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A class of infinitely divisible distributions connected to branching processes and random walks

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A class of infinitely divisible distributions connected to branching processes and random walks

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Abstract

A class of infinitely divisible distributions on \( \{0, 1, 2, \ldots \} \) is defined by requiring the (discrete) Lévy function to be equal to the probability function except for a very simple factor. These distributions turn out to be special cases of the total offspring distributions in (sub)critical branching processes and can also be interpreted as first passage times in certain random walks. There are connections with Lambert’s \( W \) function and generalized negative binomial convolutions.

Key words: infinite divisibility, branching processes, random walk, first passage time, Bürmann-Lagrange’s formula, negative binomial distribution, Borel distribution, queuing theory, Lambert’s \( W \), Hausdorff representation.

1 Introduction, definitions and preliminary results

For infinitely divisible distributions on the half-line there is a simple explicit relation between the probability measure and the corresponding canonical measure. We exploit this relation to construct a class of infinitely divisible distributions on \( \mathbb{Z}^+ := \{0, 1, 2, \ldots \} \), related to random walks and branching processes, in the following manner.

It is well known (see, e.g., Steutel, 1970) that a probability distribution \((p_n)_0^\infty\) on \( \mathbb{Z}^+ := \{0, 1, 2, \ldots \} \) with \( p_0 > 0 \) is infinitely divisible if and only if the quantities \( r_n, n \in \mathbb{Z}^+ \) uniquely defined by

\[
(n + 1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}, \quad n \in \mathbb{Z}^+,
\]

are all nonnegative; the sequence \((r_n)\) then necessarily satisfies \( \sum_0^\infty r_n/(n+1) < \infty \). Conversely, every sequence \((r_n)\) satisfying this condition by (1.1) defines an inf div distribution on \( \mathbb{Z}^+ \). Writing \( P \) and \( R \) for the (probability) generating functions, \((p)gf’s, of \( (p_n) \) and \( (r_n) \), (1.1) translates into

\[
R(z) = (\log(P(z)))' = P'(z)/P(z).
\]
Now, the class of inf div distributions we want to consider is defined by (1.1) and the following relation between \((p_n)\) and \((r_n)\)

\[
(1.2) \quad r_n = (n + c)p_n, \quad n \in \mathbb{Z}_+,
\]

where, for the time being, \(0 < c < 1\). The dependence on \(c\) of \((p_n)\) and its pgf \(P\) will be expressed by writing \(p_n(c)\) instead of \(p_n\) and \(P_c\) instead of \(P\) if desirable. A well-known special case with \(c = 1/2\) is provided by the pgf \(P\) given by

\[
P(z) = \frac{1 - \sqrt{1 - z}}{z}
\]

with \(p_n = \frac{1}{n+1} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n+1}\) and \(r_n = (n + \frac{1}{2})p_n\), as is easily verified. This is the distribution of \((T - 1)/2\) with \(T\) the time it takes to reach the point 1 in a simple symmetric random walk with start at the point 0.

Generally, taking gf's in (1.2) and using the fact that \(R = P'/P\), we obtain the following differential equation for \(P\):

\[
(1.3) \quad P'(z) = cP^2(z) + zP(z)P'(z).
\]

Clearly, there is only one solution of (1.3) with \(P'(1) = 1\). Now, multiplying both sides of (1.3) by \(\frac{1}{c}P^\frac{1}{c}-2\), we obtain

\[
\frac{1}{1 - c} (P^\frac{1}{c}-1(z))' = (zP^\frac{1}{c}_c(z))',
\]

from which it follows, by using \(P(1) = 1\), that \(P\) is a solution of

\[
(1 - c)zP^\frac{1}{c}_c(z) - P^\frac{1}{c}-1(z) + c = 0.
\]

We rewrite this in a form to be used later:

\[
(1.4) \quad z = \frac{1}{1 - c} \left(\frac{1}{P(z)} - \frac{c}{P^\frac{1}{c}_c(z)}\right);
\]

this is the equation we would like to solve for \(P(z)\). It is quite easily seen that for \(c = 1/2\) we find \(P(z) = (1 - \sqrt{1 - z})/z\) again. Before proceeding we make a couple of remarks.

