Electromagnetic diffraction by a unidirectionally conducting circular disk

Citation for published version (APA):

DOI:
10.1137/0114115

Document status and date:
Published: 01/01/1966

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 28. Jan. 2021
ELECTROMAGNETIC DIFFRACTION BY A UNIDIRECTIONALLY CONDUCTING CIRCULAR DISK*

J. BOERSMA†

Abstract. This paper is concerned with the diffraction of an electromagnetic wave by a unidirectionally conducting circular disk.

First, the behavior of a time-harmonic electromagnetic field near the edge of an arbitrary plane unidirectionally conducting screen is determined from the condition that the energy density must be integrable over any finite domain.

Secondly, the problem of the diffraction of an arbitrary time-harmonic electromagnetic wave by a plane unidirectionally conducting circular disk is treated for the low-frequency case, i.e., the product ka of the wave number k and the disk radius a is small. Expansions in powers of ka are derived for the far field, the scattered energy and the field on the disk. Some special results for the case of plane-wave excitation are presented.

1. Introduction. In his paper [17] Toraldo di Francia gave an approximate solution for the diffraction of a plane electromagnetic wave by a circular disk composed of small wires. Such a screen was idealized as an infinitely thin disk, perfectly conducting in one direction and insulating in the orthogonal direction. This formulation led to a boundary value problem which was solved approximately. The approximate solution is valid when the radius of the disk is small compared to the wavelength of the incident wave, i.e., when ka is small, where k is the wave number and a is the radius of the disk. In fact Toraldo di Francia's result is the first term in a low-frequency series expansion in powers of ka for the scattered field and for the scattering cross section. Toraldo di Francia's analysis is related to Bethe's [4] solution for the diffraction of an electromagnetic wave through a small circular hole in a perfectly conducting screen. The reason for Toraldo di Francia's investigation was to use the unidirectionally conducting disk as a device for measuring the angular momentum carried by a circularly polarized electromagnetic wave.

Toraldo di Francia's paper started further research in diffraction problems dealing with unidirectionally conducting screens, halfplanes and strips. We mention papers by Karp [9], Radlow [12], Hurd [6], [7], Seshadri [13], [14], [15], Seshadri and Wu [16], Karal and Karp [8], Karp and Karal [10].

In §2 of the present paper we investigate the behavior of an electromagnetic field near the edge of a plane unidirectionally conducting screen...
of arbitrary shape. Following the method of Meixner [11], expansions for the electromagnetic field and for the surface charge density and current density induced in the screen are derived, which are valid near the edge of the screen. It turns out that the edge behavior for a unidirectionally conducting screen is different from the edge behavior for a perfectly conducting screen. The current density vanishes at the edge of the screen. Actually, Toraldo di Francia [17], Karp [9] and others started from this result and just proposed as their edge condition that the current density should vanish at the edge of the screen.

In §3 we treat the diffraction of an arbitrary time-harmonic electromagnetic wave by a unidirectionally conducting circular disk. A method is presented which yields in a systematic manner low-frequency series expansions for the scattered field. By means of this method an arbitrary number of terms of the expansions may be determined, though in practice the calculation of the higher order terms becomes rather laborious. We derive the scattered field from a Hertz vector which has a fixed direction parallel to the direction of conduction of the disk. The Hertz vector has to satisfy the reduced wave equation and a boundary condition on the disk which contains a number of undetermined constants. As in the work of Bazer and Brown [1], Bazer and Hochstadt [2], the Hertz vector is represented by suitable integrals. These integral representations which contain certain unknown functions are designed to satisfy all conditions of the problem except the boundary condition on the disk. The latter condition leads to Fredholm integral equations of the second kind for the unknown functions which may be solved by iteration when $ka$ is sufficiently small, yielding series expansions in powers of $ka$ for the unknown functions. The undetermined constants in the boundary condition follow from the edge condition which has to be imposed on the Hertz vector in order to ensure the proper edge behavior of the electromagnetic field. The scattered field at a large distance from the disk, the scattered energy, the scattered field on the disk, the current density and charge density induced in the disk can easily be derived using the integral representations for the Hertz vector. Low-frequency expansions for the various field quantities can be calculated. Actually, these expansions were evaluated up to relative order $(ka)^3$.

Finally, in §4 the general results of §3 are specialized to the case of plane-wave excitation. The first terms of the various series expansions are in agreement with Toraldo di Francia's [17] results.

2. Edge condition for a plane unidirectionally conducting screen. We introduce rectangular coordinates $x, y, z$. An infinitely thin plane screen
S occupies the part of the plane \( z = 0 \) within the closed curve \( C \) given by
\[
(2.1) \quad x = u(s), \quad y = v(s), \quad z = 0,
\]
where \( s \) denotes the arc length along \( C \) measured from a certain fixed point. The functions \( u(s) \) and \( v(s) \) are assumed to be smooth. \( C \) is the edge of the screen \( S \). \( S \) is perfectly conducting in the direction of the \( x \)-axis and insulating in the direction of the \( y \)-axis.

We will now investigate the behavior of a time-harmonic electromagnetic field (time dependence \( e^{-i\omega t} \)) near the edge \( C \). Then, first, the electromagnetic field has to satisfy Maxwell’s equations,
\[
(2.2) \quad \nabla \times \mathbf{E} = i\omega \mu \mathbf{H}, \quad \nabla \times \mathbf{H} = -i\omega \varepsilon \mathbf{E},
\]
where \( \varepsilon \) and \( \mu \) denote the dielectric constant and the magnetic permeability of the homogeneous medium which surrounds the screen. Secondly, the following boundary conditions will hold on the unidirectionally conducting screen \( S \):
\[
(2.3) \quad (i) \quad E_x = 0, \quad (iii) \quad [H_x] = 0,
(ii) \quad [E_y] = 0, \quad (iv) \quad [H_z] = 0.
\]
The notation \([w]\) denotes the difference between the values of \( w \) at the upper and lower sides of \( S \).


**Assumption 1.** The electromagnetic field near the edge \( C \) of the screen \( S \) can be expanded in a series in terms of powers of the distance to the edge.

**Assumption 2.** The electric and magnetic energy density will be integrable over any domain of three-dimensional space, including the edge \( C \) of the screen \( S \).

We introduce special orthogonal coordinates \( \rho, s, \psi \), suitable near the edge \( C \): \( s \) is the arc length along \( C \); \( \rho, \psi \) are polar coordinates in the plane normal to \( C \) at \( s \) where the upper and lower sides of the screen \( S \) will correspond to \( \psi = 0 \) and \( \psi = 2\pi \) respectively. Then the rectangular coordinates \( x, y, z \) and the coordinates \( \rho, s, \psi \) are connected by
\[
(2.4) \quad x = u(s) + \rho u'(s) \cos \psi, \quad y = v(s) - \rho u'(s) \cos \psi, \quad z = \rho \sin \psi.
\]
The line element in \( \rho, s, \psi \) coordinates is given by
\[
(2.5) \quad (d\sigma)^2 = (d\rho)^2 + (1 + \rho \kappa \cos \psi)^2 (ds)^2 + \rho^2 (d\psi)^2,
\]
where
\[
(2.6) \quad \kappa = \kappa(s) = u'(s)v''(s) - u''(s)v'(s)
\]
denotes the curvature of $C$. In deriving (2.5) we used the relation
\[ \{u'(s)\}^2 + \{v'(s)\}^2 = 1. \] From (2.5) the metric coefficients are given by
\[ h_1 = 1, \quad h_2 = 1 + \rho \kappa \cos \psi, \quad h_3 = \rho. \]
Hence, the following expressions will hold for the curl and divergence of a
vector in the curvilinear $\rho, s, \psi$ coordinates:
\[
(\nabla \times \mathbf{A})_\rho = \{\rho(1 + \rho \kappa \cos \psi)\}^{-1} \left[ \frac{\partial}{\partial s} (\rho A_\psi) \right. \\
\left. - \frac{\partial}{\partial \psi} \{\rho(1 + \rho \kappa \cos \psi)A_s\} \right],
\]
\[
(\nabla \times \mathbf{A})_s = \rho^{-1} \left[ \frac{\partial}{\partial \psi} (A_\rho) - \frac{\partial}{\partial \rho} (\rho A_\psi) \right],
\]
\[
(\nabla \times \mathbf{A})_\psi = (1 + \rho \kappa \cos \psi)^{-1} \left[ \frac{\partial}{\partial \rho} \{1 + \rho \kappa \cos \psi\} A_s - \frac{\partial}{\partial s} (A_\rho) \right],
\]
\[
\nabla \cdot \mathbf{A} = \{\rho(1 + \rho \kappa \cos \psi)\}^{-1} \left[ \frac{\partial}{\partial \rho} \{\rho(1 + \rho \kappa \cos \psi)A_\rho\} \right. \\
\left. + \frac{\partial}{\partial s} (\rho A_\rho) + \frac{\partial}{\partial \psi} \{\rho(1 + \rho \kappa \cos \psi)A_\psi\} \right].
\]
Using these formulae (2.8), Maxwell's equations (2.2) can be expressed in
$\rho, s, \psi$ coordinates.

