for the local problem as

\[ \varphi_{1,h}^{n-1+k/\tau} = \tilde{\varphi}_{1,h}^{n-1+k/\tau}, \quad \text{on } \partial \Omega_{1,h}, \text{ for } k = 1, 2, \ldots, n_t, \]  

(2.25)

where

\[ \tilde{\varphi}_{1,h}^{n-1+k/\tau} := \begin{cases} \psi(t_{n-1+k/\tau}), & \text{on } \partial \Omega_{1,h} \setminus \Gamma_h, \text{ for } k = 1, \ldots, n_t, \\ R_k^H(\varphi_{1,h}^{n-1}|_{\Gamma_h}, P_x(\varphi_{H}^{n})), & \text{on } \Gamma_h, \text{ for } k = 1, \ldots, n_t. \end{cases} \]  

(2.26)
From (2.26) we can see that the boundary condition for the local problem depends on \( \varphi_{H}^{n} \), the approximation of \( \varphi^{n} \) that we computed on the global coarse grid solving (2.24). Note that on \( \Gamma_{H} \) the fine and coarse grid approximation coincide at \( t_{n} \). This can easily be verified, since
\[
\varphi_{l,H}^{n}|_{\Gamma_{l}} = \tilde{\Psi}_{l,H}^{n}|_{\Gamma_{l}} = P_{\Gamma}(\varphi_{H}^{n})|_{\Gamma_{l}} = \varphi_{H}^{n}|_{\Gamma_{l}}. \tag{2.27}
\]

If we now introduce a local discretization of (2.6), we are able to formulate a local problem from which we can compute \( \varphi_{l,H}^{n} \). The local approximation \( \varphi_{l,H}^{n} \) is regarded to be more accurate than \( \varphi_{H}^{n} \) since it is computed using a finer grid \( (h < H) \) and a smaller time step \( (\Delta t \leq \Delta t) \). If again the Implicit Euler scheme is used, the local problem that enables us to determine \( \varphi_{l,H}^{n} \) is
\[
\begin{cases}
\frac{T_{l}^{h}(\varphi_{l,H}^{n-1+k/t/\nu}) - T_{l}^{h}(\varphi_{l,H}^{n-1+k-1/t/\nu})}{\nu} + \nabla_{\Sigma}^{2}F_{l}^{H}(\varphi_{l,H}^{n-1+k/t/\nu}) \delta t = S_{l}^{h,n-1+k/t/\nu} \delta t, & \text{for } k = 1, 2, \ldots, n_{t}, \\
\varphi_{l,H}^{n-1+k/t/\nu} = \tilde{\Psi}_{l,H}^{n-1+k/t/\nu}, & \text{on } \partial \Omega_{l,H}, \text{ for } k = 1, 2, \ldots, n_{t}.
\end{cases} \tag{2.28}
\]

The procedure (2.28) is initialized using (2.22) if \( n > 1 \), or by a proper discretization of the original initial condition (2.3) if \( n = 1 \). The coarse and fine grid approximation computed at \( t_{n} \) define the composite grid solution
\[
\varphi_{H,H}^{n} = \begin{cases}
\varphi_{l,H}^{n}, & \text{on } \tilde{\Omega}_{l,H} \\
\varphi_{H}^{n}, & \text{on } \tilde{\Omega}_{H,H} \setminus \tilde{\Omega}_{l,H}.
\end{cases} \tag{2.29}
\]

At this point we have completed the initialization of the LDC algorithm for parabolic problems in combination with a finite volume discretization. In the next section we explain how, through a defect correction, we can obtain a more accurate composite grid solution at time \( t_{n} \).

### 3 The finite volume adapted defect term

The crucial part of the LDC algorithm is how the local solution is used to improve the global approximation \( \varphi_{H}^{n} \) through an approximation of the local discretization error or defect. In particular, like it is done in [2] for elliptic problems, we are interested in expressing the defect in such a way that the resulting composite grid discretization satisfies a discrete conservation law. The defect \( d_{H}^{n} \in G(\Omega_{H}) \) is defined as
\[
d_{H}^{n} := \nabla_{H}^{*}(\varphi^{n}|_{\Omega_{H}}) - \nabla_{H}^{*}(\varphi^{n-1}|_{\Omega_{H}}) + \nabla_{\Sigma}^{H}F_{H}(\varphi^{n}|_{\tilde{\Omega}_{H}}) \Delta t - S_{H}^{n} \Delta t. \tag{3.1}
\]

In (3.1) we have plugged the exact analytical solution \( \varphi \) into the coarse grid discretization scheme (2.24). We consider now the continuous equation (2.6), which is valid for all volumes \( V \subset \Omega \) and thus for any control volume \( V_{i,j} \subset \Omega \) too. If we apply time integration between \( t_{n-1} \) and \( t_{n} \), we obtain
\[
\int_{V_{i,j}} \varphi^{n} \, d\Omega - \int_{V_{i,j}} \varphi^{n-1} \, d\Omega + \int_{t_{n-1}}^{t_{n}} \int_{\partial V_{i,j}} f \cdot n \, d\gamma \, dt = \int_{t_{n-1}}^{t_{n}} \int_{V_{i,j}} s \, d\Omega \, dt, \quad \text{for } i, j = 1, 2, \ldots, N_{x} - 1, \tag{3.2}
\]
or
\[
T(\varphi^{n}) - T(\varphi^{n-1}) + \int_{t_{n-1}}^{t_{n}} \nabla_{\Sigma}^{H}F(\varphi) \, dt = \int_{t_{n-1}}^{t_{n}} S \, dt. \tag{3.3}
\]
Combination of (3.3) and (3.1) yields
\[
\begin{align*}
d^n_H = & \left( T^n(\varphi^n|_{\Omega_H}) - T(\varphi^n) \right) - \left( T^n(\varphi^{n-1}|_{\Omega_H}) - T(\varphi^{n-1}) \right) \\
+ & \nabla^H \left( F^n(\varphi^n|_{\Omega_H}) \Delta t - \int_{t_{n-1}}^{t_n} F(\varphi) \, dt \right) - \left( S^{H,n} \Delta t - \int_{t_{n-1}}^{t_n} S \, dt \right),
\end{align*}
\]
(3.4)
which is the definition of defect that we will use in practice. If we would know the values of \(d^n_H\), we could use them to compute a more accurate (both in space and time) approximation of \(\varphi^n\) on the global grid. This could be done by adding \(d^n_H\) on the right hand side of (2.24). However, since we do not know the exact solution of our partial differential equation, we cannot compute the values of \(d^n_H\). What we can do, though, is to use the local solution \(\varphi^n_H\) to get an estimate of \(d^n_H\). We introduce
\[
\begin{align*}
d^n_H;T & := T^n(\varphi^n|_{\Omega_H}) - T(\varphi^n), \\
d^n_H;F & := F^n(\varphi^n|_{\Omega_H}) \Delta t - \int_{t_{n-1}}^{t_n} F(\varphi) \, dt, \\
d^n_H;S & := S^{H,n} \Delta t - \int_{t_{n-1}}^{t_n} S \, dt,
\end{align*}
\]
(3.5-3.7)
so that, cf. (3.4),
\[
d^n_H = d^n_H;T - d^{n-1}_H;T + \nabla^H d^n_H;F - d^n_H;S.
\]
(3.8)
In the following section we will approximate each of the terms that appear on the right hand side of (3.8).

