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DECOUPLING OF MULTIVARIABLE CONTROL SYSTEMS OVER UNIQUE FACTORIZATION DOMAINS*

K. B. DATTA[†] AND M. L. J. HAUTUS[‡]

Abstract. Necessary and sufficient conditions are established for the existence of a state variable feedback decoupling of an m -input, m -output time invariant linear control system over a unique factorization domain. An explicit computation is provided for the feedback and the feedforward gain matrix. Also necessary and sufficient conditions for the existence of a stability-preserving state feedback decoupling are given. The results are illustrated by some examples.

Key words. decoupling, delay systems, systems over rings, multivariable systems, state feedback

1. Introduction. The design and synthesis of noninteracting control in multivariable control systems by state-variable feedback were initiated by Morgan (1964) and definitive results in this direction by establishing necessary and sufficient conditions for the existence of a decoupling feedback, as well as an explicit construction, were first given by Falb and Wolovich (1967). Their results were formulated for systems with real coefficients but they are easily seen to be extendible to systems over arbitrary fields. The extension of these results to systems over rings, however, is less obvious. On the other hand, systems over rings have shown to possess a wide range of potential applications such as delay systems, 2-D systems, parametrized systems, discrete time distributed systems, systems with integer coefficients, etc. We refer to the survey papers E. D. Sontag (1976), (1981), E. W. Kamen (1978), and the references therein.

This abundance of control systems which can conveniently be modelled as systems over rings is a motivation for a systematic investigation of systems over rings. This investigation was started with the thesis Rouchaleau (1972) and the paper Rouchaleau, Wyman and Kalman (1972) and it has received much attention recently.

The purpose of this paper is to formulate necessary and sufficient conditions for the existence of a decoupling state feedback for a linear time-invariant system over a unique factorization domain. This particular class of ring is wide enough to encompass almost all the models arising from the applications mentioned before and on the other hand, it allows a complete solution of the problem. The conditions which will be obtained reduce to the Falb-Wolovich conditions when applied to systems over a field. The method of proof, however, is completely different from the proof in Falb and Wolovich (1967). It can be regarded a generalization to systems over rings of the type of proof given in Hautus and Heymann (1980), (1983) and it is based on a characterization of feedback transformations given in Hautus and Heymann (1978).

It is possible to axiomatize the concept of stability for systems over rings in such a way that in each particular specification and application (delay systems, 2-D systems) the notion of stability customary in that field accommodates conveniently in the general framework. An example will be given in § 2. The treatment is based on what we have called "denominator set". This concept was introduced for systems over a field in

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and Khargonekar and Sontag (1981) (where the name Hurwitz set is used). In the general framework of stability thus provided we give necessary and sufficient conditions for the existence of a stability preserving decoupling state feedback.

The problem formulation and the main results are given in § 3. Examples, illustrating the results, are given in § 4, and § 5 is devoted to the proof of our main result.

2. Stability of systems over rings. Throughout the paper $\mathcal{R}$ will denote a unique factorization domain (= UFD) or factorial ring (see Samuel (1963), Barshay (1969)). We use the notations $\mathcal{R}[z]$ and $\mathcal{R}(z)$ to denote the rings of polynomials and rational functions over $\mathcal{R}$, respectively. A polynomial q is called *monic* if its leading coefficient equals 1. A rational function is called *causal* (or *proper*) if it has a representation of the form p/q , where q is a monic polynomial and $\deg p \leq \deg q$.

A *denominator set* is a subset $\mathcal{D}$ of $\mathcal{R}[z]$ satisfying the following conditions:

- (i) $\mathcal{D}$ is *multiplicative*, i.e. $1 \in \mathcal{D}$ and if $p, q \in \mathcal{D}$ then $pq \in \mathcal{D}$.
- (ii) Each polynomial $p \in \mathcal{D}$ is *monic* (in particular $0 \notin \mathcal{D}$).
- (iii) $\mathcal{D}$ is *saturated*, i.e. if $p \in \mathcal{D}$ and q is monic and divides p then $q \in \mathcal{D}$.
- (iv) There exists $a \in \mathcal{R}$ such that $z - a \in \mathcal{D}$.

Since a denominator set is multiplicative, it is possible to associate with it a *ring of fractions* to be denoted by $\mathcal{R}_{\mathcal{D}}[z]$ (see Barshay (1969, Chap. 3)). Specifically $\mathcal{R}_{\mathcal{D}}[z]$ is the set of rational functions having a representation of the form p/q , where p and q are polynomials and $q \in \mathcal{D}$. It is well known and easily seen that $\mathcal{R}_{\mathcal{D}}[z]$ is a ring, even a UFD (see Samuel (1963, Thm. 4, p. 29)). In addition, we introduce the set of causal fractions in $\mathcal{R}_{\mathcal{D}}[z]$, i.e. elements of $\mathcal{R}_{\mathcal{D}}[z]$ that are causal rational functions. This set is denoted by $\hat{\mathcal{R}}_{\mathcal{D}}[z]$, or, if the denominator set does not have to be specified, by $\mathcal{P}$.

