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STEADY STOKES FLOW IN AN ANNULAR CAVITY

By T. S. KRASNOPOLSKAYA,† V. V. MELESHKO,‡
G. W. M. PETERS and H. E. H. MEIJER

(Centre for Polymers and Composites, Eindhoven University of Technology,
Postbus 513, 5600 MB Eindhoven, The Netherlands)

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SUMMARY

This paper addresses a general analytical method of superposition for the study of two-dimensional creeping flows in a wedge-shaped cavity \( a \leq r \leq b, |\theta| \leq \theta_0 \) caused by tangential velocities of its curved walls. The method is illustrated by several numerical examples; the rate of convergence and the accuracy of fulfilling the boundary conditions are investigated. The main objective is to demonstrate the advantages of the method of superposition when analysing streamline patterns and the velocity-field distribution in the whole domain, including the Moffatt eddies near corner points. The equations for the positions of the stagnation and separation points are written analytically. The streamline patterns for uniform velocities at the top and the bottom walls are shown graphically. These patterns represent the transition from the corner eddies into internal eddies.

1. Introduction

The topic of a creeping flow of an incompressible viscous fluid has been one of constant interest in fluid dynamics. When the motion is so slow that the terms involving the squares of velocities in the Navier–Stokes equations may be omitted (the so-called Stokes flow) the problem becomes linear and admits an analytical solution for many canonical domains. In the past this subject has attracted much attention (for example, from Stokes, Reynolds, Rayleigh, Lorentz, Lamb, Filon, Oseen and other prominent scientists), and nowadays the literature is vast. The early stages of the research on this topic are summarized in the classic books of Lamb (1) and Happel and Brenner (2), while relatively recent results are reviewed in (3).

The two-dimensional velocity field in steady Stokes flows is determined by the biharmonic stream function and by the boundary values of this function and its normal derivative following from the given velocity distribution at the boundaries. As was first pointed out by Rayleigh (4) and Goodier (5), the Stokes-flow problem is mathematically similar to the problems of the bending of clamped thin elastic plates and of a plane stress

† Permanent address: Institute of Mechanics, National Academy of Sciences, 252057 Kiev, Ukraine.
‡ Permanent address: Institute of Hydromechanics, National Academy of Sciences, 252057 Kiev, Ukraine.

situation in a thin elastic disk of uniform thickness loaded by forces in its midplane. Therefore, the large number of valuable solutions developed in the field of elasticity over more than 100 years (see, for instance, (6) for references), can be used successfully for the analysis of creeping flows in various domains.

The biharmonic problem becomes much more complicated when the boundary of the domain has edges with sharp corner points. As was shown by Dean and Montagnon (7) and Moffatt (8), for an infinite wedge the eigensolution forms an infinite sequence of viscous corner eddies located near the apex. The flow near the corner of an infinite wedge is determined by the competition between the outside stirring of the fluid and the local conditions in the corner—in other words, by the competition between solutions driven by boundary conditions and the eigenfunction solutions (9 to 11). These so-called Moffatt eddies were observed experimentally in a finite wedge (12) as well as numerically for the finite rectangular cavity where the top wall has a constant velocity while the three other walls are fixed (13). The last problem is a traditional benchmark problem for testing various numerical methods for the solution of the full Navier–Stokes equations with different values of the Reynolds number. In spite of the large number of papers devoted to this problem (see, for example, (14,15) for further references, including related mathematical and elasticity problems) it still remains important to raise accuracy and reduce computational costs in determining the velocity field.

In this paper we analyse the Stokes flow in an annular cavity induced by the prescribed tangential velocities at the curved walls. A similar problem, but with a temperature gradient in the viscous fluid as a driving force, was considered in (16). Various related elastic problems of two-dimensional curved beams and thin plates can be found in (6,17 to 19, 20, 21).

Here we develop a rigorous analytical method for the solution of this biharmonic problem with 'non-canonical' (according to the terminology of (14)) boundary conditions. The main idea of our approach traces back to the famous Lamé lectures (22) on the mathematical theory of elasticity. For representing the biharmonic function on a two-dimensional rectangular domain, this method (later named the 'method of crosswise superposition' (6)) uses the sum of two ordinary Fourier series on the complete systems of trigonometric functions in \( x \) and \( y \) coordinates respectively. These series both satisfy identically the governing equation inside the domain and have sufficient functional freedom to fulfil the two boundary conditions on each of the four edges. Because of the interdependency, the expression for a coefficient in one series will depend on all coefficients of the other series and vice versa. Therefore, the final solution involves solving an infinite system of linear algebraic equations giving the relations between the coefficients and the applied boundary conditions. Similar infinite systems were obtained for the elastic curved beam or plate (17, 21).
In spite of its apparent simplicity (linear equations) the theory of infinite systems contains some difficult issues which can sometimes lead to controversies and misunderstandings. A traditional way of solving an infinite system of linear algebraic equations, already proposed by Fourier in 1819 (23), consists of a simple reduction, that is, leaving only the first \( N \) equations and putting all unknowns with suffixes greater than \( N \) equal to zero. By solving such a finite system and increasing the number \( N \), it is believed that the solution of the initial infinite system will be approached.

However, for the analysis of the stream function and velocities at the boundaries (especially near corner points), this traditional approach seems to show some serious problems and does not provide reasonable satisfaction of the boundary conditions. Besides, the local structure of the solution (the well-known Goodier (5) and Taylor (24) solution near the lid corner and Moffatt eddies (8, 10) near the quiet corner) cannot be seen easily from the numerical data. This necessitates a more careful consideration of the traditional way of obtaining a solution of an infinite system for which it has usually been assumed that coefficients with suffixes higher than some fixed number may be neglected.

The algorithm of the improved reduction as proposed in the present paper yields a high accuracy and low computational cost. Knowledge of the asymptotic behaviour of the coefficients provides a detailed description of the local structure of the stream function near corners. Based on the (stable) solution of the finite system of linear algebraic equations the amplitudes of the strongest series of Moffatt eddies can be found.

The paper is organized as follows: the formulation of the problem and the analytical method for its solution are described in section 2. Theoretical considerations about the reduction of the infinite system are presented in section 3. The stream function and velocity field for the important case of uniform constant boundary velocities are considered in section 4. Next, in section 5, numerical results concerning the streamline patterns and their transformation with a change in wall velocities are described. Finally, a brief discussion of the results and some conclusions are given in section 6.

2. Problem formulation and construction of the solution

Consider a two-dimensional creeping flow in an annular cavity, \( a \leq r \leq b \), \(-\theta_0 \leq \theta \leq \theta_0\) (Fig. 1), caused by tangential velocities \( V_{\text{bot}}(\theta) \) and \( V_{\text{top}}(\theta) \) at the curved bottom and top boundaries \( r = a \) and \( r = b \), respectively. The side walls, \( a \leq r \leq b \), \( \theta = \pm \theta_0 \), are fixed. We restrict our consideration to cases for which \( V_{\text{bot}}(\theta) \) and \( V_{\text{top}}(\theta) \) are continuous, even functions of \( \theta \). In the plane polar coordinate system \((r, \theta)\), the boundary conditions are

\[
\begin{align*}
&u_r = 0, \quad u_\theta = V_{\text{bot}}(\theta), \quad r = a, \quad |\theta| \leq \theta_0, \\
&u_r = 0, \quad u_\theta = V_{\text{top}}(\theta), \quad r = b, \quad |\theta| \leq \theta_0, \\
&u_r = 0, \quad u_\theta = 0, \quad a \leq r \leq b, \quad |\theta| = \theta_0.
\end{align*}
\]
The radial \((u_r)\) and azimuthal \((u_\theta)\) components of velocity can be expressed by means of the stream function \(\Psi(r, \theta)\) as

\[
    u_r = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \Psi}{\partial r}.
\]

(2)

The equation for the stream function governing the two-dimensional creeping steady flow of a viscous fluid is

\[
    \nabla^2 \Psi = 0,
\]

(3)

where \(\nabla^2\) stands for the Laplace operator

\[
    \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]

in polar coordinates. In terms of the stream function the problem for the Stokes flow in an annular cavity consists of finding the biharmonic function \(\Psi\) for given values of the function and its normal derivative on the boundary:

\[
    \begin{align*}
    \Psi &= 0, \quad \frac{\partial \Psi}{\partial r} = -V_{\text{bot}}(\theta), \quad r = a, \quad |\theta| \leq \theta_0, \\
    \Psi &= 0, \quad \frac{\partial \Psi}{\partial r} = -V_{\text{top}}(\theta), \quad r = b, \quad |\theta| \leq \theta_0, \\
    \Psi &= 0, \quad \frac{\partial \Psi}{\partial \theta} = 0, \quad a \leq r \leq b, \quad |\theta| = \theta_0.
    \end{align*}
\]

(4)
The main idea of the superposition method is the representation of the stream function as a sum of two functions:

$$\Psi = \Psi_1 + \Psi_2.$$  \hfill (5)

These two biharmonic functions are chosen in the form of Fourier series with sufficient freedom to separately fulfill the boundary conditions (4) at the walls $r = a$ and $r = b$ and at the walls $|\theta| = \theta_0$.