The angle between the direction of conduction of the screen $S$ (i.e., the
$x$-axis) and the inward normal to the edge $C$ at $s$ will be denoted by $\theta = \theta(s)$. It
is obvious that
\[
\cos \theta = v'(s), \quad \sin \theta = u'(s).
\]
Then the boundary conditions (2.3) can be formulated as follows:
\[
(i) \quad E_\rho \cos \theta + E_s \sin \theta = 0,
\]
\[
(ii) \quad [-E_\rho \sin \theta + E_s \cos \theta] = 0,
\]
\[
(iii) \quad [H_\rho \cos \theta + H_s \sin \theta] = 0,
\]
\[
(iv) \quad [H_\psi] = 0.
\]
The notation $[w]$ denotes the difference between the values of $w$ for $\psi = 0$
and $\psi = 2\pi$.

Now we investigate the behavior of the electromagnetic field in a point
$(\rho, s, \psi)$ for small values of $\rho$. In the following analysis it is assumed that
$\theta(s) \neq \pi/2$, i.e., the direction of conduction is not parallel to the tangent
to $C$ at $s$. Of course such a condition cannot hold at each point $s$ of $C$. At the end of this section we will make some remarks about the edge behavior of the electromagnetic field near such points $s$ for which $\theta(s) = \pi/2$.

According to Assumption 1, we expand the curvilinear components of the electromagnetic field in series in terms of powers of $\rho$, the coefficients of these powers being functions of $s$ and $\psi$. We assume that the leading terms of these expansions can be represented in the following form:

$$
E_\rho = \rho^0 \alpha(s, \psi), \quad H_\rho = \rho^0 \alpha(s, \psi),
$$
(2.11)

$$
E_\sigma = \rho^1 \beta(s, \psi), \quad H_\sigma = \rho^1 \beta(s, \psi),
$$

$$
E_\phi = \rho^0 \gamma(s, \psi), \quad H_\phi = \rho^0 \gamma(s, \psi).
$$

According to Assumption 2, we have to require $t > -1$.

The leading terms (2.11) are substituted into Maxwell's equations (2.2), using (2.8). Then we obtain the following equations for $\alpha, \beta, \gamma$:

$$
\frac{\partial \beta}{\partial \psi} = 0, \quad t \beta = 0,
$$
(2.12)

$$
\frac{\partial \alpha}{\partial \psi} - (t + 1)\gamma = 0, \quad (t + 1)\alpha + \frac{\partial \gamma}{\partial \psi} = 0.
$$

A similar set of equations holds when $\alpha, \beta, \gamma$ are replaced by $a, b, c$.

The general solution of (2.12) is given by

$$
\alpha(s, \psi) = \alpha(s) \sin (t + 1)\psi + \gamma(s) \cos (t + 1)\psi,
$$
(2.13)

$$
\gamma(s, \psi) = \alpha(s) \cos (t + 1)\psi - \gamma(s) \sin (t + 1)\psi,
$$

$$
\beta(s, \psi) = 0 \text{ if } t \neq 0, \quad \beta(s, \psi) = \beta(s) \text{ if } t = 0.
$$

The functions $\alpha(s), \beta(s), \gamma(s)$ are undetermined functions of $s$.

According to (2.10) the following boundary conditions have to be satisfied:

$$
\alpha(s, 0) \cos \theta + \beta(s, 0) \sin \theta = \alpha(s, 2\pi) \cos \theta + \beta(s, 2\pi) \sin \theta = 0,
$$

$$
\alpha(s, 0) \sin \theta - \beta(s, 0) \cos \theta = \alpha(s, 2\pi) \sin \theta - \beta(s, 2\pi) \cos \theta,
$$

which conditions are equivalent to

$$
\alpha(s, 0) = \alpha(s, 2\pi), \quad \beta(s, 0) = \beta(s, 2\pi),
$$
(2.14)

$$
\alpha(s, 0) \cos \theta + \beta(s, 0) \sin \theta = 0.
$$

When (2.13) is substituted into (2.14), nonzero solutions for $\alpha(s, \psi)$. 

\( \beta(s, \psi), \gamma(s, \psi) \) only exist if \( t = \frac{1}{2}n \), where \( n \) is an arbitrary integer, in which case \( \gamma(s) = 0 \) if \( t \neq 0 \), and \( \gamma(s) = -\beta(s) \tan \theta \) if \( t = 0 \).

The general solution of the set of equations for \( a(s, \psi), b(s, \psi), c(s, \psi) \) is also given by (2.13) when \( \alpha, \beta, \gamma \) are replaced by \( a, b, c \). The functions \( a(s), b(s), c(s) \) are again undetermined functions of \( s \). The boundary conditions (2.10) determine the values of \( t \) for which nonzero solutions for \( a(s, \psi), b(s, \psi), c(s, \psi) \) exist, viz., \( t = n \), where \( n \) is an arbitrary integer.

It is clear now that the expansions of the electromagnetic field near the edge only contain integral and half-integral powers of \( \rho \). Owing to the requirement \( t > -1 \), the leading term in the expansion of the electric field will contain a factor \( \rho^{-1/2} \), whereas the leading term in the expansion of the magnetic field will contain a factor \( \rho^0 \).

The expansions for the electromagnetic field will now be calculated up to and including terms of order \( \rho^{1/2} \). In accordance with the foregoing results we state the following expansions.

\[
E_\rho = \rho^{-1/2}a_0(s) \sin \frac{1}{2} \psi + \{\alpha_1(s) \sin \psi - \beta_1(s) \tan \theta \cos \psi\} + \rho^{1/2}a_2(s, \psi) + O(\rho),
\]

(2.15) \[E_s = \beta_1(s) + \rho^{1/2}b_2(s, \psi) + O(\rho),\]

\[E_\psi = \rho^{-1/2}a_0(s) \cos \frac{1}{2} \psi + \{\alpha_1(s) \cos \psi + \beta_1(s) \tan \theta \sin \psi\} + \rho^{1/2}\gamma_2(s, \psi) + O(\rho),\]

\[H_\rho = \{a_1(s) \sin \psi + c_1(s) \cos \psi\} + \rho^{1/2}a_2(s, \psi) + O(\rho),\]

(2.16) \[H_s = b_1(s) + \rho^{1/2}b_2(s, \psi) + O(\rho),\]

\[H_\psi = \{a_1(s) \cos \psi - c_1(s) \sin \psi\} + \rho^{1/2}c_2(s, \psi) + O(\rho),\]

where the functions \( a_2, b_2, \gamma_2, a_2, b_2, c_2 \) have to be determined. Substitution of (2.15) and (2.16) into Maxwell's equations (2.2) and into the boundary conditions (2.10) leads to a set of equations and boundary conditions for these functions which can be solved. Ultimately, the following expansions will hold for the electromagnetic field near the edge \( C \) of the plane unidirectionally conducting screen \( S \):

\[
E_\rho = \rho^{-1/2}a_0(s) \sin \frac{1}{2} \psi + \{\alpha_1(s) \sin \psi - \beta_1(s) \tan \theta \cos \psi\} + \rho^{1/2}\{\frac{3}{4} \kappa a_0(s) \sin \frac{1}{2} \psi + a_2(s) \sin \frac{3}{2} \psi\} + O(\rho),
\]

(2.17) \[E_s = \beta_1(s) + 2\rho^{1/2}a_0'(s) \sin \frac{1}{2} \psi + O(\rho),\]

\[E_\psi = \rho^{-1/2}a_0(s) \cos \frac{1}{2} \psi + \{\alpha_1(s) \cos \psi + \beta_1(s) \tan \theta \sin \psi\} + \rho^{1/2}\{\frac{3}{4} \kappa a_0(s) \cos \frac{1}{2} \psi + a_2(s) \cos \frac{3}{2} \psi\} + O(\rho);\]
\( H_p = \{a_1(s) \sin \psi + c_1(s) \cos \psi\} + 2i\omega \varepsilon_p^{1/2} \alpha_0(s) \tan \theta \cos \frac{3}{2} \psi + O(\rho), \)

\( H_s = b_1(s) - 2i\omega \varepsilon_p^{1/2} \alpha_0(s) \cos \frac{1}{2} \psi + O(\rho), \)

\( H_\psi = \{a_1(s) \cos \psi - c_1(s) \sin \psi\} - 2i\omega \varepsilon_p^{1/2} \alpha_0(s) \tan \theta \sin \frac{3}{2} \psi + O(\rho). \)

The functions \( \alpha_i(s), \beta_i(s), \gamma_i(s), \alpha_i(s), \beta_i(s), \gamma_i(s) \) are undetermined functions of \( s \).

