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A linear algebra course with PC-MATLAB: some experiences

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The authors present their views on the impact that the use of computers and software packages should have on the contents of a first service course on linear algebra. Furthermore they report on their experiences using the software package PC-MATLAB in such a course.

1. Introduction

Almost every first service course on linear algebra aims at three things. First of all the students have to be able to speak the language of linear algebra. Second, they have to be able to solve standard problems with standard techniques. And third, the students have to have enough background to be able to master new concepts and new techniques during their further academic training and in their profession. Of course the elaboration of these aims depends on the discipline for which the course is intended. But one also has to keep in mind the fact that computers and software packages on linear algebra are used increasingly in many fields of applications. We believe that because of this development the contents of the course has to change. One of the reasons is that introduction of certain concepts becomes necessary if one wants to understand the mathematics behind the algorithms used by the software, or wants to be able to read the manual in detail and really understand the computer output. In other words if one wants to work with the software in a sensible way and not use it as a black box. Another reason is that other techniques become available to solve (standard) problems. This gives rise to the question of which technique to use; a choice has to be made. In short we believe that students have to become familiar with these tools and thus one has to give the students a good background on linear algebra with the use of a computer in mind. Also from an educational point of view we find it desirable to incorporate the computer in such a course. This allows the user to concentrate on the main aspects of problems instead of the technical aspects.

At Eindhoven University of Technology, we have a few years experience with first service courses in calculus and linear algebra in which from the beginning we use the computer. These courses are given to first year students in mechanical engineering. In a recent article [1] we described in general the history of and the philosophy behind our courses as well as the role of the computer. In this paper we concentrate on the linear algebra part in more detail. We first present our starting points for such a course. Then we give a description of the contents of the course. Next we discuss the use of a computer by the students during the course and the use of PC-MATLAB in particular. We conclude with some experiences.

2. Starting points

In the introduction we discussed the influence of the computer on the elaboration of the general aims of a course in mathematics. Based upon this we chose the starting points of our course. In this section we present some of them.

We believe that in a first course on linear algebra one must not start at too abstract a level. Yet, we must provide the students with enough background to be able to master new concepts and new techniques in future. So we try to discuss concepts and techniques in such a way that they can be generalized. In some instances we go deeper into the matter. In this way we try to provide the students with a profound basic knowledge instead of a wide one. Our treatment of vectorspaces may serve as an example of this. On the one hand we do not treat the general concept of vector spaces (and then quickly descend to $\mathbb{R}^n$, as happens quite often) but stick to $\mathbb{R}^n$. On the other hand we talk about the four fundamental spaces of a matrix: the nullspace $N(A)$, the column space $R(A)$, the row space $R(A^T)$ and the left nullspace $N(A^T)$ and their mutual relation (e.g. being perpendicular to each other).

We also place mathematical insight above mathematical rigour. In many cases we can argue by a simple reasoning why a certain property must hold without presenting a detailed mathematical proof. Proofs of properties are given only when they contribute to insight.

Furthermore we try to connect the main topics to each other and show the students the links, so as to give them more insight into the matter and make it easier to internalize. We pay attention to the structure of the whole more than to individual parts. Trying to follow a systematic approach, we start with practical problems (e.g. fitting data). Then we translate the problem into linear algebra (overdetermined systems) and try to solve it (normal equations). Afterwards we discuss it in more detail leading to general concepts (projection and QR-decomposition). This provides student motivation. We only use the concepts we need, keeping in mind our aim to give the students enough background for the future. We try to avoid teaching recipes (some can be done by computer). Instead we discuss the concepts in different contexts, e.g. we discuss how the change to a basis of eigenvectors simplifies different kinds of problems: for instance difference equations, differential equations and geometric interpretation of matrices.

We keep in mind the use of certain techniques in practice. So for instance we discuss that the rule of Cramer is not very effective to solve a general system and we pay attention to the benefit of knowing the special structure of a matrix, e.g. a bandmatrix. The students do not have to become experts at choosing the best algorithm for all kinds of linear algebra problems (it is not a course on numerical linear algebra), but we do want them to be aware of the fact that there is not one single strategy to all problems that look alike.

Central in our course is the concept of matrix decompositions. One of the reasons is that the students have to understand the language of the software and have to be able to work with these decompositions as an important tool. But they also form the abstraction of certain techniques (e.g. Gauss-elimination leading to LU-decomposition and Gram-Schmidt leading to QR-decomposition). Emphasis is put on the use of these decompositions, as their practical calculation can be left to the computer. Another reason for introducing matrix decompositions is that it gives the opportunity to concentrate on matrices as a whole instead of individual matrix elements. Once you have introduced certain decompositions (e.g. spectral decomposition) it becomes easier to go into more complicated concepts (e.g. singular value decomposition).

