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Cross-country sailplane flight

as a dynamic optimization problem

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CROSS-COUNTRY SAILPLANE FLIGHT
AS A DYNAMIC OPTIMIZATION PROBLEM

SUMMARY

A minimum-time, thermal-to-thermal sailplane trajectory optimization problem is formulated as a nonlinear optimal control problem. Numerical solutions are obtained using a gradient projection algorithm which incorporates conjugate directions of search. Further insight into the nature of the solutions and the computational process is obtained through an analysis of the linearized sailplane dynamics and the necessary conditions for optimality. Numerical results are presented for two sailplane types and various values of thermal strength and distance between thermals. An additional problem is formulated and solved for the case of bounded control rate.

1. Introduction
1.1 Background

Cross-country soaring is the art of piloting a motorless glider or sailplane in such a way as to extract energy from the surrounding air currents to maintain flight over long distances. Several modes of energy extraction are known. For example, ridge soaring consists of flying low over and parallel to natural ridges which deflect strong horizontal winds upward at the ridge face. Wave soaring is possible at high altitudes in the oscillatory vertical wind distribution often found on the lee side of a mountain range. Extended areas of upward air currents may also be found under certain cloud formations known as cloud streets. When flying under such a cloud street, the pilot reduces his speed when in rising air and increases his speed when in sinking air. The resulting oscillatory trajectory is, for its obvious resemblance, called a "dolphin trajectory", and the corresponding mode of flying is known as "dolphin soaring"."}

But by far the most common mode of cross-country flight is that of "thermal-to-thermal" flight. In this case, the pilot seeks out rising columns of air called thermals. If the upward air velocity of the thermal exceeds the still-air descent rate of the sailplane, circling flight within the thermal will provide a net
altitude gain. Each cross-country flight segment then consists of a glide between thermals followed by an upward spiraling motion in the next thermal. The altitude gained in the thermal offsets the altitude loss sustained during the previous glide.

Although the sport of competitive soaring is now well established, the application of optimal control methodology to various sailplane performance problems is a relatively recent development. If the sailplane dynamics are neglected, the dolphin soaring problem has been investigated by means of the calculus of variations and optimal control techniques, respectively, by Arho and Metzger and Hedrick. Comparable dynamic analysis of the same problem have been given by Irving, Gedeon, and Kauer and Junginger. An initial treatment of a maximum-altitude sailplane winch launch problem, using an optimal-control formulation, has been presented by Pierson. One can readily predict further application of optimal control theory and associated numerical methods to these and related sailplane trajectory optimization problems in the years to come.

The optimal control problem treated in this paper is closely connected to the classical static performance problem which is briefly reviewed next.

1.2 The static MacCready problem

In still air, assuming a uniform gravity field, each fixed sailplane attitude corresponds to a particular straight-line, constant-speed descent trajectory: an equilibrium glide. Along each equilibrium glide, the aerodynamic forces, lift $L$ and drag $D$, exactly balance each other and the weight force $mg$ so that no net acceleration acts on the sailplane. From Figure 1, the force balance is given by

\[
L = \frac{1}{2} \rho V^2 C_L S = mg \cos \gamma
\]

\[
D = \frac{1}{2} \rho V^2 C_D S = -mg \sin \gamma
\]

where $\rho$ is the air density, $V$ the relative airspeed, $\gamma$ the flight path angle, $S$ the gross wing area of the sailplane, and $C_L$ and $C_D$ the dimensionless lift and drag coefficients, respectively. Note that since from (1), $C_D/C_L = -\tan \gamma$, there exists a relationship between $C_D$ and $C_L$, say
which is independent of the sailplane weight and holds for the class of all equilibrium glide trajectories. Each $C_L, C_D$-pair from (2) yields a particular equilibrium glide. The relationship (2) is called a drag polar and depends only on the size, shape, and surface condition of the sailplane.