The function $\Psi_1$ is found by the technique of separation of variables:

$$\Psi_1(r, \theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\alpha_m} \left[ A_m \left(\frac{r}{b}\right)^{\alpha_m} + B_m \left(\frac{r}{b}\right)^{\alpha_m+2} + C_m \left(\frac{a}{r}\right)^{\alpha_m} + D_m \left(\frac{a}{r}\right)^{\alpha_m-2} \right] \cos \alpha_m \theta,$$  \hfill (6)

with $\alpha_m = (2m-1)\pi/2\theta_0$. The structure of this representation is clear: the sets of coefficients $A_m$, $B_m$ and $C_m$, $D_m$ are (mainly) responsible for satisfying two boundary conditions at the curved walls $r = a$ and $r = b$, respectively. Of course, representations of $\Psi_1$ other than (6) exist belonging to the complete system of trigonometric functions on the interval $|\theta| \leq \theta_0$.

For example, we can choose the complete system of functions $\cos (k\pi \theta/\theta_0)$ or even $\cos (k \theta)$ with $k = 0, 1, 2, \ldots$ in the Fourier series. But the choice (6) appears to be the most convenient for further analytical treatment.

For construction of the solution for $\Psi_2$ on the complete system of functions of $r$, we introduce the new variable $\tau$ such that

$$\tau = \ln \frac{r}{a}.$$  \hfill (7)

Then the biharmonic equation (3), which contains coefficients that depend on $r$, can be written in the coordinates $\tau$ and $\theta$, for which the coefficients are constant:

$$\frac{\partial^4 \Psi}{\partial \tau^4} + 2 \frac{\partial^4 \Psi}{\partial \tau^2 \partial \theta^2} - 4 \frac{\partial^3 \Psi}{\partial \tau^3} - 4 \frac{\partial^3 \Psi}{\partial \tau \partial \theta^2} + 4 \frac{\partial^3 \Psi}{\partial \tau^2} + 4 \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\partial^4 \Psi}{\partial \theta^4} = 0.$$  \hfill (8)

Next, introducing in (8) the substitution $\Psi(\tau, \theta) = e^\tau F(\tau, \theta)$, we find that the function $F(\tau, \theta)$ has to satisfy the equation

$$\frac{\partial^4 F}{\partial \tau^4} + 2 \frac{\partial^4 F}{\partial \tau^2 \partial \theta^2} + \frac{\partial^4 F}{\partial \theta^4} - 2 \frac{\partial^2 F}{\partial \tau^2} + 2 \frac{\partial^2 F}{\partial \theta^2} + F = 0.$$  \hfill (9)

This equation has constant coefficients and only even derivatives with respect to $\tau$ and $\theta$. Therefore, in the 'rectangle' $0 \leq \tau \leq \tau_0$, $|\theta| \leq \theta_0$ (with
The solution of equation (9) can be written in the form of Fourier series on the complete trigonometric system \( \sin (n \pi \tau / \tau_0) \) on the interval \( 0 \leq \tau \leq \tau_0 \). The procedure of separation of variables provides the following representation:

\[
\Psi_2(\tau, \theta) = a e^{r} \sum_{n=1}^{\infty} \frac{1}{\beta_n} \left( G_n \frac{\sinh \beta_n \theta}{\sin \beta_n \theta_0} \sin \theta + H_n \frac{\cosh \beta_n \theta}{\cosh \beta_n \theta_0} \cos \theta \right) \sin \beta_n \tau, \quad (10)
\]

with \( \beta_n = n \pi / \tau_0 \).

Therefore, from the very beginning the representation of the general solution (5) with (6) and (10) has sufficient sets of arbitrary coefficients \( A_m, B_m, C_m, D_m, G_n \) and \( H_n \) to satisfy the boundary conditions (4). The determination of these constants can be carried out in the following way.

The boundary condition \( \Psi = 0 \) at all walls leads to the algebraic relations between coefficients

\[
\begin{align*}
A_m r_0^{\alpha_m} + B_m r_0^{\alpha_m^2} + C_m + D_m &= 0, \\
A_m + B_m + C_m r_0^{\alpha_m} + D_m r_0^{\alpha_m^2} &= 0, \\
G_n \sin \theta_0 + H_n \cos \theta_0 &= 0,
\end{align*}
\]

with \( r_0 = a/b < 1 \).

The boundary conditions for the normal derivative of \( \Psi \) provide three functional equations:

\[
\begin{align*}
\sum_{m=1}^{\infty} (-1)^{m-1} \left[ \frac{A_m}{b} r_0^{\alpha_m-1} + \frac{B_m \alpha_m + 2}{b} \frac{r_0^{\alpha_m^2-1}}{\alpha_m} - \frac{C_m}{a} - \frac{D_m \alpha_m - 2}{a} \right] \cos \alpha_m \theta \\
+ \sum_{n=1}^{\infty} \left( G_n \frac{\sinh \beta_n \theta}{\sin \beta_n \theta_0} \sin \theta + H_n \frac{\cosh \beta_n \theta}{\cosh \beta_n \theta_0} \cos \theta \right) &= -V_{\text{bot}}(\theta), \quad |\theta| \leq \theta_0, \\
\sum_{m=1}^{\infty} (-1)^m \left[ \frac{A_m}{b} + \frac{B_m \alpha_m + 2}{b} \frac{r_0^{\alpha_m^2-1}}{\alpha_m} - \frac{C_m}{a} - \frac{D_m \alpha_m - 2}{a} \right] \cos \alpha_m \theta \\
+ \sum_{n=1}^{\infty} (-1)^n \left( G_n \frac{\sinh \beta_n \theta}{\sin \beta_n \theta_0} \sin \theta + H_n \frac{\cosh \beta_n \theta}{\cosh \beta_n \theta_0} \cos \theta \right) &= -V_{\text{top}}(\theta), \quad |\theta| \leq \theta_0, \\
- \sum_{m=1}^{\infty} \left[ \frac{A_m}{b} \frac{\alpha_m-1}{b} + \frac{B_m \alpha_m^2 + 2}{b} \frac{\alpha_m^2}{\alpha_m} - \frac{C_m}{a} \frac{\alpha_m^2 + 1}{r} - \frac{D_m \alpha_m - 2}{a} \frac{\alpha_m^2}{r} \right] \\
+ \sum_{n=1}^{\infty} \left[ G_n \left[ \coth \beta_n \theta_0 \sin \theta_0 + \frac{\cos \theta_0}{\beta_n} \right] \right] \\
+ H_n \left[ \tanh \beta_n \theta_0 \cos \theta_0 - \frac{\sin \theta_0}{\beta_n} \right] \sin \beta_n \tau = 0, \quad a \leq r \leq b, \quad 0 \leq \tau \leq \tau_0.
\end{align*}
\]

(12)
The algebraic equations (11) can be solved by putting

$$\begin{align*}
A_m &= -\frac{Y_m}{r_0} + X_m r_0^{\alpha_m}, \\
B_m &= \frac{Y_m}{r_0} - X_m r_0^{\alpha_m - 2}, \\
C_m &= -X_m + Y_m r_0^{\alpha_m - 1}, \\
D_m &= X_m - Y_m r_0^{\alpha_m + 1}, \\
G_n &= -Z_n \cos \theta_0, \\
H_n &= Z_n \sin \theta_0,
\end{align*}$$

(13)

with the new sets of constants $X_m$, $Y_m$ ($m = 1, 2, \ldots$) and $Z_n$ ($n = 1, 2, \ldots$).