LEMMA 2.1. $\mathcal{P}$ is a UFD.

For a proof, see § 5. The set of all monic polynomials, which is denoted $\mathcal{D}_0$, is an example of a denominator set. The corresponding set of causal fractions is denoted $\mathcal{P}_0$.

A (free) linear system is identified by a quadruple (A, B, C, D) of matrices over $\mathcal{R}$ of such dimensions that the following equations are well defined

$$(2.2) \quad x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t,$$

where

$$x_t \in \mathcal{X} := \mathcal{R}^n, \quad u_t \in \mathcal{U} := \mathcal{R}^m, \quad y_t \in \mathcal{Y} := \mathcal{R}^r.$$

Equations (2.2) give a discrete time interpretation of the system $\Sigma := (A, B, C, D)$. The system is called *reachable* if the columns of the matrix $[B, AB, \dots, A^{n-1}B]$ span the total state space $\mathcal{R}^n$. (See Sontag (1976) for details.) To the system Σ a *transfer function*

$$(2.3) \quad W(z) := W_{\Sigma}(z) := C(zI - A)^{-1}B + D$$

is associated. This is a matrix whose entries are causal rational functions. For a given transfer function $W(z)$, $\Sigma = (A, B, C, D)$ is called a *realization* if (2.3) holds.

Other interpretations of Σ can be given. Systems over rings can be used to model systems with parameters, systems with delays, 2-D systems, neutral systems (see Eising (1980), Hautus and Sontag (1981), E. W. Kamen (1978), Rouchaleau (1972), Sontag (1976), (1981)). We will give an example below. By a suitable choice of $\mathcal{D}$ (and sometimes of $\mathcal{R}$, see Eising (1980, § 4.3)) one can accommodate various stability conditions one wants to impose on the transfer function. Once the ring $\mathcal{R}$ and the

denominator set $\mathcal{D}$ have been chosen, we call a rational function *stable* if it is in $\mathcal{R}_{\mathcal{D}}[z]$. A (single variable) stable transfer function is an element of $\hat{\mathcal{R}}_{\mathcal{D}}[z]$. An $n \times n$ matrix A is called a *stability matrix* if $\det(zI - A) \in \mathcal{D}$. Obviously, $W(z)$ is stable if A is a stability matrix. The converse is not always true, however, for any stable transfer function matrix, there exists a (free) realization Σ which is stable, i.e., for which A is a stability matrix (see Sontag (1976)).

Let us give some examples of interpretations of systems over rings and particular choices of denominator sets.

Example 2.4. In the case that $\mathcal{R} = \mathbb{R}$ (the field of real numbers) stability often is formulated in terms of pole location. Specifically, a set $\mathbb{C}^- \subseteq \mathbb{C}$ is given and a monic denominator $q(z)$ is in $\mathcal{D}$ iff it has no zeros outside $\mathbb{C}^-$. It is easily seen that $\mathcal{D}$, thus defined, is a denominator set provided $\mathbb{C}^- \cap \mathbb{R} \neq \emptyset$. $\square$

Example 2.5. One can model a *delay system* with delays all multiple of a given positive real number τ by a system over the ring $\mathcal{R} = \mathbb{R}[\sigma]$ of polynomials in σ , where σ stands for the delay operator

$$\sigma x(t) = x(t - \tau).$$

The system then will be of the form

$$(2.6) \quad \dot{x} = A(\sigma)x + B(\sigma)u, \quad y = C(\sigma)x + D(\sigma)u$$

where A, B, C, D are polynomial matrices. The systemic significance of the transfer function

$$(2.7) \quad W(s, \sigma) = D(\sigma) + C(\sigma)(sI - A(\sigma))^{-1}B(\sigma)$$

is described in Sontag (1976). In particular, applying a Laplace transform to (2.6) yields

$$(2.8) \quad \hat{y}(s) = W(s, e^{-\tau s})\hat{u}(s).$$

It is well known (see Hale (1977, § 7.4)) that $\Sigma = (A, B, C, D)$ is (externally) stable iff $W(s, e^{-\tau s})$ has no pole in $\text{Re } s \geq 0$. Thus, here we define

$$(2.9) \quad \mathcal{D} := \{p \in \mathbb{R}[s, \sigma] \mid p \text{ is monic with respect to } s \text{ and } p(s, e^{-\tau s}) \neq 0 \text{ for } \text{Re } s \geq 0\}.$$

When saying p is monic with respect to s we mean that p is of the form

$$p(s, \sigma) = s^n + p_1(\sigma)s^{n-1} + \cdots + p_n(\sigma)$$

where $p_1, \dots, p_n \in \mathbb{R}[\sigma]$. It is easily seen that $\mathcal{D}$ is a denominator set. In order that the system be internally stable one must require that $\det(sI - A(\sigma))$ be in $\mathcal{D}$. $\square$

Further examples demonstrating the generality of the stability concept described here can be given. Compare Datta and Hautus (1981), Eising (1980), Hautus and Sontag (1981), Kamen (1980).

