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On a Hele-Shaw type domain evolution with convected surface energy density: the third-order problem

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Abstract

We investigate a moving boundary problem with a gradient flow structure which generalizes Hele-Shaw flow driven solely by surface tension to the case of nonconstant surface tension coefficient taken along with the liquid particles at boundary. The resulting evolution problem is first order in time, contains a third-order non-linear pseudodifferential operator and is degenerate parabolic. Well-posedness of this problem in Sobolev scales is proved by showing semiboundedness of the third order operator with respect to a variable symmetric bilinear form. Moreover, we show that any shape of the liquid domain near a ball is an equilibrium for some appropriately chosen distribution of the surface tension coefficient. Finally, a numerical example is given.

Keywords: Free boundary motion, degenerate nonlocal parabolic evolution


1. Introduction

It is the aim of the present paper to consider the generalization of the well-investigated Hele-Shaw flow problem to the case of nonconstant surface tension coefficient (or surface energy density) which is convectively transported along the boundary. This generalization leads to the following moving boundary problem: For a given bounded domain $\Omega(0) \subset \mathbb{R}^m$ and a given non-negative function $\gamma_0$ defined on $\partial \Omega(0)$ one looks for a family of $C^2$-domains $\Omega(t) \subseteq \mathbb{R}^m$, $t > 0$ and functions $\varphi(\cdot, t) \in C^2(\Omega(t))$, $\psi(\cdot, t) \in C^2(\Omega(t))$,.
\[ \gamma_t \in C^2(\Gamma(t)) \] such that
\[
\begin{align*}
\Delta \varphi(\cdot, t) &= 0 & \text{in } \Omega(t), \\
\Delta \psi(\cdot, t) &= 0 & \text{in } \Omega(t), \\
\partial_n \psi(\cdot, t) &= \Delta_{\Gamma(t)} \gamma_t & \text{on } \Gamma(t), \\
\varphi(\cdot, t) &= \gamma_t \kappa(t) - \psi(\cdot, t) & \text{on } \Gamma(t), \\
V_n &= \partial_n \varphi(\cdot, t) & \text{on } \Gamma(t).
\end{align*}
\] (1.1)

Here \( \kappa(t) \) is the \((m-1)\)-fold mean curvature of \( \Gamma(t) \), with the sign taken such that \( \kappa \) is negative for convex domains, \( \partial_n \) is the outer normal derivative and \( V_n(t) \) is the (outer) normal velocity of \( \Gamma(t) \), determining its time evolution.

This problem generalizes the well-known Hele-Shaw flow with surface tension regularization in the following way: Any solution represents a gradient flow with respect to the usual energy functional
\[ E(\gamma, \Gamma) := \int_\Gamma \gamma \, d\Gamma, \]
where \( \gamma > 0 \) is now variable on \( \Gamma \), and to the Riemannian metric \( g_\Gamma \) on the infinite-dimensional manifold \( \mathcal{M} \) of surfaces \( \Gamma \) enclosing a fixed volume given by
\[ g_\Gamma(v_1, v_2) := \int_\Omega \nabla \varphi_1 \nabla \varphi_2 \, dx \] (1.2)
where the \( \varphi_i, i = 1, 2 \) are (weak) solutions of the Neumann problems
\[ \Delta \varphi_i = 0 \text{ in } \Omega, \quad \partial_n \varphi_i = v_i \text{ on } \Gamma. \]

The functions \( v_i \) can be identified with tangent vectors of \( \mathcal{M} \); note that the conservation of volume implies \( \int_\Gamma v_i \, d\Gamma = 0 \). For more details and references see [1, 4, 7]. Adding in (1.2) a boundary integral term \( \beta \int_\Gamma v_1 v_2 \, d\Gamma \) with \( \beta > 0 \) corresponds to the so-called kinetic undercooling regularization, this case is discussed in [7].

As mentioned already, we assume that the values of the function \( \Gamma \) are transported with the liquid particles: Introducing Lagrangian coordinates \( x = x(\xi, t), \xi \in \Gamma(0) \) corresponding to the velocity field via
\[ \partial_t x(\xi, t) = \nabla \varphi(x(\xi, t), t) \text{ for } t \geq 0, \quad x(\xi, 0) = \xi, \]
we obtain that \( x = x(\cdot, t) \) is a diffeomorphism from \( \Gamma(0) \) onto \( \Gamma(t) \), and the transport law for \( \gamma_t \) takes the form
\[ \gamma_t(x(\xi, t)) = \gamma_0(\xi), \quad \xi \in \Gamma(0), \ t \geq 0. \] (1.4)

This assumption is reasonable, for example, when \( \gamma \) depends on temperature and heat diffusion is negligible compared to convection. While it certainly oversimplifies the physical situation in the case when e.g. surfactants play a role, it seems that the mathematical character of problem is essentially the same there as in our case.

Our approach is based on reformulating (1.1) as a vector-valued evolution equation for a diffeomorphism mapping a fixed reference manifold to the moving boundary. In this way, the transport problem for \( \gamma \) is simply solved by prescribing a fixed smooth positive function on this reference manifold and pushing it forward to the moving boundary.
The paper is organized as follows:

After announcing our main results on short-time existence in Section 2, we start the proofs in Section 3 by investigating mapping properties of the occurring nonlocal operators in Sobolev scales. In particular, we derive flexible multilinear estimates for their Fréchet derivatives in low norms and extend them to higher norms by a generalized chain rule based on invariance properties. For related considerations concerning the analytic dependence of the Dirichlet-Neumann operator on the domain we refer to [3] and the references given there. Section 4 is devoted to the proof of the crucial estimate providing the semiboundedness of the evolution operator with respect to a specifically constructed variable inner product. As we are concerned here with an evolution equation whose right hand side is of order three, we have to refine the construction from [7] by including certain lower order terms. Differing from the situation there, here we have to demand strict positivity of \( \gamma \) because its inverse \( \gamma^{-1} \) enters one of these terms.

Technically, we use the natural decomposition of the right hand side into a second order operator mapping vectors to scalars and a first order operator mapping scalars to vectors. Furthermore, we use the fact that the right hand side is -in a sense to be made precise later - coercive with respect to the normal component. The semiboundedness enables us to invoke an abstract existence result based on Galerkin approximations and Rothe’s method. This is done in Section 5. We will omit certain details as they are parallel to the discussion in [7]. Finally, in Section 6 we investigate the existence of equilibria near the trivial equilibrium solutions given by balls with constant \( \gamma \). In this situation, any shape near a ball occurs as an equilibrium configuration for a certain function \( \gamma \) near the constant. The proof of this rests on an eigenspace decomposition into spherical harmonics and straightforward perturbation arguments.

2. Statement of the local existence results

We list some notation. \( C, C_1, \ldots \) etc. denote generic constants; their dependences on other quantities is only indicated if not obvious from the context. Let \( E \subseteq \mathbb{R}^m, m \geq 2 \) be a bounded domain with smooth boundary \( S := \partial E \) and \( \nu \) the outer unit normal on \( S \). For \( M = S \) or \( M = E \), we make constant use of the usual \( L^2 \)-based Sobolev spaces \( H^s(S), H^s(S, \mathbb{R}^m) \) of order \( s \) with values in \( \mathbb{R} \) and \( \mathbb{R}^m \), respectively. The norms of these spaces will be denoted by \( \| \cdot \|_s^M \); for \( M = S \) the upper index \( M \) is dropped in most cases. When Fréchet derivatives of operator-valued mappings are considered, the additional arguments describing the variations are written in accolades (\{\}).

Now, as already mentioned in the introduction, we reformulate the moving boundary problem (1.1) - (1.4) by describing \( \Gamma(t) \) as an embedding \( u(\cdot, t) : S \to \mathbb{R}^m \) such that the curves \( t \mapsto u(y, t) \) for fixed \( y \in S \) are trajectories belonging to the velocity field and \( \gamma_t \) is constant along these curves. This approach enables us to consider \( \gamma_t \) as a known function during the evolution at the cost of describing the moving boundary by \( m \) functions. To do so, let

\[
U := \{ u : S \to \mathbb{R}^m \mid u = w|_S \text{ with } w \in \text{Diff}(\overline{E}, \Omega_u \cup \Gamma_u) \}
\]  

(2.1)

where

\[ \Omega_u = w(E) \quad \text{and} \quad \Gamma_u = \partial \Omega_u = u(S). \]
Throughout this paper, we use the abbreviation
\[ U_s := U \cap H^s(S, \mathbb{R}^m). \]

Now, (1.1) - (1.4) is reduced to the following Cauchy problem, which will be investigated in the sequel: For given \( u_0 \in U_s \), \( s \) sufficiently large, we look for \( T > 0 \) and a mapping \([0, T] \ni t \mapsto u(t) \in U_s\), such that
\begin{align*}
\frac{d}{dt} u(t) &= \mathcal{F}(u(t)), \quad t \in [0, T], \quad (2.2) \\
u(0) &= u_0. \quad (2.3)
\end{align*}

Thereby, for \( u \in U \), we have set
\[ \mathcal{F}(u) := F(u) \mathcal{G}(u) \quad \text{with} \quad \mathcal{G}(u) := H(u) + G(u), \quad (2.4) \]
where, for any given function \( f \) on \( S \),
\[ F(u)f := \nabla \varphi(u, f) \circ u \quad (2.5) \]
and \( \varphi = \varphi(u, f) \) denotes the solution of the Dirichlet problem
\[ \Delta \varphi = 0 \quad \text{in} \ \Omega_u, \quad \varphi = f \circ u^{-1} \quad \text{on} \ \Gamma_u. \quad (2.6) \]