Substituting these relations into the functional equations (12) and using the Fourier expansions

$$r^r = a^r e^{r \theta} = \frac{2a \gamma}{\tau_0} \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n^2 + \gamma^2} [1 - (-1)^n r_0^{-\gamma}] \sin \beta_n \tau,$$

$$\frac{\cosh \beta_n \theta}{\cosh \beta_n \theta_0} \cos \theta \sin \theta_0 - \frac{\sinh \beta_n \theta}{\sinh \beta_n \theta_0} \sin \theta \cos \theta_0$$

$$= \frac{4\beta_n p_n}{\theta_0} \sum_{m=1}^{\infty} (-1)^{n-1} \frac{\alpha_m}{[(\alpha_m - 1)^2 + \beta_n^2][\alpha_m + 1)^2 + \beta_n^2]} \cos \alpha_m \theta,$$

we obtain three infinite sets of linear non-homogeneous equations to determine the constants $X_m$, $Y_m$ and $Z_n$:

$$-X_m (1 - r_0^{\alpha_m}) + Y_m \alpha_m (r_0^{\alpha_m - 1} - r_0^{\alpha_m + 1})$$

$$- \frac{a}{\theta_0} \sum_{n=1}^{\infty} Z_n \sum_{m=1}^{\infty} \frac{2\alpha_m^2 \beta_n}{\beta_n^2 + (\alpha_m - 1)^2}[\beta_n^2 + (\alpha_m + 1)^2] = -\frac{a}{2} \alpha_m V_{m1}, \quad m = 1, 2, \ldots,$$

$$-Y_m (1 - r_0^{2\alpha_m}) + X_m \alpha_m (r_0^{\alpha_m - 1} - r_0^{\alpha_m + 1})$$

$$- \frac{a}{\theta_0} \sum_{n=1}^{\infty} (-1)^n Z_n \sum_{m=1}^{\infty} \frac{2\alpha_m^2 \beta_n}{\beta_n^2 + (\alpha_m - 1)^2}[\beta_n^2 + (\alpha_m + 1)^2]$$

$$= -\frac{a}{2} \alpha_m V_{m2}, \quad m = 1, 2, \ldots,$$

$$\sum_{m=1}^{\infty} [X_m + (-1)^n Y_m] [1 + r_0^{2\alpha_m} - (-1)^n (r_0^{\alpha_m - 1} + r_0^{\alpha_m + 1})]$$

$$\times \frac{4\beta_n^2 \alpha_m}{[(\alpha_m - 1)^2 + \beta_n^2][\alpha_m + 1)^2 + \beta_n^2]}$$

$$+ Z_n \frac{a \tau_0}{2} \left(1 + \beta_n \frac{\sin 2 \theta_0}{\sinh 2 \beta_n \theta_0}\right) = 0, \quad n = 1, 2, \ldots.$$

(14)
Here $V_{m}^{\text{bot}}$ and $V_{m}^{\text{top}}$ are the coefficients of the Fourier series of the functions $V_{\text{bot}}(\theta)$ and $V_{\text{top}}(\theta)$,

$$
\begin{align*}
V_{\text{bot}}(\theta) &= \sum_{m=1}^{\infty} (-1)^{m-1} V_{m}^{\text{bot}} \cos \alpha_{m} \theta, \\
V_{m}^{\text{bot}} &= \frac{(-1)^{m-1}}{\theta_{0}} \int_{-\theta_{0}}^{\theta_{0}} V_{\text{bot}}(\theta) \cos \alpha_{m} \theta \, d\theta, \\
V_{\text{top}}(\theta) &= \sum_{m=1}^{\infty} (-1)^{m-1} V_{m}^{\text{top}} \cos \alpha_{m} \theta, \\
V_{m}^{\text{top}} &= \frac{(-1)^{m-1}}{\theta_{0}} \int_{-\theta_{0}}^{\theta_{0}} V_{\text{top}}(\theta) \cos \alpha_{m} \theta \, d\theta.
\end{align*}
$$

(15)

After solving the infinite system (14), the stream function can be calculated as

$$
\Psi(r, \theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\alpha_{m}} \left[ X_{m} \left( \frac{r^{2}}{a^{2}} - 1 \right) \left( \frac{\alpha_{m}}{r} \right) - r_{0}^{\alpha_{m}} \left( \frac{r}{b} \right)^{\alpha_{m}} \right] \\
- \frac{Y_{m}}{r_{0}} \left( 1 - \frac{r^{2}}{b^{2}} \right) \left[ \left( \frac{r}{b} \right)^{\alpha_{m}} - r_{0}^{\alpha_{m}} \left( \frac{a}{r} \right)^{\alpha_{m}} \right] \cos \alpha_{m} \theta \\
+ r \sum_{n=1}^{\infty} Z_{n} \left[ \frac{\cosh \beta_{n} \theta}{\cosh \beta_{n} \theta_{0}} \cos \theta \sin \theta_{0} - \frac{\sinh \beta_{n} \theta}{\sinh \beta_{n} \theta_{0}} \sin \theta \cos \theta_{0} \right] \frac{\sin \beta_{n} \tau}{\beta_{n}}.
$$

(16)

Thus, expression (16) represents the formal analytical solution of the problem in question. In order to find whether this solution can be used to determine accurately the velocity field in the cavity, especially near the boundaries, we must examine the convergence of the solution of the infinite system (14).

3. Reduction of the infinite system

The structure of the infinite system (14) is such that it is convenient to introduce the new unknowns $\bar{X}_{m}, \bar{Y}_{m}, \bar{Z}_{n}$ as

$$
\bar{X}_{m} = -(X_{m} - Y_{m})(1 + r_{0}^{\alpha_{n}-1})(1 + r_{0}^{\alpha_{n}+1}), \\
\bar{Y}_{m} = -(X_{m} + Y_{m})(1 - r_{0}^{\alpha_{n}-1})(1 - r_{0}^{\alpha_{n}+1}), \\
\bar{Z}_{n} = Z_{n} p_{n} \frac{\alpha_{p_{0}}}{2 \theta_{0}}
$$

and to consider instead of the first two sets (14) their difference and sum and to split the third set (14) into two subsets with odd and even $n$. In such a way we finally obtain two separate infinite systems

$$
\begin{align*}
\bar{X}_{m} s_{m} - \sum_{k=1}^{\infty} g_{m,2k-1} \bar{Z}_{2k-1} &= \frac{a}{2} \alpha_{m} (V_{m}^{\text{bot}} - V_{m}^{\text{top}}), & m &= 1, 2, \ldots, \\
\bar{Z}_{2k-1} \Delta_{2k-1} - \sum_{m=1}^{\infty} h_{2k-1,m} \bar{X}_{m} &= 0, & k &= 1, 2, \ldots
\end{align*}
$$

(18a)
and

\[
Y_m t_m - \sum_{k=1}^{\infty} g_{m,2k} \bar{Z}_{2k} = \frac{a}{2} \alpha_m (V_m^{\text{bot}} + V_m^{\text{top}}), \quad m = 1, 2, \ldots,
\]

\[
\bar{Z}_{2k} \Delta_{2k} - \sum_{m=1}^{\infty} h_{2k,m} Y_m = 0, \quad k = 1, 2, \ldots,
\]

(18b)

where the following notation is introduced:

\[
g_{m,n} = \frac{8a^2 \beta_n}{\tau_0[(a_m - 1)^2 + \beta_n^2][(a_m + 1)^2 + \beta_n^2]}
\]

\[
h_{n,m} = \frac{4a^2 \beta_n^2}{\tau_0[(a_m - 1)^2 + \beta_n^2][(a_m + 1)^2 + \beta_n^2]}
\]

\[
\Delta_n = \frac{1 + \beta_n \sin 2\theta_0 / \sinh 2\beta_n \theta_0}{\tanh \beta_n \theta_0 + 2 \cos^2 \theta_0 / \sinh 2 \beta_n \theta_0}
\]

The main reason for this transformation is to obtain a fully regular infinite system—a system for which in each row the sum of the modulae of the non-diagonal coefficients is less than the diagonal coefficient minus some positive fixed value (see (25) for a detailed review of the subject). Using the following expressions (26, pp. 105, 106):

\[
\sum_{k=1}^{\infty} \frac{1}{k(k^2 x^2 + y^2)} = \frac{1}{2y^2} \left[ \psi\left(1 + \frac{i y}{x}\right) + \psi\left(1 - \frac{i y}{x}\right) - 2\psi(1) \right],
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(2k + 1)((2k + 1)^2 x^2 + y^2)} = \frac{1}{4y^2} \left[ \psi\left(\frac{1}{2} + \frac{i y}{2x}\right) + \psi\left(\frac{1}{2} - \frac{i y}{2x}\right) - 2\psi\left(\frac{1}{2}\right) \right],
\]

\[
\sum_{k=1}^{\infty} \frac{1}{(kx + y)^2 + c^2} = \frac{1}{2xc} \left[ \psi\left(\frac{y + ic}{x}\right) - \psi\left(\frac{y - ic}{x}\right) \right]
\]
(where \( \psi(z) = d[\ln \Gamma(z)]/dz \) is the psi (digamma) function), we can write for the systems (18):

\[
S_m = \sum_{k=1}^{\infty} g_{m,2k-1} = \frac{\alpha_m}{2\pi} \left[ \left( \psi\left(\frac{1}{2} + ia_m\right) + \psi\left(\frac{1}{2} - ia_m\right) \right) - \left( \psi\left(\frac{1}{2} + i\alpha_m\right) + \psi\left(1 - i\alpha_m\right) \right) \right],
\]

\[
T_m = \sum_{k=1}^{\infty} g_{m,2k} = \frac{\alpha_m}{2\pi} \left[ \left( \psi\left(1 + ia_m\right) + \psi\left(1 - ia_m\right) \right) - \left( \psi\left(1 + i\alpha_m\right) + \psi\left(1 - i\alpha_m\right) \right) \right],
\]

\[
Q_n = \sum_{m=1}^{\infty} h_{n,m} = \frac{i\beta_n}{2\pi} \left[ \left( \psi\left(d(+) - i\beta_n\right) - \psi\left(d(+) + i\beta_n\right) \right) \right],
\]

\[
- \left[ \psi\left(\frac{d(-) - i\beta_n}{c}\right) - \psi\left(\frac{d(-) + i\beta_n}{c}\right) \right] = \frac{2}{\pi} + O(\alpha_m^{-2}), \quad \text{when } m \to \infty,
\]

\[
- \left[ \psi\left(\frac{d(-) - i\beta_n}{c}\right) - \psi\left(\frac{d(-) + i\beta_n}{c}\right) \right] = \frac{2}{\pi} + O(\alpha_m^{-2}), \quad \text{when } n \to \infty.
\]