### 3.1 Approximation of the defect

We start by considering \(d^n_H;T\). After solving the global and the local problem, the following approximations are available for an arbitrary \(T_{i,j}(\varphi^n)\):

1. the coarse grid approximation \(T^H_{i,j}(\varphi^n_H)\);
2. a coarse grid approximation that makes use of the composite grid solution (2.29), i.e. \(T^H_{i,j}(\varphi^n_{H,h}|_{\Omega_H})\);
3. a sum of fine grid approximations
\[
T^{\text{sum}}_{i,j}(\varphi^n_{H,h}) := \sum_{p=-(\sigma-1)/2}^{(\sigma-1)/2} \sum_{q=-(\sigma-1)/2}^{(\sigma-1)/2} T^h_{i,t+p/q,j+q} \varphi^n_{t,h}.
\]
(3.9)
In \(T^{\text{sum}}_{i,j}\) the subscript \(l\) reminds us that this third approximation is local and it exists only for \((x_i,y_j) \in \Omega_l\), i.e. for the coarse grid points lying in the area of refinement.

The approximations above are considered to be listed in order of increasing accuracy. The second approximation is more accurate than the first one because in the area of refinement it exploits the values computed on the fine grid with a time step \(\delta t \leq \Delta t\). The approximation number three can be considered more accurate than the second one because it computes spatial integrals using the fine grid discretization. We introduce
\( \Omega_{1,H} := \Omega_H \cap \Omega_{1,h} \) and we use the best available information to define \( T_{\text{best}}(\varphi_{1,h}) \in G(\Omega_H) \) as
\[
T_{\text{best}}(\varphi_{1,h}) := \begin{cases} T_{1,\text{sum}}^H(\varphi_{1,h}), & \text{on } \Omega_{1,H}, \\ T^H(\varphi_{1,h}), & \text{on } \Omega_H \setminus \Omega_{1,H}. \end{cases}
\]
(3.10)

Definition (3.10) is used to introduce the following approximation for \( d_n^{H,T} \):
\[
d_n^{H,T} = T^H(\varphi^n_{1,H}) - T(\varphi^n) \approx T^H(\varphi^n_{1,h} \mid \Omega_H) - T_{\text{best}}(\varphi^n_{1,h}) =: \tilde{d}_n^{H,T}. \]
(3.11)

We consider now the flux discretization error \( d_n^{l,F} \), see (3.6), and in particular
\[
F_{l+1/2,j}^n := \int_{t_{n-1}}^{t_n} F_{l+1/2,j}(\varphi) \, dt,
\]
(3.12)

where \( F_{l+1/2,j} \) is an arbitrary horizontal flux. In the expression of \( F_{l+1/2,j}^n \) both space (cf. (2.13)) and time integrals appear; these can be approximated in different ways, using the coarse or the fine grid size, the \( \Delta t \) or the \( \delta t \) time discretization. Below we list the possible approximations for \( F_{l+1/2,j}^n \):

1. an \( H-\Delta t \) approximation, i.e. an approximation based on the \( H \) space discretization and on the \( \Delta t \) time discretization:
\[
F_{l+1/2,j}^n \approx \int_{t_{n-1}}^{t_n} F_{l+1/2,j}^H(\varphi_H(t)) \, dt \approx F_{l+1/2,j}^H(\varphi^n_H) \Delta t;
\]
(3.13)

2. an \( (H,h)-\Delta t \) approximation, which makes use of the composite grid solution (2.29):
\[
F_{l+1/2,j}^n \approx \int_{t_{n-1}}^{t_n} F_{l+1/2,j}^H(\varphi_{H,h}(t)) \, dt \approx F_{l+1/2,j}^H(\varphi^n_{H,h} \mid \Omega_H \cap \Omega_{1,h}) \Delta t;
\]
(3.14)

3. a local \( h-\Delta t \) approximation based on the sum of fine grid fluxes. We introduce
\[
F_{l+1/2,j}^{\text{sum}}(\varphi_{1,h}(t)) := \sum_{q=-\lceil (\sigma-1)/2 \rceil}^{\lceil (\sigma-1)/2 \rceil} F_{l+1/2,j+q/\sigma}^h(\varphi_{1,h}(t)),
\]
(3.15)

and we write
\[
F_{l+1/2,j}^n \approx \int_{t_{n-1}}^{t_n} F_{l+1/2,j}^{\text{sum}}(\varphi_{1,h}(t)) \, dt \approx F_{l+1/2,j}^{\text{sum}}(\varphi^n_{1,h}) \Delta t;
\]
(3.16)

4. a local \( h-\delta t \) approximation based on the \( \delta t \) time discretization:
\[
F_{l+1/2,j}^n \approx \int_{t_{n-1}}^{t_n} F_{l+1/2,j}^{\text{sum}}(\varphi_{1,h}(t)) \, dt = \sum_{k=1}^{\tau} \int_{t_{n-1} + k/\tau}^{t_{n-1} + (k+1)/\tau} F_{l+1/2,j}^{\text{sum}}(\varphi_{1,h}(t)) \, dt
\]
\[
\approx \sum_{k=1}^{\tau} F_{l+1/2,j}^{\text{sum}}(\varphi_{1,h}^{n-1 + k/\tau}) \delta t.
\]
(3.17)