Further, \( H(u), G(u) \) are given by
\[ H(u) := \gamma(\kappa_{\Gamma_u} \circ u), \quad G(u) := -A(u) \Delta(u) \gamma. \quad (2.7) \]
Here \( \gamma \in C^\infty(S) \) is a fixed and given positive function, \( \kappa_{\Gamma_u} \) denotes the mean curvature of \( \Gamma_u \) with sign and scaling conventions as above and
\[ \Delta(u) w := \Delta_{\Gamma_u}(w \circ u^{-1}) \circ u \quad (2.8) \]
is the pullback to \( S \) of the Laplace-Beltrami operator \( \Delta_{\Gamma_u} \) on \( \Gamma_u \) and
\[ A(u) f := \varphi_N(u, f) \circ u \quad (2.9) \]
the Neumann-Dirichlet operator, i.e. \( \varphi_N = \varphi_N(u, f) \) solves the Neumann problem
\[ \Delta \varphi_N = 0 \quad \text{in} \ \Omega_u, \quad \partial_n \varphi_N = c + f \circ u^{-1} \quad \text{on} \ \Gamma_u, \quad \int_{\Gamma_u} \varphi_N \, dx = 0. \quad (2.10) \]
The constant \( c = c(u, f) \in \mathbb{R} \) in (2.10) is determined by the solvability condition
\[ \int_{\Gamma_u} (f \circ u + c) \, d\Gamma_u = 0; \quad (2.11) \]
clearly \( c(u, f) = 0 \) for \( f = \Delta(u) \gamma \). For fixed smooth \( \gamma \) on \( S \), the mapping \( u \mapsto H(u) \) constitutes a quasi-linear second order differential operator on \( S \). Moreover, the solutions of the boundary value problems (2.6), (2.10) depend smoothly on the domain \( \Omega_u \), i.e. on \( u \in H^s \), \( s > (m+1)/2 \) and \( f \mapsto F(u)f, f \mapsto A(u)f \) represent pseudodifferential operators of order one and minus one, respectively. In particular, \( G \) is a pseudodifferential operator of lower order than \( H \) and may be considered as a correction term to ensure the gradient flow structure of the evolution problem. We will show later that
\[ [u \mapsto \mathcal{F}(u)] \in C^\infty(U_s, H^{s-3}(S, \mathbb{R}^m)) \quad (2.12) \]
for \( s > (m + 3)/2, \ s \geq 3 \). Now we are in position to formulate our main results.
Theorem 2.1. (Short-time existence and uniqueness.)
Fix an even integer $s_0 > (m + 7)/2$, $s_0 \geq 6$ and assume $\gamma \in C^\infty(S)$ strictly positive on $S$. Let $s \geq s_0$ be an even integer and $u_0 \in U_s$. Then there exist $T > 0$ and an unique solution
\begin{equation}
 u \in C([0, T], U_s) \cap C^1([0, T], H^{s-3}(S, \mathbb{R}^m))
\end{equation}
of the initial value problem (2.2), (2.3). Additionally, any given $\bar{u}_0 \in U_{s_0}$ has a suitable $H^{s_0}$-neighborhood $K$, such that for initial values $u_0$ varying in $K \cap H^s$, there are $T > 0$ and $C$ independent of $u_0$ such that
\begin{equation}
 \|u(t)\|_s \leq C(1 + \|u(0)\|_s) \text{ for all } t \in [0, T].
\end{equation}

Theorem 2.2. (Regularity and continuous dependence on initial values.)
Under the assumptions of Theorem 2.1 let $u$ be a any solution to (2.2) in the class (2.13) with some $T > 0$. Then there holds:

(i) $u(0) \in H^{s+1}(S, \mathbb{R}^m)$ implies $u(t) \in H^{s+1}(S, \mathbb{R}^m)$ for all $t \in [0, T]$.

(ii) Assume $u^n_0 \to u_0$ in $H^s(S, \mathbb{R}^m)$ for $n \to \infty$. Then, for $n$ sufficiently large, there exist solutions $u_n$ of (2.2) in the class (2.13) with initial values $u_n(0) = u^n_0$ and there holds $u_n \to u$ in $C([0, T], H^s(S, \mathbb{R}^m))$.

The proof of both theorems is given in Section 5.

Remarks: The restriction to even integers $s$ is due purely to the construction of our bilinear form involving integer powers of a generalized Laplacian. This restriction can be lifted afterwards by using the nonlinear interpolation result given in [2], Proposition A.1 and Remark A.2. The dimension independent restriction $s_0 \geq 6$ is needed as we use dual estimates for elliptic boundary value problems in norms with negative index.

3. Smooth domain dependence of the non-local operators
We start by gathering some properties of the operator $F$ defined by (2.5), (2.6). They are essentially parallel to Corollary 4.4 and Lemma 4.5 in [7], therefore proofs will be omitted. Note, however, that $f \mapsto F(u)f$ is an operator of order one here as (2.6) is a Dirichlet problem. In fact, the normal component of $F$ is given by the Dirichlet-Neumann operator while the tangential component is given by the tangential gradient of $f$.

Lemma 3.1. Let $s > (m + 1)/2$, $u \in U_s$ and $t \in [1, s]$ be given. Then
\begin{equation}
 F \in C^\infty(U_s, Z'(H^t(S), H^{t-1}(S)), \mathbb{R}^m),
\end{equation}
and for any choice of $s_1, \ldots, s_{k+1} \in [1, s]$ with $s_1 + \ldots + s_{k+1} \geq t + ks$ there exists a constant $C > 0$ such that for all $f \in H^s(S)$ and all $u_1, \ldots, u_k \in H^s(S, \mathbb{R}^m)$ there holds
\begin{equation}
 \|F^{(k)}(u_1, \ldots, u_k)f\|_{t-1} \leq C\|u_1\|_{s_1} \cdots \|u_k\|_{s_k} \|f\|_{s_{k+1}}.
\end{equation}
The constant may be chosen independently of $u$ as $u$ varies in bounded and weakly closed subsets of $U_s$.

Remark 3.2. Note that a bounded subset of $H^s(S)$ is weakly closed if and only if it is closed in $H^t(S)$ for some $t < s$. Then it is compact in all $H^t(S)$ with $t < s$. 
Next, we prove some estimates on the operator $A$ defined by (2.8)–(2.11) and its Fréchet derivatives which will be needed later. In structure, they are similar to the estimates on $F$, but here we have to work with norms with negative index which implies a loss of flexibility.

**Lemma 3.3.** Assume $s > (m + 1)/2$, $s \geq 4$, $t \in [-3, s - 1]$. Then

$$A \in C^\infty(U_s, \mathcal{L}(H^{t-1}(S), H^t(S)))$$

and for $u \in U_s$

$$\|A(u)f\|_t \leq C\|f\|_{t-1},$$

$$\|A^{(k)}(u)\{v_1, \ldots, v_k\}f\|_t \leq C\|v_1\|_s \cdots \|v_k\|_s \|f\|_{t-1},$$

$$\|A^{(k)}(u)\{v_1, \ldots, v_k\}f\|_s \leq C\|v_1\|_s \cdots \|v_{k-1}\|_s \|v_k\|_s \|f\|_s,$$ \hspace{1em} (3.2)-(3.4)

$v_1, \ldots, v_k \in H^s(S, \mathbb{R}^m)$, $k \in \mathbb{N}$. The constants $C$ can be chosen independently of $u$ as $u$ varies in bounded, weakly closed subsets of $U_s$.

**Proof.** Fix $s_0 \in ((m + 1)/2, s)$ and an extension operator $\mathcal{E} \in \mathcal{L}(H^s(S), H^{s+1/2}(E))$, $t > 0$. As in the proof of Lemma 4.3 in [7], we pick $v \in U_s$ and choose an $H^{s_0}$-neighborhood $V_{s_0} \subset U_s$ and $u_0 \in C^\infty(E, \mathbb{R}^m)$ such that

$$\tilde{u} := u_0 + \mathcal{E}(u - u_0) \in \text{Diff}(E, \overline{\Omega}_u).$$

This is possible by Lemma 4.1 in [7].

For $u \in V_{s_0}$, let the transformed operators $L(u)$ and $\mathcal{B}(u)$ be defined by

$$L(u)\psi := \partial_i(\sqrt{g}g^{ij}\partial_j\psi), \quad \mathcal{B}(u)\psi = \nu_i\sqrt{g}g^{ij}\partial_j\psi,$$

where $\sqrt{g}$, $g^{ij}$ are the volume element and the (inverse) coefficients of the metric on $E$ induced by $\tilde{u}$, respectively, and $\nu$ is the outer unit normal on $S$. We consider the transformed boundary value problem

$$L(u)\psi = \Phi_1, \quad \mathcal{B}(u)\psi = \omega(u)(\Phi_2 + c), \quad \int_s \omega(u)(\Phi_2 + c)\, dS = \int_E \sqrt{g}\Phi_1\, dx, \hspace{1em} (3.5)$$

$c = c(u, \Phi_1, \Phi_2) \in \mathbb{R}$. Here $\omega(u) = d\Gamma_u/dS$ is the surface element belonging to the transformation induced by $u$ which is given by a nonlinear first-order differential operator in $u$ (see [7]).

It can be shown as in [6], Lemma 3.1., that (3.5) is uniquely solvable and $\psi$ satisfies an estimate

$$\|\psi\|_t + \|\omega\|_t \leq C(\|\Phi_1\|_t^{3/2} + \|\Phi_2\|_t^2)$$ \hspace{1em} (3.6)

with $C$ independent of $u$ as $u$ varies in bounded subsets of $V_{s_0}$ if $V_{s_0}$ is chosen sufficiently small. (For the definition of $\| \cdot \|_t$ if $t < 0$ by duality and the corresponding properties we refer to [6].)

As $A(u)f$ is the trace of the solution $\psi$ of (3.5) with $\Phi_1 = 0$, $\Phi_2 = f$, we get (3.2) immediately from (3.6).

