Note on the singularity exponents for complimentary sectors

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NOTE ON THE SINGULARITY
EXPONENTS FOR
COMPLEMENTARY SECTORS

by

J. Boersma
Note on the Singularity Exponents for Complementary Sectors

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Abstract. By use of Babinet’s principle it is proved that the electric singularity exponent for a conducting plane sector is identical to the magnetic singularity exponent for the complementary sector.

I. INTRODUCTION

This note deals with the singularities of the electromagnetic field at the tip of a perfectly conducting, plane sector. In Cartesian coordinates x, y, z, the sector S lies in the plane z = 0, its tip is at the origin, and the sector is symmetric with respect to the y-axis. We shall also employ spherical coordinates r, θ, φ, defined in the usual manner.

It is known [1], [2], that two basic singularities must be considered for the electromagnetic field at the tip of the sector S:

i) the electric singularity, in which the electric field becomes infinite like $r^{v-1}$ at the tip of S;

ii) the magnetic singularity, in which the magnetic field becomes infinite like $r^{\tau-1}$ at the tip of S.

Numerical results for the singularity exponents $v$ and $\tau$ as functions of the opening angle $\alpha$ of S, were presented in [2]-[4]. There it was observed, more or less casually, that

$$v(\alpha) = \tau(2\pi - \alpha),$$

(1)

i.e. the electric singularity exponent for a sector of opening angle $\alpha$ is identical to the magnetic singularity exponent for the complementary sector of angle $2\pi - \alpha$. It is the aim of this note to rigorously prove property (1), by use of Babinet’s principle.
II. FIELD REPRESENTATION IN TERMS OF DEBYE POTENTIALS

In free space outside the sector $S$, the time-harmonic electromagnetic field (with time-dependence factor $e^{j\omega t}$ suppressed throughout) is represented by

$$E = \nabla \times \nabla \times (r \Pi_e \hat{r}) - j \omega \mu_0 \nabla \times (r \Pi_m \hat{r}),$$

$$H = j \omega \epsilon_0 \nabla \times (r \Pi_e \hat{r}) + \nabla \times \nabla \times (r \Pi_m \hat{r}) ,$$

in terms of two Debye potentials $\Pi_e$ and $\Pi_m$ [5, Sec. 1.2.6]. These scalar potentials must satisfy the Helmholtz equation

$$(\nabla^2 + k^2) \Pi_{e,m} = 0$$

and the boundary conditions

$$\Pi_e = 0, \quad \frac{\partial \Pi_m}{\partial n} = 0 \quad \text{on} \quad S .$$

Using separation of variables in spherical coordinates, we obtain the following solutions of (4) and (5).

In the case of the electric singularity, one may take $\Pi_m = 0$ and the electromagnetic field is derived from a potential $\Pi_e$ of the form

$$\Pi_e = \Pi_e(r, \theta, \phi) = j_v (kr) Y(\theta, \phi) ,$$

where $j_v$ denotes the spherical Bessel function of the first kind. The quantities $v$ and $Y(\theta, \phi)$ in (6) are obtained from the solution of an eigenvalue problem for the Laplacian on the unit sphere, subject to the Dirichlet boundary condition $Y = 0$ on the curve $C$ that is the intersection of the sector $S$ and the unit sphere. The singularity exponent $v$ is related to the smallest eigenvalue (which is equal to $v(v+1)$), and $Y(\theta, \phi)$ is the associated eigenfunction which is known to have even-even symmetry [2, Fig. 3]. Hence, the potential $\Pi_e = \Pi_e(x,y,z)$ is even in $z$ and even in $x$, and for the e.m. field components we have the symmetries:

$$E_x, E_y, H_z \quad \text{even in} \quad z ; \quad E_z, H_x, H_y \quad \text{odd in} \quad z .$$

In the case of the magnetic singularity, one may take $\Pi_e = 0$ and the electromagnetic field is derived from a potential $\Pi_m$ of the form

$$\Pi_m = \Pi_m(r, \theta, \phi) = j_\tau (kr) Z(\theta, \phi) .$$

The quantities $\tau$ and $Z(\theta, \phi)$ in (8) are again obtained from the solution of an eigenvalue problem for the Laplacian on the unit sphere, but now subject to the Neumann boundary condition $\partial Z/\partial n = 0$ on the curve $C$. The singularity exponent $\tau$ is related to the smallest positive
eigenvalue (which is equal to $\tau(\tau + 1)$), and $Z(\theta, \phi)$ is the associated eigenfunction which is known to have odd-even symmetry [2, Fig. 3]. Hence, the potential $\Pi_m = \Pi_m(x, y, z)$ is odd in $z$ and even in $x$, and for the e.m. field components we have the symmetries:

$$E_x, E_y, H_z \text{ even in } z; \ E_z, H_x, H_y \text{ odd in } z,$$

identical to (7).