As before the different approximations are listed in order of increasing accuracy. The third and the fourth approximation can only be expressed for points \( (x_{i+1/2}, y_j) \in \Omega_t \).
Analogous approximations are available for the terms $F_{i-1/2,j}$, $F_{i+1/2,j}$, $F_{i,j-1/2}$. We can therefore define $F_{i,j}^{\text{best}} \in G(V_H)$ and $F_{i,j}^{\text{best}} \in G(V_H)$ as

$$
F_{i,j}^{\text{best}}(\varphi_{H,H}^n) := \begin{cases} 
\sum_{k=1}^{\tau} F_{i,j}^{\text{sum}}(\varphi_{1,H}^{n-1+k/\tau}) \delta t, & \text{on } V_H \cup \Omega_t, \\
F_{i,j}^H(\varphi_{H,H}^n|_{\Omega_t}) \Delta t, & \text{elsewhere}. 
\end{cases}
$$

We can now approximate $d_{i,j}^n$ as

$$
d_{i,j}^n = F_{i,j}^H(\varphi_{H,H}^n|_{\Omega_t}) \Delta t - \int_{t_{n-1}}^{t_n} F(\varphi) \, dt \approx F_{i,j}^H(\varphi_{H,H}^n|_{\Omega_t}) \Delta t - F_{i,j}^{\text{best}}(\varphi_{H,H}^n) \Delta t =: d_{i,j}^n. 
$$

Similar considerations hold for an arbitrary source term

$$
S_{i,j}^n := \int_{t_{n-1}}^{t_n} S_{i,j}(t) \, dt, 
$$

which can be approximated through:

1. an $H$-$\Delta t$ approximation

$$
S_{i,j}^n \approx \int_{t_{n-1}}^{t_n} S_{i,j}^H(t) \, dt \approx S_{i,j}^H \Delta t.
$$

This approximation is global and holds for all $1 \leq i, j \leq N_x - 1$;

2. a local $h$-$\Delta t$ approximation based on a sum of fine grid approximations and on a $\Delta t$ time integration. We introduce

$$
S_{i,j}^{\text{sum}}(t) := \sum_{p=-(\sigma-1)/2}^{(\sigma-1)/2} \sum_{q=-\sigma/2}^{\sigma/2} S_{i,p,q}(t),
$$

and we write

$$
S_{i,j}^n \approx \int_{t_{n-1}}^{t_n} S_{i,j}^{\text{sum}}(t) \, dt \approx S_{i,j}^{\text{sum}} \Delta t.
$$

The approximation (3.23) holds for $(x_i, y_j) \in \Omega_{1,H}$;

3. a local $h$-$\delta t$ approximation

$$
S_{i,j}^n \approx \int_{t_{n-1}}^{t_n} S_{i,j}^{\text{sum}}(t) \, dt
$$

$$
= \sum_{k=1}^{\tau} \int_{t_{n-1}+(k-1)/\tau}^{t_{n-1}+(k+1)/\tau} S_{i,j}^{\text{sum}}(t) \, dt \approx \sum_{k=1}^{\tau} S_{i,j}^{\text{sum}, n-1+k/\tau} \delta t,
$$

defined for points $(x_i, y_j) \in \Omega_{1,H}$.

Gathering the best available information, we can define $S^{\text{best}}_{i,j} \in G(\Omega_H)$ and $S^{\text{best}}_{i,j} \in G(\Omega_H)$ as

$$
S^{\text{best}, n} := S^{\text{best}, n} \Delta t := \begin{cases} 
\sum_{k=1}^{\tau} S_{i,j}^{\text{sum}, n-1+k/\tau} \delta t, & \text{on } \Omega_{1,H}, \\
S_{H,n} \Delta t, & \text{on } \Omega_H \setminus \Omega_{1,H}. 
\end{cases}
$$

It is now possible to provide an approximation for $d_{i,j}^n$:

$$
d_{i,j}^n := S_{H,n} \Delta t - \int_{t_{n-1}}^{t_n} S \, dt \approx S_{H,n} \Delta t - S^{\text{best}, n} \Delta t =: d_{i,j}^n. 
$$

10
3.2 The finite volume adapted LDC algorithm

In the previous section we have found an approximation for all the terms that appear on the right hand side of (3.8). At this point we can thus estimate the defect by \( \tilde{d}_H^n \approx \tilde{d}_H^n \), where \( \tilde{d}_H^n \in G(\Omega_H) \) is defined by (cf. (3.11), (3.19) and (3.26))

\[
\tilde{d}_H^n := \tilde{d}_H^n - \tilde{d}_H^{n-1} + \nabla^H \tilde{d}_H^n - \tilde{d}_H^S
\]

\[
= \left( T^H(\varphi_{H,h}^n | \Omega_H) - T^{\text{best}}(\varphi_{H,h}^n) \right) - \left( T^H(\varphi_{H,h}^{n-1} | \Omega_H) - T^{\text{best}}(\varphi_{H,h}^{n-1}) \right) + \Delta t \nabla^H \left( F^H(\varphi_{H,h}^n | \Omega_H) - F^{\text{best}}(\varphi_{H,h}^n) \right) - \Delta t \left( S^H,n - S^{\text{best},n} \right).
\]

(3.27)

The approximation \( \tilde{d}_H^n \) clearly depends on the solution computed on the fine grid in the time interval \([t_{n-1}, t_n]\). We can finally compute a more accurate global approximation of \( \varphi^n \) solving the modified coarse grid problem

\[
T^H(\varphi_{H,h}^n | \Omega_H) - T^{\text{best}}(\varphi_{H,h}^n) + \nabla^H T^H(\varphi_{H,h}^n) \Delta t = S^{H,n} \Delta t + \tilde{d}_H^n.
\]

(3.28)

with \( \varphi_{H,h}^1 = \psi(t_n) \), on \( \partial \Omega_H \). In (3.28) we called the new approximation \( \varphi_{H,h,1}^n \), where the new subscript is used to distinguish the new approximation from the previous one \( \varphi_{H,h}^n \), from now on referred as \( \varphi_{H,h,0}^n \). Once \( \varphi_{H,h,1}^n \) is computed, we are able to define new boundary conditions, see (2.25), for a new local problem on \( \Omega_{1,h} \) and this triggers an iterative procedure which is formalized in Algorithm 3.1. As for the global grid solution, in Algorithm 3.1 an extra subscript is added to number the different approximations computed locally; the same is done for the defect term.