Alternately, for a given $C_L$ and the corresponding $C_D$ from (2), one can solve for $V$ and $\gamma$ from (1) and thereby construct a velocity polar, as illustrated in Figure 2. $C_L$ would then appear as a parameter along this curve. Although the information contained in a velocity polar is equivalent to that in a drag polar, the velocity polar depends also on the air density and the sailplane weight. In subsequent discussion, the velocity polar will be denoted by

$$g_1(V, \gamma) = 0$$  \hspace{1cm} (3a)

or, equivalently,

$$g_2(V_x, V_d) = 0$$  \hspace{1cm} (3b)

or

$$V_d = f_2(V_x).$$  \hspace{1cm} (3c)

Consider now a single trajectory segment between two fixed vertical thermals of equal and prescribed strength which are separated by the fixed distance $X_f$ as shown in Figure 3. The objective is to minimize the total flight time from A to C. Only equilibrium glides will be considered for the still-air trajectory from A to B. The altitude loss $\Delta h$ incurred during the glide from A to B must be regained by climbing in the thermal from B to C. Motion in the thermal will be modeled by pure vertical motion of constant speed $V_T > 0$, where $V_T$ is the difference between the upward speed of the thermal air mass and the estimated minimum still-air descent speed of the sailplane in circular flight appropriate for thermalling. If a steep equilibrium glide is flown, the glide time will be small, but the time spent in the thermal will be excessive. If a shallow glide is flown, little time will be spent in the thermal, but the glide itself will consume a lot of time.
Thus, from physical grounds, one can expect a well-defined minimum for some intermediate glide path.

Since the time during the glide from A to B is just $X_f/V_x$, and since the time spent in the thermal is $\Delta h/V_T = V_d X_f/(V_x V_T)$, the problem may be stated as follows.

\[
\begin{align*}
\text{minimize} & \quad L(V_x, V_d) = \frac{X_f}{V_T} \left[ \frac{V_T + V_d}{V_x} \right] \\
\text{subject to} & \quad g_2(V_x, V_d) = 0 \quad \text{and specified } X_f \quad \text{and } V_T
\end{align*}
\]

Note that the solution will be independent of $X_f$. If the pair $(V, \gamma)$ is used to define the equilibrium glide, rather than the pair $(V_x, V_d)$, problem (4) becomes

\[
\begin{align*}
\text{minimize} & \quad \frac{X_f}{V_T} \left[ \frac{V_T - V \sin \gamma}{V \cos \gamma} \right] \\
\text{subject to} & \quad g_1(V, \gamma) = 0 \quad \text{and specified } X_f \quad \text{and } V_T
\end{align*}
\]

Returning now to problem (4), note that the equilibrium glide constraint (3c) may be used to eliminate $V_d$ so that the problem becomes simply

\[
\begin{align*}
\min & \quad \left[ \frac{V_T + f_2(V_x)}{V_x} \right] \\
\text{subject to} & \quad \text{specified } X_f \quad \text{and } V_T
\end{align*}
\]

After setting the first derivative to zero, one obtains the solution equation for the minimizing $V_x$, say $V_{xM}$, as

\[
V_{xM} f_2'(V_{xM}) - f_2(V_{xM}) - V_T = 0
\]

Although (7) is nonlinear in $V_{xM}$, it has a particularly simple geometric interpretation as shown in Figure 4.

For a given sailplane's velocity polar, this graphical solution can be used to construct and calibrate a so-called MacCready ring which is installed on the sailplane instrument panel around the variometer (an accurate rate-of-climb meter). The MacCready ring simply shows $V_{xM}$ values corresponding to a fixed linear scale.
for the sum $V_T + f_2(V_{xM})$ which scale corresponds to the linear variometer scale for $V_d$. The pilot then estimates a value for $V_T$ for the next thermal, sets the zero value of the MacCready ring to that value on the variometer, and adjusts his velocity until his rate of descent equals $f_2(V_{xM})$ (and his velocity accordingly equals $V_{xM}$). Since typical glide ratios are in the range $V_x/V_d = 20$ to $40$, $V_x = V_x$ and the pilot attempts to fly the sailplane between thermals so that the reading on the airspeed indicator ($V$) matches the prescribed MacCready speed $V_{xM}$. If vertical winds are encountered during the glide, the pilot adjusts his speed to agree with the $V_{xM}$ value on the MacCready ring opposite the actual descent rate being experienced. Thus, the solution to problem (4) results in a very practical scheme for choosing the "best" speed to fly between thermals. The use of the MacCready ring, or more sophisticated equivalents of it, is commonplace today.