3. Problem formulation and statement of the main results. First, we give a general formulation of the decoupling problem. We introduce the i/s -map corresponding to system Σ (i.e., system (2.2)) by (see Fig. 3.1)

$$(3.1) \quad W_s(z) := (zI - A)^{-1}B,$$

so that $W = CW_s + D$. Let F and G be dynamical systems with dimensions such that the formula

$$(3.2) \quad u = -F(z)x + G(z)y$$

is well defined. Then this formula defines a (combined) compensator, which transforms

FIG. 3.1

Σ into a system $\Sigma_{F,G}$ with transfer matrix

$$(3.3) \quad W_{F,G}(z) = W(z)L_{F,G}(z),$$

where

$$(3.4) \quad L_{F,G}(z) = (I + F(z)W_s(z))^{-1}G(z).$$

We notice that the same transfer matrix is obtained if one replaces (F, G) by $(0, L_{F,G})$. A compensator in which $F = 0$ is called a *precompensator*. If G is *static* (no dynamics in the precompensator part) then we say that (F, G) is *pure (dynamic) feedback* and if, in addition, F is static then (F, G) is called a *static state feedback*. Our objective is to find a compensator of a specified class (precompensator, pure dynamic feedback, static feedback) such that the resulting transfer matrix $W_{F,G}$ is diagonal, in which case we call the resulting system *decoupled*. In order to guarantee that each output can effectively be controlled, we require in addition that the diagonal elements of $W_{F,G}$ be nonzero. This is equivalent to requiring G to be nonsingular. Sometimes one wants to impose stronger conditions, such as the invertibility (over $\mathcal{R}$) of G (compare Datta and Hautus (1981)). Finally, assuming that the original system is internally stable, we try to find (F, G) such that the resulting system is internally stable. Such a compensator will be called *stability preserving*. Notice that we do not attempt to stabilize and to decouple the system simultaneously. Rather, we try to decouple it while maintaining its stability. If the system is not stable at the outset, it has to be stabilized first and afterwards one has to design the decoupling compensator. It will follow from the results of this paper, that one cannot destroy the existence of such a decoupling compensator when applying the stabilizing feedback. We assume that $\mathcal{U} = \mathcal{Y}$, i.e., the number of input and output variables are equal.

It turns out that the problem of decoupling by precompensation or combined compensation (i.e., no restrictions on F, G) is very simple even if $\mathcal{R}$ is an arbitrary integral domain.

THEOREM 3.5. *In the situation described above, the following statements are equivalent.*

- (i) *There exists a (stability preserving) decoupling combined compensator (F, G) .*
- (ii) *There exists a (stability preserving) decoupling precompensator $(0, G)$.*
- (iii) *W is nonsingular, i.e., $\det W$ is not identically zero.*

Proof. (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii). According to (3.3) we have

$$\det W \cdot \det L_{F,G} = \det W_{F,G} \neq 0,$$

since the diagonal elements of $W_{F,G}$ are nonzero.

(iii) $\Rightarrow$ (ii). Let $\text{adj } W$ denote the adjoint of W (occurring in Cramér's rule). Then

$$W \text{ adj } W = (\det W)I.$$

Choose $a \in \mathcal{R}$ such that $z - a \in \mathcal{D}$. Then, for sufficiently high k , $(z - a)^{-k} \text{adj } W = G$ is causal and stable since the entries of W are stable. Hence, by

$$W \cdot G = (z - a)^{-k} (\det W) \cdot I,$$

G is a stable decoupling compensator, which has an internally stable realization. If G is internally stable, then the total realization is internally stable. $\square$

The condition for the existence of pure feedback decoupling compensators is more involved. To formulate it we need some notation. We write

$$(3.6) \quad W(z) = \begin{bmatrix} w_1(z) \\ \vdots \\ w_m(z) \end{bmatrix},$$

where $w_i(z)$ denotes the i th row of W . Let $d_i(z)$ denote a GCD over $\mathcal{P}$ of the entries of $w_i(z)$. Such a GCD exists because $\mathcal{P}$ is a UFD. An explicit construction of such a GCD is given in Lemma 3.11. We can write $w_i = d_i w_i^*$ for suitable w_i^* with entries in $\mathcal{P}$. Hence

$$(3.7) \quad W(z) = \Delta(z) W^*(z),$$

where $\Delta(z) = \text{diag}(d_1, \dots, d_m)$ and W^* is the matrix consisting of the rows $w_1^*, \dots, w_m^*$.