**Algorithm 3.1**

(Time-dependent LDC algorithm with a finite volume adapted defect term)

For loop, \( n = 1, 2, \ldots, N_t \)

Initialization

- **Compute a global approximation** \( \varphi_{H,0}^n \in G(\Omega_H) \) solving problem (2.24), with \( \varphi_{H,0}^n = \psi(t_n) \) on \( \partial \Omega_H \).
- **Compute a local approximation** \( \varphi_{h,0}^n \in G(\Omega_{h}) \) solving the local problem (2.28).
- **Define a composite grid solution** \( \varphi_{H,h,0}^n \in G(\Omega_{H,h}) \) as in (2.29).
- **Set** \( w = 1 \).

Until convergence

- **Compute** \( \tilde{d}_{H,w}^n \in G(\Omega_H) \), **an approximation of the local discretization error** \( d_{H}^n \in G(\Omega_H) \), **through**

\[
\tilde{d}_{H,w}^n = \left( T^H(\varphi_{H,h,w-1}^n | \Omega_H) - T^{\text{best}}(\varphi_{H,h,w-1}^n) \right) - \left( T^H(\varphi_{H,h}^{n-1} | \Omega_H) - T^{\text{best}}(\varphi_{H,h}^{n-1}) \right) + \Delta t \nabla^H \left( F^H(\varphi_{H,h,w-1}^n | \Omega_H) - F^{\text{best}}(\varphi_{H,h,w-1}^n) \right) - \Delta t \left( S^{H,n} - S^{\text{best},n} \right).
\]

(3.29)
• Compute a more accurate global approximation \( \varphi^n_{H,w} \in G(\Omega_H) \) solving the modified global problem

\[
\begin{cases}
  T^H(\varphi^0_{H,w}) - T^H(\varphi^{n-1}_{H,w}|_{\Omega_H}) + \nabla^H T^H(\varphi^n_{H,w}) \Delta t = S^{H,n} \Delta t + \tilde{d}^n_{H,w-1}, \\
  \varphi^n_{H,w} = \psi(t_n), \quad \text{on } \partial \Omega_H.
\end{cases}
\] (3.30)

• Use \( \varphi^n_{H,w} \) to update the boundary condition \( \tilde{\psi}_{t,n,h,w} \) on \( \partial \Omega_{1,h} \).

• Solve the following local problem with updated boundary conditions

\[
\begin{cases}
  T^l_h(\varphi^{n-1+1/k}/\tau) - T^l_h(\varphi^{n-1+(k-1)/\tau}) \\
  \quad + \nabla^l H^l(\varphi^{n-1+1/k}/\tau) \delta t = S^{l,n-1+1/k}/\tau \delta t, \quad \text{for } k = 1, 2, \ldots, n_t, \\
  \varphi^{n-1+1/k}/\tau = \psi^{n-1+1/k}/\tau, \quad \text{on } \partial \Omega_{1,h}, \\
  \text{for } k = 1, 2, \ldots, n_t.
\end{cases}
\] (3.31)

• Define the composite grid approximation

\[
\varphi^n_{H,h,w} = \begin{cases}
  \varphi^n_{l,h,w}, \quad \text{on } \Omega_{1,h} \\
  \varphi^n_{H,w}, \quad \text{on } \hat{\Omega}_H \setminus \Omega_{1,H}.
\end{cases}
\] (3.32)

• Set \( w = w + 1 \).

END UNTIL

• Call \( \varphi^n_{l,h} \) and \( \varphi^n_{H} \) the latest solutions that have been found on the local and global grid respectively (remove the last subscript); the solution on the composite grid at time \( t_n \) is:

\[
\varphi^n_{H,H} = \begin{cases}
  \varphi^n_{l,h}, \quad \text{in } \hat{\Omega}_{1,h}, \\
  \varphi^n_{H}, \quad \text{in } \hat{\Omega}_{H,H} \setminus \hat{\Omega}_{1,h}.
\end{cases}
\] (3.33)

END FOR

This is the LDC algorithm for time-dependent problems as presented in [7], but now adapted to a setting with a finite volume discretization. In particular, like it is done in [2] for stationary cases, the expression of the defect term \( \tilde{d}^n_{H,w-1} \) is such that the resulting composite grid discretization is still conservative. This property of the finite volume adapted LDC method is discussed in Section 4. Before that, Section 3.3, we give a practical way to compute the defect term (3.29).

3.3 Practical considerations on the defect term

In practice the defect term \( \tilde{d}^n_{H,w-1} \) is not computed through a straightforward application of (3.29). That would in fact be quite onerous, since it would require to sum fine grid values at all the points \( \Omega_{1,h} \) and for all the times \( t_{n-1+k}/\tau \), with \( k = 1, 2, \ldots, n_t \). In the perspective of simplifying the computation of \( \tilde{d}^n_{H,w-1} \), in Lemma 3.2 we state a local conservation law that holds just for the points \( (x_i, y_j) \in \Omega_{1,H} \). This is obtained summing the fine grid balance of each fine grid control volume at each time \( t_{n-1+k}/\tau \), with \( k = 1, 2, \ldots, n_t \).
Proof. We consider the fine grid balance in (3.31) and we sum it for all the simplication stated in Theorem 3.3 holds for 
which is equivalent to (3.34).

\[
T_{l,i,j}^{\text{sum}}(\varphi_{l,h,w}^n) - T_{l,i,j}^{\text{sum}}(\varphi_{H,h,w}^{n-1} | \Omega_{l,h}) + \left( \nabla_{x} \sum_{k=1}^{\tau} F_{l,i,j}^{\text{sum}}(\varphi_{l,h,w}^{n-1+k/\tau}) \right)_{i,j} \delta t = \sum_{k=1}^{\tau} S_{l,i,j}^{\text{sum},n-1+k/\tau} \delta t \quad \text{for } (x_l, y_j) \in \Omega_{l,H}. \quad (3.34)
\]

Using definitions (3.9), (3.15), (3.22), and the fact that internal fluxes cancel, we can rewrite (3.35) as

\[
\frac{1}{2} \sum_{p=-(\sigma-1)/2}^{(\sigma-1)/2} \sum_{q=-(\sigma-1)/2}^{(\sigma-1)/2} \left( T_{l,i,j}^{h}(\varphi_{l,h,w}^{n-1+k/\tau}) - T_{l,i,j}^{h}(\varphi_{l,h,w}^{n-1+k/\tau}) \right) + \frac{1}{2} \sum_{p=-(\sigma-1)/2}^{(\sigma-1)/2} \sum_{q=-(\sigma-1)/2}^{(\sigma-1)/2} \left( \nabla_{x} F_{l,i,j}^{h}(\varphi_{l,h,w}^{n-1+k/\tau}) \delta t \right)_{i+p,j+q} = \sum_{p=-(\sigma-1)/2}^{(\sigma-1)/2} \sum_{q=-(\sigma-1)/2}^{(\sigma-1)/2} \left( S_{l,i,j}^{h,n-1+k/\tau} \delta t \right)_{i+p,j+q} \quad \text{for } (x_l, y_j) \in \Omega_{l,H}, k = 1, 2, \ldots, \tau. \quad (3.35)
\]