Here the notation

\[
a_m^{(+)} = \frac{\alpha_m + 1}{2\pi} \tau_0, \quad a_m^{(-)} = \frac{\alpha_m - 1}{2\pi} \tau_0, \quad d^{(+)} = \frac{c}{2} + 1,
\]

\[
d^{(-)} = \frac{c}{2} - 1, \quad c = \frac{\pi}{\theta_0}
\]

is introduced and the asymptotic expansions (27)

\[
\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + O(z^{-3}), \quad \ln \frac{z + 1}{z - 1} = \frac{2}{z} + \frac{2}{3z^3} + O(z^{-5}),
\]

\[|z| \to \infty, \ \arg z \neq \pi\]

are used.

Now, either applying tables (27) for the digamma function for direct evaluation of \( S_m, T_m, Q_n \) for small values of \( m \) and \( n \) or taking into account their asymptotic behaviour according to (19) to (21) as well as the asymptotics

\[
s_m = 1 + O(\alpha_m r_0^{\alpha_m}), \quad t_m = 1 + O(\alpha_m r_0^{\alpha_m}), \quad \Delta_n = 1 + O(\beta_n e^{-2\beta_n \theta_0}),
\]

for large values of \( m \) and \( n \) we obtain

\[
s_m - S_m > \delta, \quad t_m - T_m > \delta, \quad \Delta_n - Q_n > \delta,
\]

with \( \delta = 1 - 2/\pi > 0 \).

Restricting further consideration to the case for which the functions \( V_{bot}(\theta) \) and \( V_{top}(\theta) \) are continuous up to the second derivative, we can write
the Fourier coefficients, using the procedure of integration by parts, in the form
\[ V^\text{bot}_m = \frac{2V^\text{bot}_0(\theta_0)}{\theta_0 \alpha_m} + \tilde{V}^\text{bot}_m, \quad V^\text{top}_m = \frac{2V^\text{top}_0(\theta_0)}{\theta_0 \alpha_m} + \tilde{V}^\text{top}_m, \tag{23} \]

where
\[ \tilde{V}^\text{bot}_m = \frac{(-1)^m}{\theta_0 \alpha_m^2} \int_{-\theta_0}^{\theta_0} V^\text{bot}_0(\theta) \cos \alpha_m \theta \, d\theta, \]
\[ \tilde{V}^\text{top}_m = \frac{(-1)^m}{\theta_0 \alpha_m^2} \int_{-\theta_0}^{\theta_0} V^\text{top}_0(\theta) \cos \alpha_m \theta \, d\theta, \]
\[ \tilde{V}^\text{bot}_m = O(\alpha_m^{-3}), \quad \tilde{V}^\text{top}_m = O(\alpha_m^{-3}), \quad m \to \infty. \]

Therefore, the right-hand side terms in the systems (18) are bounded. Consequently, taking into account (22), we may conclude that these systems are fully regular. It was proved (25) that the unique bounded solution of the fully regular systems can be found by the method of reduction, namely, considering first the finite systems of \( M + K \) unknowns \( X_m, Z_{2k-1} \) and \( Y_m, Z_{2k} \), and putting
\[
\begin{align*}
X_m &= 0, \quad m > M, \\
Y_m &= 0, \quad m > M, \\
Z_{2k-1} &= 0, \quad k > K, \\
Z_{2k} &= 0, \quad k > K.
\end{align*}
\tag{24}
\]

Obviously, the values of the unknowns \( X^{(M)}_m, Y^{(M)}_m \) \((m = 1, 2, \ldots, M)\) and \( Z_{2k-1}^{(K)}, Z_{2k}^{(K)} \) \((k = 1, 2, \ldots, K)\)—the solutions of these finite systems—can vary as we increase the numbers \( M \) and \( K \). These values tend to the fixed values \( X_m, Y_m \) \((m = 1, 2, \ldots)\) and \( Z_{2k-1}, Z_{2k} \) \((k = 1, 2, \ldots)\) when \( M \to \infty \) and \( K \to \infty \), and these limits become the true values of the unknowns which satisfy the infinite systems (18).

However, there exists a more important question which was first pointed out for such a system in a remarkable paper of Koialovich (28): can we have a priori knowledge about the asymptotic behaviour of the unknowns \( X_m, Y_m, Z_{2k-1}, Z_{2k} \) when \( m \to \infty \) and \( k \to \infty \)? The importance of this question is obvious: the construction of expression (16) for the stream function is such that it represents the biharmonic function for any values of the Fourier coefficients. Therefore, the only criterion for the correctness of the solution is the accuracy of satisfying the prescribed boundary conditions. The knowledge of the asymptotic behaviour of the Fourier coefficients allows for considerable improvement of the convergence of all series at the boundary by extracting the poorly converging part and writing it down in a closed form. Of course, such knowledge is less important inside the domain: the radial and angular functions in the expression (16) become negligibly small with the increase in the numbers \( m \) and \( n \), respectively. Therefore, it is believed that any small variation in the coefficients \( X_m, Y_m, Z_{2k}, Z_{2k-1} \)
(even putting them all equal to zero when \( m > M \) and \( k > K \)) will not change the main field inside the domain significantly. Moreover, it was found that increasing the numbers \( M \) and \( K \) does not significantly affect the coefficients with small suffixes. In any case, the question of accuracy of satisfying the boundary conditions remains unanswered.

It follows from (23) that if the value of \( V_{\text{bot}}(\theta_0) \) and/or \( V_{\text{top}}(\theta_0) \) is not equal to zero, the right-hand side terms in the infinite systems (18) tend to constant values. Although this fact does not contradict the general theorem (25), experience shows that the finite systems must contain hundreds of equations to obtain stable values, even for the first half of the unknowns. In order to get more convenient systems, (18) is rewritten using:

\[
\begin{align*}
\bar{X}_m &= X + \bar{x}_m, \\
\bar{Y}_m &= Y + \bar{y}_m, \\
\bar{Z}_{2k-1} &= Z_{\text{odd}} + \bar{z}_{2k-1}, \\
\bar{Z}_{2k} &= Z_{\text{even}} + \bar{z}_{2k},
\end{align*}
\] (25)

with some arbitrary (for the moment) constants \( X, Y, Z_{\text{odd}}, Z_{\text{even}} \). Then the following systems are obtained:

\[
\begin{align*}
\bar{x}_m - \sum_{k=1}^{\infty} g_{m,2k-1} \bar{z}_{2k-1} &= \left\{ \frac{a}{\theta_0} [V_{\text{bot}}(\theta_0) - V_{\text{top}}(\theta_0)] - X + \frac{2}{\pi} Z_{\text{odd}} \right\} \\
&+ \frac{a}{2} \alpha_m (V_{\text{bot}} - \bar{V}_m) + X (1 - s_m) - Z_{\text{odd}} \left( \frac{2}{\pi} - S_m \right), \\
\bar{z}_{2k-1} \Delta_{2k-1} - \sum_{m=1}^{\infty} h_{2k-1,m} \bar{x}_m &= \left\{ \frac{2}{\pi} X - Z_{\text{odd}} \right\} + X \left[ Q_{2k-1} - \frac{2}{\pi} \right] + Z_{\text{odd}}[1 - \Delta_{2k-1}], \\
&\text{(26a)}
\end{align*}
\]

\[
\begin{align*}
\bar{y}_m - \sum_{k=1}^{\infty} g_{m,2k} \bar{z}_{2k} &= \left\{ \frac{a}{\theta_0} [V_{\text{bot}}(\theta_0) + V_{\text{top}}(\theta_0)] - Y + \frac{2}{\pi} Z_{\text{even}} \right\} \\
&+ \frac{a}{2} \alpha_m (\bar{V}_m - V_{\text{top}}) + Y (1 - t_m) - Z_{\text{even}} \left( \frac{2}{\pi} - T_m \right), \\
\bar{z}_{2k} \Delta_{2k} - \sum_{m=1}^{\infty} h_{2k,m} \bar{y}_m &= \left\{ \frac{2}{\pi} Y - Z_{\text{even}} \right\} + Y \left[ Q_{2k} - \frac{2}{\pi} \right] + Z_{\text{even}}[1 - \Delta_{2k}], \\
&\text{(26b)}
\end{align*}
\]

which still remain fully regular.