III. SINGULARITY EXPONENTS FOR COMPLEMENTARY SECTORS

In the plane $z = 0$ we introduce the complementary sectors $S(\alpha)$ and $S(2\pi - \alpha)$ with opening angles $\alpha$ and $2\pi - \alpha$. Both sectors have their tip at the origin and are symmetric with respect to the $y$-axis. The sectors $S(\alpha)$ and $S(2\pi - \alpha)$ completely cover the plane $z = 0$.

Let the sector $S(\alpha)$ be perfectly conducting (metal), while the sector $S(2\pi - \alpha)$ is open (aperture). For the electric singularity at the tip of $S(\alpha)$, the associated e.m. field $E, H$ is described by the potential $\Pi_e$ in (6) with singularity exponent $\nu = \nu(\alpha)$. This field satisfies the boundary conditions

$$E_x(x, y, 0) = E_x(x, y, 0) = H_x(x, y, 0) = 0, \ (x, y) \in S(\alpha); \quad \text{(10)}$$

$$E_x(x, y, 0) = H_x(x, y, 0) = H_y(x, y, 0) = 0, \ (x, y) \in S(2\pi - \alpha). \quad \text{(11)}$$

Here, the condition (10) is the usual boundary condition for a perfect conductor, whereas the condition (11) results from (7) and the requirement that the field be continuous across the aperture.

Guided by Babinet's principle [6, pp. 45-46], we introduce the "complementary" e.m. field $E^{(1)}, H^{(1)}$, defined by

$$E^{(1)} = \frac{\varepsilon(\varepsilon_0/\mu_0)^{\frac{\nu}{4}}}{\varepsilon_0} H, \quad H^{(1)} = \pm(\varepsilon_0/\mu_0)^{\frac{\nu}{4}} E, \quad (z \geq 0) \quad \text{(12)}$$

which also satisfies Maxwell's equations. Then the boundary conditions (10) and (11) turn into

$$E_x^{(1)}(x, y, 0) = H_x^{(1)}(x, y, 0) = H_y^{(1)}(x, y, 0) = 0, \ (x, y) \in S(\alpha); \quad \text{(13)}$$

$$E_x^{(1)}(x, y, 0) = E_y^{(1)}(x, y, 0) = H_z^{(1)}(x, y, 0) = 0, \ (x, y) \in S(2\pi - \alpha). \quad \text{(14)}$$

The condition (13) expresses the continuity of $E^{(1)}, H^{(1)}$ across the sector $S(\alpha)$, which now acts as the aperture. From (14) it is recognized that the sector $S(2\pi - \alpha)$ is perfectly conducting with respect to the field $E^{(1)}, H^{(1)}$. Clearly, the e.m. field $E^{(1)}, H^{(1)}$ is associated with the magnetic singularity at the tip of $S(2\pi - \alpha)$. Thus the field is described by the potential $\Pi_m$ in (8) with singularity exponent $\tau = \tau(2\pi - \alpha)$. From (2), (3) and (12) it is readily seen that the Debye potentials $\Pi_m$ and $\Pi_e$ are related by
\[ \Pi_m = \pm (\varepsilon_0 / \mu_0)^{1/2} \Pi_x, \quad (z \geq 0). \]  

(15)

Notice that because of the even-even symmetry of \( \Pi_x \), the potential \( \Pi_m \) has the correct odd-even symmetry in \( z \) and \( x \), respectively. Finally, we conclude from (15) that \( v(\alpha) = \tau(2\pi - \alpha) \), which proves property (1).

The eigenvalue problems for the potentials \( \Pi_x \) and \( \Pi_m \) can be solved exactly by separation of variables in spheroidal coordinates, whereby the eigenfunctions \( Y \) and \( Z \) are represented by products of Lamé functions [7]. Some numerical results (correct to 5 decimal places) for the singularity exponents \( v \) and \( \tau \) over the range \( 0 < \alpha \leq \pi \) are presented in Table I. The present

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<th>( \alpha )</th>
<th>( v )</th>
<th>( \tau )</th>
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<td>10°</td>
<td>0.13004</td>
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</tr>
<tr>
<td>180°</td>
<td>0.50000</td>
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</tr>
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</table>

TABLE I. SINGULARITY EXPONENTS \( v \) and \( \tau \)

FOR A SECTOR OF OPENING ANGLE \( \alpha \)

results are in perfect agreement with those in [3, Tab. 3]. The singularity exponents in [2, Tables I and III] were calculated by a variational method; a comparison with Table I shows agreement up to 3 decimal places. Accurate numerical results for \( v = v(\alpha) \) over the range \( 0 < \alpha < 2\pi \) were also presented in [8].
REFERENCES


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