If we now sum over \(k\), we get

\[
T_{l,i,j}^{\text{sum}}(\varphi_{l,h,w}^n) - T_{l,i,j}^{\text{sum}}(\varphi_{H,h,w}^{n-1} | \Omega_{l,h}) + \sum_{k=1}^{\tau} \left( \nabla_{x} F_{l,i,j}^{\text{sum}}(\varphi_{l,h,w}^{n-1+k/\tau}) \delta t \right)_{i,j} = \sum_{k=1}^{\tau} \left( S_{l,i,j}^{\text{sum},n-1+k/\tau} \delta t \right)_{i,j}, \quad \text{for } (x_l, y_j) \in \Omega_{l,H}, \quad (3.37)
\]

which is equivalent to (3.34). \( \square \)

The results in Lemma 3.2 are used in the proof of Theorem 3.3, which gives us a practical way to compute \( \Delta t_{l,w-1} \). First we define the new set \( \Omega_{c,H} := \Omega_H \setminus (\Omega_{l,H} \cup \Gamma_{H}) \). In this way the coarse grid points \( \Omega_{H} \) are divided into three distinct groups, namely \( \Omega_{l,H}, \Gamma_{H} \) and \( \Omega_{c,H} \). From the definitions of the three subsets, it is easy to verify that \( \Omega_H = \Omega_{l,H} \cup \Gamma_{H} \cup \Omega_{c,H} \). In Figure 3 the coarse grid points \( \Omega_{l,H} \) are marked with triangles, while squares and circles denote points \( \Gamma_{H} \) and \( \Omega_{c,H} \) respectively. The simplification stated in Theorem 3.3 holds for \( \Omega_{l,H} \) and \( \Omega_{c,H} \).
Figure 3: The coarse grid points $\Omega_H$ can be divided into three complementary subsets: $\Omega_{l,H}$ (triangles), $\Gamma_H$ (squares) and $\Omega_{c,H}$ (circles).

Theorem 3.3

The defect term $\tilde{d}^n_{H,w-1} \in G(\Omega_H)$ can be written as

$$
(\tilde{d}^n_{H,w-1})_{i,j} = \begin{cases} 
T^H_{i,j}(\varphi_{n,h,w-1}^n|\Omega_H) - T^H_{i,j}(\varphi_{n-1,h}^n|\Omega_H) & 
\text{for } (x_i, y_j) \in \Omega_{l,H}, \\
+ \left( \nabla^H \varphi_{n}^n \right)_{i,j} \Delta t & 
\text{for } (x_i, y_j) \in \Omega_{c,H}, \\
- S^H_{i,j} \Delta t, & 
\text{for } (x_i, y_j) \in \Gamma_H \\
0 & 
\text{otherwise}
\end{cases}
$$

(3.38)

Proof. Consider a point $(x_i, y_j) \in \Omega_{l,H}$. We have

$$
(\tilde{d}^n_{H,w-1})_{i,j} \overset{(3.29)}{=} \left( T^H_{i,j}(\varphi_{n,h,w-1}^n|\Omega_H) - T^H_{i,j}(\varphi_{n-1,h}^n|\Omega_H) \right)
- \left( T^H_{i,j}(\varphi_{n-1,h}^n|\Omega_H) - T^H_{i,j}(\varphi_{n-1,h}^n|\Omega_{l,H}) \right)
+ \left( \nabla^H \varphi_{n}^n \right)_{i,j} \Delta t - \sum_{k=1}^{r} F^\text{sum}_{l}(\varphi_{n-1+k/h}^n|\Omega_H, h) \delta t

(3.39)

This proves the first part of (3.38). The second part can be proved analogously, i.e. combining (3.29) with (3.10), (3.18) and (3.25) for a point $(x_i, y_j) \in \Omega_{c,H}$. 

Theorem 3.3 gives us formulas to compute the finite volume adapted defect term on $\Omega_{l,H} \cup \Omega_{c,H}$. Therefore we need the original definition (3.29) for points $\Gamma_H$ only. We note however that even on $\Gamma_H$ things can be simplified significantly. Combining again (3.29)
with definitions (3.10) and (3.25), we can in fact see that

\[ T^H(\phi_{n+1}^n|\Omega_n^H) - T_{\text{best}}(\phi_{n+1}^n) = 0, \quad \text{on } \Gamma_H, \]  
\[ T^H(\phi_{n+1}^n|\Omega_n^H) - T_{\text{best}}(\phi_{n+1}^n) = 0, \quad \text{on } \Gamma_H, \]  
\[ S^H,n - S_{\text{best},n} = 0, \quad \text{on } \Gamma_H. \]  

The only term in (3.29) that we explicitly have to compute on \( \Gamma_H \) is thus the sum of the fine grid fluxes

\[ \tilde{d}_{n,w-1}^H = \Delta t \nabla_d^H \left( F^H(\phi_{n,w}^n|\Omega_n^H) - F_{\text{best}}(\phi_{n,w}^n) \right), \quad \text{on } \Gamma_H. \]  

Note that Theorem 3.3 not only gives us a practical way to compute the finite volume adapted defect term (3.29), but it also establishes the connection between the LDC method with the finite volume adapted defect term introduced in this paper and the LDC method with the standard defect term presented in [7]. On \( \Omega_{i,H} \) the standard defect term is computed plugging the fine grid solution at time \( t_n \) into the coarse grid discretization scheme. This is what happens in the first part of (3.38) too. Elsewhere the standard defect is zero. This is not true for the finite volume adapted defect term, which is zero on \( \Omega_{c,H} \) only. On \( \Gamma_H \) it is not zero and the balance of integrated fluxes guaranteed by (3.43) is such that the finite volume adapted LDC method produces a solution that satisfies discrete conservation law holds on the composite grid. This is discussed in Section 4.