In these infinite systems we equate the constant expressions in braces to zero, and, consequently, choose

\[
\begin{align*}
X &= \frac{\pi^2 a}{(\pi^2 - 4) \theta_0} [V_{\text{bot}}(\theta_0) - V_{\text{top}}(\theta_0)], \\
Z_{\text{odd}} &= \frac{2}{\pi} X, \\
Y &= \frac{\pi^2 a}{(\pi^2 - 4) \theta_0} [V_{\text{bot}}(\theta_0) + V_{\text{top}}(\theta_0)], \\
Z_{\text{even}} &= \frac{2}{\pi} Y,
\end{align*}
\] (27)
to establish that the rest of the right-hand-side terms in the systems (26) decrease at least as $O(\alpha_m^{-2})$ and $O(\beta_n^{-2})$, in accordance with the asymptotic behaviour of $S_m$, $T_m$, $Q_n$.

Next, usage Mellin's transform method (see (29, 30) for additional details) for the analysis of these systems provides the following asymptotic behaviour:

$$\begin{align*}
\bar{x}_m &= \frac{E_0}{\alpha_m} + \text{Re} \left( \frac{E_1}{\alpha_m^{\gamma_1+1}} \right) + o(\alpha_m^{-\text{Re} \gamma_1-1}), \quad m \to \infty, \\
\bar{z}_{2k-1} &= \frac{E_0}{\beta_{2k-1}} - \text{Re} \left( \frac{E_1}{\beta_{2k-1}^{\gamma_1+1}} \right) + o(\beta_{2k-1}^{-\text{Re} \gamma_1-1}), \quad k \to \infty, \\
\bar{y}_m &= \frac{F_0}{\alpha_m} + \text{Re} \left( \frac{F_1}{\alpha_m^{\gamma_1+1}} \right) + o(\alpha_m^{-\text{Re} \gamma_1-1}), \quad m \to \infty, \\
\bar{z}_{2k} &= \frac{F_0}{\beta_{2k}} - \text{Re} \left( \frac{F_1}{\beta_{2k}^{\gamma_1+1}} \right) + o(\beta_{2k}^{-\text{Re} \gamma_1-1}), \quad k \to \infty,
\end{align*}$$

where

$$E_0 = \frac{\pi}{4\theta_0} [aV'_{\text{bot}}(\theta_0) + bV'_{\text{top}}(\theta_0)], \quad F_0 = \frac{\pi}{4\theta_0} [aV'_{\text{bot}}(\theta_0) - bV'_{\text{top}}(\theta_0)],$$

and $E_1$, $F_1$ are complex constants and $\gamma_1 = 1.739593 + i1.119025$ is the root with the lowest positive real part of the transcendental equation

$$\cos \frac{\pi \gamma}{2} + \gamma + 1 = 0. \quad (30)$$

The asymptotic behaviour (28) of the unknowns $\bar{x}_m$, $\bar{z}_{2k-1}$, $\bar{y}_m$, $\bar{z}_{2k}$ together with the proper decrease of the free terms in the infinite systems (26) provides a considerable improvement of the method of reduction. Namely, putting in all infinite series of the systems (18)

$$\begin{align*}
\bar{X}_m &= X + \frac{E_0}{\alpha_m}, \quad m > M, \quad \bar{Z}_{2k-1} = Z_{\text{odd}} + \frac{E_0}{\beta_{2k-1}}, \quad k > K, \\
\bar{Y}_m &= Y + \frac{F_0}{\alpha_m}, \quad m > M, \quad \bar{Z}_{2k} = Z_{\text{even}} + \frac{F_0}{\beta_{2k}}, \quad k > K,
\end{align*}$$

and solving the finite systems of $M + K$ equations, we obtain the values of all coefficients that, in general, are not equal to zero.

Thus, we conclude that, although the method of reduction expressed by (24) is valid in principle when $M \to \infty$ and $K \to \infty$, it cannot provide the accurate values for the unknowns with large suffixes after solving the finite systems. The improved reduction approach (31) gives us, after solving the finite systems, the accurate values for all coefficients in the Fourier series.
The difference between the two approaches becomes even more clear when the expressions for the stream function and velocity components are considered.

If we want to have a more precise estimate than (31) for the coefficients with large suffixes, we have to take into account the next terms in the asymptotic expansions (28). The equations for defining the (complex) constants $E_1$ and $F_1$ can be written as

$$\begin{align*}
\text{Re} \left( \frac{E_1}{\alpha_j^{+1}} \right) &= X_M - \frac{E_0}{\alpha_M}, \\
\text{Re} \left( \frac{F_1}{\alpha_j^{+1}} \right) &= Y_M - \frac{F_0}{\alpha_M}, \\
\text{Re} \left( \frac{E_1}{\beta_j^{+1}} \right) &= -\bar{Z}_{2K-1} + Z_{\text{odd}} + \frac{E_0}{\beta_{2K-1}}, \\
\text{Re} \left( \frac{F_1}{\beta_j^{+1}} \right) &= -\bar{Z}_{2K} + Z_{\text{even}} + \frac{F_0}{\beta_{2K}}.
\end{align*}$$

(32)

These values are important for finding the amplitude of Moffatt eddies near the corners of the cavity.

4. Stream function and velocity field

For simplicity, we restrict our further considerations to the important case of uniform constant velocities at the walls

$$V_{\text{bot}}(\theta) = V_a, \quad V_{\text{top}}(\theta) = V_b.$$  

(33)

More general velocity distributions at the boundaries (when, in particular, $V_{\text{bot}}(\theta_0) \neq 0$ and/or $V_{\text{top}}(\theta_0) \neq 0$) can be treated in a similar manner.

First we consider expression (16) for which the finite number of bounded coefficients are defined by the simple reduction method (24). It is easily concluded that the Fourier series for the velocity components converge very poorly at the boundary, and, moreover, taken separately they diverge at the corner points $r = a$, $r = b$, $|\theta| = \theta_0$.

The solution of the finite reduced systems corresponding to (18) using the improved reduction approach (31) provides the asymptotic behaviour at infinity for all coefficients in the Fourier series (16). Thus we have the possibility of considerably improving the convergence of these series for the stream function and velocity components.

Using equations (17) the coefficients $X_m, Y_m, Z_n$ can be written as

$$X_m = R_a + x_m, \quad Y_m = r_0 R_b + r_0 y_m, \quad Z_n = F_a + (-1)^n F_b + z_n,$$

(34)

with

$$\begin{align*}
R_b &= -\frac{\pi^2 b V_b}{\theta_0(\pi^2 - 4)}, \quad R_a = -\frac{\pi^2 a V_a}{\theta_0(\pi^2 - 4)}, \\
F_b &= \frac{4\pi V_b}{\tau_0(\pi^2 - 4)}, \quad F_a = \frac{4\pi V_a}{\tau_0(\pi^2 - 4)}.
\end{align*}$$

(35)
The expressions for \( x_m, y_m \) and \( z_n \) are not written out explicitly; they are long but straightforward. Substituting (34) into (16) and separating some series with \( R_a, R_b, F_a, F_b \) we can transform the expression for the stream function into

\[
\Psi(r, \theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\alpha_m} \left\{ \left( \frac{r^2}{a^2} - 1 \right) x_m \left( \frac{a}{r} \right)^{\alpha_m} - X_m r \left( \frac{a}{b} \right)^{\alpha_m} \right\} \\
- \left( 1 - \frac{r^2}{b^2} \right) \left\{ y_m \left( \frac{r}{b} \right)^{\alpha_m} - Y_m r \left( \frac{a}{b} \right)^{\alpha_m} \right\} \cos \alpha_m \theta + r \sum_{n=1}^{\infty} Z_n P_n(\theta) \frac{\sin \beta_n \tau}{\beta_n}
\]

\[
+ R_a \left( \frac{r^2}{a^2} - 1 \right) s \left( \frac{a}{r}, \theta \right) - R_b \left( 1 - \frac{r^2}{b^2} \right) s \left( \frac{r}{b}, \theta \right)
+ rF_a P_+ (\tau, \theta) + rF_b P_- (r, \theta),
\]

where (31, p. 16)

\[
S(\eta, \theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\alpha_m} \eta^{\alpha_m} \cos \alpha_m \theta = \frac{\theta_0}{\pi \tan \left[ \frac{2 \eta \pi}{2 \theta_0} \cos \left( \pi \theta/2 \theta_0 \right) \right]}
\]

and

\[
P_+(\tau, \theta) = \sum_{n=1}^{\infty} P_n(\theta) \frac{\sin \beta_n \tau}{\beta_n}
\]

\[
P_-(\tau, \theta) = \sum_{n=1}^{\infty} \frac{(-1)^n P_n(\theta)}{\beta_n} \frac{\sin \beta_n \tau}{\beta_n}
\]

\[
= \sin \left( \theta_0 - \theta \right) \sum_{v=0}^{\infty} \left[ S_1(\tau, \gamma_v - \theta) - S_1(\tau, \delta_v + \theta) \right]
+ \sin \left( \theta_0 + \theta \right) \sum_{v=0}^{\infty} \left[ S_1(\tau, \gamma_v + \theta) - S_1(\tau, \delta_v - \theta) \right],
\]

with \( \gamma_v = (4v + 1)\theta_0, \delta_v = (4v + 3)\theta_0. \)