There are, however, two obvious shortcomings to the static minimum-time problem just discussed. First, the sailplane dynamics have been neglected entirely. The static solution implies an instantaneous change in the $(V_x, \gamma)$-state between circular flight in the thermal and the equilibrium glide between thermals. How does the actual aircraft behave during this transition? Secondly, only equilibrium glides are allowed between thermals. Would a more general glide trajectory result in lower flight times? These two questions are related and serve as the primary motivation for an examination of the corresponding optimal control problem which follows.

2. The dynamic MacCready problem
2.1 Problem statement

The assumptions used in the problem formulation developed here may be listed as follows:

1. constant gravity acceleration $g$
2. constant density atmosphere
3. point mass vehicle dynamics; no rotational dynamics
4. planar flight in still air between thermals
5. vertical constant-strength thermal at a specified distance $X_f$ from the previous thermal; motion within the thermal equivalent to a vertical ascent at fixed vertical speed $V_T$
The equations of motion for planar flight can be written as

\[
\dot{V} = - \frac{D(V, C_L)}{m} - g \sin \gamma \\
\dot{\gamma} = \frac{[L(V, C_L)/m - g \cos \gamma]}{V} \\
\dot{Y} = V \sin \gamma \\
\dot{X} = V \cos \gamma
\]

where the state variables are: \( V \), the speed relative to the surrounding air; \( \gamma \), the flight path angle; \( Y \), altitude; \( X \), range. Since range is expected to be monotonically increasing, it is advantageous to replace \( t \) by \( X \) as the independent variable. This will reduce the number of state variables by one and will also result in a "fixed terminal-time" optimal control problem which is generally easier to solve numerically. The resulting third-order dynamic system becomes

\[
\dot{V}' = - \frac{[D(V, C_L)/m + g \sin \gamma]}{V \cos \gamma} \\
\dot{\gamma}' = \frac{[L(V, C_L)/(m \cos \gamma) - g]}{V^2} \\
\dot{Y}' = \tan \gamma
\]

where the prime denotes differentiation with respect to \( X \). Furthermore, because of assumptions (1) and (2), \( Y \) does not appear in the right hand side of any of the differential equation. Thus, (9c) can be uncoupled from (9a) and (9b).

The performance index \( J \) is the total flight time for one segment of cross-country flight and therefore consists of the still-air glide time between thermals plus the time-to-climb in the subsequent thermal to the original altitude.

\[
J = \int_0^{t_f} dt + \frac{[Y(0) - Y(t_f)]}{V_T}
\]
\[
\int_0^{X_f} \frac{dX}{V \cos \gamma} - \frac{1}{V_T} \int_0^{Y(t_f)} dY - \int_0^{X_f} \frac{dX}{V \cos \gamma} - \frac{1}{V_T^2} \int_0^{X_f} \tan \gamma \, dX \\
\int_0^{X_f} \left[ \frac{V_T - V \sin \gamma}{V_T V \cos \gamma} \right] \, dX \tag{10}
\]

Note the obvious similarity of the objective functions in (10) and (5).

Finally, the control function is chosen to be the lift coefficient \(C_L\), and the problem is nondimensionalized by defining

\[
x \triangleq \frac{X}{X_f} \quad \text{and} \quad v \triangleq \frac{V}{(gX_f)^{\frac{1}{2}}} \tag{11}
\]

so that the problem may be stated as follows. Find that control \(u(x), 0 \leq x \leq 1\), which minimizes

\[
J = \int_0^1 \left[ \frac{V_T - v \sin \gamma}{V_T v \cos \gamma} \right] \, dx \tag{12}
\]

subject to the second-order dynamic constraints

\[
v' = - (\eta \ C_D(u) \ v^2 + \sin \gamma) / (v \cos \gamma) \tag{13a}
\]
\[
\gamma' = (\eta \ u \ v^2 / \cos \gamma - 1) / v^2 \tag{13b}
\]

with boundary conditions

\[
v(0) = v(1) = V_o \tag{14a}
\]
\[
\gamma(0) = \gamma(1) = \gamma_o \tag{14b}
\]

where \(C_D(u)\) is a prescribed drag polar (2) and the nondimensional aerodynamic
parameter $\eta$ is given by

$$\eta = \frac{1}{2} \rho g x_f / (mg)$$

(15)

Note that the initial and final states are specified and equal. The pair $(v_0, \gamma_0)$ should correspond to minimum rate of sink values for steady circular flight within the thermal. The optimal trajectory can be generated after solving this optimal control problem by numerically integrating (8c) and (8d).