4 Conservation properties of the finite volume adapted LDC method

In this section, we discuss a few properties of the LDC algorithm described in Sections 2 and 3. The time-dependent LDC technique is an iterative procedure that implicitly gives a discretization of the convection-diffusion problem on a composite grid at the discrete time levels \( t_n = n \Delta t \), with \( n = 1, 2, \ldots, N_t \). Throughout this section, we will assume that at each time \( t_n \) the LDC iteration converges. A sufficient condition for the iterative procedure to be convergent is that for every \( n = 1, 2, \ldots, N_t \) the vector norm

\[ \| \phi_{n,w}^n - \psi_{n,w-1}^n \| \to 0, \quad (w \to \infty). \]  

Condition (4.1) implies that

\[ \| \tilde{\psi}_{n,w}^{n-1+(k-1)/\tau} - \tilde{\psi}_{n,w}^{n-1+(k-1)/\tau} \| \to 0, \quad (w \to \infty), \forall k = 1, 2, \ldots, n_t, \]  

and therefore

\[ \| \phi_{t,n,w}^n - \psi_{t,n,w-1}^n \| \to 0, \quad (w \to \infty). \]  

When the LDC iteration at \( t_n \) has converged, the subscript \( w \) is removed, and the converged global and local solution are called \( \phi_{n}^n \in G(\tilde{\Omega}_H) \) and \( \psi_{n}^n \in G(\tilde{\Omega}_{i,H}) \) respectively. They define a composite grid approximation \( \phi_{n}^n \in G(\tilde{\Omega}_{i,H}) \) as in (3.33). In Lemma 4.1 we show that the converged coarse grid solution \( \phi_{n}^n \) and the converged fine grid solution \( \phi_{n}^n \) coincide in \( \Omega_{i,H} \) for \( n > 1 \). Note that this property is automatically verified for \( n = 0 \) (initial condition).
Lemma 4.1
Assume that the local coarse grid stationary homogeneous system
\[
\begin{aligned}
&\left\{ \begin{array}{l}
T_{i,j}^{H}(\varphi) + \left( \nabla_{\Sigma} F^H(\varphi) \right)_{i,j} = 0, \quad \text{for } (x_i, y_j) \in \Omega_{1,H}, \\
\varphi = 0, \quad \text{for } (x_i, y_j) \in \Gamma_H,
\end{array} \right.
\end{aligned}
\]  
(4.4)
has only the zero solution in \( G(\Omega_{1,H} \cup \Gamma_H) \). Then the limit solution \((\varphi^*_H, \varphi^*_{1,h})\) of the LDC iteration at time \( t_n \) \((n \geq 1)\) satisfies
\[
\varphi^n_H = \varphi^n_{1,h}, \quad \text{on } \Omega_{1,H}. 
\]  
(4.5)

Proof. From (3.30) and (3.38), we obtain
\[
T_{i,j}^{H}(\varphi^n_H) + \left( \nabla_{\Sigma} F^H(\varphi^n_H) \right)_{i,j} = T_{i,j}^{H}(\varphi^n_{H,h}|\Omega_H) + \left( \nabla_{\Sigma} F^H(\varphi^n_{H,h}|\Omega_H) \right)_{i,j} \quad \text{for } (x_i, y_j) \in \Omega_{1,H}. 
\]  
(4.6)
or
\[
T_{i,j}^{H}(\varphi^n_H - \varphi^n_{H,h}|\Omega_H) + \left( \nabla_{\Sigma} F^H(\varphi^n_H - \varphi^n_{H,h}|\Omega_H) \right)_{i,j} = 0 \quad \text{for } (x_i, y_j) \in \Omega_{1,H}. 
\]  
(4.7)

Moreover, from (2.27) and (3.33), we can deduce that \( \varphi^n_{H,h} - \varphi^n_{H,h}|\Omega_H \in G(\Omega_{1,H} \cup \Gamma_H) \) satisfies system (4.4). From the assumption, \( \varphi \) is null on \( \Omega_{1,H} \cup \Gamma_H \), which is equivalent to (4.5) because \( \varphi^n_{H,h} \equiv \varphi^n_{1,h} \) on \( \Omega_{1,H} \) (see (3.33)).

The results of Lemma 4.1 will be used in the proof of Theorem 4.2, which gives the discrete conservation law that is satisfied by the converged composite grid solution \( \varphi^*_{1,h} \). Before that, we consider for a while what happens when problem (2.6) is discretized on a global grid \( \Omega_H \) by the finite volume method. Again, for simplicity, we consider the implicit Euler scheme for time discretization. The following conservation law
\[
T_{i,j}^{H}(\varphi^n_H) - T_{i,j}^{H}(\varphi^{n-1}_H) + \left( \nabla_{\Sigma} F^H(\varphi^n_H) \right)_{i,j} \Delta t = S^{H,n}_{i,j} \Delta t 
\]  
(4.8)
holds for any control volume \( V_{i,j} \subset \Omega_H \). Formula (4.8) is a discrete equivalent of the continuous conservation law (2.6). Summation of the discrete conservation laws that hold for individual control volumes \( V_{i,j} \) leads to a conservation law on the union of these control volumes. This is because fluxes over internal faces cancel: in a time step \( \Delta t \), the influx into \( V_{i,j} \subset \Omega_H \) out a neighboring control volume \( V_{l,m} \subset \Omega_H \) is balanced by the outflux from \( V_{i,j} \) to \( V_{l,m} \) in the same time step. Theorem 4.2 states that this property is also verified for the limit of the LDC iteration \( \varphi^*_{1,h} \), which is computed on a composite grid using two different time integration steps \( \Delta t \) and \( \delta t \). This is the generalization to time-dependent problems of the conservation property presented in [2] for the LDC method in stationary cases.

Theorem 4.2
Under the assumption of Lemma 4.1, the composite grid solution satisfies the following system of discrete conservation laws:
\[
\nabla_{\Sigma} \text{best}^H(\varphi^n_{1,h}) - \nabla_{\Sigma} \text{best}^H(\varphi^{n-1}_{1,h}) + \nabla_{\Sigma} \text{best}^H(\varphi^n_{1,h}) \Delta t = S^{\text{best},n}_{i,j} \Delta t. 
\]  
(4.9)
Proof. Combination of (3.30) and (3.29) yields

\[ T^H(\varphi^n_h) + \nabla^H_\Sigma f^H(\varphi^n_h) \Delta t = T^H(\varphi^n_{H,h} | \Omega_h) - T^{\text{best}}(\varphi^n_{H,h}) + T^{\text{best}}(\varphi^{n-1}_H) \]

\[ + \nabla^H_\Sigma \left( f^H(\varphi^n_{H,h} | \Omega_h) - T^{\text{best}}(\varphi^n_{H,h}) \right) \Delta t + S^{\text{best},n} \Delta t. \tag{4.10} \]