**Remark 1.** Slightly more general than (1.2), one may consider

\[
r_n = (an + \beta)\bar{p}_n = \alpha(n + c)\bar{p}_n, \quad n \in \mathbb{Z}_+,
\]

with \(\alpha > 0\) and \(0 < c < 1\). These equations can be solved by first solving the case \(\alpha = 1\) with the solution \((\bar{p}_n)\). The solution \((\bar{p}_n)\) then takes the form \(\bar{p}_n = a^n p_n / \alpha P(a)\), with \(a\) such that \(aP(a) = \alpha\). Whereas the case \(\alpha = 1\) will turn out to be connected to a critical branching process, the case \(\alpha < 1\) corresponds to a subcritical process. It is not possible
to have $\alpha > 1$. It can be shown that, for $\alpha > 1$, $\sum \tilde{p}_n = \infty$.

**Remark 2.** It is also possible to consider (1.2) for numbers $c > 1$ (the case $c = 1$ will be dealt with later). Replacing $c$ by $1/c$ in (1.4) and denoting the solution of the resulting equation by $P_{1/c}$, it is easily seen that we must have $P_{1/c} = P_c^{1/c}$ or

$$P_c(z) = (P_{1/c}(z))^c. \quad (1.5)$$

Note that the right hand side is a well-defined pgf since $P_{1/c}$ is inf div. For $c = 2$, for instance, we obtain:

$$P_2(z) = (P_{1/2}(z))^2 = \left( \frac{1 - \sqrt{1 - z}}{z} \right)^2 = \frac{zP_{1/2}(z) - 1}{z}. \quad (2.1)$$

\section{Connection with branching processes; $0 < c < 1$}

Consider a (sub)critical branching process starting with one individual at time zero and offspring distribution with pgf $Q$ and $0 < Q'(1) \leq 1$. The process gets extinct with probability one and has a finite number $T$, say, of total offspring ($T$ may be zero). We use the following result from Jagers (1975); we give a short proof of the formula we need.

**Lemma 2.1** Let $T$ denote the total number of offspring in a branching process with offspring pgf $Q$ as described above, and let $P_T$ be the pgf of $T$. Then $P_T$ is the smallest positive solution to the following equation (in $y$)

$$y = Q(yz). \quad (2.1)$$

**Proof.** Let $y = E(z^T)$. Then by the independence of the offspring of different individuals, we have

$$y = \sum_{j=0}^{\infty} q_j E(z^T | X = j) = \sum_{j=0}^{\infty} q_j z^j y^j = Q(yz). \Box$$

The idea is now to choose $Q$ such that $y = P(z)$ satisfies (1.4). As a first and very simple example we put $Q = \frac{1/2}{1 - z^{1/2}}$, a geometric pgf with parameter $p = 1/2$. Solving for $y = P$ in (2.1) again yields $P(z) = P_{1/2}(z) = (1 - \sqrt{1 - z})/z$. Encouraged by this result we did some numerical calculations and conjectured that $P_c$ would be the solution of (2.1) if we would take $Q$ negative binomial as follows:

$$Q_c(z) = \frac{c}{1 - (1 - c)z}^{c/(1-c)}. \quad (2.2)$$
Indeed, substitution of $Q_c$ into (2.1) leads to

$$z = \frac{1}{1-c} \left( \frac{1}{y} - \frac{c}{y^{1/c}} \right),$$

which is exactly equation (1.4) with $P_c = y$. So we now have the following result.

**Proposition 2.2** Let $0 < c < 1$. The inf div pgf $P_c$ of the distribution $(p_n)$ satisfying (1.1) and (1.2) is precisely the pgf of the total offspring $T$ in a branching process as in Lemma 2.1 and with offspring pgf $Q_c$ given by (2.2).