Now we are in the position to formulate the main result of this paper.

THEOREM 3.8. *Let Σ be a reachable, internally stable system with respect to the denominator set $\mathcal{D}$. Then the following statements are equivalent:*

- (i) Σ can be decoupled by a stability preserving static state feedback with G invertible over $\mathcal{R}$.
- (ii) Σ can be decoupled by a stability preserving, stable dynamic state feedback with G invertible over $\mathcal{R}$ (and $F(z)$ stable).
- (iii) Σ can be decoupled by a stable precompensator L which is invertible over $\mathcal{P}$.
- (iv) W^* , as given in (3.7), is invertible over $\mathcal{P}$.

The proof of this result will be given in § 5.

In the theorem it is assumed that the gain matrix G is invertible, although this is not necessary in the original problem formulation: G nonsingular would do. It is possible to generalize the theorem to this more general case, but the formulation becomes more involved. There are two remarkable consequences of Theorem 3.8, already noted for systems over fields in Hautus and Heymann (1980), (1983). In the first place, if decoupling is possible by dynamic state feedback, it is also possible by static feedback. In the second place, the condition for the existence of a decoupling state feedback does not depend on the realization, provided the realization is reachable (for systems over fields this latter restriction is not necessary).

The GCD's used in defining W^* are not unique and consequently so is not W^* itself. However, the invertibility of W^* is independent of the particular choice of the GCD's, as easily can be seen. The condition on W^* can be checked by computing $w(z) = \det W^*$ and checking whether $(w(z))^{-1} \in \mathcal{P}$. Whether this condition can be verified effectively depends on $\mathcal{D}$. In the particular case that $\mathcal{D} = \mathcal{D}_0$, the set of all monic polynomials, the condition $(w(z))^{-1} \in \mathcal{P}_0$ can be checked very easily. To this

extent, expand $w(z)$ in powers of z^{-1} ,

$$w(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \dots,$$

and $(w(z))^{-1} \in \mathcal{P}_0$ iff w_0 is invertible over $\mathcal{R}$. It is even not necessary to compute $w(z)$.

COROLLARY 3.9. *If $\mathcal{D} = \mathcal{D}_0$ in Theorem 3.8 and if W^* is expanded as*

$$W^*(z) = W_0^* + W_1^* z^{-1} + \dots$$

then the condition (iv) of Theorem 3.8 may be replaced by: W_0^ is invertible over $\mathcal{R}$.*

When restricted to the case that $\mathcal{R}$ is a field, this is exactly the condition given in Falb and Wolovich (1967).

Although the particular choice of the GCD's used in the definition of W^* is of no relevance for condition (iv) of Theorem 3.8, it will turn out that for the actual construction of a decoupling feedback it is imperative that the GCD's satisfy an additional condition. To express this condition we use the notation $(p, q | \mathcal{R}[z])$ for the GCD's of elements in $\mathcal{R}[z]$.

DEFINITION 3.10. Let $w_1, \dots, w_m \in \mathcal{P}$. A GCD d of $w_1, \dots, w_m$ is called *admissible* if there exist polynomials q and p such that pd and qw_i/pd , $i = 1, \dots, m$ are polynomials, and $(p, q | \mathcal{R}[z]) = 1$.

The existence and the construction of an admissible GCD is guaranteed by the following result.

LEMMA 3.11. *Let $w_1, \dots, w_m \in \mathcal{P}$ and let q be a least common denominator of $w_1, \dots, w_m$. Define $\tilde{p}_i := qw_i$ and let $v := (\tilde{p}_1, \dots, \tilde{p}_m | \mathcal{R}[z])$. Finally, write $p_i := \tilde{p}_i/v$. Choose $a \in \mathcal{R}$ such that $z - a \in \mathcal{D}$ and define $\mu := \min \{\deg q - \deg p_i | i = 1, \dots, m\}$. Then*

$$d := v \cdot (z - a)^{-\mu}$$

is an admissible GCD of $w_1, \dots, w_m$.

For a proof see § 5. The lemma provides us with an actual construction of an admissible GCD, at least if we have an algorithm for computing GCD's in $\mathcal{R}[z]$. This is for instance the case when $\mathcal{R} = \mathbb{R}[\sigma]$, see Bose (1976).

Now we can specify Theorem 3.8:

PROPOSITION 3.12. *If the elements d_i used in the construction of W^* are admissible GCD's of w_i then there exists a static feedback (F, G) such that $L_{F,G} = W^{*-1}$.*

For a proof see § 5.