From the discontinuities of the normal component of the electric field and of the tangential components of the magnetic field across the screen we derive the following expansions for the surface charge density \( \sigma \) and for the current density \( I = (I_x, I_y) \) induced in the screen.

\[ \sigma = [\varepsilon E_\psi] = 2\varepsilon \omega_p^{1/2} \alpha_0(s) + 2\varepsilon \omega_p^{1/2} \{\frac{1}{4} \alpha_0(s) + \alpha_2(s)\} + O(\rho), \]

\( I_x = [-H_s] = 4i\omega \varepsilon_p^{1/2} \alpha_0(s) + O(\rho), \)

\( I_s = [H_\psi] = 4i\omega \varepsilon_p^{1/2} \alpha_0(s) \tan \theta + O(\rho). \)

It follows easily from (2.19) that the rectangular components \( (I_x, I_y) \) of the current density \( I \) are given by

\[ I_x = 4i\omega \varepsilon_p^{1/2} \alpha_0(s) \sec \theta + O(\rho), \quad I_y = O(\rho), \]

in agreement with the screen being perfectly conducting in the direction of the \( x \)-axis and insulating in the direction of the \( y \)-axis. In fact, the component \( I_y \) will be zero. According to (2.19) the current density vanishes at the edge of the screen where \( \rho = 0 \). We remark that Toraldo di Francia [17], Karp [9], and others just stated their edge condition to be the requirement that the current density should vanish at the edge of the screen.

For comparison we quote the edge behavior of an electromagnetic field near the edge of a perfectly conducting screen. According to Meixner [11] the leading terms of the expansions for the \( \rho, s, \psi \)-components of the electromagnetic field and for the charge and current density are given by these estimates:

\[ E_\rho = O(\rho^{-1/2}), \quad H_\rho = O(\rho^{-1/2}), \quad \sigma = O(\rho^{-1/2}), \]

\[ E_s = O(\rho^{1/2}), \quad H_s = O(1), \quad I_\rho = O(\rho^{1/2}), \]

\[ E_\psi = O(\rho^{-1/2}), \quad H_\psi = O(\rho^{-1/2}), \quad I_\rho = O(\rho^{-1/2}). \]

The behavior of the electric field and of the charge density is rather the same for the unidirectionally conducting screen and for the perfectly con-
ducting screen. However, the magnetic field will be finite and the current density will even vanish at the edge of a unidirectionally conducting screen, whereas at the edge of a perfectly conducting screen only the tangential component of the magnetic field is finite and only the normal component of the current density vanishes, the other components becoming infinite at the edge.

The expansions (2.17) and (2.18) refer to the total electromagnetic field, so when dealing with diffraction of an incident wave \((E^i, H^i)\) by the unidirectionally conducting screen \(S\), the vectors \(E, H\) in (2.17), (2.18) stand for the sum of the incident wave and the scattered wave \((E^s, H^s)\). Now we assume that the incident wave is due to sources which are not located on \(S\); then \(E^i, H^i\) are certainly finite at the edge of \(S\). Hence, the scattered field \((E^s, H^s)\) will show the edge behavior as prescribed by (2.17), (2.18):

\[
E^s_\rho = O(\rho^{-1/2}), \quad E^s_\sigma = O(1), \quad E^s_\psi = O(\rho^{1/2}),
\]
\[
H^s_\rho = O(1), \quad H^s_\sigma = O(1), \quad H^s_\psi = O(1).
\]

The scattered wave \((E^s, H^s)\) can be understood to be due to the current density \(I\) in the screen \(S\). Because this current density has a fixed direction, the scattered field can be derived from a Hertz vector \(\Pi\) which has the same direction, parallel to the direction of conduction of \(S\), according to

\[
E^s = (1/\varepsilon) \nabla \times \nabla \times \Pi, \quad H^s = -i\omega \nabla \times \Pi.
\]

Let \(\Pi\) denote the length of the vector \(\Pi\); then the \(\rho, \sigma, \psi\)-components of \(\Pi\) are given by

\[
\Pi_\rho = \Pi \cos \theta \cos \psi, \quad \Pi_\sigma = \Pi \sin \theta, \quad \Pi_\psi = -\Pi \cos \theta \sin \psi.
\]

We will now derive an edge condition to be imposed on the function \(\Pi(\rho, s, \psi)\) in order to ensure that the corresponding scattered wave as given by (2.23) shows the proper edge behavior. It is clear that the function \(\Pi(\rho, s, \psi)\) can be expanded in a series in terms of integral and half-integral powers of \(\rho\), valid near the edge of the screen \(S\). We assume that the leading term of this expansion is given by

\[
\Pi = \rho^s \delta(s, \psi).
\]

From (2.25) we calculate the leading term in the expansion of \(\nabla \times \Pi\) using (2.8) and we require that it agree with the leading term in the expansion of \(H^s\). It turns out that only the following solutions are possible: \(t = 0, \delta(s, \psi) = d(s)\) or \(t = 1\). Hence, the function \(\Pi(\rho, s, \psi)\) can be represented by

\[
\Pi(\rho, s, \psi) = d(s) + O(\rho)
\]
where \(d(s)\) is an undetermined function of \(s\). The characteristic feature of the expansion (2.26) is the fact that the term of order \(\rho^{1/2}\) is lacking. This feature will be used as the edge condition to be imposed on \(\Pi(\rho, s, \psi)\). So, in §3 we will require that the expansion of the Hertz vector component \(\Pi(\rho, s, \psi)\) near the edge of the unidirectionally conducting disk does not contain a term of order \(\rho^{1/2}\). It will turn out that this condition determines uniquely the solution of the problem of the diffraction by a unidirectionally conducting circular disk.

The foregoing results can be checked by comparison to the edge behavior of certain exact solutions to diffraction problems for a unidirectionally conducting halfplane: cf. Karp [9], Radlow [12], Hurd [6], Seshadri [14]. We will examine Karp's [9] solution to the diffraction of a plane electromagnetic wave,

\[
\begin{align*}
H^i &= A \exp \{i(k_1x + k_2y + k_3z - \omega t)\}, \\
E^i &= -(c/\omega) \nabla \times H^i,
\end{align*}
\tag{2.27}
\]

by a halfplane \(x \geq 0, -\infty < y < \infty, z = 0\), which is only conducting in a direction which makes an angle \(\alpha, 0 \leq \alpha < \pi/2\), with the positive \(x\)-axis. Karp derives the scattered wave from a scalar function \(u(x, y, z)\), according to

\[
\begin{align*}
H^s &= \nabla \times u, \\
E^s &= -(c/\omega) \nabla \times \nabla \times u,
\end{align*}
\tag{2.28}
\]

where \(u\) is a vector of length \(u(x, y, z)\) parallel to the direction of conduction. Actually, the function \(u(x, y, z)\) corresponds to our function \(\Pi(\rho, s, \psi)\). Karp formulates boundary conditions similar to (2.3) and an edge condition which requires that the current density vanish at the edge of the halfplane. The exact solution has a somewhat complicated form, but as Karp mentions the solution can be reduced to an expression in terms of Fresnel integrals. We made this reduction, yielding

\[
u(x, y, z) = D_1 \sqrt{\frac{2(K + k_1)}{\pi K}} e^{-\pi i/4} e^{ik_2y} \\
\cdot \left[ \frac{1}{2 \cos \frac{1}{2} \psi_0} \left\{ \exp \left[iK\rho \cos \left(\frac{\phi - \psi_0}{2}\right)\right] F\left(\sqrt{2K\rho} \sin \frac{\phi - \psi_0}{2}\right) \right. \\
+ \exp \left[iK\rho \cos \left(\frac{\phi + \psi_0}{2}\right)\right] F\left(\sqrt{2K\rho} \sin \frac{\phi + \psi_0}{2}\right) \right] \\
- \frac{1}{2 \cos \frac{1}{2} \psi_1} \left\{ \exp \left[iK\rho \cos \left(\frac{\phi - \psi_1}{2}\right)\right] F\left(\sqrt{2K\rho} \sin \frac{\phi - \psi_1}{2}\right) \right. \\
+ \exp \left[iK\rho \cos \left(\frac{\phi + \psi_1}{2}\right)\right] F\left(\sqrt{2K\rho} \sin \frac{\phi + \psi_1}{2}\right) \right],
\tag{2.29}
\]
where
\[ x = \rho \cos \phi, \quad z = \rho \sin \phi, \quad 0 \leq \phi \leq 2\pi, \]
(2.30) \[ \cos \psi_0 = k_1/K, \quad \sin \psi_0 = \sqrt{K^2 - k_1^2}/K, \quad 0 \leq \psi_0 \leq \frac{\pi}{2}, \]
\[ \cos \psi_1 = c_1/K, \quad \sin \psi_1 = i\sqrt{c_1^2 - K^2}/K, \quad \psi_1 = i\phi_1, \quad \phi_1 > 0. \]
The symbols \( D_1, K, c_1 \) are introduced in [9]. The Fresnel integral \( F(w) \) is defined by
(2.31) \[ F(w) = \int_w^\infty e^{i\lambda x} d\lambda. \]