We would now like to obtain explicit expressions for the distributions with pgf’s $P_c$ (the $P_c$ themselves are mostly intractable). Fortunately, there is another result in branching process theory giving the answer.

**Lemma 2.3** The total offspring $T$ in a branching process as in Lemma 2.1 and with offspring distribution $(q_n)$ has distribution given by

$$(2.3) \quad Pr(T = n) = \frac{1}{n+1} q_n^{n(n+1)},$$

where $(q_n^k)$ denotes the k-fold convolution of $(q_n)$ with itself.

**Remark.** In Lemmas 2.1 and 2.3, the quantity $T+1$ can be interpreted as the first-passage time from 0 to 1 in a random walk having step-length $Y$ satisfying $Y \sim 1 - X$, where $X$ has pgf $Q$. The random walk is now 'skip-free' to the right. For details, we refer to Grimmet and Stirzaker (1992, Section 5.3). We can now combine the results of Lemma 2.2: $P_c = P_T$, and Lemma 2.3: see (2.3), to obtain explicit expressions for the distributions $(p_n(c))$ with pgf's $P_c$.

**Theorem 2.4** Let $0 < c < 1$ and let $(p_n(c))_0^\infty$ denote the inf div distributions satisfying $(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}$ and $r_n = (n+c)p_n$. Then

$$(2.4) \quad p_n(c) = c^\gamma \frac{1}{n+1} \binom{n+1}{\gamma + n - 1} \beta^n,$$

where $\gamma = c/(1-c)$ and $\beta = (1-c)c^\gamma$.

**Proof.** Using Proposition 2.2 and Lemma 2.3 we have $(c = \gamma/(1 + \gamma))$

$$p_n(c) = \frac{1}{n+1} \{\text{coefficient of } z^n \text{ in } \left( \frac{c}{1 - (1-c)z} \right)^{(n+1)\gamma} \}$$

$$= \frac{1}{n+1} c^{(n+1)\gamma} (n+1)^n (1-c)^\gamma \frac{(-\gamma + n - 1)}{n} = c^\gamma \frac{1}{n+1} \left( \left( \frac{(n+1)\gamma + n - 1}{n} \right) \beta^n. \right.$$

As a check we put $c = 1/2$ i.e. $\gamma = 1$, $\beta = 1/4$, and find $p_n(1/2) = \frac{1}{2^{n+1}} (2n)! \binom{1}{2}^{2n+1}$ as before.
3 The cases \( c \geq 1 \)

The case \( c = 1 \) can be dealt with by letting \( c \rightarrow 1 \). Then \( Q_c \rightarrow Q_1 \), where \( Q_1 = \exp\{z - 1\} \) (Poisson distribution with mean 1), and using Lemma 2.3 we obtain

\[
(3.1) \quad p_n(1) = \frac{1}{n+1} e^{-\frac{(n+1)}{n}} e^{-\frac{(n+1)}{n} \left( \frac{n+1}{n} \right)^n} = e^{-\frac{(n+1)}{n}} \frac{(n+1)^{n-1}}{n!}.
\]

Had we started from \( Q(z) = e^{\lambda(z-1)} \), with \( \lambda < 1 \), we would have obtained

\[
p_n(1) = e^{-\lambda(n+1)} \frac{\lambda^{n+1} (n+1)^{n-1}}{n!},
\]

the well-known Borel distribution which arrives in applications like queueing theory. See e.g. Johnson, Kotz, and Kemp (1993, Chapter 1, Section 5 and pp. 394-395). Alternatively, taking \( c \rightarrow 1 \) in (1.4), we get

\[
(3.2) \quad P_1(z) = \exp\{zP_1(z) - 1\},
\]

and we can recover (3.1) by use of Bürmann-Lagrange's formula in series form; see, e.g., Grimmett and Stirzaker (1992, p.146). Formula (3.2) is very similar to the defining equation for Lambert's function \( W \), viz.

\[
z = W(z) e^{W(z)},
\]

see Corless et al. (1996). From this it follows that \( P_1(z) \) can be written as

\[
P_1(z) = -W(-z/c)/z.
\]