2.2 Linearized dynamics

It will be instructive to examine briefly the properties of the dynamic system (13) in the vicinity of an equilibrium glide state. By defining the state vector $z = (v, \gamma)^T$, (13) may be written as

$$z' = F(z, u)$$

(16)

The corresponding homogeneous linearized system for constant $C_L$ is then given by

$$\delta z' = A \delta z$$

(17)

where

$$A = F_z = \begin{bmatrix}
\sin \gamma - \eta C_D v^2 & 1 + \eta C_D v^2 \sin \gamma \\
\frac{v^2 \cos \gamma}{\cos^2 \gamma} & \frac{v \cos^2 \gamma}{\cos^2 \gamma}
\end{bmatrix}$$

(18)

The partial derivatives in (18) are to be evaluated along an equilibrium glide. For an equilibrium glide,

$$\begin{cases}
\eta C_D v^2 + \sin \gamma = 0 \\
\eta C_L v^2 - \cos \gamma = 0
\end{cases}$$

(19a, 19b)
or,

\[
\sin \gamma = -C_D (C_L^2 + C_D^2)^{-\frac{1}{2}} = -C_D/C_R \quad (20a)
\]

\[
\cos \gamma = C_L (C_L^2 + C_D^2)^{-\frac{1}{2}} = C_L/C_R \quad (20b)
\]

\[
v = \eta^{-\frac{1}{2}} (C_L^2 + C_D^2)^{-\frac{1}{2}} = 1/(\eta C_R)^{\frac{1}{2}} \quad (20c)
\]

Using (20), (18) simplifies considerably. The characteristic equation for (17),
\[
\text{det}(\lambda I - A) = 0,
\]
then becomes

\[
\lambda^2 + 3\eta C_R (C_D/C_L) \lambda + 2\eta^2 C_R^2/C_L^2 = 0 \quad (21)
\]

From (21) it follows that the natural frequency \( \omega_n \) for this second-order nondimensional system is given by

\[
\omega_n = \sqrt{2} \eta C_R^2/C_L = \sqrt{2} (C_R/C_L)/v^2 \quad (22a)
\]

and the damping ratio \( \zeta \) is given by

\[
\zeta = (3\sqrt{2} / 4) (C_D/C_R) \quad (22b)
\]

Note that since \( C_D << C_L \)

\[
C_R = (C_L^2 + C_D^2)^{\frac{1}{2}} = C_L (1 + (C_D/C_L)^2)^{\frac{1}{2}} \approx C_L \quad (23)
\]

so that (22) can be approximated by

\[
\omega_n \approx \sqrt{2} \eta C_L = \sqrt{2} v^{-2} \quad (24a)
\]

and

\[
\zeta \approx (3\sqrt{2} / 4)/(C_L/C_D) \quad (24b)
\]
Thus, in (24b) the damping ratio is proportional only to the inverse lift-to-drag ratio. Typical values for $\zeta$ are quite small: perhaps 0.025 to 0.05. The higher the performance of the sailplane, the lower will be the damping ratio. Similarly, from (24a) the nondimensional frequency increases linearly with both $C_L$ and, via (15), the final range $X_f$, or equivalently, with the inverse nondimensional airspeed squared.

Therefore, it may be observed that the basic nature of the dynamic system under consideration is that of a very lightly damped, highly oscillatory system. In practical terms, a relatively small numerical integration step size will be required to maintain stability of the integration process and to provide sufficient accuracy for the gradient information used in the optimization process. Furthermore, since $\omega_n$ is proportional to $\eta$, it will be necessary to decrease the step size proportionately as the distance between thermals is increased.