Note that $A'(u)\{v\}f$ is given as the solution $\psi'$ of

$$L(u)\psi' = -L'(u)\{v\}\psi, \quad \mathcal{B}(u)\psi' = -\mathcal{B}'(u)\{v\}\psi + \omega'(u)\{v\}(f + c(u, 0, f)) + \omega(u)\partial_u c(u, 0, f)\{v\}.$$
As $f \mapsto c(u,0,f)$ and $v \mapsto \partial_u c(u,0,f)\{v\}$ are given by smoothing operators, to obtain (3.3) and (3.4) it is sufficient to use (3.6) and estimate either
$$
\|L'(u)\{v\}\mathbf{\psi}\|^{\Omega}_{t-3/2} \leq C(\|L\|^{\Omega}_{t+1/2} + \|v\|_t)\|\mathbf{\psi}\|_{s+1} \leq C\|v\|_t\|f\|_s
$$
or
$$
\|L'(u)\{v\}\mathbf{\psi}\|^{\Omega}_{t-3/2} \leq C(\|\mathbf{\psi}\|^{\Omega}_{t+1/2} + \|v\|_t)\|\mathbf{\psi}\|_s \leq C\|v\|_s\|f\|_{t-1},
$$
together with analogous estimates for $\|B'(u)\mathbf{\psi}\|_{t-1}$ and $\|\omega'(u)\{v\}f\|_{t-1}$. The general case follows now by induction over $k$, cf. [7], Lemma 4.5.

Finally, the uniformity of the estimates follows from the fact that bounded, weakly closed subsets of $U_s$ are compact in $H^{s_0}(S)$.

In a similar fashion and under the same assumptions on $u$ and $s$, one can show that
$$
\|F'(u)\{v\}f\|_t \leq C\|v\|_s\|f\|_{t+1}, \tag{3.7}
$$
t $\in [-1, s-1]$.

We choose $m$ smooth vector fields $D_1, \ldots, D_m$ on $S$ such that
$$
\text{span}\{D_1, \ldots, D_m\} = T_x \text{ for all } x \in S
$$
and use the multi-index notation $D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m}$, $\alpha = (\alpha_1, \ldots, \alpha_m)$ for higher order derivatives. Note that, for $s \geq 0$ integer, we can use
$$
(u,v)_s = \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2(S)}
$$
as scalar product generating the norm in $H^s(S)$. Moreover, as consequence of certain invariance properties, we have a differentiation rule which resembles Leibniz’ rule at an abstract level, cf. [7]: For any multi-index $\alpha$ and $u \in U_s$, $f \in H^s(S)$, $s > |\alpha| + (m+1)/2$ there holds
$$
D^\alpha F(u) f = \sum c_{\beta_1, \ldots, \beta_{k+1}} F^{(k)}(u) \{D^{\beta_1} u, \ldots, D^{\beta_k} u\} D^{\beta_{k+1}} f
$$
where the sum has to be extended over all integers $k$ and systems of non-negative multi-indices $\beta_1, \ldots, \beta_{k+1}$ with
$$
0 \leq k \leq |\alpha|, \quad 1 \leq |\beta_1|, \ldots, |\beta_k|, \quad \beta_1 + \ldots + \beta_{k+1} = \alpha. \tag{3.9}
$$
The coefficients are non-negative integers, in particular, $c_\alpha = c_0, 0 = 1$. Combining the differentiation rule with the estimate of the derivatives in lower norms we obtain

**Proposition 3.4.** (i) Let $s \geq s_0 > (m+1)/2$, $s$ integer, $u \in U_s$. Then
$$
\|F(u)\|_{s-1} \leq C(\|u\|_s\|f\|_{s_0} + \|f\|_s)
$$
with an uniform constant as long as $u$ varies in $H^{s_0}$- bounded and weakly $H^{s_0}$- closed subsets of $U_s$.

(ii) Assume additionally $s \geq s_0 + 2$ and let $\alpha$ be any multi-index with $|\alpha| = s$. Writing $D^\alpha = D^{\alpha_1} \cdots D^{\alpha_s}$ with $|\alpha_1| = \ldots = |\alpha_s| = 1$, we have
$$
D^\alpha F(u) f = F(u) D^\alpha f + F'(u) \{D^\alpha u\} f + \sum_{i=1}^s F'(u) \{D^\alpha u\} D^{\beta_i} f + R_{\alpha}(u) f
$$

\[3.11\]
where $\alpha = \alpha_i + \beta_i$ and the remainder term allows the estimate
\[ \|R_{\alpha}(u)f\|_0 \leq C\left(\|u\|_s \|f\|_{s_0 + 1} + \|f\|_{s-1}\right). \]
The constant can be chosen uniformly as $u H^{s_0 + 2}$ bounded, weakly $H^{s_0 + 2}$-closed subsets of $U_s$.

**Proof.** We consider the more complicated situation (ii) only. According to (3.8), the remainder term has a representation as a sum of terms
\[ I_\beta = F^{(k)}(u)\{D^{\beta_1}u, \ldots, D^{\beta_k}u\}D^{\beta_{k+1}}f, \]
where the multi-indices satisfy (3.9) and additionally
\[ |\beta_1, \ldots, |\beta_k| \leq s - 1, \quad |\beta_{k+1}| \leq s - 2. \]
Hence $k \geq 1$. For each of the terms $I_\beta$, we will choose numbers $\theta_1, \ldots, \theta_{k+1} \in [0,1]$ such that $\theta_1 + \ldots + \theta_{k+1} = 1$ and set
\[ s_i := (1 - \theta_i)s_0 + \theta_i. \]
If $k = 1$ we choose $\theta_1, \theta_2$ such that $\theta_1 + \theta_2 = 1$ and $|\beta_{2}| = \theta_1 + \theta_2(s - 2)$. If $k = 2$ and $|\beta_3| = 0$ we choose $\theta_i := (|\beta_i| - 1)/(s - 2)$ for $i = 1,2$ and $\theta_3 := 0$. If $k = 2$ and $|\beta_3| \geq 1$ or $k \geq 3$ we choose
\[ \theta_i := (|\beta_i| - 1)/(s - 3) \quad \text{for } i = 1,2,3, \quad \theta_i := |\beta_i|/(s - 3) \quad \text{for } i \geq 4. \]
In all cases, we have
\[ \begin{align*}
|\beta_i| + s_i &\leq (1 - \theta_i)(s_0 + 2) + \theta_is, \quad i = 1,\ldots,k, \\
|\beta_{k+1}| + s_{k+1} &\leq (1 - \theta_{k+1})(s_0 + 1) + \theta_{k+1}(s - 1).
\end{align*} \tag{3.12} \]
Set $\lambda := \theta_1 + \ldots + \theta_k$. Using (3.1) with $t = 1$, $s = s_0$, (3.12), norm convexity, and Young’s inequality we get
\[ \begin{align*}
\|I_\beta\|_0 &\leq C\|u\|_{|\beta_1| + s_1} \cdots \|u\|_{|\beta_{k+1}| + s_{k+1}} \|f\|_{|\beta_{k+1}| + s_{k+1}} \\
&\leq C\|u\|_{s_0 + 2}^{-1}\left(\|u\|_{s_0 + 2} \|f\|_{s-1}\right)^{1-\lambda}\left(\|u\|_s \|f\|_{s_0 + 1}\right)^\lambda \\
&\leq C\|u\|_{s_0 + 2}^{-1}\left(\|u\|_{s_0 + 2} \|f\|_{s-1} + \|u\|_s \|f\|_{s_0 + 1}\right),
\end{align*} \]
and the result follows.

The following lemma provides an explicit characterization for the linearization of $F$, namely, up to terms of order zero in $v$,
\[ F'(u)f = -F(u)(v \cdot F(u)f). \]
This structure will be important later. It can be verified in an informal way by performing the variation on $\Omega_u$ itself instead of transforming the problem to the reference domain.

**Lemma 3.5.** Let $s > (m + 3)/2$. Then for $u \in U_s$, $v \in H^s(S, \mathbb{R}^m)$ and $f \in H^s(S)$ there holds
\[ \|F'(u)v f + F(u)(v \cdot F(u)f)\|_0 \leq C\|f\|_s \|v\|_0. \]
Proof. From (2.5) we get
\[ F'(u)(v)f = \partial_i \phi' \circ u + v_j \partial_i \partial_j \phi \circ u \]
with \( \phi = \phi(u, f) \) from (2.6) and \( \phi' = \phi'(u, f)(v) \) given by
\[ \phi'(u, f)(v) = \partial_\varepsilon(\phi(u + \varepsilon v, f))(x) |_{\varepsilon=0}, \quad x \in \Omega_u. \]
The function \( \phi' \) satisfies
\[ \Delta \phi' = 0 \text{ in } \Omega_u, \quad \phi' = -\nabla \phi \cdot v \text{ on } \Gamma_u, \]
therefore \( \partial_i \phi' \circ u = -F_i(u)(v \cdot F(u)f) \).

Parallel to the proof of Lemma 5.1 in [7] one obtains
\[ \|v_i \partial_i \partial_j \phi \circ u\|_0 \leq C\|v\|_0 \|\phi(u, f)\|_{C^2(\Omega_u)} \leq C\|f\|_s \|v\|_0. \]
This proves the assertion.

4. The main estimate

In this section we prove \( H^s \)- a priori estimates for the non-linear operator \( \mathcal{F} \) w.r. to variable bilinear forms which we define in the sequel. As already mentioned in the introduction, these estimates are the main ingredient in the existence proof.