The identities for \( P_+ (\tau, \theta) \) and \( P_- (\tau, \theta) \), providing rapidly convergent series in \( v \), can be obtained by using the expansions

\[
\frac{1}{\cosh \xi} = 2 \sum_{v=0}^{\infty} (-1)^v e^{-(2v+1)\xi}, \quad \frac{1}{\sinh \xi} = 2 \sum_{v=0}^{\infty} e^{-(2v+1)\xi},
\]
changing the sequence of summation in $n$ and $v$ and taking into account the expressions for Fourier series (31, p. 16):

$$S_1(\tau, \xi) = \sum_{n=1}^{\infty} e^{-\beta_n \tau} \sin \frac{\beta_n \tau}{\beta_n} \frac{\tau_0}{\pi} \arctan \frac{\sin (\pi \tau/\tau_0)}{\pi e^{\pi i/\tau_0} - \cos (\pi \tau/\tau_0)},$$

$$S_2(\tau, \xi) = \sum_{n=1}^{\infty} (-1)^n e^{-\beta_n \tau} \sin \frac{\beta_n \tau}{\beta_n} \frac{\tau_0}{\pi} \arctan \frac{\sin (\pi \tau/\tau_0)}{\pi e^{\pi i/\tau_0} + \cos (\pi \tau/\tau_0)}.$$

Now the terms in the series for $P_+(\tau, \theta)$ and $P_-(\tau, \theta)$ decay exponentially with $v$, even at the boundary, and in practical calculations it appears sufficient to use only a few of the first terms (typically five only).

Due to the factor $r_0^\alpha$ and the asymptotic behaviour

$$x_m = O(\alpha_m^{-\alpha_1}), \quad y_m = O(\alpha_m^{-\alpha_1}), \quad m \to \infty,$$

$$z_n = O(\beta_n^{-\alpha_1}), \quad n \to \infty,$$

all Fourier series in (36) converge uniformly and absolutely in the whole domain $a \leq r \leq b$, $|\theta| \leq \theta_0$ along with their first and second derivatives. Therefore, it is sufficient to leave only the first $M$ and $N = 2K$ terms when performing numerical simulations.

The function $S(\eta, \theta)$ is infinitely differentiable when $r_0 \leq \eta < 1$. In order to calculate the input of this function into the velocity components at the parts of the boundary $r = a$ or $r = b$, where $\eta = 1$, it is necessary to take the first derivative with respect to $r$ and then to consider some obvious limits. After the transformations (37) we can differentiate term by term the series in $v$ for $P_+(\tau, \theta)$ and $P_-(\tau, \theta)$, taking into account some obvious limits for the terms with $v = 0$ at the boundary.

Now, the components of the velocity can be calculated as the corresponding first derivatives of the stream function (36). Thus, one obtains a very rapidly converging series in the whole domain, including the boundary, for the stream function and velocity field and numerical calculations do not show any problems.

The accuracy of fulfilling the boundary conditions is the only way to estimate the quality of the whole representation of the biharmonic stream function for the Stokes flow in the annular cavity. Table 1 presents the tangential velocity $u_\theta$ at the curved boundaries for the case of a cavity with $\theta_0 = \pi/4$ and $r_0 = a/b = 0.5$ and $V_a = 0.5$, $V_b = 1$. The summations in the Fourier series were performed for different values of $M$ and $N$. The results show that the boundary conditions can be well satisfied even for $M = 3$ and $N = 6$ ($K = 3$). It is worth noting that the terms with $R_a$, $R_b$, $F_a$, $F_b$ corresponding to $M = 0$, $N = 0$ in (36) accurately describe the discontinuities of $u_\theta$ at the corner points $r = a$, $r = b$, $|\theta| = \theta_0$ but are inaccurate at $\theta = 0$.

The analytical expression (36) for the stream function also provides information about points at the boundaries where the zero streamline is
Table 1. Tangential velocity $u_\theta$ on the bottom ($r = a$) and top ($r = b$) walls for the uniform velocity, for various numbers $(M, N)$ of coefficients in the Fourier series. $V_a = 0.5$, $V_b = 1.0$

<table>
<thead>
<tr>
<th>$\theta/\theta_0$</th>
<th>$u_\theta(a, \theta)$</th>
<th>$u_\theta(b, \theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>(0, 0)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>0.20</td>
<td>0.376</td>
<td>0.463</td>
</tr>
<tr>
<td>0.40</td>
<td>0.451</td>
<td>0.504</td>
</tr>
<tr>
<td>0.60</td>
<td>0.518</td>
<td>0.552</td>
</tr>
<tr>
<td>0.80</td>
<td>0.562</td>
<td>0.577</td>
</tr>
<tr>
<td>1.00</td>
<td>0.500</td>
<td>0.500</td>
</tr>
</tbody>
</table>

$u_e(a, \theta)$ $u_e(b, \theta)$ $u_e(0, 0)$ $u_e(1, 2)$ $u_e(3, 6)$ $u_e(7, 14)$ $u_e(0, 0)$ $u_e(1, 2)$ $u_e(3, 6)$ $u_e(7, 14)$

0.00 0.376 0.463 0.496 0.500 1.794 0.953 0.997 1.000
0.20 0.401 0.472 0.502 0.500 1.755 0.967 1.001 1.000
0.40 0.451 0.504 0.502 0.500 1.673 1.010 1.002 1.000
0.60 0.518 0.552 0.495 0.500 1.537 1.060 0.997 1.000
0.80 0.562 0.577 0.504 0.500 1.324 1.077 1.003 1.000
1.00 0.500 0.500 0.500 0.500 1.000 1.000 1.000 1.000

These separation points define the existence of cellular structures which are often observed experimentally in streamline patterns. The positions of these points at the boundaries of the cavity as well as the angles of inclination of the dividing zero streamline to the wall can be found in the following way.

Let the point $W(r_w, \theta_0)$ on the right-hand wall be the point of separation of the zero streamline (Fig. 1). Let us expand the stream function $\Psi(r, \theta)$ near this point into the Taylor series in $\Delta r = r_w - r$, $\Delta \theta = \theta_0 - \theta$. Using the definition (2) of the velocity components, we can write

$$\Psi(r_w - \Delta r, \theta_0 - \Delta \theta) = \Psi(r_w, \theta_0) + [u_\theta|_w \Delta r - (ru_r)|_w \Delta \theta]$$

$$+ \frac{1}{2} \left[ - \frac{\partial u_\theta}{\partial r} \right]_w (\Delta r)^2 - 2 \frac{\partial (ru_r)}{\partial r} \left|_w \Delta r \Delta \theta + \frac{\partial (ru_r)}{\partial \theta} \right|_w (\Delta \theta)^2$$

$$+ \frac{1}{6} \left[ - \frac{\partial^2 u_\theta}{\partial r^2} \right]_w (\Delta r)^3 + 3 \frac{\partial^2 u_\theta}{\partial r \partial \theta} \left|_w (\Delta r)^2 \Delta \theta$$

$$+ 3 \frac{\partial^2 u_\theta}{\partial \theta^2} \left|_w \Delta r (\Delta \theta)^2 - \frac{\partial^2 (ru_r)}{\partial \theta^2} \right|_w (\Delta \theta)^3 \right] + O((\Delta r)^4, (\Delta \theta)^4).$$

(38)

Now, the components of the velocities $u_r$, $u_\theta$ on the wall as well as their partial derivatives with respect to $r$ are equal to zero. This is used in (38), and if the point $(r - \Delta r, \theta_0 - \Delta \theta)$ then belongs to the zero streamline, the following equations must hold:

$$\frac{\partial u_r}{\partial \theta} \bigg|_w = 0, \quad 3 \frac{\partial^2 u_\theta}{\partial \theta^2} \bigg|_w \Delta r - \frac{\partial^2 (ru_r)}{\partial \theta^2} \bigg|_w \Delta \theta = 0.$$

(39)

The first equation defines the value of $r_w$ and the second provides the value of the inclination angle as

$$\tan \chi_w = \frac{r_w \Delta \theta}{\Delta r} = 3 \frac{\partial^2 u_\theta}{\partial r^2} \bigg|_w \left/ \frac{\partial^2 u_r}{\partial \theta^2} \bigg|_w \right..$$

(40)
Due to the proper convergence of the Fourier series, the first and the second derivatives of the velocity components at the boundary can be calculated by term-by-term differentiation.