Once it has been proved that a given precompensator L can be implemented by a (static) feedback (F, G) , the actual computation of F and G is straightforward. Let us start from L^{-1} , rather than L (recall that $L^{-1} = W^*$). The relation $L^{-1} = L_{F,G}^{-1}$ reads

$$G^{-1}(I + FW_s) = L^{-1} = M_0 + M_1 z^{-1} + \dots,$$

where we have expanded L^{-1} into powers of z^{-1} . Invertibility of L^{-1} over $\mathcal{P}$ implies that M_0 is invertible. Since W_s is strictly causal we must have

$$(3.13) \quad G = M_0^{-1}.$$

Then it follows that $GL^{-1} = I =: V(z)$ is strictly causal and the map F has to be computed from

$$FW_s(z) = V(z).$$

On substitution of $W_s = (zI - A)^{-1}B$ into this equation and expanding both sides as a series in z^{-1} , one obtains

$$(3.14) \quad F[B, AB, \dots, A^{n-1}B] = [V_1, V_2, \dots, V_n],$$

where we have used the expansion $V(z) = V_1 z^{-1} + V_2 z^{-2} + \dots$. Because of reachability, the map $[B, AB, \dots, A^{n-1}B]$ has a right inverse and hence F can be solved uniquely from (3.14).

Notice that the maps F and G are uniquely determined by L . In particular, the stability-preservation property of (F, G) is automatically guaranteed by the invertibility of W^* over $\mathcal{P}$.

4. Examples.

Example 4.1. Consider the system $\Sigma = (A, B, C, D)$ over $\mathcal{R} = \mathbb{R}[\sigma]$, where

$$A = \begin{bmatrix} -1 & 1 \\ 1 & \sigma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & \sigma \end{bmatrix}, \quad C = \begin{bmatrix} -\sigma+1 & 1 \\ -\sigma^2+\sigma+1 & \sigma \end{bmatrix}, \quad D = 0.$$

Suppose that $\mathcal{D} = \mathcal{D}_0$. It is easily seen that Σ is reachable. The transfer matrix has the following expansion

$$\begin{aligned} W(z) &= CBz^{-1} + CABz^{-2} + \dots \\ &= \begin{bmatrix} 1 & 1 \\ \sigma & \sigma+1 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 2\sigma \\ \sigma+1 & 2\sigma^2+\sigma-1 \end{bmatrix} z^{-2} \\ &\quad + \begin{bmatrix} 2\sigma & 2\sigma^2-\sigma+1 \\ 2\sigma^2+\sigma-1 & 2\sigma^3+2 \end{bmatrix} z^{-3} + \dots \end{aligned}$$

The system can be decoupled by state feedback since $W_0^* = \begin{bmatrix} 1 & 1 \\ \sigma & \sigma+1 \end{bmatrix}$ is invertible. According to (3.13), we have

$$G = W_0^{*-1} = \begin{bmatrix} \sigma+1 & -1 \\ -\sigma & 1 \end{bmatrix}.$$

Furthermore, F has to be computed from (3.14), where V_i is the coefficient of z^{-i-1} in the expansion of $W_0^{*-1}W(z)$. (Notice that $W^*(z) = z^{-1}W(z)$.) It follows that

$$F \begin{bmatrix} 0 & 1 & 1 & \sigma-1 \\ 1 & \sigma & \sigma & \sigma^2+1 \end{bmatrix} = \begin{bmatrix} 0 & \sigma+1 & \sigma+1 & \sigma^2-1 \\ 1 & \sigma-1 & \sigma-1 & \sigma^2-\sigma+2 \end{bmatrix}.$$

Equating the first two columns we obtain

$$F = \begin{bmatrix} \sigma+1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The transfer function of the resulting system is $z^{-1}I$.

Example 4.2. The purpose of this example is to show that the reachability condition in Theorem 3.8 is essential. Let $\mathcal{R} = \mathbb{R}[\sigma]$, $\mathcal{D} = \mathcal{D}_0$

$$A = \begin{bmatrix} \sigma & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad C = I, \quad D = 0.$$

The system is not reachable, since

$$[B, AB] = \begin{bmatrix} \sigma & 0 & \sigma^2 & \sigma \\ 0 & \sigma & 0 & 0 \end{bmatrix}$$

and hence all reachable vectors have to be divisible by σ . The transfer function is

$$W = W_s = \frac{\sigma}{z(z-\sigma)} \begin{bmatrix} z & 1 \\ 0 & z-\sigma \end{bmatrix}.$$