The function \( u(x, y, z) \) can be expanded in powers of \( \rho \), the distance to the edge:
\[ u(x, y, z) = D_1 \sqrt{\frac{2(K + k_1)}{\pi K}} e^{-\frac{r}{\sqrt{2}}} e^{ik_2 y} \]
\[ \cdot \left[ \frac{e^{i\rho^4/2} \sqrt{\pi}}{2} \left( \frac{1}{\cos \frac{1}{2} \psi_0} - \frac{1}{\cos \frac{1}{2} \psi_1} \right) \right. \]
\[ + iK\rho \cos \phi \left( \frac{\cos \psi_0}{\cos \frac{1}{2} \psi_0} - \frac{\cos \psi_1}{\cos \frac{1}{2} \psi_1} \right) \]
\[ - \frac{K^2 \rho^2}{4} \left( \frac{1 + \cos 2\phi \cos 2\psi_0}{\cos \frac{1}{2} \psi_0} - \frac{1 + \cos 2\phi \cos 2\psi_1}{\cos \frac{1}{2} \psi_1} \right) \]
\[ \left. - \frac{i}{3} (2K\rho)^{3/2} \sin \frac{3}{2} \phi \left( \cos \psi_0 - \cos \psi_1 \right) + O(\rho^{5/2}) \right]. \]

Note that no term of order \( \rho^{1/2} \) occurs in this expansion. In fact the requirement of a vanishing term of order \( \rho^{1/2} \) determined the unknown coefficient \( a_1 \) in [9].

Similar expansions may be derived for the scattered field near the edge of the halfplane using (2.28) and (2.32). It was found that these expansions agreed with the results (2.17) and (2.18).

Radlow [12] treated the diffraction of a dipole field by a unidirectionally conducting halfplane. He also investigated the edge behavior of the diffracted field. His results agree with the expansions derived from Karp's solution except that \( H_z \) will be of order \( \rho^0 \) and not of order \( \rho^{1/2} \) near the edge of the halfplane.

Finally, we have to return to the case \( \theta(s) = \pi/2 \) for a certain point \( s \) of the edge \( C \). In that case it turns out that the boundary conditions (2.14) do not determine any longer the exponent \( t \) occurring in the leading terms of the expansions of the electromagnetic field near the edge. As a matter of fact, any value of \( t > -1 \) would be compatible with the boundary conditions (2.14). Hence, using Meixner's method we cannot make any state-
ment about the behavior of the electromagnetic field and of the Hertz vector near a point \( s \) of the edge \( C \) for which \( \theta(s) = \pi/2 \).

In order to overcome this difficulty we make a third assumption.

**Assumption 3.** The behavior of the electromagnetic field and of the Hertz vector near the edge of a unidirectionally conducting screen is the same along the whole edge.

Hence near a point \( s \) for which \( \theta(s) = \pi/2 \), the scattered electromagnetic field and the Hertz vector will show an edge behavior as given by (2.22) and (2.26) respectively. It is clear that Assumption 3 cannot lead to inadmissible solutions, because, the edge behavior of the scattered electromagnetic field being given by (2.22), the electric and magnetic energy density are certainly integrable up to the edge of the screen. In §3, when dealing with diffraction by unidirectionally conducting circular disk, we will see that the edge condition for the Hertz vector valid along the whole edge determines the solution uniquely. Hence, assuming that the boundary value problem of §3 has a unique solution, the solution of §3 is the correct one and it satisfies Assumption 3.

We remark that the results (2.22) and (2.26) describing the edge behavior of the scattered electromagnetic field and of the Hertz vector do not hold for a halfplane which is only conducting in a direction parallel to its edge, i.e., for a case in which \( \theta(s) = \pi/2 \) for each point \( s \) of the edge. Using Karp's [9] method, we derived the following solution for the diffraction of the plane wave (2.27) by such a halfplane. The function \( u(x, y, z) \) expressed in terms of Fresnel integrals is given by the following expression:

\[
\begin{align*}
  u(x, y, z) &= D_1 \sqrt{\frac{2(K + k_1)}{\pi K}} \frac{e^{-\pi i/4} e^{i k_2 y}}{2 \cos \frac{1}{2} \psi_0} \\
  &\cdot \left\{ \exp \left[ i K_\rho \cos \left( \phi - \psi_0 \right) \right] F \left( \sqrt{2 K_\rho} \sin \frac{\phi - \psi_0}{2} \right) \\
  &+ \exp \left[ i K_\rho \cos \left( \phi + \psi_0 \right) \right] F \left( \sqrt{2 K_\rho} \sin \frac{\phi + \psi_0}{2} \right) \right\}.
\end{align*}
\]

The same symbols have been used as in [9] and (2.29). The function \( u(x, y, z) \) can be expanded in powers of \( \rho \) yielding this expansion:

\[
\begin{align*}
  u(x, y, z) &= D_1 \sqrt{\frac{2(K + k_1)}{\pi K}} \frac{e^{-\pi i/4} e^{i k_2 y}}{2 \cos \frac{1}{2} \psi_0} \\
  &\cdot \left[ e^{\pi i/4} \sqrt{\pi} \left\{ 1 + i K_\rho \cos \phi \cos \psi_0 - \frac{K_\rho^2}{4} (1 + \cos 2\phi \cos 2\psi_0) \right\} \\
  &- (2 K_\rho)^{1/2} \sin \frac{1}{2} \phi \cos \frac{1}{2} \psi_0 \\
  &- \frac{i}{3} (2 K_\rho)^{3/2} \sin \frac{3}{2} \phi \cos \frac{3}{2} \psi_0 + O(\rho^{5/2}) \right].
\end{align*}
\]
We notice that in this case the expansion of \( u(x, y, z) \) does contain a term of order \( \rho^{1/2} \). Similar expansions can be derived for the scattered and for the total electromagnetic field near the edge, using (2.28) and (2.34). It turns out that the behavior of the total electromagnetic field near the edge of the halfplane is the same as in the case of a perfectly conducting halfplane, i.e., the edge behavior is given by (2.21).

3. Diffraction by a unidirectionally conducting circular disk. Introducing rectangular coordinates \( x, y, z \) and cylindrical coordinates \( r, \phi, z \) connected by \( x = r \cos \phi, y = r \sin \phi, 0 \leq \phi < 2\pi \), a plane infinitely thin circular disk occupies the region \( x^2 + y^2 \leq a^2, z = 0 \), or \( r \leq a, 0 \leq \phi < 2\pi, z = 0 \). The disk is perfectly conducting in the direction of the \( x \)-axis and insulating in the direction of the \( y \)-axis. An electromagnetic wave \( (E', H') \) impinges upon the disk. The vectors \( E', H' \) will show a time dependence \( e^{-i\omega t} \), this factor being omitted in what follows. The scattered electromagnetic wave is denoted by \( (E^s, H^s) \). Then the following boundary value problem can be formulated for \( (E^s, H^s) \):

(i) \( (E^s, H^s) \) satisfies Maxwell’s equations,
\[
\nabla \times E^s = i\omega H^s, \quad \nabla \times H^s = -i\omega E^s,
\]
\[
\nabla \cdot E^s = 0, \quad \nabla \cdot H^s = 0;
\]

(ii) \( E^s_x = -E^s', \quad [E^s_y] = 0, \quad [H^s_x] = 0, \quad [H^s_y] = 0 \) on the disk, i.e., when \( z = 0, \quad x^2 + y^2 \leq a^2 \);  

(iii) \( (E^s, H^s) \) satisfies Sommerfeld’s radiation condition at infinity;  

(iv) the behavior of \( (E^s, H^s) \) near the edge of the disk is given by (2.22).

The boundary conditions (ii) are a consequence of the conditions (2.3), where \( (E, H) \) stands for the total electromagnetic wave, viz., \( (E, H) = (E' + E^s, H' + H^s) \).