2.3 An observation

As a final preliminary before the presentation of the numerical results, consider the following equality-constrained static or finite-dimensional optimization problem.

$$\begin{align*}
\text{minimize} & \quad L(z,u) \\
\text{subject to} & \quad F(z,u) = 0
\end{align*}$$  \hspace{1cm} (25)

Here, $z$ and $F$ are $n$-vectors, and $u$ is an $m$-vector. The problem variables have been split into "independent" variables ($u$) and "dependent" variables ($z$) the latter of which can be chosen to satisfy the equality constraints. Assume the problem (25) is normal and possesses a unique solution ($z^*, u^*$). Then, this solution satisfies the necessary conditions for optimality [see Ref. 12, pp. 6,7 ]:

$$\begin{align*}
L_z^T (z^*, u^*) + F_z^T (z^*, u^*) \lambda^* &= 0 \\
L_u^T (z^*, u^*) + F_u^T (z^*, u^*) \lambda^* &= 0 \\
F(z^*, u^*) &= 0
\end{align*}$$ \hspace{1cm} (26a, b, c)
where $\lambda^*$ is the unique $n$-vector of Lagrange multipliers.

Next, consider a related but special class of dynamic optimization problems

\[
\begin{align*}
\text{minimize} & \quad \int_0^1 L(z, u) \, dx \\
\text{subject to} & \quad z' = F(z, u), \, z(0) = z(1) = z^0
\end{align*}
\]

(27)

In particular, note the following two distinguishing features of (27): (1) the initial and final states are specified and equal, and (2) the independent variable $x$ does not appear explicitly in either $L$ or $F$. Now observe that the first-order necessary conditions [see Ref. 12, p. 66]

\[
\begin{align*}
z' &= F(z, u), \, z(0) = z(1) = z^0 \\
\lambda' &= -F^T_z \lambda - L^T_z \\
0 &= L^T_u + F^T_u \lambda, \quad 0 \leq x \leq 1
\end{align*}
\]

are satisfied by

\[
z(x) \equiv z^*, \, u(x) \equiv u^*, \text{ and } \lambda(x) \equiv \lambda^*,
\]

(29)

i.e., the solution to the static problem (25), if $z^0 = z^*$.

Of course, in general (29) may not be a minimizing solution for (27) since only the necessary conditions (28), and not the sufficient conditions, are known to be satisfied at this point. But since the static MacCready problem (5) and the "dynamic MacCready" problem (12)-(14) qualify as the "static" problem (25) and the "dynamic" problem (27), respectively, one might reasonably expect the static MacCready solution to play a major role in the solution of the optimal control problem treated here.

3. Numerical results

For the numerical results presented here, the gradient projection algorithm
of Ref. 13 has been used. This is a direct method in the sense that successive
alterations are made in the control function so as to monotonically reduce the
performance index value. Conjugate gradient directions of search are used with
a one-dimensional minimization along these directions of search based on parabolic
interpolation among function values for which the terminal state constraints
\( v(1) = v_o; \gamma(1) = \gamma_o \) are satisfied. The terminal state constraints are enforced
on each iteration using a projection operator in the control space. The imple-
mentation of the projection operator for this problem requires the backward numerical
integration of a fourth-order system of adjoint or influence function equations
and the inversion of an associated 2 x 2 matrix. The conjugate gradient process
is arbitrarily restarted periodically; for the results given here, each conjugate
gradient cycle consists of four iterations. The \( L^2 \)-norm of the projected gradient
function, which is theoretically zero at the constrained minimum, is used as the
automatic measure of convergence, although the difference in performance index
values and the \( L^2 \)-norm of the difference in control functions over each conjugate
gradient cycle are computed and displayed as well.

Numerical solutions for two specific sailplanes, the Nimbus II and the ASW-15B,
have been obtained. A parabolic drag polar

\[
C_D = 0.009278 - 0.009652 C_L + 0.022288 C_L^2
\]

has been adopted for the Nimbus II and was obtained from a least-squares fit based
on data taken from the manufacturer's velocity polar. For the ASW-15B, a sixth-
order polynomial

\[
C_D = 0.01277 - 0.01776 C_L + 0.06344 C_L^2
\]

\[- 0.09215 C_L^3 + 0.15168 C_L^4 - 0.13759 C_L^5
\]

\[+ 0.04767 C_L^6\]