Substitution of the local coordinates $\Delta r = \rho_1 \cos \chi_1$, $r_w \Delta \theta = \rho_1 \sin \chi_1$ into the expression (38) leads to the expansion

$$
\Psi(\rho_1, \chi_1) = \rho_1^2 [C(\cos \chi_1 - \cos 3\chi_1) + D(\sin \chi_1 - \frac{1}{3} \sin 3\chi_1)],
$$

where

$$
C = \frac{1}{8r_w^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \bigg|_w, \quad D = -\frac{1}{8r_w^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \bigg|_w.
$$

This local expression coincides with the solution found by Rayleigh (32) for the half-plane and also supports the opinion of Michael and O'Neill (33) about the strongest mode of separation at a smooth boundary.

In a similar way, if $V_a = 0$ the positions of the point $U(a, \theta_a)$ and the angle $\chi_a$ of the dividing streamlines at the bottom boundary are defined by the equations

$$
\frac{\partial u_\theta}{\partial r} \bigg|_U = 0, \quad \tan \chi_a = 3 \frac{\partial^2(r u_\theta)}{\partial r^2} \bigg|_U \left/ \frac{\partial^2 u_\theta}{\partial r^2} \right|_U.
$$

Finally, the analytical expression (36) for the stream function provides an opportunity to analyse the local behaviour near the corners. For example, introducing the local polar coordinates $(\rho, \chi)$ near the lower right corner point $O(a, \theta_0)$ (see Fig. 1) as

$$
\rho \cos \chi = r \sin (\theta_0 - \theta), \quad \rho \sin \chi = r \cos (\theta_0 - \theta) - a,
$$

and expressing $r$ and $\theta$ in terms of $\rho$ and $\chi$, substituting them into (36) and expanding all the Fourier series in $\rho$ (directly, or based on Mellin transformation techniques (34)), we obtain

$$
\Psi = \rho \frac{4V_a}{\pi^2 - 4} \left[ \chi \cos \chi - \frac{\pi}{2} \left( \frac{\pi}{2} - \chi \right) \sin \chi \right]
$$

$$
+ \text{Re} \left\{ \rho^{\gamma_1 + 2} A_M \left[ \sin \left( (\gamma_1 + 1) \left( \frac{\pi}{2} - \chi \right) \right) \sin \chi + \sin (\gamma_1 + 1) \chi \cos \chi \right] \right\}
$$

$$
+ O(\rho^4),
$$

with $A_M = E_1 \Gamma(-\gamma_1 - 1)/\pi$. The principal point of our approach is the explicit value $A_M$ of the eigenmode, which is based upon a (stable) solution of the above-mentioned finite systems.

The linear term in $\rho$ in this expansion is formed by those parts in (36) which are connected with the constants $R_a$ and $F_a$. It represents the well-known solution by Goodier (5) and Taylor (24) for Stokes flow in the wedge with a constant tangential velocity along the side $\chi = 0$. This term corresponds to the discontinuous tangential velocity at the corner points of
the annular cavity. Of course, this discontinuity at the boundary does not violate the continuity of the velocity field near the corner inside the cavity.

The relative competition of the Goodier–Taylor flow (the term linear in \( p \) in (43)) and the Moffatt eddies (the term in the curly braces) determines the distance from the corner at which free eddies appear. This situation is the same as the one considered by Jeffrey and Sherwood (9), who studied the streamlines in an infinite corner with one wall sliding parallel to itself at speed \( V_a \).

5. Streamline patterns for the quarter annular cavity

The general solution (36) for Stokes flow in an annular cavity shows that the biharmonic stream function \( \Psi(r, \theta) \) is governed by three dimensionless parameters \( r_0 = a/b, \theta_0 \) and \( v = V_a/V_b \). Rather than analysing the whole range of these parameters, we restrict our consideration to the case of \( a = 1, b = 2 \) \( (r_0 = 0.5) \), \( \theta_0 = \frac{1}{4}\pi \) and study the influence of the ratio \( v \) on the streamline patterns. The primary interest is connected with the general structure of the streamlines and the evolution process of the corner Moffatt eddies and their relation to the main eddy at the centre of the cavity.

To illustrate the typical tangential velocity distribution we have plotted in Fig. 2a three velocity profiles \( u_\theta(r) \) along the centreline \( \theta = 0 \). Two ‘basic’ profiles \( u_\theta^{\text{b}}(r) \) and \( u_\theta^{\text{op}}(r) \) in this figure correspond to the boundary

![Fig. 2a. Tangential velocity profiles \( u_\theta(r) \) along the centreline \( \theta = 0 \). The solid line corresponds to the boundary velocities \( V_a = 0.5 \) and \( V_b = 1.0 \) \((v = 0.5)\), the dashed line describes the profile \( u_\theta^{\text{op}}(r) \), with \( V_a = 0 \) and \( V_b = 1 \) \((v = 0)\) and the dotted line represents the profile \( u_\theta^{\text{b}}(r) \), with \( V_a = 1 \) and \( V_b = 0 \) \((v = \infty)\).](image-url)
velocities \( V_a = 1, V_b = 0 \) and \( V_a = 0, V_b = 1 \), respectively. All three curves reveal the points where the velocity \( u_\theta \) is equal to zero. At these so-called stagnation points the fluid is totally stationary (due to the symmetry of the problem, \( u_r = 0 \) when \( \theta = 0 \)). The position \((r_c, 0)\) of such a point on the \( \theta \)-axis for any values of \( V_a \) and \( V_b \) can be found by solving the equation

\[
V_a u_{\theta}^{\text{top}}(r) + V_b u_{\theta}^{\text{bot}}(r) = 0,
\]

or equivalently

\[
\frac{u_{\theta}^{\text{top}}(r)}{u_{\theta}^{\text{bot}}(r)} = -v. \tag{44}
\]

Plotting the ratio \( u_{\theta}^{\text{top}}(r)/u_{\theta}^{\text{bot}}(r) \) as a function of \( r \) (Fig. 2b), we can find the roots of (44) for any value of \( v \). These roots provide the locations of the stagnation points on the centreline. It is clear from Fig. 2b that for \( v > 0 \) two stagnation points exist, while for \( v < 0 \) there is only one. Of course, this conclusion only holds for the specific ratio of \((b - a)/2a\theta_0\) used here. When the length ratio of the side wall \( b - a \) to the bottom wall \( 2a\theta_0 \) increases, additional zeros of the basic functions \( u_{\theta}^{\text{top}}(r) \) and \( u_{\theta}^{\text{bot}}(r) \) may appear, and consequently the number of the stagnation points will also increase.

The character of the flow near the stagnation point \((r_c, 0)\) can be classified by expanding the stream function into a Taylor series. Using the relation between the stream function and the velocity components (2), we can write

\[
\Psi(r_c + \Delta r, \Delta \theta) = \Psi(r_c, 0) - \frac{1}{2} \left[ \frac{\partial^2 \Psi}{\partial r^2} \right]_{(r_c, 0)} (\Delta r)^2 - \frac{1}{r_c} \frac{\partial \Psi}{\partial \theta} \left. \right|_{(r_c, 0)} (\Delta \theta)^2 + \ldots.
\]

**Fig. 2b.** The function \( u_{\theta}^{\text{top}}(r)/u_{\theta}^{\text{bot}}(r) \)
The derivatives $\partial \Psi / \partial r$, $\partial \Psi / \partial \theta$, $\partial^2 \Psi / \partial r \partial \theta$ are absent here because they are proportional to the components of the velocity, which are zero at the point $(r_c, 0)$. Now, if the values $\partial u_\theta / \partial r$ and $\partial u_r / \partial \theta$ at the point $(r_c, 0)$ have opposite signs, then this stagnation point is an elliptical one, because nearby the streamlines surrounding it have the form of ellipses. On the other hand, if these coefficients have the same signs, the stagnation point is hyperbolic. Nearby streamlines now form two sets of hyperbolae which are locally not closed and these lines do not surround the stagnation point. There is, however, a closed streamline (separatrix) which passes through the hyperbolic point.