The scattered wave is now derived from a Hertz vector \( \Pi \) which is parallel to the \( x \)-axis according to (2.23). When the length of the Hertz vector is denoted by \( \Pi(x, y, z) \), (2.23) can be written out as
\[
E^s_x = \frac{1}{\epsilon} \left( \frac{\partial^2 \Pi}{\partial x^2} + k^2 \Pi \right), \quad E^s_y = \frac{1}{\epsilon} \frac{\partial \Pi}{\partial x} \frac{\partial \Pi}{\partial y}, \quad E^s_z = \frac{1}{\epsilon} \frac{\partial^2 \Pi}{\partial x \partial z}, \quad (3.1)
\]
\[
H^s_x = 0, \quad H^s_y = -i\omega \frac{\partial \Pi}{\partial z}, \quad H^s_z = i\omega \frac{\partial \Pi}{\partial y},
\]
where we used the fact that \( \Pi(x, y, z) \) is a solution of the reduced wave equation \( \Delta \Pi + k^2 \Pi = 0, \quad k = \omega (\epsilon \mu)^{1/2} \). Now the Hertz vector component \( \Pi(x, y, z) \) has to satisfy the following conditions:

(i) \( \Pi \) is a solution of the reduced wave equation, \( \Delta \Pi + k^2 \Pi = 0 \);
(ii) on the disk, i.e., when $z = 0$, $x^2 + y^2 \leq a^2$,
\[(3.2)\quad \frac{\partial^2 \Pi}{\partial x^2} + k^2 \Pi = -\epsilon E_x^i;\]

(iii) $\Pi$ satisfies Sommerfeld's radiation condition at infinity;

(iv) $\partial \Pi / \partial z = 0$ when $z = 0$, $x^2 + y^2 > a^2$;

(v) the expansion of $\Pi$ near the edge of the disk does not contain a
term of order $\rho^{1/2}$ where $\rho$ is the distance to the edge.

The boundary condition (ii) is implied by (3.1) and the condition
$E_x^s = -E_x^i$ on the disk. Further, it follows from a well-known relationship
between the Hertz vector and the current density induced in the disk that
$\Pi(x, y, z)$ is an even function of $z$. Hence, the boundary values $\Pi(x, y, +0)$,
$\Pi(x, y, -0)$ ($x^2 + y^2 \leq a^2$) assumed at the upper and lower sides of the disk
are equal and the remaining boundary conditions for the scattered wave are
automatically fulfilled. Condition (iv) also follows from the function $\Pi$
being even in $z$. The edge condition (v) to be imposed on $\Pi$ was derived in
§2, cf. (2.26).

It is our goal to derive in a systematic manner series expansions in powers
of $ka$ for the Hertz vector $\Pi$, for the scattered field on the disk and at a large
distance from the disk, for the scattered energy and for the current density
and surface charge density induced in the disk. The method will be illus-
trated by actually calculating the first few terms of the various expansions.

We assume that the $x$-component of the incident electric field on the disk
can be expanded in a Taylor series of the following form:
\[(3.3)\quad E_x^i(x, y, 0) = E \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e_{mn}(ikx)^m(iky)^n.\]

The double series will be convergent over the whole disk $x^2 + y^2 \leq a^2$ and
the coefficients $e_{mn}$ will be real and independent of $x$ and $y$. For several
practical examples, e.g., for the case of plane-wave excitation these assump-
tions are fulfilled.

A double series of similar type will represent $\Pi(x, y, 0)$,
\[(3.4)\quad \Pi(x, y, 0) = -\frac{\epsilon E/k^2}{\Pi_{mn}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}(ikx)^m(iky)^n,\]

where the coefficients $a_{mn}$ have to be determined. It is assumed that (3.4) is
also convergent over the whole disk $x^2 + y^2 \leq a^2$. As a matter of fact, it can
be shown that the convergence of (3.3) implies the convergence of (3.4); hence we may assume the convergence of the latter series at the outset.

In order to satisfy the boundary condition (3.2) the coefficients $a_{mn}$ and
$e_{mn}$ must be connected by the relation
\[(3.5)\quad a_{mn} - (m + 1)(m + 2)a_{m+2,n} = e_{mn}, \quad m, n = 0, 1, 2, \ldots .\]
In the following analysis we take into account only a finite number of terms of the series (3.4).

$$\Pi(x, y, 0) = -\left(\frac{\epsilon E}{k^2}\right)[a_{00} + ik(a_{10}x + a_{01}y)]$$
$$- k^2(a_{20}x^2 + a_{11}xy + a_{02}y^2)$$
$$- ik^3(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3)$$
$$+ k^4(a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4) + O(\alpha^5)],$$

where $\alpha = ka$.

Changing to polar coordinates $r, \phi$ we get

$$\Pi(r, \phi, 0) = -\left(\frac{\epsilon E}{k^2}\right)$$
$$\cdot \left[ a_{00} - \frac{k^2 r^2}{2} (a_{20} + a_{02}) + \frac{k^4 r^4}{8} (3a_{40} + a_{22} + 3a_{04})$$
$$+ ikr \cos \phi \left\{ a_{01} - \frac{k^2 r^2}{4} (3a_{30} + a_{12}) \right\}$$
$$+ ikr \sin \phi \left\{ a_{01} - \frac{k^2 r^2}{4} (a_{21} + 3a_{03}) \right\}$$
$$+ \frac{k^2 r^2}{2} \cos 2\phi \left\{ -a_{20} + a_{02} + k^2 r^2 (a_{40} - a_{04}) \right\}$$
$$+ \frac{k^2 r^2}{2} \sin 2\phi \left\{ -a_{11} + \frac{k^2 r^2}{2} (a_{31} + a_{13}) \right\}$$
$$+ \frac{ikr^3}{4} \cos 3\phi (-a_{30} + a_{12}) + \frac{ikr^3}{4} \sin 3\phi (-a_{21} + a_{03})$$
$$+ \frac{k^4 r^4}{8} \cos 4\phi (a_{40} - a_{22} + a_{04})$$
$$+ \frac{k^4 r^4}{8} \sin 4\phi (a_{31} - a_{13}) + O(\alpha^5) \right \}. \tag{3.6}$$

Now the Hertz vector component $\Pi(x, y, z)$ or $\Pi(r, \phi, z)$ will be represented by

$$\Pi(r, \phi, z)$$
$$= -\left(\frac{\epsilon E}{k^2}\right) a_{00} F_0(r, z) - \frac{1}{2} (a_{20} + a_{02}) G_0(r, z)$$
$$+ \frac{1}{6} (3a_{40} + a_{22} + 3a_{04}) H_0(r, z)$$
$$+ i \cos \phi \{ a_{10} F_1(r, z) - \frac{1}{4} (3a_{30} + a_{12}) G_1(r, z) \}$$
$$+ i \sin \phi \{ a_{01} F_1(r, z) - \frac{1}{4} (a_{21} + 3a_{03}) G_1(r, z) \}$$
ELECTROMAGNETIC DIFFRACTION

(3.7) \[ + \frac{1}{2} \cos 2\phi \{(a_{10} + a_{02})F_2(r, z) + (a_{40} - a_{04})G_2(r, z)\} \]
\[ + \frac{1}{2} \sin 2\phi \{-a_{10}F_2(r, z) + \frac{1}{2}(a_{81} + a_{13})G_2(r, z)\} \]
\[ + \frac{i}{4} \cos 3\phi \{-a_{10} + a_{12}\}F_3(r, z) + \frac{i}{4} \sin 3\phi \{-a_{20} + a_{03}\}G_3(r, z) \]
\[ + \frac{1}{2} \cos 4\phi (a_{40} - a_{22} + a_{04})F_4(r, z) \]
\[ + \frac{1}{2} \sin 4\phi (a_{31} - a_{12})F_4(r, z) + O(\alpha^5) \],

where the functions \( F_m(r, z), G_m(r, z), H_m(r, z) \) have to satisfy the following conditions.

(i) \( F_m \sin m\phi, F_m \cos m\phi, G_m \sin m\phi, G_m \cos m\phi, H_m \sin m\phi, H_m \cos m\phi \)

are solutions of the reduced wave equation; hence,

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2 - \frac{m^2}{r^2} \right) F_m(r, z) = 0, \]
\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2 - \frac{m^2}{r^2} \right) G_m(r, z) = 0, \]
\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2 - \frac{m^2}{r^2} \right) H_m(r, z) = 0. \]

(ii) \( F_m = (kr)^m, G_m = (kr)^{m+2}, H_m = (kr)^{m+4} \) when \( z = 0, r \leq a \).

(iii) \( F_m, G_m, H_m \) satisfy Sommerfeld’s radiation condition at infinity.

(iv) \( F_m, G_m, H_m \) are even functions of \( z \); hence,

\[ \frac{\partial F_m}{\partial z} = \frac{\partial G_m}{\partial z} = \frac{\partial H_m}{\partial z} = 0 \text{ when } z = 0, \quad r > a. \]

Owing to (iv) it is sufficient to consider \( F_m, G_m, H_m \) only for \( z \geq 0 \).