3. Description of the contents of the course

This first year course in linear algebra is divided into two blocks. A block covers nine weeks, each containing a three hour lecture followed by a tutorial. We use as a textbook Strang's *Linear Algebra and its Applications* [2], accompanied by a study-guide we developed that contains additional reading and computer exercises. In this section we give a short description of the contents of the course.

In the first term the solving of linear equations plays a prominent part. We start with the true meaning of the two words 'linear algebra': the solving of linear equations. The students already know the algorithm of repeated substitution. We show how to do this systematically and in a well-organized way, leading to Gaussian elimination, matrix notations and matrix multiplication. We discuss whether solving linear equations can be done more efficiently and so introduce the counting of flops, the method of Gauss-Jordan and the recording of the elimination steps leading to the LU-decomposition. We consider the breakdown of elimination and talk about non-singular matrices in general. Because of their importance positive definite matrices and the Cholesky decomposition are discussed. As for numerical aspects we introduce the concepts of well- and ill-conditioned matrices, partial pivoting and the iterative methods of Jacobi and Gauss-Seidel. Up till this point the systems of equations have been square. Next rectangular systems are treated. The LU-decomposition gives us the general solution. Then we introduce the concepts of nullspace, column space, linear independence, basis and dimension. We discuss it, not in general but linked to linear systems. At this point students have enough background to look at the following practical problem: models of relations between unknowns with some measurements. The translation of these kinds of problems into linear algebra leads to overdetermined systems. We solve these problems of regression introducing normal equations. Afterwards we discuss it in more detail, that is projections, Gram-Schmidt and as an abstraction the QR-decomposition: its construction and its use. We conclude with the fact that the four fundamental subspaces are perpendicular to each other. In the last week of the first term determinants are treated.

In the second term we concentrate on eigenvalues and eigenvectors. We introduce their concept, starting with a dynamical system. The eigenvalue decomposition for matrices that are diagonalizable is treated, together with some of its applications, e.g. matrixpowers and their use in difference equations and in the calculation of eigenvalues and eigenvectors by power methods. Then linear transformations are introduced, where we concentrate on their matrix representation and geometrical interpretation. In this context the influence of a change of basis is treated: with respect to suitable bases, e.g. a basis of eigenvectors, it is possible to find the matrix representation in a simple way or to give an interpretation of a transformation, represented by a certain matrix. We then discuss properties of special linear transformation and matrices, e.g. symmetric, Hermitian, orthogonal and unitary types. At the end of the pure linear algebra part of the course we try to give the students the idea that they master the general concept of matrix-decompositions: there we treat, though not in an extensive way, the singular value decomposition and the polar decomposition with their applications in the direction of pseudo-inverses and the interpretation of general transformation-matrices.

In the last three weeks of the course systems of ordinary differential equations form the topic. In fact it is an extension of the earlier treatment of differential equations in the calculus course and in this way it is some kind of mixture of calculus

and linear algebra. Linear and almost linear systems of differential equations are studied qualitatively by their behaviour in the phase plane. Based on these ideas the concepts of linearization and of the stability of solutions are worked out more carefully, together with the analytical solution on the basis of a decoupling of the equations by a suitable choice of coordinates. We conclude with types of differential equations that can be converted to linear systems, e.g. higher-order linear equations or non-autonomous equations. The emphasis is on the fact that this is a way to get information about a broader class of differential equations, more than on the specific properties of these types of equations.

4. Actual use of the computer

Up till now we have concentrated on the influence of computers and software with respect to the mathematical contents of our course. However this is only one aspect and one also has to think about the actual use of a computer during the course. In this section we discuss and illustrate some of the mathematical and didactical considerations that might play a role in respect to this. We pay no attention to the practical realization, as this will be the topic of the next section.

In situations where a lot of standard calculations have to be done before one can come to a conclusion, the use of a computer can be very effective. Even if these calculations can in principle be carried out by hand the fact that they are time-consuming and apt to calculational errors may be demotivating and they may obscure the real aspects of the problem. An example of this is the determination of the four fundamental subspaces of a matrix: the real problem is the interpretation of the row-echelon form of the matrix, not its calculation. By leaving the calculation to the computer one can concentrate on this interpretation and have a lot more practise at the same time.

Another advantage is that with a computer it is no longer necessary to search for examples and exercises that lead to nice and easy calculations or results. In this way one can treat more realistic problems. For instance with the geometric interpretation of the matrix of a linear transformation there are virtually no restrictions, as the eigenvalues and eigenvectors of the matrices that play a role in the problem can be calculated by numerical techniques. The same holds for a least-square fit through a set of data.