To begin with, for \( u \in U_s \), \( s > (m+1)/2 \) we define
\[ P(u)v := v \cdot (n(u) \circ u), \quad N(u)w := w(n(u) \circ u), \]
\[ \Lambda(u)w := \nabla_{\Gamma_u}(w \circ u^{-1}) \circ u \]
as the euclidean scalar product and multiplication with outer normal \( n(u) \) of \( \Gamma_u \) and pullback of tangential gradient \( \nabla_{\Gamma_u} \) along \( \Gamma_u \), respectively. Considered as operators in \( v \) and \( w \), the coefficients of \( P(u) \), \( N(u) \) and \( \Lambda(u) \) are smooth functions of \( u \) and its first derivatives. Thus,
\[ P(u) \in \mathcal{L}(H^t(S, \mathbb{R}^m), H^t(S)), \quad N(u) \in \mathcal{L}(H^t(S, H^t(S, \mathbb{R}^m))), \]
\[ \Lambda(u) \in \mathcal{L}(H^t(S), H^{t-1}(S, \mathbb{R}^m)) \]
depend smoothly on \( u \in U_s \) for \( -2 \leq t \leq s \). Clearly, the operators \( P, N, \Lambda \) satisfy invariance properties as stated for \( F \) in [7]. As a consequence, the differentiation rule (3.8) is also true for \( P, N, \Lambda \); we make use of that without explicit mention. Further recall that the pullback \( \Delta(u)w \) of the Laplace Beltrami operator \( \Delta_{\Gamma_u} \) on \( \Gamma_u \) according to (2.8) and the operator \( H(u) \) according to (2.7) may be expressed as
\[ \Delta(u)w = \Lambda_i(u)(\Lambda_i(u)w), \quad H(u) = -\gamma \Lambda_i(u)(n_i(u) \circ u) \]
respectively. As in [7], we get from this
\[ \mathcal{G} \in C^\infty(U_s, H^{s-2}(S)) \]
for the operator \( \mathcal{G} \) defined by (2.4), (2.7), provided \( s > (m+3)/2 \). Together with Lemma 3.1 this implies the smoothness of \( \mathcal{F} \) as stated in (2.12).

In the further considerations of this section we fix \( s_0 \) to be the smallest integer such
that \( s_0 \geq 6 \) and \( s_0 > (m+7)/2 \) and set

\[
\widetilde{U}_s := U_s \cap K \quad \text{for all } s \geq s_0
\]

with a fixed \( H^{s_0}\)–bounded and weakly \( H^{s_0}\)-closed subset \( K \subseteq U_{s_0} \). Note that

\[1 \leq C\|u\|_{s_0} \leq C'\|u\|_s, \quad \|u\|_{C^2(S)} \leq C\]

for all \( u \in \widetilde{U}_s, s \geq s_0 \).

Furthermore, note the estimates

\[
\|\mathcal{I}(u)\|_{s-2}, \|\mathcal{F}(u)\|_{s-3} \leq C\|u\|_s \quad \text{for all } u \in \widetilde{U}_s, s \geq s_0.
\]

On the compact reference manifold \( S \) we define the Laplace-type operator

\[
\Delta_0 := -D_1D_i.
\]

As \( \Delta_0 \) is elliptic, it has an approximate inverse, i.e. there is an operator

\[
\Delta_0^+ \in \mathcal{L}(H^r(S), H^{r+2}(S)),
\]

\( \tau \in \mathbb{R} \), such that \( \Delta_0^+\Delta_0 = \text{id} + Q_1, \Delta_0^+\Delta_0 = \text{id} + Q_2 \), where \( Q_1, Q_2 \) are smoothing operators; in particular, \( Q_1, Q_2 \in \mathcal{L}(H^r(S), H^{r}(S)) \) for any \( \sigma, \tau \in \mathbb{R} \).

**Lemma 4.1.** Let \( s \geq s_0 \) with \( s = 2k, k \in \mathbb{N} \) and \( u \in U_s \). Then we have

\[
\Delta_0^k\mathcal{F}(u) = (F(u) + F_0(u))(\gamma\Delta(u)(P(u)\Delta_0^k u) + G_1(u)(\Delta_0^k u)) + R(u).
\]

(4.6)

Here \( f \mapsto F_0(u)f \) and \( v \mapsto G_1(u)v \) are operators of order zero and one, respectively,

\[
F_0(u) \in \mathcal{L}(H^1(S), H^1(S, \mathbb{R}^m)), \quad G_1(u) \in \mathcal{L}(H^1(S, \mathbb{R}^m), H^{t-1}(S)),
\]

\( t \in [-1, s] \) and \( t \in [-2, s] \), respectively, and the remainder term \( R \) satisfies

\[
\|R(u)\|_0 \leq C(\|u\|_s + \|P(u)\|_1)
\]

(4.8)

with a constant independent of \( u \) as long as \( u \) varies in a set \( \widetilde{U}_s \).

**Proof.** Using Proposition 3.4, (ii), we write \( \Delta_0^k\mathcal{F}(u) \) in the form

\[
F(u)(\Delta_0^k\mathcal{G}(u)) + F'(u)\{\Delta_0^k u\}\mathcal{G}(u) + 2 \sum_{j=0}^{k-1} F'(u)\{D_j u\}\Delta_0^jD_i\Delta_0^{k-1-j}\mathcal{G}(u) + R_1(u)
\]

\[
= F(u)(\Delta_0^k\mathcal{G}(u)) + F'(u)\{\Delta_0^k u\}\mathcal{G}(u) + 2kF'(u)\{D_1 u\}D_i\Delta_0^{k-1}\mathcal{G}(u) + R_1(u) + R_2(u).
\]

According to this proposition, \( R_1(u) \) allows the estimate

\[
\|R_1(u)\|_0 \leq C(\|u\|_s\|\mathcal{G}(u)\|_{s_0-2} + \|\mathcal{G}(u)\|_{s-1}).
\]

For \( R_2(u) \) we find from Lemma 3.1 (with \( t = 1 \))

\[
\|R_2(u)\|_0 \leq 2 \sum_{j=0}^{k-1} \|F'(u)\{D_j u, [\Delta_0^j, D_i]\Delta_0^{k-1-j}\mathcal{G}(u)\}\|_0
\]

\[
\leq C \sum_{i,j} \|[\Delta_0^j, D_i]\Delta_0^{k-1-j}\mathcal{G}(u)\|_1 \leq C\|\mathcal{G}(u)\|_{s-1}.
\]
Applying now the estimate
\[ \|D^\alpha G(u)\|_{-1} \leq C(\|\gamma^{1/2} P(u) D^\alpha u\|_1 + \|u\|_s) \]
from [7], Lemma 5.5, we obtain
\[ \|G(u)\|_{s-1} \leq C \left( \|D_0^k G(u)\|_{-1} + \|G(u)\|_{-1} \right) \leq C \left( \|P(u)(D_0^0 u)\|_1 + \|u\|_s \right). \]
Consequently, both \( R_1(u) \) and \( R_2(u) \) satisfy an estimate parallel to (4.8).

By Lemma 3.5 we have
\[ F'(u)\{D_0^k u\} G(u) = -F(u)(D_0^k u \cdot G(u)) + R_3(u) \]
with
\[ \|R_3(u)\|_0 \leq C\|D_0^k u\|_0 \|G(u)\|_{s_0-2} \leq C\|u\|_s. \]
Thus, defining \( F_0(u) \) by
\[ F_0(u)v := 2kF'(u)\{D_0^k u\} D_1 D_0^0 v \]
we get
\[ D_0^k G(u)G = F'(u)\{D_0^k u\} \gamma + F_1(u) + R_5(u), \]
where \( R_4(u) \) satisfies an estimate parallel to (4.8).

Recall that \( G(u) \) depends linearly on \( \gamma \). Slightly abusing notation, we write \( G(u)\gamma \) etc. in the remaining part of this proof (see (6.2) below). Applying a differentiation rule parallel to (3.8) to \( G \) we get
\[ D_0^k G(u)\gamma = F'(u)\{D_0^k u\} \gamma + F_1(u) + R_5(u), \]
where \( F_1(u) \) contains all terms where derivatives of \( u \) of order \( s-1 \) and \( s-2 \) occur. It is a sum of terms of the forms
\[ F'(u)\{D_0^k D_1 D_0^{k-1-j} u\} D_1 \gamma, \quad F''(u)\{D_0^k D_1 D_0^{k-2-j} u\} \gamma, \]
j \in \{0, \ldots, k-1\}, and
\[ F'(u)\{z\} D_1 \gamma, \quad F''(u)\{D_1 D_0^k u, z\} D_1 \gamma, \quad F''(u)\{D_1 D_0^k u, z\} \gamma, \]
z := \( D_0^k D_1 D_0^k D_0^{k-2-j} u \), \( j, \mu \in \{0, \ldots, k-2\} \), \( j + \mu \leq k-2 \).

From analogous arguments as in the proof of Proposition 3.4 one obtains
\[ \|R_5(u)\|_1 \leq C\|u\|_s. \]

Writing in the above terms
\[ D_0^{k-1-j} u = (D_0^+)^{j+1} D_0^k u + R_{6,j} u, \]
\[ D_0^{k-2-j-\mu} u = (D_0^+)^{j+\mu+2} D_0^k u + R_{7,j,\mu} u \]
with smoothing remainder terms, we get
\[ D_0^k G(u)\gamma = F'(u)\{D_0^k u\} \gamma + F_2(u)D_0^k u + R_8(u) \]
with a first-order operator \( v \rightarrow F_2(u)v \) and a remainder term \( R_8(u) \) satisfying an estimate parallel to (4.9) again. Using now the splitting
\[ F'(u)\{v\} \gamma = \gamma D(u) P(u)v + F_3(u)v \]
with a first-order operator \( v \mapsto \mathcal{G}_0(u)\{v\} \) and adding and subtracting the term \( F_0(\Delta^k_0 u \cdot \mathcal{F}(u)) \) we get (4.6) and (4.8), setting
\[
G_1(u)v := \mathcal{G}_2(u)v + \mathcal{G}_3(u)v - v \cdot \mathcal{F}(u)
\]
and using
\[
\|F_0(\Delta^k_0 u \cdot \mathcal{G}(u))\|_0 \leq C\|u\|.
\]
The statements (4.7) are consequences of Lemma 3.3 and of (3.7). This becomes clear if of \( G_1 \) is written out explicitly in terms of differential operators and Fréchet derivatives of \( A \).