The boundary value problems for \( F_m, G_m, H_m \) are closely related to certain boundary value problems which arise at the diffraction of a scalar wave by a circular aperture in a rigid screen. An interesting method of solution for these problems was presented by Bazer and Brown [1], Bazer and Hochstadt [2]. We will now apply their method to the present problem. Therefore we introduce the following Bazer and Hochstadt type integral representations for the functions \( F_m, G_m, H_m \):

\[
F_m(r, z) = r^m \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^m \int_{-1}^{1} \frac{\exp\{ik\sqrt{r^2 + (z + i\alpha t)^2}\}}{\sqrt{r^2 + (z + i\alpha t)^2}} f_m(t) \, dt,
\]

where the unknown function \( f_m(t) \) is required to be an even function of \( t \).
and to satisfy the conditions

\[(3.9) \quad \frac{d^j f_m(t)}{dt^j} = 0, \quad j = 0, 1, \ldots, m - 1.\]

Similar representations hold for the functions $G_m$, $H_m$ after replacing $f_m(t)$ by the unknown functions $g_m(t)$, $h_m(t)$ respectively, which are required to have similar properties as $f_m(t)$.

These integral representations automatically satisfy conditions (i), (iii) and (iv). Using the technique developed by Boersma [5], condition (ii) leads to Fredholm integral equations of the second kind for the functions $f_m(t)/(1 - t^2)^m$, $g_m(t)/(1 - t^2)^m$, and $h_m(t)/(1 - t^2)^m$. The kernel of these integral equations is small when $\alpha$ is small. In the latter case the integral equations can be solved by iteration yielding expansions in powers of $\alpha$ for the functions $f_m$, $g_m$, $h_m$.

Actually, the present boundary value problems are contained within a boundary value problem treated in [5]. In [5, 2.4] we calculated a function $f^{(m)}(t)$, $m \geq 1$, occurring in a Bazer and Hochstadt type integral representation, which integral assumed the boundary value $J_m(kr \sin \gamma)$ for $z = 0, r \leq a$. From this function $f^{(m)}(t)$ we derive the following expansions for the functions $f_m$, $g_m$, $h_m$ when $m = 1, 2, 3 \cdots$:

\[
f_m(t) = \frac{(-1)^m \alpha^{m+1}}{2^m \sqrt{\pi} \Gamma(m + \frac{3}{2})} (1 - t^2)^m \left[ 1 - \frac{2m - (2m - 1)t}{2(2m - 1)(2m + 1)} \alpha^2 - \frac{4i}{9\pi} \alpha^3 \delta_{m,1} + O(\alpha^4) \right],
\]

\[
g_m(t) = \frac{(-1)^m \alpha^{m+1}}{2^m \sqrt{\pi} \Gamma(m + \frac{3}{2})} \alpha^{m+2}(1 - t^2)^m[m + t^2 + O(\alpha^2)],
\]

\[
h_m(t) = \frac{(-1)^m \alpha^{m+1}}{2^m \sqrt{\pi} \Gamma(m + \frac{3}{2})} (1 - t^2)^m O(\alpha^{m+4}),
\]

where $\delta_{m,1} = 1$ if $m = 1$, $\delta_{m,1} = 0$ if $m \neq 1$.

The functions $f_0$, $g_0$, $h_0$ can be quoted in a similar manner from Bazer and Hochstadt [2, 7]:

\[
f_0(t) = \frac{a}{\pi} \left[ 1 - \frac{2i}{\pi} \alpha + \left(-\frac{4}{\pi^2} + \frac{1}{2} t^2\right) \alpha^2 + i \left(\frac{4}{9} + \frac{8}{\pi^2} - \frac{1}{3} t^2\right) \alpha^3 \right. \\
\left. + \left(-\frac{4}{3\pi^2} + \frac{16}{\pi^4} - \frac{2}{3\pi^2} t^2 + \frac{1}{24} t^4\right) \alpha^4 + O(\alpha^5) \right],
\]

\[
g_0(t) = \frac{2a}{\pi} \alpha^2 \left[ t^2 - \frac{2i}{3\pi} \alpha + \left(-\frac{4}{3\pi^2} + \frac{1}{6} t^4\right) \alpha^2 + O(\alpha^3) \right],
\]

\[
h_0(t) = \frac{8a}{3\pi} \alpha^4 \left[ t^4 + O(\alpha) \right].
\]
We remark that
\[ f_m(t)/a^{m+1} = O(\alpha^m), \quad g_m(t)/a^{m+1} = O(\alpha^{m+2}), \]
\[ h_m(t)/a^{m+1} = O(\alpha^{m+4}), \]
from which follows by means of (3.8),
\[ F_m(r, z) = O(\alpha^m), \quad G_m(r, z) = O(\alpha^{m+2}), \quad H_m(r, z) = O(\alpha^{m+4}), \]
at a finite distance from the disk.

We still have to satisfy the edge condition for the Hertz vector \( \Pi(r, \phi, z) \): In a point at distance \( \rho \) from the edge, the expansion of \( \Pi \) in powers of \( \rho \) does not contain a term of order \( \rho^{1/2} \). We use the following formula of Bazer and Hochstadt [2], which describes the behavior of \( F_m(r, z) \) when \( r = a + \rho \cos \gamma, z = \rho \sin \gamma, \rho > 0, 0 \leq \gamma \leq \pi \); i.e., in a point at a distance \( \rho \) from the edge, \( F_m(a + \rho \cos \gamma, \rho \sin \gamma) \)
\begin{equation}
F_m(a + \rho \cos \gamma, \rho \sin \gamma) = F_m(a, 0) - \frac{2\sqrt{2}}{a^{m+1}} f_m^{(m)}(1) \left( \frac{\rho}{a} \right)^{1/2} \cos \frac{1}{2} \gamma + O \left( \frac{\rho}{a} \right),
\end{equation}
where \( f_m^{(m)}(1) \) denotes the \( m \)th derivative of \( f_m(t) \) at \( t = 1 \). Similar expansions hold for the functions \( G_m, H_m \). Using this formula the expansion for \( \Pi(r, \phi, z) \) near the edge of the disk can easily be obtained. The term of order \( \rho^{1/2} \) in this expansion is set equal to zero leading to the following equations:
\[ a_{00} f_0(1)/a - \frac{1}{2} (a_{20} + a_{a0}) g_0(1)/a \]
\[ + \frac{1}{8} (3a_{40} + a_{22} + 3a_{a4}) h_0(1)/a^2 = O(\alpha^5); \]
\[ a_{10} f_1(1)/a^2 - \frac{1}{4} (3a_{10} + a_{12}) g_1(1)/a^2 = O(\alpha^5); \]
\[ a_{01} f_1(1)/a^2 - \frac{1}{4} (a_{21} + 3a_{03}) g_1(1)/a^2 = O(\alpha^5); \]
\begin{equation}
(\begin{array}{c}
-a_{20} + a_{a0} f_2(1)/a^2 + (a_{40} - a_{a0}) g_2(1)/a^2 = O(\alpha^5); \\
-a_{11} f_2(1)/a^2 + \frac{1}{2} (a_{31} + a_{13}) g_2(1)/a^2 = O(\alpha^5); \\
(-a_{30} + a_{12}) f_3(1)/a^4 = O(\alpha^5); \\
(a_{40} - a_{22} + a_{a0}) f_4(1)/a^5 = O(\alpha^5); \\
(a_{31} - a_{13}) f_4(1)/a^5 = O(\alpha^5).
\end{array}
\end{equation}
These nine equations for the fifteen unknown coefficients $a_{mn}$, $m, n = 0, 1, 2, \cdots, m + n \leq 4$, have to be supplemented by six equations (3.5). Then, the final solution is given by

$$a_{00} = -\epsilon_{00}\alpha^2 - \frac{4i}{3\pi}\epsilon_{00}\alpha^3 + \left( -\frac{8}{15}\epsilon_{00} + \frac{\epsilon_{02}}{5} + \frac{\epsilon_{02}}{15} \right)\alpha^4 + O(\alpha^5);$$

$$a_{10} = -\frac{2}{9}\epsilon_{00}\alpha^2 + O(\alpha^4); \quad a_{01} = -\frac{2}{3}\epsilon_{01}\alpha^2 + O(\alpha^4);$$

$$a_{20} = -\frac{\epsilon_{00}}{2} - \frac{\epsilon_{00}}{2}\alpha^2 + O(\alpha^3); \quad a_{11} = -\frac{\epsilon_{11}}{5}\alpha^2 + O(\alpha^3);$$

$$a_{02} = -\frac{\epsilon_{00}}{2} + \left( -\frac{7}{10}\epsilon_{00} + \frac{\epsilon_{02}}{5} - \frac{3}{5}\epsilon_{02} \right)\alpha^2 + O(\alpha^3);$$