Following the procedure described at the beginning of this section we obtain

$$W^* = \begin{bmatrix} \frac{z}{z-\sigma} & \frac{1}{z-\sigma} \\ 0 & 1 \end{bmatrix} = W_0^* + \dots$$

where

$$W_0^* = I$$

is invertible. We show that, nevertheless, decoupling by dynamic state feedback is impossible. Suppose that (F, G) is a dynamic state feedback decoupling the system. Then $W(I + FW)^{-1}G = \Delta$ for some diagonal matrix Δ (recall that $W = W_s$). Equivalently,

$$(4.3) \quad W = \Delta G^{-1}(I + FW).$$

Expanding the matrices D, W, F in powers of z^{-1} we have

$$\begin{aligned} W &= W_1 z^{-1} + W_2 z^{-2} + \dots, \\ \Delta &= D_0 + D_1 z^{-1} + D_2 z^{-2} + \dots, \\ F &= F_0 + F_1 z^{-1} + \dots, \end{aligned}$$

where

$$W_1 = B = \sigma I, \quad W_2 = AB = \begin{bmatrix} \sigma & \sigma^2 \\ 0 & 0 \end{bmatrix}.$$

Substitution into (4.3) yields

$$W_1 z^{-1} + W_2 z^{-2} + \dots = (D_0 + D_1 z^{-1} + \dots) G^{-1} (I + (F_0 + \dots)(W_1 z^{-1} + \dots)).$$

The coefficients of z_0 yields: $D_0 = 0$, and of z^{-1} : $W_1 = D_1 G^{-1}$ hence $\sigma G = D_1$. In particular, G is diagonal. Finally, equating the coefficients of z^{-2} we obtain

$$\begin{bmatrix} \sigma^2 & \sigma \\ 0 & 0 \end{bmatrix} = W_2 = D_1 G^{-1} F_0 W_1 + D_2 G^{-1} = \sigma^2 F_0 + D_2 G^{-1}.$$

The matrix $D_2 G^{-1}$ is diagonal and hence $\sigma^2 F_{0,12} = \sigma$, which has no solution in $\mathbb{R}[\sigma]$.

Further examples can be found in Datta and Hautus (1981).

5. Proofs.

Proof of Lemma 2.1. It is well known that $\mathcal{R}_{\mathcal{D}}[z]$ is a UFD if $\mathcal{D}$ is a multiplicative set (see Samuel (1969, Thm. 4, p. 29)). We define an isomorphism which maps $\mathcal{P}$ onto $\mathcal{R}_{\mathcal{D}_1}[z]$ for some $\mathcal{D}_1$. The isomorphism is

$$\varphi: r(z) \mapsto \hat{r}(z) := r\left(a + \frac{1}{z}\right)$$

defined on $\mathcal{R}(z)$, where $a \in \mathcal{R}$ is chosen such that $z - a \in \mathcal{D}$, and

$$\mathcal{D}_1 := \{z^n \hat{p}(z) \mid p \in \mathcal{D}, n = \deg p\}.$$

$\mathcal{D}_1$ is easily seen to be a multiplicative set in $\mathcal{R}[z]$. The map φ is invertible, and $\varphi^{-1}r(z) = r((z - a)^{-1})$. The homomorphism properties are readily verified. It remains to be shown that φ maps $\mathcal{P}$ into $\mathcal{R}_{\mathcal{D}_1}[z]$ and φ^{-1} maps $\mathcal{R}_{\mathcal{D}_1}[z]$ into $\mathcal{P}$.

Let $r = p/q \in \mathcal{P}$. Then

$$\varphi r = \hat{r} = \frac{\hat{p}}{\hat{q}} = \frac{z^n \hat{p}(z)}{z^n \hat{q}(z)} \in \mathcal{R}_{\mathcal{D}_1}[z],$$

where $n = \deg q$. Notice that because of the causality of r , the numerator $z^n \hat{p}(z)$ is a polynomial. Conversely, let $r \in \mathcal{R}_{\mathcal{D}_1}[z]$, say

$$r(z) = \frac{p(z)}{z^n \hat{q}(z)}$$

for some $q \in \mathcal{D}$ with $\deg q = n$. Then

$$\varphi^{-1} r(z) = \frac{(z-a)^n}{q(z)} p\left(\frac{1}{z-a}\right) \in \mathcal{P}$$

since $(z-a)^{-1} \in \mathcal{P}$ and hence $p((z-a)^{-1}) \in \mathcal{P}$, and $q \in \mathcal{D}$ and hence $(z-a)^n/q(z) \in \mathcal{P}$. Since $\mathcal{R}_{\mathcal{D}_1}[z]$ is a UFD and isomorphic to $\mathcal{P}$, it follows that $\mathcal{P}$ is a UFD. $\square$

Remark. $\mathcal{D}_1$ need not be a denominator set, in particular, the elements of $\mathcal{D}_1$ need not be monic.

Remark. The isomorphism φ is a standard device for transferring properties known about quotient rings to rings of causal quotients (Eising (1980, § 4.2), Hautus and Sontag (1980)).