In the sequel, we will need an approximate inverse for \( \Delta(u) \), denoted by \( \Delta(u)^+ \), which is defined by \( \Delta(u)^+ \phi := w \) where
\[
\Delta(u)w = \phi - \int_S \omega(u) \phi dS, \quad \int_S \omega(u) w dS = 0.
\]
It is straightforward to show that \( u \mapsto \Delta(u)^+ \) is smooth from \( U_s \) to \( \mathcal{L}(H^1(S), H^{1+2}(S)) \), \( t \in [0, s-2] \). Furthermore,
\[
\Delta(u)\Delta(u)^+ = \text{id} + Q_1(u), \quad \Delta(u)^+\Delta(u) = \text{id} + Q_2(u),
\]
where \( Q_1(u), Q_2(u) \in \mathcal{L}(H^1(S), H^2(S)) \) for any \( \sigma \in \mathbb{R}, \tau \geq 1 - s \), and the corresponding norms are bounded independently of \( u \in \tilde{U}_s \).

Now fix \( s \geq s_0 \) with \( s = 2k, k \in \mathbb{N} \). Letting \( F_0 \) and \( G_1 \) as in Lemma 4.1 we set for \( u \in U_s \)
\[
\tilde{\mathcal{F}}(u)v := (F(u) + F_0(u))\tilde{G}(u)v, \quad \tilde{G}(u)v := \gamma \Delta(u)P(u)v + G_1(u)v.
\]
For \( u \in U_s \) let \( M(u) \) be the operator defined by
\[
M(u)v := M_0(u)v + \tilde{M}_0(u)v, \quad \tilde{M}_0(u)v = M_1(u)v + M_2(u)v + N(u)M_2(u)v \quad (4.10)
\]
Here, the main part \( M_0 \) of \( M \) is given by
\[
M_0(u)v := v - \Lambda(u)A(u)P(u)v \quad (4.11)
\]
with \( A \) from (2.9) (cf. [7]), whereas the lower order terms are given by
\[
M_1(u)w := -M_0(u)F_0(u)A(u)w, \quad (4.12)
\]
\[
M_2(u)v := \Delta(u)^+ (\gamma^{-1} G_1(u)v) \quad (4.13)
\]
From (3.2) and (4.7) we get
\[
M_0(u) \in \mathcal{L}(H^1(S, \mathbb{R}^m), H^1(S, \mathbb{R}^m)), \quad M_1(u) \in \mathcal{L}(H^t(S, \mathbb{R}^m)), \quad M_2(u) \in \mathcal{L}(H^t(S, \mathbb{R}^m), H^{t+1}(S)), \quad 2 \leq t \leq s-1.
\]
The operators depend smoothly on \( u \in U_s \) and have uniformly bounded norms as \( u \) varies in \( \tilde{U}_s \).

Because of \( P(u)A(u) = 0 \) the operator \( M_0(u) \) constitutes an isomorphism in \( L^2(S, \mathbb{R}^m) \) with inverse
\[
M_0(u)^{-1}v = v + \Lambda(u)A(u)P(u)v. \quad (4.14)
\]
In particular, we have
\[ c\|v\|_0 \leq \|M_0(u)v\|_0 \leq C\|v\|_0, \quad (4.15) \]
\[ c\|v\|_0 - C\|v\|_1 \leq \|M(u)v\|_0 \leq C\|v\|_0, \quad (4.16) \]
with suitable positive constants \( c, C \) independent of \( u \in \tilde{U}_s \) and \( v \in L^2 \). Moreover, after a simple calculation we obtain
\[ (M_0(u)f, M_0(u)v)_0 = (B(u)f, P(u)v)_0, \quad (4.17) \]
where \( f \mapsto B(u)f := P(u)(F(u)f) \) is the Dirichlet-Neumann operator.

For the sake of completeness, we gather some properties of \( B \) which we will need in the sequel. We will use the commutator notation \([Q_1, Q_2] := Q_1Q_2 - Q_2Q_1 \) for operators, in particular, if \( f \) is a function we will write \([f, Q]w := fw - Q(fw)\). Note that property b) is in fact the \( L^2 \)-symmetry of \( B(u) \) with respect to the measure induced from \( \Gamma_u \).

**Lemma 4.2.** Assume \( u \in \tilde{U}_s, f \in C^1(S), w \in H^2(S), v \in H^1(S) \). Then:

a) If \( f \geq \alpha > 0 \) then
\[ \int_S fwB(u)w dS \geq c\|w\|_1^2 - C\|w\|_0^2 \]
with \( c = c(\alpha) > 0, C = C(\|f\|_{C^1}) \). Moreover,
b) \[ \int_S w B(u)v dS = \int_S \omega(u)v B(u)(\omega(u)^{-1}w) dS, \]
c) \[ \|B(u)w\|_{-2} \leq C\|w\|_{-1}, \]
d) \[ ||[f, B(u)]w||_0 \leq C\|w\|_0 \]
with \( C = C(\|f\|_{C^1}) \),
e) \[ ||[\Lambda_i(u), B(u)]w||_0 \leq C\|w\|_1. \]
All constants are independent of \( u \in \tilde{U}_s \).

**Proof.** a) As in the proof of Lemma 3.3 we extend \( u \) to a diffeomorphism from \( E \) to \( \Omega_u \) and denote the coefficients of the corresponding induced metric by \( g^{ij} \) and the corresponding volume element by \( \sqrt{g} \). Let \( \nu \) denote the outer unit normal on \( S \) and let \( \delta \) denote the harmonic extension from \( S \) into \( E \). Let \( \phi \) be the solution of the Dirichlet problem
\[ L(u)\phi := \partial_i(\sqrt{g}g^{ij}\partial_j\phi) = 0 \text{ in } E, \quad \phi|_S = w. \]
Then
\[ B(u)w = \omega(u)^{-1}\nu_i\sqrt{g}g^{ij}\partial_j\phi, \]
and by integration by parts
\[ \int_S f \omega B(u) w \, dS = \int_E \partial_i (\mathcal{E}'(f \omega(u)^{-1}) \sqrt{g} g^{ij} \partial_j \phi) \, dx \]
\[ = \int_E \mathcal{E}'(f \omega(u)^{-1}) \sqrt{g} g^{ij} \partial_j \phi \, dx + \int_E \partial_i (\mathcal{E}'(f \omega(u)^{-1}) \sqrt{g} g^{ij} \partial_j \phi) \, dx \]
\[ \geq c \left\| \phi \right\|^2_E - C \left\| \phi \right\|^2 \left( \mathcal{E}' \phi \right)_E \left( \mathcal{E} \phi \right)_E \geq c \left\| \phi \right\|^2 \mathcal{E} + C \left\| \phi \right\| \mathcal{E} \geq c \left\| \phi \right\|^2 - C \left\| \phi \right\|^2_0. \]

The uniformity of these estimates with respect to \( u \in \hat{U} \) follows by a compactness argument as in [7].

b) The assertion follows from transforming the integral to \( \Gamma_u \), applying Green’s formula and transforming back.

c) Using b), the assertion follows from a standard duality argument and the fact that \( B(u) \in \mathscr{L}(H^2(S), H^1(S)) \).

d) Maintaining the notation from the proof of a), we have
\[ [f, B(u)]w = \omega(u)^{-1} \nu \sqrt{g} g^{ij} \partial_j (f \partial_i \phi - \partial_j \psi) \]
where \( \psi \) satisfies
\[ L(u) \psi = 0 \text{ in } E, \quad \psi|_S = f w = f \phi|_S. \]

Therefore, by estimates parallel to Lemma 4.3 in [7],
\[ \left\| [f, B(u)]w \right\|_{1/2} \leq \left\| \omega(u)^{-1} \nu \sqrt{g} g^{ij} \partial_j (\mathcal{E} \phi \psi - \psi) \right\|_{1/2} + C \left\| \phi \right\|_{1/2} \]
\[ \leq C \left( \left\| L(u) \mathcal{E} \phi \right\| + \left\| \phi \right\|_{1/2} \right) \leq C \left( \left\| \phi \right\| \mathcal{E} + \left\| \phi \right\|_{1/2} \right) \leq C \left\| \phi \right\|_{1/2}. \]

As both multiplication by \( f \) and \( B(u) \) are symmetric with respect to the \( L^2 \)-inner product induced from \( \Gamma_u \), we get by duality
\[ \left\| [f, B(u)]w \right\|_{-1/2} \leq C \left\| w \right\|_{-1/2}, \]
and the result follows by interpolation.

e) We have, by the chain rule for the operators \( D_k \),
\[ [\Lambda_i(u), B(u)] = [\alpha_k(u) D_k, B(u)] = [\alpha_k(u) D_k + \alpha_k(u) B', B(u)] = [\alpha_k(u), B(u)] D_k + [\alpha_k(u), B'] D_k. \]
The result follows now from b) and the estimate
\[ \left\| B'(u) D_k \right\| \leq C \left\| w \right\| \]
which is a simple consequence of (3.1).

The next lemma will be crucial in the proof of the main estimate as it will provide coercivity for the normal component.

**Lemma 4.3.** There are positive constants \( c, C \) such that
\[ (\Delta(u) + \gamma^{-1} w, B(u) w) \leq -c \left\| w \right\|^2_{-1/2} + C \left\| w \right\|^2 \]
for all \( u \in \tilde{U}_s, \ w \in H^1(S) \).