For \( c > 1 \) we shall use (1.5) for obtaining explicit expressions for \( p_n(c) \); see below. For some special cases we have interpretations and we can then use Lemma 2.1 again together with Lemma 2.3. Taking

\[
Q(z) = \left( \frac{1}{c} + (1 - \frac{1}{c})z \right)^{c/(c-1)},
\]

with \( c/(c-1) \) integer, i.e. \( c = N/(N-1) \), with \( N \) a positive integer. Now the offspring distribution is Binomial \( \left( \frac{c}{c-1}, 1 - \frac{1}{c} \right) \) with mean 1. We obtain from (2.3)

\[
p_n = \frac{1}{n+1} \left\{ \text{coefficient of } z^n \text{ in } \left( \frac{1}{c} + (1 - \frac{1}{c})z \right)^{(n+1)c/(c-1)} \right\}
\]

\[
= \frac{1}{n+1} \left\{ \binom{n+1}{n} \frac{c/c}{n+1} \right\} \left( \frac{c}{n+1} \right)^{(n+1)c/(c-1)z-n}
\]

\[
= c^\gamma \frac{1}{n+1} \binom{n+1}{n} \left( \frac{c}{n+1} \right)^{\gamma + n - 1} \beta^n,
\]

which is the same function of \( \gamma = c/(1-c) \) and \( \beta = (1-c)c^\gamma \) as in (2.4), but \( \gamma \) and \( \beta \) are now negative. It turns out that this is true for all \( c > 1 \), i.e. formula (2.4) is also the formula for \( p_n(c) \) when \( c > 1 \). We give this result as a formal statement.
Proposition 3.1 Let $c > 0$, $c \neq 1$. Then the solution $p_n(c)$ of the set of equations

$$(n + 1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}; \quad r_n = (n + c)p_n, \quad n \in \mathbb{Z}_+,$$

is given by the right-hand side of (2.4).

Proof. We must show that (2.4) not only holds for $0 < c < 1$, but also for $c > 1$. To this end we use (1.5) from which it follows that for $c > 1$

$$(3.3) \quad r_n(c) = cr_n(1/c);$$

this is a consequence of the fact that the sequence $(r_n)$ corresponding to $P^c$ has generating function $c \cdot R$. From (3.3) and (1.2) we obtain

$$(n + c)p_n(c) = r_n(c) = cr_n(1/c) = (cn + 1)p_n(1/c),$$

or

$$p_n(c) = \frac{cn + 1}{n + c} p_n(1/c), \quad c > 1.$$

Now we have to do two things; first use (2.4) for $p_n(1/c)$ with $0 < 1/c < 1$, and then (see above) show that $\frac{cn+1}{n+c} p_n(1/c)$ is equal to $p_n(c)$ as in (2.4), but now with $c > 1$. Consequently we have to prove the following identity for $c > 1$

$$\frac{cn + 1}{n + c} \left( \frac{1}{c} \right)_{n+1} (1 - \frac{1}{c})^n \left( \frac{n+1}{c-1} + n - 1 \right) = c^{1-c} (1 - c)^n \left( \frac{n-1+2c}{n} \right).$$

This can be done by writing out the binomial coefficients and carefully comparing the expressions on both sides; they are indeed equal. We do not give the details.

So far we do not have an interpretation for $p_n(c)$ with $c > 1$ and $c/(c-1)$ non-integer, or for the relation $P_c = P_{1/c}$. We add that for $c/(c-1)$ integer, the distributions we have met are known as Consul's distributions; see e.g. Johnson, Kotz, and Kemp (1993, p. 98).