(3.14) $$a_{30} = -\frac{\epsilon_{01}}{6} + O(\alpha^2), \quad a_{21} = -\frac{\epsilon_{01}}{2} + O(\alpha^2);$$

$$a_{12} = -\frac{\epsilon_{01}}{6} + O(\alpha^2), \quad a_{03} = -\frac{\epsilon_{01}}{2} + O(\alpha^2);$$

$$a_{40} = -\frac{\epsilon_{00}}{24} - \frac{\epsilon_{02}}{12} + O(\alpha); \quad a_{31} = -\frac{\epsilon_{11}}{6} + O(\alpha);$$

$$a_{22} = -\frac{\epsilon_{00}}{4} - \frac{\epsilon_{02}}{2} + O(\alpha); \quad a_{13} = -\frac{\epsilon_{11}}{6} + O(\alpha);$$

$$a_{04} = -\frac{5}{24}\epsilon_{00} + \frac{\epsilon_{02}}{12} - \frac{\epsilon_{02}}{2} + O(\alpha).$$

These results are substituted into (3.7) together with the functions $f_m, g_m, h_m$ as given by (3.10), (3.11). Then we obtain the following expansion for the Hertz vector component $\Pi(r, \phi, z)$:

$$\Pi(r, \phi, z) = \left( \epsilon E/\pi k^2 \right) \left[ \alpha \int_{-1}^{1} \exp \left\{ ik \sqrt{r^2 + (z + iat)^2} \right\} \frac{\partial}{\partial t} \int_{-1}^{1} \exp \left\{ ikr^2 + (z + iat)^2 \right\} \left\{ \frac{1}{r^2 + (z + iat)^2} \right\} dt \right]$$

$$- ia^2 \cos \phi \frac{\partial}{\partial r} \int_{-1}^{1} \exp \left\{ ik \sqrt{r^2 + (z + iat)^2} \right\} \frac{\epsilon_{00}}{9} \alpha^3(1 - t^2)^2 dt$$

(3.15) $$- ia^2 \sin \phi \frac{\partial}{\partial r} \int_{-1}^{1} \exp \left\{ ik \sqrt{r^2 + (z + iat)^2} \right\} \frac{\epsilon_{00}}{3} \alpha^3(1 - t^2)^2 dt$$

$$+ a^3 \cos 2\phi \frac{1}{r} \left( \frac{\partial}{\partial r} \right)^2 \int_{-1}^{1} \exp \left\{ ik \sqrt{r^2 + (z + iat)^2} \right\} \frac{\epsilon_{00}}{r^2 + (z + iat)^2}$$
Next, we investigate the behavior of the Hertz vector component $\Pi$ at a large distance $R$ from the disk. According to Bazer and Hochstadt [2] their integral representation assumes the following asymptotic value at a point $r = R \sin \theta, z = R \cos \theta$ for large values of $R$:

$$
\Pi(R \sin \theta, \phi, R \cos \theta) \sim \frac{4\pi E\alpha}{3\pi k^2} \alpha^2 \left[ \epsilon_{\phi} + \left( \frac{3}{5} \epsilon_{\phi} - \frac{\epsilon_{\phi}}{5} \right) + \frac{4}{15} \epsilon_{\phi} \sin \theta \cos \phi \right.
\left. + \frac{4}{15} \epsilon_{\phi} \sin \theta \sin \phi - \frac{1}{10} \epsilon_{\phi} \sin^2 \theta \right] \alpha^2 + O(\alpha^3) \frac{e^{ikR}}{R}.
$$

In the case that $f(t)$ is an even function of $t$. This result holds for $0 \leq \theta \leq \pi$. Using this formula we derive an expansion in powers of $\alpha$ for the asymptotic value of the Hertz vector component $\Pi$ at a large distance $R$ from the disk:

$$
\Pi(R \sin \theta, \phi, R \cos \theta) \sim 0, \quad H_R \sim 0,
$$

$$
E_{\theta} \sim \sqrt{\mu/\epsilon}, \quad H_\phi \sim \left(\frac{k^2}{\epsilon}\right) \Pi \cos \theta \cos \phi,
$$

$$
E_\phi \sim -\sqrt{\mu/\epsilon}, \quad H_\theta \sim -\left(\frac{k^2}{\epsilon}\right) \Pi \sin \phi.
$$

Substitution of $\Pi$ as given by (3.16) yields series expansions in powers of $\alpha$ for the asymptotic values of the scattered field components.

Integration of the Poynting vector over a sphere of infinitely increasing
radius yields the following result for the average scattered energy $E_{sc}$:

$$E_{sc} = \lim_{\kappa \to \infty} \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \int_0^\pi \int_0^{2\pi} \left\{ |E^\theta|^2 + |E^\phi|^2 \right\} R^2 \sin \theta \, d\phi \, d\theta$$

$$\tag{3.18} = \frac{64}{27\pi} \sqrt{\frac{\varepsilon}{\mu}} \alpha^4 \left[ e_{00} + e_{00} \left( \frac{27}{25} e_{00} - \frac{2}{5} e_{20} - \frac{2}{5} e_{02} \right) \alpha^2 + O(\alpha^4) \right].$$

In order to calculate the scattered field on the circular disk we need the values of $\Pi$ and $\partial^2 \Pi / \partial z^2$ for $z = 0$, $r < a$. From Bazer and Hochstadt [2] we quote that their integral representation and its normal derivative assume the following limiting values on the circular disk:

$$\lim_{z \to +0} \frac{1}{r^m} \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^m \int_{-1}^1 \exp \left\{ \frac{ik\sqrt{r^2 + (z + i\alpha t)^2}}{\sqrt{r^2 + (z + i\alpha t)^2}} \right\} f(t) \, dt$$

$$= r^m \left( \frac{1}{r} \frac{d}{dr} \right)^m \left[ \frac{2}{a} \int_0^{\sqrt{r/a}} \cos \left\{ \alpha \sqrt{\frac{(r/a)^2 - t^2}{(r/a)^2 - \ell^2}} \right\} f(t) \, dt \right]$$

$$+ \frac{2i}{a} \int_0^{\sqrt{r/a}} \frac{\sinh \left\{ \alpha \sqrt{\frac{t^2 - (r/a)^2}{\ell^2 - (r/a)^2}} \right\} f(t) \, dt}{\sqrt{\ell^2 - (r/a)^2}};$$

$$\lim_{z \to +0} \frac{\partial}{\partial z} \frac{1}{r^m} \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^m \int_{-1}^1 \exp \left\{ \frac{ik\sqrt{r^2 + (z + i\alpha t)^2}}{\sqrt{r^2 + (z + i\alpha t)^2}} \right\} f(t) \, dt$$

$$= 2r^m \left( \frac{1}{r} \frac{d}{dr} \right)^{m+1} \int_{r/a}^1 \cosh \left\{ \alpha \sqrt{\frac{t^2 - (r/a)^2}{\ell^2 - (r/a)^2}} \right\} f(t) \, dt;$$

valid for $r < a$ in the case that $f(t)$ is an even function of $t$. Using these formulae, (3.15) and (3.1) and transforming back from polar coordinates $r, \phi$ to rectangular coordinates $x, y$, the following expansions have been derived for the scattered field on the disk:

$$E_z^{\pm} = -E[\varepsilon_{00} + ik(\varepsilon_{10} x + \varepsilon_{01} y) - k^2(\varepsilon_{20} x^2 + \varepsilon_{11} xy + \varepsilon_{02} y^2)$$

$$+ O(\alpha^3)] = -E_z^i \quad \text{(in agreement with (2.3))};$$

$$E_y^{\pm} = E \left[ -ik \left( \varepsilon_{01} x + \frac{1}{3} \varepsilon_{10} y \right) + k^2 \left\{ \varepsilon_{11} \left( -\frac{1}{5} a^2 + \frac{1}{2} x^2 \right) \right. \right.$$

$$\left. + \left( \varepsilon_{01} + 2\varepsilon_{02} \right) xy + \frac{1}{2} \varepsilon_{11} y^2 \right\} + O(\alpha^3) \right];$$

$$E_x^{\pm} = \pm \frac{4E}{\pi} \alpha^2 \left( a^2 - x^2 - y^2 \right)^{-1/2}.$$
\[
\begin{align*}
(3.19) & \quad \cdot \left[ e_{00} x + ik \left( e_{01} \left( \frac{a^2}{9} + \frac{8}{9} x^2 \right) + \frac{4}{3} e_{00} x y + \frac{4}{9} e_{10} y^2 \right) \right] \\
& + k^2 \left\{ \left( e_{00} + \frac{1}{3} e_{20} + \frac{1}{3} e_{02} \right) a^2 x + \frac{8}{15} e_{11} a^2 y \\
& - \left( \frac{11}{30} e_{00} + \frac{4}{5} e_{20} + \frac{4}{15} e_{02} \right) x^3 - \frac{16}{15} e_{11} x^2 y \\
& - \left( \frac{9}{10} e_{00} + \frac{4}{15} e_{20} + \frac{28}{15} e_{02} \right) x y^2 - \frac{8}{15} e_{11} y^3 \right\} + O(\alpha^3) \right];
\end{align*}
\]

\[ H_x^{\pm} = 0; \]