**Proof.** Set \( z := \Delta(u)^+ \gamma^{-1} w \). Then \( w = \gamma \Delta(u)z + Rz \) with a smoothing operator \( R \). By Lemma 4.2 b),

\[
I := (\Delta(u)^+ \gamma^{-1} w, B(u)w)_0 \leq (z, B(u)\gamma \Delta(u)z)_0 + \|z\|_1 \|B(u)Rz\|_{-1} \leq (\omega(u)\gamma \Delta(u)z, B(u)(\omega(u)^{-1}z))_0 + C\|w\|_{-1}^2.
\]

Setting \( \tilde{z} := \omega(u)^{-1}z, \ \tilde{\gamma} := \omega(u)^2 \gamma \) and using (4.5) we get

\[
I \leq (\tilde{\gamma} \Delta(u)\tilde{z}, B(u)\tilde{z})_0 + (\omega(u)\gamma[\omega(u), \Delta(u)] \tilde{z}, B(u)\tilde{z})_0 + C\|w\|_{-1}^2 \\
\leq (\tilde{\gamma} \Lambda_i(u)\Lambda_i(u)\tilde{z}, B(u)\tilde{z})_0 + C\|z\|_1^2 + C\|w\|_{-1}^2.
\]

By integration by parts, one obtains an estimate

\[
\left| \int_S \Lambda_i(u)f \ dS \right| \leq C \int_S |f| \ dS,
\]

cf. [7], Eq. (5.5). This yields

\[
I \leq -(\Lambda_i(u)\tilde{z}, \Lambda_i(u)\tilde{\gamma} B(u)\tilde{z})_0 + C\|\tilde{z}\|_1^2 + C\|w\|_{-1}^2 \\
\leq -(\Lambda_i(u)\tilde{z}, \tilde{\gamma} \Lambda_i(u)B(u)\tilde{z})_0 + C\|\tilde{z}\|_1 \sum_i ||[\Lambda_i(u), \tilde{\gamma}] B(u)\tilde{z}\|_0 + C\|w\|_{-1}^2 \\
\leq -(\Lambda_i(u)\tilde{z}, \tilde{\gamma} B(u)\Lambda_i(u)\tilde{z})_0 + C\|\tilde{z}\|_1 \sum_i \|[B(u), \Lambda_i(u)]\tilde{z}\|_0 + C\|w\|_{-1}^2 \\
\leq -c\|\Lambda_i(u)\tilde{z}\|_{1/2}^2 + C\|\tilde{z}\|_1^2 + C\|w\|_{-1}^2 \\
\leq -c\|\tilde{z}\|_{1/2}^2 + C\|w\|_{-1}^2 \leq -c\|w\|_{-1/2}^2 + C\|w\|_{-2}^2,
\]

where parts a) and e) of Lemma 4.2 have been used together with interpolation in the scale \( H^t(S) \).

\[\square\]

In view of (4.15), (4.16) for every fixed \( u \in U_s, \ s \geq s_0, \ s = 2k, \ k \in \mathbb{N} \) and \( \lambda \) sufficiently large (independent of \( u \in \tilde{U}_s \))

\[
(v, w)_{s,u} := \lambda(M_0(u)v, M_0(u)w)_0 + (M(u)\Delta_b^s v, M(u)\Delta_b^s w)_0 \tag{4.18}
\]

defines a scalar product on \( H^t(S, \mathbb{R}^m) \) which is equivalent to the usual one.

Now we are prepared to formulate and prove the following a-priori estimates for \( \mathcal{F} \) w.r. to the bilinear forms \( (\cdot, \cdot)_{s,u} \).

**Proposition 4.4.** Let \( s \geq s_0 \) be an even integer. Then

\[
(u, \mathcal{F}(u))_{s,u} \leq C\|u\|_s^2 \tag{4.19}
\]

for all \( u \in \tilde{U}_s \cap C^\infty(S, \mathbb{R}^m) \) with a constant independent of \( u \).

**Proof.** Setting \( v := \Delta_b^s u \) and using the notations of Lemma 4.1 it suffices to prove

\[
I(u)v^2 := (M(u)v, M(u)\mathcal{F}(u)v)_0 \leq C\|v\|_0^2 - c\|P(u)v\|_1^2 \tag{4.20}
\]
with positive constants $c$ and $C$. To begin with, we decompose $I$ according to

$$I(u)v^2 = I_1(u)v^2 + \ldots + I_4(u)v^2$$

with

$$I_1(u)v^2 = (M_0(u)v, M_0(u)\tilde{\mathcal{F}}(u)v)_0, \quad I_2(u)v^2 = (\tilde{M}_0(u)v, M_0(u)\tilde{\mathcal{F}}(u)v)_0$$

$$I_3(u)v^2 = (M_0(u)v, M_0(u)\tilde{\mathcal{F}}(u)v)_0, \quad I_4(u)v^2 = (\tilde{M}_0(u)v, \tilde{M}_0(u)\tilde{\mathcal{F}}(u)v)_0.$$  

Further, by (4.17) the term $I_1$ may be written as

$$I_1(u)v^2 = (M_0(u)v, M_0(u)F(u)\tilde{G}(u)v)_0 + (M_0(u)v, M_0(u)F_0(u)\tilde{G}(u)v)_0$$

$$= (P(u)v, B(u)\tilde{G}(u)v)_0 + (M_0(u)v, M_0(u)F_0(u)\tilde{G}(u)v)_0.$$  

As

$$P(u)v = \Delta(u)^+ \left( \gamma^{-1}(\tilde{G}(u)v - G_1(u)v) \right)$$

we have by (4.7)

$$\|P(u)v\|_1 \leq C \left( \|\tilde{G}(u)v\|_{-1} + \|v\|_0 \right)$$  \hspace{1cm} (4.21)

and by Lemma 4.3

$$\left( \Delta(u)^+(\gamma^{-1}\tilde{G}(u)v), B(u)\tilde{G}(u)v \right)_0 \leq -c_0\|\tilde{G}(u)v\|_{-1/2}^2 + C\|v\|_0^2.$$  

Consequently

$$I_1(u)^2v \leq (M_0(u)v, M_0(u)F_0(u)\tilde{G}(u)v)_0$$

$$- \left( \Delta(u)^+(\gamma^{-1}G_1(u)v), B(u)\tilde{G}(u)v \right)_0 - c_0\|\tilde{G}(u)v\|_{-1/2}^2 + C\|v\|_0^2.$$  \hspace{1cm} (4.22)

From Lemma 4.2 c) we get $F \in \mathcal{L}(H^{-1}(S), H^{-2}(S, \mathbb{R}^m))$ and therefore for $I_4$ we have the estimate

$$|I_4(u)v^2| \leq C\|\tilde{M}_0(u)v\|_1\|\tilde{M}_0(u)\tilde{\mathcal{F}}(u)v\|_{-1}$$

$$\leq C\|v\|_0\|(F(u) + F_0(u))\tilde{G}(u)v\|_{-2} \leq C\|v\|_0\|\tilde{G}(u)v\|_{-1},$$

where (4.7) has been applied again. Further, concerning $I_3$ we have

$$I_3(u)v^2 = (M_0(u)v, \tilde{M}_0F(u)\tilde{G}(u)v)_0 + (M_0(u)v, \tilde{M}_0F_0(u)\tilde{G}(u)v)_0$$

where the last summand allows the estimate

$$\left( \|M_0(u)v, \tilde{M}_0F_0(u)\tilde{G}(u)v \right)_0 \leq C\|v\|_0\|\tilde{G}(u)v\|_{-1}.$$  

Remembering $\tilde{M}_0 = M_1(u)P(u) + N(u)M_2(u)$ we have

$$(M_0(u)v, \tilde{M}_0F(u)\tilde{G}(u)v)_0$$

$$= (M_0(u)v, M_1(u)B(u)\tilde{G}(u)v)_0 + (P(u)v, M_2(u)F(u)\tilde{G}(u)v)_0.$$  

Using (4.21) we get

$$\left( \|P(u)v, M_2(u)F(u)\tilde{G}(u)v \right)_0 \leq C\|P(u)v\|_1\|\tilde{G}(u)v\|_{-1} \leq C\left( \|v\|_0^2 + \|\tilde{G}(u)v\|_{-1}^2 \right),$$

and consequently

$$I_3(u)v^2 \leq (M_0(u)v, M_1(u)B(u)\tilde{G}(u)v)_0 + C\left( \|v\|_0^2 + \|\tilde{G}(u)v\|_{-1}^2 \right).$$  \hspace{1cm} (4.23)
Arguing along the same lines for $I_2$ we obtain

$$I_2(u)v^2 = \langle \tilde{M}_0(u)v, M_0(u)vF(u)\tilde{G}(u)v \rangle_0 + \langle \tilde{M}_0(u)v, M_0(u)vF_0(u)\tilde{G}(u)v \rangle_0,$$

where again

$$\|\langle \tilde{M}_0(u)v, M_0(u)vF(u)\tilde{G}(u)v \rangle_0 \| \leq C\|v\|_0\|\tilde{G}(u)v\|_{-1}$$

and

$$\langle \tilde{M}_0(u)v, M_0(u)vF(u)\tilde{G}(u)v \rangle_0 = \langle M_2(u)v, B(u)\tilde{G}(u)v \rangle_0 + \langle M_1(u)v, M_0(u)vF(u)\tilde{G}(u)v \rangle_0$$

with

$$\|\langle M_1(u)v, M_0(u)vF(u)\tilde{G}(u)v \rangle_0 \| \leq \|M_1(u)v\|_2\|M_0(u)vF(u)\tilde{G}(u)v\|_{-2} \leq C\|P(u)v\|_1\|\tilde{G}(u)v\|_{-1} \leq C\left(\|v\|_0^2 + \|\tilde{G}(u)v\|_{-1}^2\right).$$

Thus we have

$$I_2(u)v^2 \leq \langle M_2(u)v, B(u)\tilde{G}(u)v \rangle_0 + C\left(\|v\|_0^2 + \|\tilde{G}(u)v\|_{-1}^2\right). \quad (4.24)$$

By the definitions (4.12), (4.13) of $M_1$ and $M_2$ we get

$$\langle M_0(u)v, M_1(u)vF(u)\tilde{G}(u)v \rangle_0 = -\langle M_0(u)v, M_0(u)vF_0(u)\tilde{G}(u)v \rangle_0$$

and

$$\langle M_2(u)v, B(u)\tilde{G}(u)v \rangle_0 = (\Delta(u)^+\gamma^{-1}G_1(u)v, B(u)\tilde{G}(u)v)_0.$$ 

Hence, summarizing (4.22) - (4.24) we have

$$I(u)v^2 \leq -c\|\tilde{G}(u)v\|_{-1/2}^2 + C\|v\|_0^2$$

with some positive constant $c$. As (4.21) implies

$$\|P(u)v\|_1 \leq C\left(\|\tilde{G}(u)v\|_{-1/2} + \|v\|_0\right)$$

we get the assertion. \qed

**Remark:** Reinspecting the estimates in the previous proofs it is straightforward to check that the constant $C$ in (4.19) is independent of $\gamma$ as long as $\gamma$ varies in some fixed set

$$\{\gamma \in H^{s+2}(S) \mid \gamma \geq \gamma^* > 0, \|\gamma\|_{s+2} \leq M\}.$$