4 Further properties

In this section we shall be concerned with complete monotonicity of sequences. A sequence $\{c_n\}, n \in \mathbb{Z}_+$, is said to be completely monotone, if it is nonnegative and has differences that alternate in sign. Equivalently, $\{c_n\}$ is completely monotone, if it has a representation of the form

$$(4.1) \quad c_n = \int_0^1 x^n \mu(dx),$$
where \( \mu \) is a finite measure on \([0,1]\). Representation (4.1) is called the Hausdorff representation of \( \{c_n\} \). Considering the special cases \( c = 1/2 \) and \( c = 1 \), we find that \( r_n(c) \), and hence also \( p_n(c) \), is completely monotone in \( n \). For \( c = 1/2 \) we easily have the Hausdorff representation

\[
p_n(\frac{1}{2}) = \frac{1}{\pi} \int_0^1 y^n(1-y)^{1/2}y^{-1/2}dy,
\]

from which, by integration by parts,

\[
r_n(\frac{1}{2}) = (n + \frac{1}{2})p_n(\frac{1}{2}) = \frac{1}{2\pi} \int_0^1 y^{n+1}y^{-1/2}(1-y)^{-1/2}dy.
\]

For \( c = 1 \) it is known (Hansen & Steutel, 1988) that

\[
p_n(1) = \frac{1}{\pi(n+1)} \int_0^\pi \frac{1}{(h(\theta))^{n+1}}d\theta,
\]

where

\[
h(\theta) = \frac{\theta}{\sin(\theta)} \exp\{1 - \theta \cot(\theta)\}
\]

increases from 1 to \( \infty \) on \([0, \pi]\). The complete monotonicity of \( r_n(1) = (n+1)p_n(1) \) trivially follows. We now try to prove the complete monotonicity of \( r_n(c) \) for all \( c \in (0,1) \). In order to do this we shall use a lemma, for which we need the following definition.

Let \( D \subset \mathbb{C} \) be the region bounded by the curves \( w = \rho(\theta)e^{i\theta} \), \( 0 \leq \theta \leq c\pi \) (in the upper half-plane) and \( w = \rho(\theta)e^{-i\theta} \), \( 0 \leq \theta \leq c\pi \) (in the lower half-plane), where

\[
\rho(\theta) = \left(\frac{c\sin(\theta/c)}{\sin(\theta)}\right)^{1-c}.
\]

It is easily verified that \( \rho \) is decreasing from 1 to 0 on \((0,c\pi)\). Both the curves start at the point 1 and end at 0. We now formulate our lemma, of which we give only a sketch of a proof.

**Lemma 4.1** The function \( g \) defined by

\[
g(w) = \frac{1}{1-c} \left( \frac{1}{w} - \frac{c}{\rho w^{1-c}} \right) = z
\]

is a bijective holomorphic mapping from the region \( D \) onto the complex plane \( \mathbb{C} \) cut along \((1,\infty)\). Further, if \( w \in D \) is in the upper half-plane, so is its image \( z \) and vice versa. Thus the inverse function \( w = P(z) \) satisfies \( 0 < \Im(P(z)) < c\pi \) for \( \Im z > 0 \).

**Sketch of proof.** Set \( w = \rho e^{i\theta} \). For \( z = x > 1 \), the equation \( g(w) = x \) can be written as

\[
\frac{1}{1-c} \left( \frac{\cos(\theta)}{\rho} - \frac{c\cos(\theta/c)}{\rho^{1-c}} \right) = x, \quad \frac{\sin(\theta)}{\rho} - \frac{c\sin(\theta/c)}{\rho^{1-c}} = 0.
\]
Obviously \( \theta \) can be both positive and negative. Concentrating on the case \( \theta > 0 \) and solving the second equation for \( \rho \), we get \( \rho = \rho(\theta) \). It can be verified that \( \rho(\theta) \) is decreasing on \((0, c\pi)\) whereas

\[
h(\theta) = \frac{1}{1 - c} \left( \cos(\theta) - c \frac{\cos(\theta/c)}{\rho(\theta)^{1/c}} \right)
\]

increases from 1 to \( \infty \) on \((0, c\pi)\). Thus the two equations \((4.2)\) can be solved. For \( z = x < 1 \), the equation \( g(w) = x \) has exactly one real solution \( w \in (0, 1) \) since \( g \) increases from \(-\infty\) to 1 on \((0, 1)\]. By using the principle of argument it can be ascertained that the equation \( g(w) = z, z \notin (1, \infty) \) has exactly one solution in \( D \), which is in the upper half-plane if \( z \) is.