\[ H_y^{\pm} = \pm \frac{4ikE}{\pi} \sqrt{\frac{\varepsilon}{\mu}} \left( a^2 - x^2 - y^2 \right)^{1/2} \left[ e_{00} + ik \left( \frac{4}{9} e_{10} x + \frac{4}{3} e_{00} y \right) \right. \\
+ k^2 \left\{ \left( \frac{34}{45} e_{00} - \frac{1}{5} e_{20} + \frac{7}{45} e_{02} \right) a^2 - \left( \frac{11}{90} e_{00} + \frac{4}{15} e_{20} + \frac{4}{45} e_{02} \right) x^2 \\
- \frac{8}{15} e_{11} x y - \left( \frac{59}{90} e_{00} - \frac{4}{15} e_{20} + \frac{76}{45} e_{02} \right) y^2 \right\} + O(\alpha^3) \left];
\]

\[ (3.20) H_z^{\pm} = -ikE \sqrt{\frac{\varepsilon}{\mu}} \left[ e_{00} y + ik \left\{ e_{01} \left( -\frac{2}{3} a^2 + \frac{1}{2} x^2 \right) + \frac{1}{3} e_{10} x y \right. \\
+ \frac{3}{2} e_{00} y^2 \right\} + k^2 \left\{ \frac{1}{5} e_{11} a^2 x + \left( \frac{7}{5} e_{00} - \frac{2}{5} e_{20} + \frac{6}{5} e_{02} \right) a^2 y \\
- \frac{1}{6} e_{11} x^2 - \left( \frac{1}{2} e_{00} + e_{02} \right) x^2 y - \frac{1}{2} e_{11} x y^2 \\
- \left( \frac{5}{6} e_{00} + \frac{1}{3} e_{20} + 2e_{02} \right) y^3 \right\} + O(\alpha^3) \right].
\]

The upper and lower signs refer to the upper and lower sides of the circular disk.

Expansions for the current density \( I = (I_x, I_y) \) and for the surface charge density \( \sigma \) induced in the disk can be derived from the discontinuities in the tangential magnetic field and in the normal electric field across the circular disk, viz.,

\[ (3.21) \quad I_x = -2H_y^{\pm}, \quad I_y = 0, \quad \sigma = 2\varepsilon E_z^{\pm}. \]

It is clear that the results of this section can be extended, yielding an arbitrary number of terms of the series expansions for the various field quantities. However, the calculation of the higher order terms in these expansions requires an increasing amount of labor.
Finally, we compare the solution of the present boundary value problem with the solutions of some related boundary value problems, presented by Bazer and Rubenfeld [3], Boersma [5]. Bazer and Rubenfeld [3] treat the diffraction of an arbitrary time-harmonic electromagnetic wave through a circular aperture in an infinite plane perfectly conducting screen. Boersma [5] considers the diffraction of a plane electromagnetic wave by a perfectly conducting circular disk. In both papers the transmitted or scattered wave respectively is derived from a Hertz vector. The components of this Hertz vector and their boundary values in the aperture or on the disk can be expanded in Fourier series with respect to $\phi$, leading to separate boundary value problems for the harmonics of the Hertz vector components. These boundary value problems are solved by means of suitable Bazer and Hochstadt [2] type integral representations. For each of these boundary value problems the boundary conditions contain an unknown constant which follows from the edge condition. Each of these constants can be determined independently of the remaining ones.

The present boundary value problem does not show this separability because (3.2) is not separable in polar coordinates $r, \phi$; hence, the value of the Hertz vector component $\Pi$ on the disk does not have a simple Fourier series expansion. Therefore the boundary value problem cannot be split into separate boundary value problems for the various harmonics of $\Pi$. Likewise, the unknown constants $a_{mn}$ have to be determined simultaneously. Probably for the same reason no exact solution in terms of spheroidal wave functions has been given until now for the present boundary value problem. These considerations make it clear that the diffraction problem for a unidirectionally conducting disk is in a certain sense more complicated than the diffraction problem for a perfectly conducting disk.

4. The special case of plane-wave excitation. In this section we apply the foregoing results to the special case where the incident wave is an arbitrary plane wave. The plane of incidence will make an angle $\beta$ with the $x$-axis, and the angle of incidence is denoted by $\gamma$. Two cases must be distinguished according to whether the electric vector is polarized parallel or perpendicular to the plane of incidence.

In the case of parallel polarization, the rectangular components of the incident electromagnetic field are given by

\begin{align*}
\mathbf{E}_{1}^i &= (E \cos \gamma \cos \beta, E \cos \gamma \sin \beta, -E \sin \gamma) \\
&\cdot \exp \{ik(x \sin \gamma \cos \beta + y \sin \gamma \sin \beta + z \cos \gamma) - i\omega t\},
\end{align*}

\begin{align*}
\mathbf{H}_{1}^i &= \sqrt{\epsilon/\mu}(-E \sin \beta, E \cos \beta, 0) \\
&\cdot \exp \{ik(x \sin \gamma \cos \beta + y \sin \gamma \sin \beta + z \cos \gamma) - i\omega t\}.
\end{align*}
The component $E_{z}^i$ on the disk $z = 0$, $x^2 + y^2 \leq a^2$, has the following double series expansion:

$$E_{z}^i = E \cos \gamma \cos \beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\sin \gamma \cos \beta)^m}{m!} \frac{(\sin \gamma \sin \beta)^n}{n!} (ikx)^m (iky)^n;$$

(4.2)

hence (compare (3.3)),

$$e_{mn} = \cos \beta \cos \gamma \frac{(\sin \gamma)^{m+n}(\sin \beta)^n(\cos \beta)^m}{m! n!}.$$

Substitution of these coefficients $e_{mn}$ as given by (4.3) into the results of §3 leads to certain special expansions for the various field quantities, these expansions being extensions to Toraldo di Francia's [17] results. We only state the expansion for the scattered energy $E_{sc}$ (cf. (3.18)), from which we derive the following result for the scattering cross section $A_{sc}$:

$$A_{sc} = \frac{E_{sc}}{\frac{1}{2}E^2 \sqrt{\epsilon/\mu}}$$

$$= \frac{128a^2}{27\pi} \alpha^4 \cos^2 \beta \cos^2 \gamma \left[ 1 + \left( \frac{27}{25} - \frac{1}{5} \sin^2 \gamma \right) \alpha^2 + O(\alpha^4) \right].$$

(4.4)

In the case of perpendicular polarization, the rectangular components of the incident electromagnetic field are given by

$$E_{\perp}^i = (E \sin \beta, -E \cos \beta, 0)$$

$$\cdot \exp \{ik(x \sin \gamma \cos \beta + y \sin \gamma \sin \beta + z \cos \gamma) - i\omega t\},$$

$$H_{\perp}^i = \sqrt{\epsilon/\mu} (E \cos \gamma \cos \beta, E \cos \gamma \sin \beta, -E \sin \gamma)$$

$$\cdot \exp \{ik(x \sin \gamma \cos \beta + y \sin \gamma \sin \beta + z \cos \gamma) - i\omega t\}.$$

The coefficients $e_{mn}$ (cf. (3.3)) will be

$$e_{mn} = \sin \beta \frac{(\sin \gamma)^{m+n}(\sin \beta)^n(\cos \beta)^m}{m! n!}.$$

(4.6)

Substitution of these coefficients again leads to certain expansions for the various field quantities, which are in agreement with Toraldo di Francia's [17] results. We only give the following expansion for the scattering cross section $A_{sc}$:

$$A_{sc} = \frac{128a^2}{27\pi} \alpha^4 \sin^2 \beta \left[ 1 + \left( \frac{27}{25} - \frac{1}{5} \sin^2 \gamma \right) \alpha^2 + O(\alpha^4) \right].$$

(4.7)

Finally, similar to Toraldo di Francia [17], we deal with the case of a
circularly polarized normally incident plane wave. Then the scattering cross section $A_{sc}$ is given by

$$A_{sc} = \frac{64a^2}{27\pi} \alpha^4 \left[ 1 + \frac{27}{25} \alpha^2 + O(\alpha^4) \right],$$

which is an extension of Toraldo di Francia's result. As Toraldo di Francia has shown, this scattering cross section is equal to the angular momentum cross section, i.e., the average mechanical moment exerted by the incident wave on the disk divided by the average angular momentum carried by the wave across a unit surface of the $x$, $y$ plane per unit time, both moments being taken about the $z$-axis.

REFERENCES


\(^1\) Added in proof. In a second paper, viz., G. Toraldo di Francia, On a macroscopic measurement of the spin of electromagnetic radiation, Nuovo Cimento, 6 (1957), pp. 150-167, the expansions for the scattered field as derived in [17] are extended for the case of plane-wave excitation. The present author verified that these extended results are in perfect agreement with his results as presented in §4 of this paper.