Using Lemma 4.1, we have

\[ T^H(\varphi^n_h) = T^H(\varphi^n_{H,h} | \Omega_h), \tag{4.11} \]

\[ F^H(\varphi^n_h) = F^H(\varphi^n_{H,h} | \Omega_h). \tag{4.12} \]

Substitution of (4.11) and (4.12) into (4.10) gives (4.9). □

We note that, for \((x_i, y_j) \in \Omega_{1,h}\), the conservation laws (4.9) reduce to

\[ T^{\text{sum}}_{1,i,j}(\varphi^n_{1,h}) - T^{\text{sum}}_{1,i,j}(\varphi^{n-1}_{1,h}) \]

\[ + \left( \nabla^H_\Sigma \sum_{k=1}^{\tau} F^{\text{sum}}_{l}(\varphi^{n-1+k/\tau}_{l,h}) \delta t \right)_{i,j} = \sum_{k=1}^{\tau} S^{\text{sum},n-1+k/\tau}_{l,i,j} \delta t. \tag{4.13} \]

This is the same relation we deduced in Lemma 3.2 summing the conservation laws that hold for each of the \(\sigma^2\) fine grid control volume that partition \(V_{i,j} \subset \Omega_1\) for all the times \(t_{n-1+k/\tau}\), with \(k = 1, 2, \ldots, n_t\). For \((x_i, y_j) \in \Omega_{c,h}\), balance (4.9) becomes

\[ T^H_{i,j}(\varphi^n_h) - T^H_{i,j}(\varphi^{n-1}_H) + \left( \nabla^H_\Sigma f^H(\varphi^n_H) \right)_{i,j} \Delta t = S^H_{i,j} n \Delta t, \tag{4.14} \]

which is clearly the set of conservation laws that correspond to the coarse grid discretization with time step \(\Delta t\). For points \((x_i, y_j) \in \Gamma_H\), Theorem 4.2 guarantees that, in a time step \(\Delta t\), the discrete influx into \(V_{i,j}\) out of a neighboring volume \(V_{l,m}\), with \((x_l, y_m) \in \Omega_l\), matches the total discrete outflux from \(V_{l,m}\) to \(V_{i,j}\) in the same time step \(\Delta t\). The latter is the sum of all the fine grid fluxes at all the intermediate times \(t_{n-1+k/\tau}\), with \(k = 1, 2, \ldots, n_t\).

In conclusion, when we sum discrete conservation laws that hold on individual control volumes \(V_{i,j}\), the finite volume adapted LDC method is such that internal fluxes cancel like it happens for the finite volume discretization on a single grid. We note that (4.9) holds for the fully converged composite grid solution \(\varphi^n_{H,h}\). However, as shown in [1] for stationary cases, the convergence of the LDC algorithm is very fast and one iteration per time step is generally sufficient. This is for example what happens in the numerical experiments we present in Section 5.

5 Numerical experiments

In this section we present two numerical experiments that illustrate the accuracy and the efficiency of the LDC method with a finite volume adapted defect term. In Section 5.1, we compare the method to a single uniform grid solver. We show that LDC can achieve the same accuracy as the uniform grid solver, while computing a numerical approximation of the solution in a significantly smaller number of grid points and thus being a more efficient method. In Section 5.2, we compare the LDC algorithm with the standard choice for the defect term and the LDC algorithm with the finite volume adapted defect term. Only for the latter a discrete conservation property holds on the composite grid.
5.1 Example 1: comparison with a uniform grid solver

In this section we consider a two-dimensional convection-diffusion equation. We choose \( \Omega = (0, 1) \times (0, 1) \) and \( \Theta = (0, \text{t}_{\text{end}}] \), with \( \text{t}_{\text{end}} = 1.5 \), and we solve the following problem

\[
\begin{align*}
\frac{\partial \varphi}{\partial \text{t}} + \nabla \varphi &= \nabla^2 \varphi + s \quad \text{in } \Omega \times \Theta, \\
\varphi &= \psi, \quad \text{on } \partial \Omega \times \Theta, \\
\varphi &= \eta, \quad \text{in } \partial \Omega, \text{ } \text{t} = 0.
\end{align*}
\] (5.1)

The source term \( s(x, t) \), the initial condition \( \eta(x) \) and the Dirichlet boundary conditions are chosen in such a way that the exact analytical solution of the problem is

\[
\varphi_{\text{ex}}(x, y, t) = \left(1 - \tanh \left(100((x - 0.1) + (y - 0.1))\right)\right) e^t \sin^2(3\pi t). \] (5.2)

At each time level in \( \Theta \) the exact solution (5.2) has a region of high activity in the bottom left corner of \( \Omega \). Figure 4 shows the plot of the exact solution (5.2) at the final time \( \text{t}_{\text{end}} \) and the temporal evolution of \( \varphi_{\text{ex}} \) at the boundary point \((0, 0)\).

We solve problem (5.1) by means of the LDC method with a finite volume adapted defect term. For this problem the application of LDC with the standard choice for the defect term would give similar results. In fact, in this case, the balance of fluxes across the interface between coarse and fine grid at every time step \( \Delta \text{t} \) is not crucial to obtain an accurate numerical approximation of the solution. On the coarse grid \( \Omega_H \) we use a cell-centered finite volume approach. The fluxes (2.13) and (2.14) are approximated by centered differences, while the midpoint rule is used to compute integrals (2.11)-(2.14). The local region is chosen as \( \Omega_t = (0, 0.275) \times (0, 0.275) \). The time integration is performed using the implicit Euler method both globally and locally. The operators \( P_x \) and \( R_k^t \) perform linear interpolation in space and time respectively. We perform several runs with different values of grid sizes \( H \) and \( h \), and time steps \( \Delta \text{t} \) and \( \delta \text{t} \). In all the tests only one LDC iteration is performed at each time step \( \Delta \text{t} \). As a measure of the accuracy of the various numerical solutions, for each run we measure the scaled Euclidean norm

\[
\epsilon_2 = \frac{\|\varphi_{\text{ex}}(x, y, t_{\text{end}}) - \varphi_{\text{ex}}|_{\Omega_H} \|_2}{N_x^2}
\] (5.3)
Grid & time step | $e_2$ | Grid points per time step $\Delta t$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
<td>H</td>
<td>$\Delta t$</td>
<td>$\sigma = \tau$</td>
<td>LDC</td>
<td>Unif.</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$\Delta t_0$</td>
<td>3</td>
<td>3.98 $10^{-3}$</td>
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<td>1.83 $10^{-3}$</td>
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<td></td>
<td></td>
<td>7</td>
<td>1.37 $10^{-3}$</td>
<td>1.17 $10^{-3}$</td>
</tr>
<tr>
<td>$\frac{H}{3}$</td>
<td>$\Delta t_0$</td>
<td>3</td>
<td>4.90 $10^{-4}$</td>
<td>4.87 $10^{-4}$</td>
</tr>
<tr>
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<td>2.31 $10^{-4}$</td>
<td>2.24 $10^{-4}$</td>
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<td></td>
<td>7</td>
<td>1.55 $10^{-4}$</td>
<td>1.46 $10^{-4}$</td>
</tr>
</tbody>
</table>