Having Lemma 4.1 in mind we will now use Cauchy's integral formula to obtain expressions for \( P(z) \) and \( R(z) \). We use a contour which consists of a very large circle around the origin, a small circle around the point 1, and two lines along \((1, \infty)\); by the inverse mapping, this contour is essentially the boundary of the domain \( D \) defined above. Since \( P(z) \) tends to zero as \( z \to \infty \) and so does \( R(z) = \frac{cP(z)}{1-zP(z)} \), and both \( P(z) \) and \( R(z) \) take on conjugate values for conjugate \( z \), we get

\[
P(z) = \frac{1}{\pi} \int_{1}^{\infty} \frac{\Im(P(x))}{x - z} \, dx \quad \text{and} \quad R(z) = \frac{1}{\pi} \int_{1}^{\infty} \frac{\Im(R(x))}{x - z} \, dx,
\]

where now \( x \) denotes a point on the upper side of the cut \((1, \infty)\). Substituting then \( x = h(\theta), 0 < \theta < c\pi \), with \( h(\theta) \) as above, we get, noticing that \( \theta = \arg(P(x)) \) and \( \Im(R(x)) = \frac{d}{d\theta} \Im(\log(P(x))) = \frac{d}{d\theta} \arg(P(x)) \),

\[
P(z) = \frac{1}{\pi} \int_{0}^{c\pi} \frac{\Im(P(h(\theta)))}{h(\theta) - z} h'(\theta) \, d\theta = \frac{1}{\pi} \int_{0}^{c\pi} \frac{k(\theta)}{h(\theta) - z} \, d\theta,
\]

where \( k(\theta) = \rho(\theta) \sin(\theta) h'(\theta) > 0 \), and

\[
R(z) = \frac{1}{\pi} \int_{0}^{c\pi} \frac{d\theta}{h(\theta) - z}.
\]

Thus

\[
(4.3) \quad p_n(c) = \frac{1}{\pi} \int_{0}^{c\pi} \frac{k(\theta)}{(h(\theta))^{n+1}} \, d\theta \quad \text{and} \quad r_n(c) = \frac{1}{\pi} \int_{0}^{c\pi} \frac{1}{(h(\theta))^{n+1}} \, d\theta.
\]

Since \( h(\theta) \geq 1 \) it follows that both \( p_n(c) \) and \( r_n(c) \) are completely monotone. The complete monotonicity of \( p_n(c) \) also trivially follows from that of \( r_n(c) \) as \( p_n(c) = r_n(c)/(n+c) \). Discrete inf div distributions with \( (rn) \) completely monotone are called generalized negative binomial convolutions (GNBC's) in Bondesson (1992, Chapter 8). They are also characterized by having \( \Im(R(z)) > 0 \) for \( \Re z > 0 \). We have thus obtained the following result.
Theorem 4.2 For $0 < c < 1$ and hence for all $c > 0$, the probability distributions $(p_n(c))$ are GNBC’s; so their canonical sequences $(r_n(c))$ are completely monotone, i.e. $r_n(c)$ can be represented as $r_n(c) = \int_0^1 y^{n+1} V(dy)$, where $V$ is a nonnegative measure on $(0,1]$. Moreover, $(p_n(c))$ is completely monotone for all $c > 0$.

The measure $V$ can be obtained from $R$ in the following way (cf. Bondesson, 1992, pp. 127 and 129): If $V$ has a density $v$ it is given by

$$v(y) = \frac{1}{\pi y^2} \lim_{z \to 1/y, z > 0} \Im(R(z)).$$

In our case, it readily follows from (4.3) that, for $0 < c < 1$,

$$v(y) = \frac{1}{\pi y^2 h'(h^{-1}(1/y))}.$$

It is not hard to see that the total $V$-measure $\int_0^1 V(dy)$ equals $c$ if $c < 1$. From (3.3) it then follows that the total $V$-measure equals 1 for $c > 1$.