Using Lemma 3.5, we write for $u \in \tilde{U}_s$, $v \in H^s$, $s \geq s_0$

$$\mathcal{F}'(u)v = F(u)(\gamma\Delta(u)(P(u)v) + G_2(u)v) + R(u)v \quad (4.25)$$

where $v \mapsto G_2(u)v := \mathcal{F}_3(u)v - \mathcal{F}(u) \cdot v$ and $v \mapsto R(u)v$ are operators of order one and zero, respectively. Note that $\mathcal{F}'(u)$ coincides with $\mathcal{F}(u)$ if $G_1$ is replaced by $G_2$ and $F_0$ is replaced by 0. Hence, defining (cf. (4.10)-(4.13))

$$M_3(u) := \Delta(u)^+\gamma^{-1}G_2(u)v, \quad \tilde{M}(u) := \tilde{M}_0(u) + M_3(u),$$

we find that $\tilde{M}$ has the same properties as $M$ above, and, parallel to (4.20), the following estimate which will be used in the uniqueness proof:
Lemma 4.5. Let $s \geq s_0$. Then there exists a constant $C$ such that for all $u \in \hat{U}_s$, $v \in H^s$ we have
\[ (\hat{M}(u)\mathcal{F}'(u)v, \hat{M}(u)v) \leq C\|v\|^2_0. \]

5. Proof of short time existence and uniqueness

We are ready now to prove our main results as announced in Theorems 2.1 and 2.2. As the existence proof is in some respects analogous to the corresponding considerations in [7], we restrict ourselves to an outline and refer to that paper for the details.

Fix an even integer $s_0 > (m + 7)/2$, $s_0 \geq 6$ and let $s \geq s_0$ be an even integer as well. Let $\hat{U}_s$ be defined as above. The notations $C_w([0,T],X)$ and $C^1_w([0,T],X)$ will denote the spaces of weakly continuous and weakly continuously differentiable functions, respectively, with values in some subset $X$ of a normed space.

At first an estimate which provides uniqueness and Lipschitz continuous dependence on the initial value in the $L^2$-norm is given:

Proposition 5.1. Let $u, v \in C_w([0,T],\hat{U}_{s_0}) \cap C^1([0,T],H^{s_0-3}(S,\mathbb{R}^m))$ be two solutions of (2.2). Then
\[ \|v(t) - u(t)\|_0 \leq C\|v(0) - u(0)\|_0 \tag{5.1} \]
with $C$ depending only on $\hat{U}_{s_0}$ and on $T$.

Proof. Set $w(t) := v(t) - u(t)$ and note that
\[ w(t) \in C([0,T],H^\sigma(S,\mathbb{R}^m)) \cap C^1([0,T],H^{\sigma-3}(S,\mathbb{R}^m)) \]
for $\sigma < s_0$. In particular, the map
\[ t \mapsto g(t) := \|\hat{M}(u(t))w(t)\|^2_0 \]
is differentiable and has the derivative
\[ g'(t) = 2(\hat{M}'(u(t))\{u'(t)\}w(t), \hat{M}(u(t))w(t))_0 \]
\[ + 2(\hat{M}(u(t))(\mathcal{F}(v(t)) - \mathcal{F}(u(t))), \hat{M}(u(t))w(t))_0. \]

To estimate the first term we note that, parallel to the estimates in Lemma 3.1,
\[ \|\hat{M}'(u(t))\{u'(t)\}w(t)\|_0 \leq C\|u'(t)\|_{s_0-3}\|w(t)\|_0 \]
\[ \leq C\|\mathcal{F}(u(t))\|_{s_0-3}\|w(t)\|_0 \leq C\|w(t)\|_0. \]

The second term can be estimated by using
\[ \mathcal{F}(v(t)) - \mathcal{F}(u(t)) = \mathcal{F}'(u(t))w(t) + R, \]
where
\[ R := \int_0^1 \int_0^\tau \mathcal{F}''(\theta v(t) + (1 - \theta)u(t))\{w(t), w(t)\} d\theta d\tau \]
al lows an estimate
\[ \|R\|_0 \leq C\|w(t)\|_3\|w(t)\|_{s_0-3} \leq C\|w(t)\|_{s_0}\|w(t)\|_0 \leq C\|w(t)\|_0, \]
where estimates on $\mathcal{F}''$ parallel to Lemma 3.1 and norm convexity have been used. Thus, by Lemma 4.5,

$$g'(t) \leq 2(\dot{M}(u(t))\mathcal{F}'(u(t))w(t), \dot{M}(u(t))w(t))_0 + C\|w(t)\|_0^2 \leq \|w(t)\|_0^2 \leq Cg(t).$$

Therefore, by Gronwall’s inequality,

$$\|v(t) - u(t)\|_0^2 \leq Cg(t) \leq Cg(0) \leq C\|v(0) - u(0)\|_0^2.$$  

$\square$

To prove existence, we will rely on an abstract existence theorem whose proof has been given in [7]. It generalizes an existence theorem concerning evolution equations with semibounded operators by Kato and Lai [8] to the case of variable bilinear forms. The setting is the following:

Let $X \subseteq Y \subseteq Z$ be real, separable Banach spaces with dense and continuous embeddings and $\mathcal{U} \subseteq Y$ open. For every $u \in \mathcal{U}$ let $\langle \cdot, \cdot \rangle_u : X \times Z \to \mathbb{R}$ be a continuous and nondegenerate bilinear form, such that with fixed constants $C \geq 1$, $M \geq 0$:

\begin{align*}
(H1) & \quad (v, w)_u = \langle v, w \rangle_u \text{ for all } v, w \in X; \\
(H2) & \quad C^{-1}\|v\|_Y^2 \leq \langle v, v \rangle_u \leq C\|v\|_Y^2 \text{ for all } v \in X, u \in \mathcal{U}; \\
(H3) & \quad \langle v, v \rangle_u \leq \langle v, v \rangle_u \left(1 + M\|u - w\|_Z\right) \text{ for all } v \in X, u, w \in \mathcal{U}; \\
(H4) & \quad \text{weak convergences } u_n \to u \text{ in } Y, u_n, u \in \mathcal{U}, \text{ and } w_n \to w \text{ in } Z \text{ imply } \langle v, w_n \rangle_{u_n} \to \langle v, w \rangle_u \text{ for all } v \in X.
\end{align*}

Assuming (H) to hold, by the dense embedding $X \subseteq Y$ and

$$\left|\langle v, w \rangle_u\right|^2 \leq \langle v, v \rangle_u \langle w, w \rangle_u \leq C^2\|v\|_Y^2\|w\|_Y^2 \text{ for } v, w \in X,$$

there exists to each $u \in \mathcal{U}$ a scalar product $\langle \cdot, \cdot \rangle_u$ on $Y$, which is compatible with $\langle \cdot, \cdot \rangle_u$, i.e. we have

$$(v, w)_u = \langle v, w \rangle_u \quad \text{for } v \in X, w \in Y.$$ 

Moreover, for $u_n, u \in \mathcal{U}, u_n \to u, w_n \to w$ in $Y$ implies

$$(v, w_n)_{u_n} \to (v, w)_u \quad \text{for all } v \in X.$$ 

For the sake of brevity we put

$$\|v\|_u = (v, v)_u^{1/2}, \quad \|u\| = (u, u)^{1/2}_u.$$ 

**Theorem 5.2.** Assume (H) is satisfied with some ball

$$\mathcal{U} = B := \{u \in Y \mid \|u\|_Y < R\}, \quad R > 0,$$

and $\mathcal{F} : B \to Z$ is a weakly sequentially continuous mapping such that

$$2(u, \mathcal{F}(u))_u + M\|\mathcal{F}(u)\|_Z\|u\| \leq \beta(\|u\|^2) \quad \text{for all } u \in X \cap B$$

with a $C^1$-function $\beta : \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$. Let $u_0 \in B$, $\|u_0\| < r := R/(2C^3)^{1/2}$,
and $T > 0$ such that the solution $\rho$ of the scalar Cauchy problem

$$\frac{d\rho}{dt} = \beta(\rho(t)), \quad \rho(0) = \|u_0\|^2$$

exists on $[0, T]$ and satisfies $\rho(t) < r^2$ there. Then the Cauchy problem

$$u'(t) = \mathcal{F}(u(t)), \quad u(0) = u_0,$$

possesses a solution $u \in C_w([0, T], \mathcal{W}) \cap C^1_w([0, T], \mathcal{Z})$ for which additionally

$$\|u(t)\|^2 < \rho(t) \text{ for all } t \in [0, T],$$

$$u(t) \to u_0 \text{ in } Y \text{ for } t \to +0.$$

**Proof of Theorems 2.1, 2.2 (outline):** Instead of (2.2), (2.3) we consider the Cauchy problem

$$\left\{ \begin{align*}
v'(t) &= \hat{T}(v) := \mathcal{F}(v + w_0), \\
v(0) &= u_0 - w_0,
\end{align*} \right. \tag{5.5}$$

where $w_0$ is smooth and near $u_0$.