Table 1: Results of the 2D numerical experiment. In the table: $H_0 = 0.05, \Delta t_0 = 0.1$.

In (5.3), the term $\varphi_{H_0,\Delta t_0}^{N_x}$ represents the restriction on the coarse grid of the composite grid solution at the final time $t_{end}$, while $N_x$ is the square root of the total number of coarse grid points $N_x^2$.

The results of each of the LDC runs are compared to the numerical solution found solving problem (5.1) on a single global uniform grid. The grid size and the time step in each uniform grid run are the same as the fine grid size $h = H/\sigma$ and the local time step $\delta t = \Delta t/\sigma$ of the corresponding LDC test. Also for the single uniform grid runs we measure the scaled Euclidean norm (5.3). Results in Table 1 show that in all the cases we considered, LDC can achieve the same order of accuracy as the uniform grid solver. Of course LDC is a more efficient method than the uniform grid solver, since a fine grid size and a small time step are adopted only where the large variations occur. To give an estimate of the complexity of the two methods, in Table 1 we also report the total number of grid points in which a numerical approximation of the solution is computed at every time step $\Delta t$. For LDC this value is the sum of two terms: the number of coarse grid points and the factor of time refinement $\tau$ multiplied by the number of fine grid points. For the uniform grid solver it is the product of the number of grid points in the uniform grid and the factor of time refinement $\tau$ of the corresponding LDC run. The last column of Table 1 shows that, in our example, LDC computes the solution at a number of grid points one order of magnitude smaller than the uniform grid solver.

### 5.2 Example 2: comparison with the standard LDC

In this section we present a two-dimensional problem for which conservation of fluxes across the interface between coarse and fine grid plays a crucial role. For such a problem, we expect the finite volume adapted LDC algorithm to be more accurate than the LDC method presented in [7], where no special precautions are taken in order to guarantee conservation on the composite grid.

We choose $\Omega = (0, 1) \times (0, 1)$ and $\Theta = (0, t_{end})$, with $t_{end} = 10$, and we solve the following problem

$$\begin{align*}
\frac{\partial \varphi}{\partial t} + \nabla \cdot (u \varphi - \nabla \varphi) &= 0, & \text{in } \Omega \times \Theta, \\
(u \varphi - \nabla \varphi) \cdot n &= 0, & \text{on } \partial \Omega \times \Theta, \\
\varphi &= 1, & \text{in } \Omega, \ t = 0.
\end{align*}$$

(5.4)

The velocity field is $u = (u_x, u_y) = (0.5 + 5 \sin(\pi t), 0.5 + 5 \sin(2\pi t))$. Problem (5.4) models the behavior of water held inside a basin. The partial differential equation expresses the tendency of the water level $\varphi$ to follow the velocity field $u$ and to level out. The
choice of the boundary conditions is such that water cannot flow in or out the basin. Integration of the partial differential equation over \( \Omega \) yields in fact the following global conservation law

\[
\frac{d}{dt} M(t) = 0,
\]

where \( M(t) \) is defined by

\[
M(t) := \int_{\Omega} \varphi(x, y, t) \, dx \, dy.
\]

Because of the choice of \( \mathbf{u} \) and of boundary layer effects, the water level \( \varphi \) varies most in the regions of \( \Omega \) next to the corners \( P_0 = (0, 0) \) and \( P_1 = (1, 1) \). Figure 5 shows the solution \( \varphi \) at two different time levels. The solution \( \varphi \) is periodic in time with period \( T = 2 \).

Because of the two local regions of high activity, problem (5.4) is solved by means of LDC. On the global coarse grid we apply the finite volume method with a cell-centered approach. Two local fine grids are placed in the corners next to \( P_0 \) and \( P_1 \). The time integration is performed using the implicit Euler method both globally and locally. Also in this case \( P_x \) and \( R_k^t \) perform linear interpolation in space and time respectively. Figures 6 and 7 report the numerical results for both the LDC method with the standard choice of the defect term and the finite volume adapted defect term. The results are obtained using a coarse grid size \( H = 0.05 \) and a time step \( \Delta t = 0.1 \). The local regions are \( \Omega_{1,0} = (0,0.175) \times (0,0.175) \) and \( \Omega_{1,1} = (0.775,1) \times (0.775,1) \). On both local regions the grid size is \( h = 0.01 \), while the time step is \( \delta t = 0.02 \). Only one LDC iteration is performed at each time step \( \Delta t \). Figure 6 shows that the LDC method with the finite volume adapted correction term can guarantee the global conservation of the water level; on the contrary the standard choice for the defect term produces an error in \( M \) which is of order 100% after only five periods \( T \). As a consequence, the amplitude of the oscillations of the water level at the two corners \( P_0 \) and \( P_1 \) increases unrealistically, see Figure 7.

We note that in [2] a one-dimensional version of the problem illustrated in this section is presented. There is, however, a crucial difference between the method presented here and in [2]: in our approach a small time step \( \delta t \) is adopted only on the local grid(s) to resolve the relatively fast phenomena occurring there. On the global coarse grid a
Figure 6: $M(t)$ for the LDC algorithm with the standard choice of the defect term (dashed line) and with the finite volume adapted defect term (solid line).

Figure 7: Time evolution of water level $\varphi$ in $P_0$ (a) and in $P_1$ (b) for the LDC algorithm with the standard choice of the defect term (dashed line) and with the finite volume adapted defect term (solid line).
larger time step $\Delta t$ is sufficient to catch the relatively slow variations of the solution. In [2] the same time step is used on the different grids; in this way the time step on the coarse might be forced to be unnecessary small, making the method less efficient than the one presented in this paper.

References