Proof of Lemma 3.11. Because of the assumption that $\mathcal{R}$ is a unique factorization domain, the ring $\mathcal{R}[z]$ is also a unique factorization domain (see Barshay (1969, Thm. 4.7)). Let us denote by $\tilde{\mathcal{D}}$ the set of polynomials v of the form $v = uw$, where u is a unit and $w \in \mathcal{D}$. If v is any polynomial in $\mathcal{R}[z]$, we can factorize it into primes $v = p_1 \cdots p_m$. As a consequence, we can decompose v into $v = v^+ v^-$, where v^- is the product of the prime factors of v which are in $\tilde{\mathcal{D}}$ and v^+ consists of the other factors. Except for unit factors this decomposition is unique. If we insist on a unique decomposition we can achieve this by requiring that v^- be monic. We call v^- the $\mathcal{D}$ -part and v^+ the non- $\mathcal{D}$ part of v . The following results follow easily from this definition.

PROPOSITION 5.1.

- (i) $(ab)^+ = a^+ b^+$, $(ab)^- = a^- b^-$.
- (ii) $p|q$ (in $\mathcal{R}[z]$) iff $p^+|q^+$ and $p^-|q^-$.
- (iii) $(\text{GCD}(v_1, \dots, v_n))^+ = \text{GCD}(v_1^+, \dots, v_n^+)$.
- (iv) If $p \in \tilde{\mathcal{D}}$, $q|p$ then $q \in \tilde{\mathcal{D}}$.

In view of the saturation condition imposed on $\mathcal{D}$, property (iv) follows from the fact that divisors of elements of $\tilde{\mathcal{D}}$ are monic up to a unit factor.

After this preparation we are in the position to prove that $d := v(z-a)^{-\mu}$ is an admissible GCD of $w_1, \dots, w_m$ over $\mathcal{P}$. In the first place d is a divisor of $w_1, \dots, w_m$ because (recall that $w_i = p_i v/q$)

$$\frac{w_i}{d} = \frac{p_i v}{q v} (z-a)^\mu = \frac{p_i (z-a)^\mu}{q} \in \mathcal{P},$$

since $q \in \tilde{\mathcal{D}}$ and $\mu + \deg p_i \leq \deg q$. Now let d_1 be also a divisor of $w_1, \dots, w_m$. We have to show that $d/d_1 \in \mathcal{P}$. Let $d_1 = \alpha/\beta$, $w_i/d_1 =: a_i = b_i/c_i$, with $\beta, c_i \in \mathcal{D}$.

We notice that

$$w_i = a_i d_1 = \frac{\alpha b_i}{\beta c_i} = \frac{\tilde{p}_i}{q},$$

and hence

$$q\alpha b_i = \tilde{p}_i \beta c_i.$$

Taking the non- $\mathcal{D}$ part we obtain $\alpha^+ b_i^+ = \tilde{p}_i^+$. This shows that $\alpha^+ | \tilde{p}_i^+$ in $\mathcal{R}[z]$. Observing that v is a GCD of the $\tilde{p}_i$'s and hence v^+ is a GCD of the $\tilde{p}_i^+$'s, we conclude that $\alpha^+ | v^+$, say $v^+ = \gamma \alpha^+$ with $\gamma \in \mathcal{R}[z]$. It follows that

$$\frac{d}{d_1} = \frac{\beta v}{\alpha(z-a)^\mu} = \frac{\beta \gamma v^-}{\alpha^-(z-a)^\mu} \in \mathcal{R}_{\mathcal{D}}[z]$$

since the denominator is in $\mathcal{D}$. It remains to be shown that d/d_1 is causal. Choose i such that $\deg q = \deg p_i + \mu$ (recall the definition of μ). Then we have that

$$\frac{d}{d_1} = \frac{d}{w_i} \cdot \frac{w_i}{d_1} = \frac{v}{(z-a)^\mu} \cdot \frac{q}{p_i v} \cdot a_i$$

is causal, since a_i is. It follows that d is a GCD of $w_1, \dots, w_m$. Finally, we show that d is admissible. We choose $p = (z-a)^\mu$, and q as already defined. Then $pd = v$ is a polynomial and pd and q are coprime. In addition, $qw_i/pd = p_i$ is also a polynomial. This completes the proof. $\square$

Proof of Theorem 3.8 (and Proposition 3.12).

(i) $\Rightarrow$ (ii) is evident.

(ii) $\Rightarrow$ (iii). If (F, G) is a stability preserving, stable dynamic state feedback and if G is invertible, then, according to (3.3), W is decoupled also by the precompensator $L_{F,G}$ defined by (3.4). It remains to be shown that the entries of $L_{F,G}$ are in $\mathcal{P}$ and that $L_{F,G}$ is invertible over $\mathcal{P}$. It is easily seen that $L_{F,G}$ is invertible as a rational matrix and that $L_{F,G}^{-1} = G^{-1}(I + FW_s)$ has entries in $\mathcal{P}$. So, only the stability of $L_{F,G}$ itself has to be shown. By assumption, the resulting system is internally stable. This implies that the matrix $V = W_s L_{F,G}$ is stable. Since

$$(I + FW_s)^{-1} = I - FW_s(I + FW_s)^{-1}$$

it follows that

$$L_{F,G} = G - FV$$

is stable.