To apply Theorem 5.2, we set for $\varepsilon \in (0, 1]$

$$X := H^{s+3}(S, \mathbb{R}^m), \quad \| \cdot \|_X := \| \cdot \|_{s+3} + \varepsilon \| \cdot \|_{s+3},$$

$$Y := H^s(S, \mathbb{R}^m), \quad \| \cdot \|_Y := \| \cdot \|_{s+1} + \varepsilon \| \cdot \|_s,$$

$$Z := H^{s-3}(S, \mathbb{R}^m), \quad \| \cdot \|_Z := \| \cdot \|_{s-3} + \varepsilon \| \cdot \|_{s-3}.$$  

For $u \in \bar{U}_{s_0}$, let $\langle \cdot, \cdot \rangle_u$ be the bilinear form compatible to the inner product on $Y$ given by

$$\langle v, w \rangle_u := (v, w)_{s_0, u} + \varepsilon^2 (v, w)_{s, u}$$

with $(v, w)_{s_0, u}$, $(v, w)_{s, u}$ given by (4.18). Using Proposition 4.4 and arguments parallel to the ones given in the proof of Lemma 5.4 in [7] one checks that the assumptions (H) are satisfied if $\varepsilon$ is chosen small enough. Thus Theorem 5.2 yields existence of a solution

$$u \in C_w([0, T], \bar{U}_s) \cap C^1_w([0, T], H^{s-3}(S, \mathbb{R}^m))$$

and an estimate

$$\|u(t)\|_s \leq C(1 + \|u_0\|_s), \tag{5.6}$$

with $C$ independent of $u_0$ and $t$.

The uniqueness result from Proposition 5.1 enables us to define an evolution operator $T_t$ by setting $T_t u_0 := u(t)$. By a nonlinear interpolation result given in [2], Proposition A.1 and Remark A.2, the estimates (5.1) and (5.6) imply $H^\tau$-continuity of $T_t$ for $\tau \in [0, s]$, uniformly in $t \in [0, T]$. Approximation of the initial value $u_0$ by $u_0^n \in H^{s+1}$ and of the solution $u$ by the corresponding solutions $u_0^n \in C([0, T], \bar{U}_s) \cap C^1([0, T], H^{s-3}(S, \mathbb{R}^m))$ yields then

$$u \in C([0, T], \bar{U}_s) \cap C^1([0, T], H^{s-3}(S, \mathbb{R}^m))$$

by uniform convergence. Finally, the existence time $T$ can be shown to be independent of $s$ by standard continuation arguments. For further details we refer to [7].
6. Nontrivial equilibria

Finally, we want to give some partial results concerning the equilibria of our problem. Our considerations are restricted to the situation near trivial equilibria; i.e. we will assume that the domain is near a ball and \( \gamma \) is near a constant. Therefore we specialize the reference domain to

\[
E := \{ x \in \mathbb{R}^m \mid |x| < 1 \}, \quad S := \partial E = \{ x \in \mathbb{R}^m \mid |x| = 1 \}
\]

and set

\[
\tilde{U}_s := \{ u \in H^s(S, \mathbb{R}^m) \mid \| u - w_0 \|_s \leq \delta \}
\]

\[
w_0(x) := x \quad \text{for} \quad x \in S.
\]

(6.1)

In the sequel we assume \( \delta > 0 \) sufficiently small, whenever necessary. Moreover, to stress the dependency on \( \gamma \), we consider now \( \mathcal{F}(u) \) and \( \mathcal{G}(u) \) as linear operators defined by

\[
\mathcal{F}(u)v := F(u)(\mathcal{G}(u)v), \quad \mathcal{G}(u)v := vH(u) + A(u)\Delta(u)v
\]

(6.2)

For \( s \geq s_0 \) and \( 2 \leq t \leq s \) the operators

\[
\mathcal{G}(u) \in \mathcal{L}(H^t(S), H^{t-1}(S)), \quad \mathcal{F}(u) \in \mathcal{L}(H^t(S), H^{t-2}(S))
\]

(6.3)

depend smoothly on \( u \in \tilde{U}_s \). Now, for given surface energy density \( \gamma \in C^\infty(S) \) and \( u_0 \in \tilde{U}_s \) the Cauchy problem (2.2), (2.3) reads

\[
\dot{u} = \mathcal{F}(u)\gamma, \quad u(0) = u_0.
\]

(6.4)

Lemma 6.1. (i) We have

\[
\int_S \omega(u)N_i(u)u_j dS = \delta_{ij}|\Omega_u|,
\]

where \( \delta_{ij} \) denotes the Kronecker symbol and

\[
\int_S \omega(u)N(u)(\mathcal{G}(u)\gamma) dS = 0.
\]

(6.6)

(ii) For any solution \( u = u(t) \) of (2.2) the center of gravity

\[
M(t) := \int_{\Omega_u(t)} x dx
\]

(6.7)

is independent of \( t \).

Proof. (i) After retransformation onto \( \Gamma_u \), the equation (6.5) follows from

\[
\int_{\Gamma_u} n_i x_j d\sigma = \int_{\Omega_u} \partial_i x_j dx = \delta_{ij}|\Omega_u|,
\]

whereas (6.6) reads

\[
\int_{\Gamma_u} n(\gamma\kappa - \psi) d\Gamma = 0,
\]

where \( \psi \) is harmonic in \( \Omega_u \) with Neumann condition \( \partial_n \psi = \Delta_{\Gamma_u} \gamma \) on \( S \). By Green’s formula we get

\[
\int_{\Gamma_u} n \psi d\Gamma = \int_{\Gamma_u} x\partial_n \psi d\Gamma,
\]
hence writing \( n\kappa = \Delta_{\Gamma_u}x \) we obtain
\[
\int_{\Gamma_u} n(\gamma\kappa - \psi) \, d\Gamma = \int_{\Gamma_u} (\gamma\Delta_{\Gamma_u}x - x\Delta_{\Gamma_u}\gamma) \, d\Gamma = 0 \tag{6.8}
\]
by an integration by parts.

(ii) Consider the solution to (1.1) corresponding to \( u \). We have, using Green’s formula and (6.8),
\[
\dot{M}(t) = \int_{\Gamma(t)} x V_n \, d\Gamma(t) = \int_{\Gamma(t)} x \partial_n \phi \, d\Gamma(t) = \int_{\Gamma(t)} \partial_n x \phi \, d\Gamma(t) = \int_{\Gamma(t)} n(\gamma\kappa - \psi) \, d\Gamma(t) = 0.
\]

**Remark:** Of course, the conservation of the center of gravity is not restricted to evolutions near equilibrium. Note that the correction term \( G \) plays a crucial role here as well as for the generalized gradient flow property.

It is natural to call a function \( \gamma \) on \( S \) an *equilibrium surface energy density* for a given \( u \in \tilde{U}_s \) if
\[
\mathcal{F}(u)\gamma = 0 \text{ on } S \quad \text{or equivalently } \quad \mathcal{G}(u)\gamma = \text{const. on } S. \tag{6.9}
\]
The following result ensures the existence of such an equilibrium surface energy density, uniquely determined up to a scaling factor and a linear combination of \( m \) functions.

**Proposition 6.2.** For given \( u \in \tilde{U}_s \) there exists a uniquely determined \( \gamma(u) \in H^{s-1}(S) \) such that
\[
\mathcal{G}(u)\gamma(u) = 1 \text{ on } S, \quad \int_S N(u)\gamma(u) \, dS = 0.
\]

**Proof.** If \( a \in \mathbb{R}^m \), \( u \in \tilde{U}_s \) and \( \gamma \in H^{s-1}(S) \) such that
\[
\mathcal{G}(u)\gamma = 1 + a \cdot u \text{ on } S,
\]
then (6.6) implies
\[
\int_S \omega(u)N(u)(a \cdot u) \, dS = 0,
\]
and further by (6.5)
\[
a_1 = \ldots = a_m = 0.
\]
Hence it suffices to show the invertibility of the operator
\[
L(u) \in \mathcal{L}(H^{s-1}(S) \times \mathbb{R}^m, H^{s-2}(S) \times \mathbb{R}^m)
\]
given by
\[
L(u)(\gamma, a) := (\mathcal{G}(u)\gamma - a \cdot u, c), \quad c := \int_S N(u)\gamma \, dS.
\]
As \( L(u) \) depends smoothly on \( u \in \tilde{U}_s \) it remains to show the invertibility for \( u = u_0 \). In this case we have
\[
\mathcal{G}(u_0)\gamma = -(m-1)\gamma - A_S\Delta_S\gamma
\]
where \( A_S \) and \( \Delta_S \) denote the Neumann-to-Dirichlet operator and the Laplace-Beltrami
operator on the unit sphere $S \subset \mathbb{R}^m$, respectively. Using an eigenspace decomposition
\[
\gamma = \sum_{l=0}^{\infty} \gamma_l,
\]
where $\gamma_l$ is a spherical harmonic of order $l$ we get
\[
L(w_0)(\gamma, a) = \left( -(m-1)\gamma_0 - a \cdot x + \sum_{l=2}^{\infty} (l-1)\gamma_l, \frac{|S|b}{m} \right),
\]
where $\gamma_1 = b \cdot x$. This proves the assertion.

Of course, the opposite question is more natural for our evolution: Given a surface energy density $\gamma$, find a corresponding class of equilibrium shapes satisfying (6.9) and, for given initial shape $u_0$, show global existence of the solution in time and convergence to some member of this class. While this is relatively straightforward in the case of constant $\gamma$ (see e.g. [5]), it is an open problem in our case.

To illustrate possible equilibrium shapes $u^*$ we have performed several numerical test calculations for $m = 2$. Starting from $S = \Gamma_{w_0}$ and a surface energy density of form
\[
\gamma(x) = 1 + 0.8\cos(6 \varphi), \quad x = (\cos \varphi, \sin \varphi) \in S, \quad 0 \leq \varphi < 2\pi,
\]
we obtain $u^*$ as pictured in Figure 1. In contrast, if the correction term $G(u)$, which ensures the gradient flow structure of the evolution problem, is dropped in the definition (2.4), then the resulting $u^*$ is given by Figure 2. Clearly, in the latter situation every equilibrium configuration $(u, \gamma)$ is characterized by $\gamma \kappa = \text{const.}$ on $\Gamma_u$, hence $\Gamma_u$ must be convex. (This is similar to a Hele-Shaw evolution where the values of $\gamma$ are transported only in normal direction to the moving boundary, as this also leads to a dropping of the term $G(u)$.) As Figure 1 shows, this is not true for the full problem.

The numerical evidence indicates that it is reasonable to conjecture that the equilibria (near the trivial one) form an attractive limit set although a proof for this is lacking.
References