Figures 3a and 3b show the global structure of the streamline pattern for

![Figure 3](image-url)

**Fig. 3.** Contour plot of the stream function: (a) global picture, $v = 0$, (b) global picture, $v = \infty$ and (c) local picture near the bottom corner, $v = 0$
$v = 0$ (top wall moving) and $v = \infty$ (bottom wall moving), respectively. A main eddy is seen surrounded by a zero streamline and two small regions near the lower or upper corners containing the infinite sequences of the Moffatt eddies. The large eddy contains one elliptical critical point $((1.646, 0)$ for $v = 0$ and $(1.295, 0)$ for $v = \infty)$—the so-called 'vortex centre'. The positions of the separation points at the side and bottom (or top) walls can be defined according to the equations (40) and (42). The local picture of the streamlines of the corner eddy is presented in Fig. 3c. Different scales for the graphical representations of the stream function levels inside and outside the corner eddy domain are used. From this figure it is clear that the primary (corner) eddies are not symmetrical: the centre of the eddy does not lie on the bisector of the corner. This effect of the departure from symmetry for the centre of the primary eddies was also shown by Shankar (15) for the rectangular cavity.

When $v \neq 0$ (Fig. 4) the picture of the streamline patterns is changed drastically. In Fig. 4a it is seen that the former primary corner eddies 'survive' while other Moffatt eddies disappear. This can be explained using the representation (43) for the local structure of the stream function, where Goodier–Taylor flow dominates as $V_a \neq 0$. There are no separation points at the bottom wall. Instead, an additional hyperbolic stagnation point appears on the centreline (compare with Fig. 2b when $v = 0.01$), and the two corner eddies, surrounding two elliptical points, are in fact 'satellites' of this central hyperbolic point. Figure 4b (with $v = 0.5$) provides a clearer picture of the division of the whole domain into two subdomains enclosed by zero streamlines. In the upper subdomain there is one elliptical stagnation point $(1.701, 0)$ while in the lower subdomain there are one hyperbolic point $(1.116, 0)$ and two elliptical points $(1.126, \pm 0.405)$. The fact that, for any two-dimensional flow the number of elliptical points inside a closed streamline should always be greater than the number of hyperbolic points by one, was known in 1886 by Joukovsky (35) and used in his lectures on hydrodynamics.

When $v$ increases further, the elliptical points move to the hyperbolic point (Fig. 4c), and these points merge at some value $v = v_1$, with $0.9 < v_1 < 1$. For the case when $v = 1$ (Fig. 4d) we have two subdomains with one elliptical point in each. Next, when $v = v_2$, with $1 < v_2 < 1.1$ the upper elliptical point on the centreline turns into a hyperbolic one and gives birth to two additional elliptical points (Fig. 4e). Figures 4e and 4f show how these elliptical points gradually move to the upper corners and end up as shown in Fig. 3b.

If the velocities $V_a$ and $V_b$ are in opposite directions ($v < 0$) there are neither corner eddies nor dividing zero streamlines—the whole domain is a single vortex region with one elliptical stagnation point, the so-called 'vortex centre'. This vortex centre lies on the centreline closer to the upper or the bottom wall depending on the absolute value of $v$. 
Fig. 4. The evolution of the free eddies with increasing $\nu$. (a) $\nu = 0.01$, (b) $\nu = 0.5$, (c) $\nu = 0.9$, (d) $\nu = 1.0$, (e) $\nu = 1.4$, (f) $\nu = 10$. STEADY STOKES FLOW IN AN ANNULAR CAVITY
6. Discussion and conclusion

The results of the present study show that the method of superposition as proposed by Lamé in 1852 for elastic problems, appears to be very efficient when dealing with the two-dimensional steady Stokes flow in an annular cavity. The algebraic work involved is cumbersome but the final formulae are simple for numerical treatment. The algorithm provides very accurate numerical results using only a few terms in the Fourier series for the stream function and velocity components.

There exists another analytical approach for solving the linear biharmonic equation in a domain like $a \leq r \leq b$, $-\theta_0 \leq \theta \leq \theta_0$, that is restricted to domains with boundaries that coincide with coordinate lines for any general orthogonal system of (curvilinear) coordinates $r, \theta$. This so-called method of homogeneous solutions is widely used for two-dimensional problems in the theory of elasticity (6,18,19) when some similar boundaries ($\theta = \pm \theta_0$ for our case) are free of loading. Within this method a class of separated variable solutions of the form $\Psi(r, \theta) = f\alpha(r)g\lambda(\theta)$ is sought, where $\lambda$ is a (complex) parameter. The function $g\lambda(\theta)$ and the value $\lambda$ are chosen in such a way that all homogeneous boundary conditions at $\theta = \pm \theta_0$ are satisfied. This leads to a transcendental equation with an infinite number of roots $\lambda_n$. The biharmonic equation reduces to a fourth-order differential equation for $f\alpha \lambda(r)$ with four linearly-independent solutions. The coefficients in such a representation are found from the full solution

$$\Psi(r, \theta) = \sum_{\alpha=1}^{\infty} f\alpha(r)g\lambda\alpha(\theta), \quad (45)$$

with two boundary conditions at $r = a$ and $r = b$, respectively.

For the case of Stokes flow in the annular cavity Liu and Joseph (16) constructed a solution in the form

$$\Psi(r, \theta) = \text{Re} \sum_{\alpha=1}^{\infty} \left[ C_{\alpha} \left(\frac{r}{b}\right)^{\lambda_{\alpha}} + D_{\alpha} \left(\frac{r}{b}\right)^{-\lambda_{\alpha}+2} \right] \frac{g\lambda\alpha(\theta)}{\lambda_n(\lambda_n - 2)}, \quad (46)$$

with $g\lambda\alpha(\theta) = \cos(\lambda_n - 2)\theta_0 \cos \lambda_n \theta - \cos \lambda_n \theta_0 \cos(\lambda_n - 2)\theta$, where $\lambda_n$ are the roots of the equation

$$\sin 2(\lambda - 1)\theta_0 + (\lambda - 1)\sin 2\theta_0 = 0 \quad (47)$$

with $\text{Re} \lambda > 0$. The system of functions $g\lambda\alpha(\theta)$ is non-orthogonal and, therefore, raises the important question of completeness: can any function prescribed at the boundary be approximated to within any arbitrary accuracy by taking a sufficient number of terms in the series (46). It was reported that for the specific case when the values of the stream function and its normal derivatives are equal to zero at the corner points, this approach provides good results for a small number of terms in (46). But, when these values are non-zero, the results are not very satisfactory.
Shankar (15), using 100 terms in an eigenfunction expansion similar to (46) for a rectangular cavity to solve a system of 400 linear algebraic equations with the matrix depending on the complex roots $\lambda_n$, could not obtain a reasonable approximation of the boundary condition near the corner points. He obtained a tangential velocity 0.6 instead of the prescribed value 1. Therefore, the structure of the Goodier–Taylor solution near the lid corner remains unclear for this type of solution.

The advantage of our approach is the low computational cost for determining the velocity field. For example, using the finite-difference method to describe the flow in a rectangular cavity, as reported by Pan and Acrivos (14), a system of 10000 linear equations had to be solved. In our simulations the numbers $M$ and $K$ are quite small. It appeared that for the case of uniform constant wall velocities it is sufficient to solve two systems of $M + K$ equations, where $3 \leq \min(M, K) \leq 7$, $M/K = [4b\theta_0/(b - a)]$ (here $[ ]$ means the integer part of the number) in order to have an error in satisfying the boundary conditions of less than 0.05% per cent. This low computational effort makes the method of superposition very attractive for the analytical/numerical study of mixing problems in an annular cavity.

Another advantage of our approach is the analytical expression for the local behaviour of the stream function near the corner points. It appears to be possible to express the amplitude of Moffatt eddies in terms of the boundary conditions. The general idea about the connection of this amplitude to flow conditions far away from a corner, which was proved by Moffatt (8), Jeffrey and Sherwood (9) and others for some infinite two-dimensional domains of a canonical form, appears to be true also for the finite cavity.

For the interesting question on separation at a smooth boundary, which was considered by several authors (3, 9, 32) based on a local expression for the stream function, it was shown that the global expression for the stream function provides the equations for the position of the separation points and the angle of inclination of the dividing zero streamline to the wall.

Using the proposed method of superposition we have shown the evolution of the streamline patterns while changing the ratio between the uniform bottom and top velocities. The growth and decay of the Moffatt corner eddies, the main eddies in the centre of the cavity, and the side eddies were investigated in detail. For the geometry considered (when the side wall is shorter than the bottom wall) with a zero top (bottom) tangential velocity, there exists one main eddy and the Moffatt corner eddies at the top (bottom) wall. When the top and the bottom wall velocities are equal, there are two main large eddies and no corner eddies. For the case of unequal non-zero velocities on the top and the bottom wall there are one main large eddy and two free eddies (in the region near the wall with smaller boundary velocity) and no corner eddies. The primary corner eddies act as the source for the free eddies and vice versa. With the increase of the difference
the number of main eddies in the cavity will increase and, therefore, the question of the relationship between the changes in geometry and boundary velocities deserves further consideration.

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