(iii) $\Rightarrow$ (iv). Suppose that for some nonsingular diagonal matrix $E = \text{diag}(e_1, \dots, e_m)$ we have $W = EL$, where L is a matrix invertible over $\mathcal{P}$. The first row of this matrix equation reads

$$[w_{11}, \dots, w_{1m}] = e_1[l_{11}, \dots, l_{1m}],$$

which implies that e_1 is a divisor of $w_1 = [w_{11}, \dots, w_{1m}]$. Since d_i is a GCD of w_1 it follows that e_1 divides d_1 , i.e., $d_1 = h_1 e_1$ for some $h_1 \in \mathcal{P}$. Similar results hold for the other rows, so that we can write $\Delta = EH$ where $H = \text{diag}(h_1, \dots, h_m)$. It follows that

$$EL = W = \Delta W^* = EHW^*$$

and hence $L = HW^*$. Therefore $W^{*-1} = HL^{-1}$ is causal and stable. (The nonsingularity of the matrices involved is obvious).

(iv) $\Rightarrow$ (i). If W^* is invertible we define $L = W^{*-1}$, and formula (3.7) reads

$$WL = \Delta.$$

Hence, the system is decoupled by the precompensator L which is a matrix over $\mathcal{P}$, hence causal and stable. We have to show that F and G exist such that $L = L_{F,G}$ (see (3.4)). To this extent, we formulate a generalization to systems over $\mathcal{R}$ of a result given for systems over a field in Hautus and Heymann (1978). The result in question is:

THEOREM 5.2. *Let (A, B, C, D) denote a reachable system and L be a bicausal isomorphism (i.e., causal and with a causal inverse). Then there exists a static state feedback compensator (F, G) , with G invertible, and $L = L_{F,G}$ iff for each polynomial $u \in \mathcal{R}^m[z]$ we have: If $W_s u$ is polynomial then $L^{-1}u$ is polynomial.*

The proof of this result is completely analogous to the proof in the field case, so it will not be repeated here. Contrary to the field case, however, the reachability of the system is essential. (A counterexample in the nonreachable case can be deduced from Example 4.2.) In order to apply the theorem to our L , we have to prove that $L^{-1}u = W^*u$ is polynomial whenever u and $W_s u$ are polynomial. We show that the following stronger statement: *u and Wu are polynomial implies W^*u is polynomial* holds, provided W^* is constructed via admissible GCD's. (This will also prove Proposition 3.12.) Let us assume that Wu is polynomial. The first entry of this vector is $w_1 u$, which is a polynomial. Let d_1 be an admissible GCD of w_1 , and let $w_1^* = d_1^{-1} w_1$. We have to show that $w_1^* u$ is polynomial. According to Definition 3.10, there exist polynomials p and q such that $a := p d_1$ and v_1/a , where $v_1 := q w_1$ are polynomials and $(a, q | \mathcal{R}[z]) = 1$. Since $w_1 u = q^{-1} v_1 u$ is polynomial we have $q | v_1 u$. Also, $a | v_1 u$, since u and v_1/a are polynomial. Hence $q a | v_1 u$, q and a being coprime. But this means that $w_1^* u = d_1^{-1} w_1 u = (q a)^{-1} v_1 u p$ is polynomial. The same argument applies to each row. This shows that L can be realized by state feedback. It remains to be shown that the resulting system is internally stable. Since $W_{s,F,G}$, the i/s -map which results after feedback is applied, is equal to $W_s L_{F,G}$ and hence stable, the desired result follows from:

LEMMA 5.3. *Let $W_s = (zI - A)^{-1} B$ be a reachable i/s -map. Then W_s is stable iff $\det(zI - A) \in \mathcal{D}$.*

Proof. Since $(zI - A)^{-1} = \text{adj}(zI - A) / \det(zI - A)$, the “if” part is obvious. To prove the “only-if” part we note that because of the reachability of (A, B) , there exist polynomial matrices $P(z)$ and $Q(z)$ such that

$$(zI - A)P(z) + BQ(z) = I$$

(see Khargonekar and Sontag (1981, Lemma 3.2)). It follows that

$$(zI - A)^{-1} = P(z) + W_s(z)Q(z)$$

is stable whenever $W_s(z)$ is. But then also $\det(zI - A)^{-1} = 1 / \det(zI - A)$ is stable, and hence $\det(zI - A) \in \mathcal{D}$ (since $\mathcal{D}$ is saturated). $\square$

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