(31)

has been adopted for the drag polar (2). It was obtained by a least-squares fit
of data taken from Waibel and from the manufacturer's velocity polar. The boundary
condition values \( (V_o, \gamma_o) \) have been chosen as the minimum rate of sink equilibrium
glide values corresponding to the respective drag polars (30) and (31):
(23.5566 m/s, -0.020963 rad) for the Nimbus II and (20.5379 m/s, -0.028751 rad)
for the ASW-15B. The values chosen for the dimensionless aerodynamic constant
\( \eta \) correspond to a gravity acceleration \( g \) of 9.81 m/s\(^2\), a standard sea level air
density of 0.125 g kg/m\(^3\), and a wing loading \( mg/S \) of 32 g nt/m\(^2\) and 28 g nt/m\(^2\),
respectively, for the Nimbus II and ASW-15B.

The computational work was done on a Burroughs 6700/7700 computer using single
precision arithmetic and a FORTRAN compiler. The required numerical integrations
were performed using the standard fourth-order Runge-Kutta single step method
with a fixed equivalent step size of 5 m for the Nimbus II runs and 10 m for the
ASW-15B runs. The terminal state constraints were satisfied within tolerances
defined by

\[
\left[ \left( \frac{V}{V_0} - 1 \right)^2 + \left( \frac{Y}{Y_0} - 1 \right)^2 \right]^{\frac{3}{2}} < 10^{-5}
\]

Typical optimal trajectories for the ASW-15B sailplane are shown in Figures
5-7 for \( V_T = 2 \) m/s and \( X_f = 500, 1000, \) and \( 2000 \) m, respectively. It may be observed
that the minimum-time, thermal-to-thermal trajectory consists of a transient dive
followed by the mid-range equilibrium glide predicted by the static MacCready solu­
tion. The terminal portion of the optimal trajectory consists of a pull-up maneuver
to attain the required minimum rate-of-sink boundary conditions at the thermal.
As the final specified range \( X_f \) is increased, the initial and terminal transient
segments of the optimal trajectory rather quickly reach "limit arcs" which depend
only upon \( V_T \), air density, wing loading, \( \eta \), the drag polar and the boundary condi-
tions. Thus, the main effect of increasing \( X_f \) is to increase the range over which
the optimal trajectory and the static MacCready solution nearly coincide. Note,
of course, that the dynamic solution is flown at altitudes substantially below
those predicted by the static solution since potential energy must be exchanged for
kinetic energy and later reclaimed in order to transfer from the boundary condition
state to the static MacCready state and back again.

The optimal control (lift coefficient) history corresponding to the optimal
trajectory of Figure 7 is presented in Figure 8. Note again the presence of static
MacCready \( C_L \) values over a broad middle range. The transient portions of Figure 8
at the start and end of the range are nearly symmetrical about the midpoint. It should also be observed that this optimal control history begins and ends with negative $C_L$ values. Thus, the optimal trajectory begins with an abrupt pitch-over into the dive and ends also with an abrupt pitch-over to meet the specified terminal boundary conditions.

Solution data for the ASW-15B drag polar are listed in Table I for $V_T = 2 \text{ m/s}$ and various final range values. Note that the flight times corresponding to the optimal control problem exceed those predicted by the static MacCready problem. The per cent difference between the two decreases, however, with increasing $X_f$ since less of the total flight is spent in the initial and final transient trajectories as the range is increased. The amount of computational effort required per iteration is roughly proportional to $X_f$. However, due to the nature of the optimal control history (see Figure 8), the total number of iterations required for a solution can be reduced dramatically by a proper choice of starting data. In particular, the initial control function may be advantageously chosen to be the optimal control function from a previous problem for smaller $X_f$ with additional static MacCready solution values inserted as needed in the middle range.

Similar solution data are listed in Table 2 for $X_f = 1000 \text{ m}$ and various $V_T$ values. Of course, the flight times decrease with increasing thermal strength. But note that the per cent difference between the flight times predicted by the dynamic and static problem models increases with increasing $V_T$ since the boundary condition state is farther removed from the static MacCready state.