Remark 1. For $c = 1/2$, $\rho(\theta) = \cos(\theta)$ and $h(\theta) = 1/\cos^2(\theta)$. One can verify that as $c \to 1$, $\rho(\theta) \to \exp\{\theta \cot(\theta) - 1\}$ and $h(\theta) \to \frac{\theta}{\sin(\theta)} \exp\{1 - \theta \cot(\theta)\}$. cf. Hansen & Steutel’s (1988) result.

Remark 2. Since $(n+1)p_n(c) = r_n(c) + (1-c)p_n(c)$, we get from the explicit formula for $p_n(c)$ in Theorem 2.5 the curious representation

$$c^\gamma \Gamma\left(\frac{(n+1)\gamma + n - 1}{n}\right) \beta^n = \frac{1}{\pi} \int_0^\infty \frac{1 + (1-c)k(\theta)}{(h(\theta))^{n+1}} d\theta,$$

where $\gamma = c/(1-c)$, $\beta = c(1-c)^\gamma$, and $k(\theta)$ is defined earlier. The formula seems to hold also for real $n \geq 0$.

5 Poisson-mixtures

We now look at another representation of the $p_n(c)$’s. It is known that a GNBC also is a Poisson-mixture, i.e.

$$p_n = \int_0^\infty \frac{\lambda^n}{n!} e^{-\lambda} F(d\lambda),$$

where $F$ is the cdf of a probability distribution, which is a generalized gamma convolution (GGC). The class of such distributions is studied in Bondesson (1992); we do not give any details here. The Laplace transform $\phi(s) = \int_0^\infty e^{-s\lambda} F(d\lambda)$ of $F$ is given by $\phi(s) = P(1-s)$. For $c = 1/2$ we see that $\phi(s) = 1/(1+\sqrt{s})$ which means that $F$ is the distribution function for $XY^2$, where $X$ is a positive stable random variable with index 1/2 and independent of $Y$ which has an exponential distribution.
The case $c = 1$ is more involved. Obviously $\phi(s) = W((s - 1)/e)/(s - 1)$, where $W$ is the Lambert W function. Since $p_n(1) = e^{-n-1}(n + 1)^{n+1}/n!$, we get

$$\int_0^\infty \lambda^n e^{\lambda - 1} F(d\lambda) = (n + 1)^{n+1} \frac{1}{(n + 1)!} e^{-n}.$$ 

Replacing $n$ by $it$ in the right-hand side, one can recognize it as the characteristic function of $X - Y - 1$ where $X$ has an exponentially tilted (by a factor $e^X$) extreme stable distribution with index 1 (and with most of its mass on $(-\infty, 0)$) and is independent of $Y$ which has a Gamma$(2,1)$ distribution. Thus $F$, exponentially tilted, corresponds to the variable $e^X e^{-Y} e^{-1}$.

One may guess that representations in terms of stable distributions are possible also for other values of $c$.

We end this paper with another open problem. For $c = 1/2$ the density of the corresponding GGC with Laplace transform $\phi(s) = 1/(1 + \sqrt{s})$ is hyperbolically completely monotone; see Bondesson (1992, pp. 70 and 86). For the discrete distribution $p_n(1/2)$ this means that not only is it true that, for $N = 0$, $\Im(P^{(N+1)}(z)/P^{(N)}(z)) \geq 0$ for $\Im z > 0$, this is true for all integers $N \geq 0$. Equivalently, the $r$-sequence corresponding to

$$(n + N)(n + N - 1) \cdots (n + 1)p_{n+N+1}(1/2), \ n \in \mathbb{Z}_+,$$

is completely monotone for every $N$. There is a lot of numerical evidence for the conjecture that this result continues to be true for all $c \in (0,1]$. One way of proving it might be to make use of a Bürmann-Lagrange type formula (see e.g. Pólya & Szegő (1964, p.125); we do not dwell upon this.

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