Finally, a limited comparison has been made between sailplane types. A standard-class, 15 m span sailplane, the ASW-15B, is compared to a high-performance, open-class sailplane with a span of 20.3 m, the Nimbus II. For the case of $V_T = 2 \text{ m/s}$ and $X_f = 1000 \text{ m}$, the minimum time is reduced from 47.36 sec for the ASW-15B to only 40.38 sec for the Nimbus II. The static MacCready values are 45.09 and 38.20 sec, respectively. The Nimbus II, during the middle of the optimal trajectory, must fly 58.7 m below the static MacCready trajectory compared to an altitude difference of only 47.3 for the ASW-15B. These results do not account for the differences in thermalling performance between the two types. Minimum rate-of-sink boundary conditions have been used in each case.
4. Lift coefficient rate control

4.1 Problem statement

It is desirable to eliminate the end-point discontinuity which occurs in the present optimal $C_L$ history: that is, the $C_L$ for a minimum rate-of-sink equilibrium glide does not equal either $u(0)$ or $u(1)$ from the optimal control. Furthermore, since rotational dynamics have not been included in the dynamic model assumed here, a more realistic problem statement should include bounds on the slope of the $C_L$ history:

$$|dC_L/dx| \leq (dC_L/dx)_{\text{max}}$$

Both improvements can be facilitated by regarding $C_L$ as a state variable, rather than the control function, and by regarding $dC_L/dx$ as the control instead. In this way, the proper boundary conditions can be enforced on $C_L$, and (33) becomes a control inequality constraint.

With these changes, the optimal control problem may be stated as follows: Find that control function $u(x)$, $0 \leq x \leq 1$, which minimizes the performance index (12) subject to the third-order dynamic system

$$v' = [\eta C_D(C_L) v^2 + \sin \gamma]/(v \cos \gamma) \quad (34a)$$

$$\gamma' = (\eta C_L v^2 / \cos \gamma - 1)/v^2 \quad (34b)$$

$$C_L' = (dC_L/dx)_{\text{max}}(2 \sin^2 u - 1) \quad (34c)$$

with (minimum rate-of-sink) boundary conditions (14) and

$$C_L(0) = C_L(1) = (C_L)_{o} \quad (35)$$

where the drag polar $C_D(C_L)$ in (34a) is given and the aerodynamic parameter $\eta$ is defined in (15) and is fixed in value.

Observe that a simple transformation has been adopted in (34c). The control inequality constraint (33) will automatically be satisfied for any value of the
new unconstrained control function $u(x)$.

4.2 Numerical results

For the Nimbus II, the value used for $C_L$ for the minimum rate-of-sink equilibrium glide is $(C_L)_0 = 0.922463$. With $(dC_L/dx)_{\text{max}} = 4$, $V_T = 2 \text{ m/s}$, and $X_f = 1000 \text{ m}$, the minimum time becomes 41.83 sec. This represents a 3.6% increase over the value reported earlier for $C_L$ control. A portion of the optimal $C_L$ histories for each case is shown in Figure 9. The apparent tangency observed in Figure 9 is only a coincidence. The corresponding optimal trajectories differ only slightly.

5. Conclusions and discussion

The optimal trajectories for this dynamic MacCready problem have been shown to consist of the corresponding static MacCready equilibrium glide preceded and followed by a transient maneuver which is required to change the state of the system from thermalling conditions to MacCready glide conditions and back again. Since the dynamic model cannot complete this transition in zero time, the minimum times predicted by this optimal control problem are larger than those predicted by the static MacCready theory. If the initial and final states were required to be equal but unspecified (that is, treated as control parameters), the solution would coincide with the static MacCready equilibrium glide.

In distinction to the static MacCready solution, the dynamic solution depends on the range between thermals. If this range is sufficiently small, say less than about 600 m for $V_T = 2 \text{ m/s}$, the entire optimal trajectory consists of the transient arcs with no intermediate equilibrium glide. For large ranges, the optimal trajectory is dominated by the presence of the intermediate equilibrium glide. In any case, these transient arcs at the beginning and end of the optimal trajectory exhibit oscillatory but highly damped controlled motion. During the equilibrium glide portion of the optimal trajectory, the sailplane must fly at a lower altitude than that predicted by static MacCready theory.

In broad terms, the results presented here tend to confirm the utility of the static MacCready theory. The optimal dynamic trajectories are not radically different; they only provide optimal transition paths to and from the "best speed" equilibrium glide between thermals. In particular, these optimal trajectories
show that one should, upon leaving the thermal, push-over sharply and attain the MacCready speed quickly from a smooth dive. This coincides with currently accepted practice.

Since the computational effort involved in obtaining solutions for large $X_f$ remains high, it would be of interest to examine suboptimal solutions which could be obtained more readily. In light of the "boundary layer" nature of these optimal trajectories, it may be possible to employ matched asymptotic expansions in this regard.

It would also be interesting to modify the dynamic model used here to allow for the presence of specified vertical velocity distributions in the atmosphere. One could then examine the effect of non-zero width thermals and study optimal dynamic "dolphin soaring" flight modes. Additional research in these areas is currently underway.

Acknowledgements

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REFERENCES


TABLE 1. Variation of solution parameters with final range: ASW-15B sailplane, $V_T = 2$ m/s.

<table>
<thead>
<tr>
<th>$X_f$, m</th>
<th>$\delta h$, m*</th>
<th>Optimal Control Solution</th>
<th>Static MacCready Solution</th>
<th>Per Cent Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>46.6</td>
<td>24.82</td>
<td>22.54</td>
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<tr>
<td>1000</td>
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<td>47.36</td>
<td>45.09</td>
<td>5.03</td>
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<tr>
<td>2000</td>
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<td>92.44</td>
<td>90.18</td>
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<tr>
<td>5000</td>
<td>47.0</td>
<td>227.71</td>
<td>225.45</td>
<td>1.00</td>
</tr>
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</table>

* Distance below static MacCready trajectory at $X = X_f/2$. 
TABLE 2. Variation of solution parameters with $V_T$: ASW-15B sailplane, $X_f = 1000$ m.

<table>
<thead>
<tr>
<th>$V_T$, m/s</th>
<th>$\delta h$, m*</th>
<th>Optimal Control Solution</th>
<th>Static MacCready Solution</th>
<th>Per Cent Difference</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>61.24</td>
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<td>92.9</td>
<td>34.06</td>
<td>30.49</td>
<td>11.7</td>
</tr>
</tbody>
</table>

* Distance below static MacCready trajectory at $X = X_f/2$. 
FIGURE CAPTIONS

Figure 1. Force and velocity diagrams for an equilibrium glide.

Figure 2. Velocity polar for a particular sailplane in equilibrium glide (sketch only).

Figure 3. Trajectory segment for the static MacCready problem.

Figure 4. Tangent construction of the solution to the static MacCready solution.

Figure 5. Optimal trajectory: ASW-15B sailplane, $V_T = 2$ m/s, $X_f = 500$ m.

Figure 6. Optimal trajectory: ASW-15B sailplane, $V_T = 2$ m/s, $X_f = 1000$ m.

Figure 7. Optimal trajectory: ASW-15B sailplane, $V_T = 2$ m/s, $X_f = 2000$ m.

Figure 8. Optimal lift coefficient history: ASW-15B sailplane, $V_T = 2$ m/s, $X_f = 2000$.

Figure 9. Partial optimal lift coefficient history with and without bounded $C_L$ rate control: Nimbus II sailplane, $V_T = 2$ m/s, $X_f = 1000$ m.
FIGURE 1
\[ V_x \]

max range

\[ C_L \]

max endurance

\[ \min V_d \]

\[ V_p \]

\[ g(V_x, V_p) = 0 \]

\[ V_d = f(V_x) \]

or

\[ g(V_x, V_p) = 0 \]

\[ V_d = \alpha \]

or

\[ g(V_x, V_p) = 0 \]

\[ V_d = \beta \]
\[ \Delta h = \frac{VX_t}{\lambda} \]
\[-V_d = \left[ f' (V_{MW}) \right]_V \]

\[ f' (V_{MW}) = f_{\mu} (V) \]
Figure 7

Graph showing the relationship between $X/X_f$ and altitude loss 'm'. The graph includes two curves: static MacCready and optimal.
bounded $C_L$ rate control

static MacCready $C_L = 0.30042$