The actual use of the computer is also helpful to illustrate some aspects of the theory or to introduce unknown techniques or concepts. We think of an approach where, with the help of the computer as a tool, the student forms conjectures on the basis of some (guided) experiments. In this way the student can play an active role and it will be more motivating and effective than a mere discussion of the results. Moreover it introduces them to a more experimental approach to mathematics. This can be important in situations where there is not one single way to get to an answer and one has to obtain some global information first to choose a strategy for the solution. Results that may influence this strategy can also be obtained quickly. In this way we introduce, for example, the rules of multiplication or inversion of partitioned matrices or the effects of small changes in ill-conditioned systems of linear equations.

To get the students acquainted with the real use of the mathematical techniques, as is one of the aims of our course, it is necessary to have some practise with the computer. In this way one gets a better view of the (im)possibilities of the techniques and on the differences between theory and reality. Non-diagonalizable matrices serve as a good example for this, as numerically they may seem diagonalizable.

The use of a computer also has some drawbacks. If it comes in action too early it may distract attention from the mathematics that play a role and it may result in an attitude of just 'pushing the buttons' to get an answer. Even more dangerous for students can be the use of specialized techniques in situations where they do not apply. As a reaction to this it can be very tempting for us to discuss all kinds of specialized techniques offered by the software package, at the expense of the basic principles. However, if in this way one gets fixed on techniques, recipes and numerical results only, there is no sound basis of concepts to build upon in future, and one of the goals of a first course in linear algebra will not be reached.

5. Some practical aspects

In this section we describe some of the choices we have made with respect to the realization of the course. We concentrate first on the organization of the tutorials and on the software we use, starting from our present situation.

As described earlier the course covers two blocks of nine weeks each. In every week there is a three hour lecture for a group of over 300 students and a tutorial session where we work in groups of 40 students under the guidance of a member of staff. These tutorials take one afternoon and are scheduled a few days after the three-hour lecture. In the meantime the students have to do some homework. The goal of this is to become familiar with the new concepts, taught at the lecture, and to actively work through some additional examples. The homework exercises are chosen in such a way that the use of a computer is not necessary, as for the moment only a minority of the students have a PC at their disposal at home.

During the first part of the tutorial session problems encountered with these exercises are discussed by the tutor with the group of students. However we avoid a systematic treatment of all the exercises as this appears to demotivate the students to work through the problems at home and as it may be too time-consuming.

In the second part of the session the students have to work on a set of exercises that require the use of the computer. In fact these exercises aim at a further practise, illustration or extension of the mathematical topics at hand and are in line with the considerations as described in the preceding section. The problems that one may encounter while working on the exercises are no longer discussed with the group but they are answered individually by the tutor. We choose this form as it stimulates students to an active attitude. For practical reasons, namely the available number of PCs, we let the students work in pairs, each couple having an IBM-AT compatible computer linked through a local network, at their disposal.

With respect to the software that is used in the course we decided in favour of a professional standard software package that is widely available, with PC-MATLAB as our specific choice.

One of the considerations that played a role in this decision is that we did not want students to lose too much time on getting started with real calculations on the computer, for this would distract their attention from the mathematics. As an implication we found it undesirable to let students develop programs by themselves, as most students do not have a large experience in computer programming in their first year at university. Furthermore we found it important to let students get some experience with software they may encounter during the rest of their career. With a package developed by ourselves or a standard software package that is supplemented with programs specifically adapted to the didactical needs of our course, this would not be the case because in this form it would only be available locally.

Our specific choice of PC-MATLAB is in line with this, as it is user-friendly and based on the well-known algorithms of LINPACK and EISPACK. Moreover it is used in courses on signal analysis that the students have to take in their second year. However, our course is not concentrated around PC-MATLAB as we only use facilities that are directly needed and do not go into specific details. The use of other software packages or program libraries with the same facilities, e.g. GAUSS, NAG, or IMSL, will have no influence on the main points of our course. Only some minor adjustments are needed then, e.g. in the study-guide in which we now introduce the relevant MATLAB commands.

6. Experiences

At the time of writing, we have taught our course on linear algebra with PC-MATLAB for three years. During this period we gained a lot of experience. Partly specific to our linear algebra course and its organization which we will consider in this section. Partly more general with respect to the use of PCs in mathematics courses. That is reported elsewhere [1].

On evaluating the course with students it appears that they appreciate the course in general. They are positive about our emphasis on the relation between different topics and about the fact that we pay attention to the use of certain techniques in practice. More than in 'traditional' courses on linear algebra they seem aware that the subject will be important in their own discipline and that passing the exam, though useful for the short term, is not the only aim. At the start they feel uncomfortable about the fact that the emphasis is not on a training of standard techniques for standard problems. As here they do not have a set of solution recipes at their disposal, they are uncertain at first about the things they have to do to solve a problem. However, in the course of time, most of them get a feel for how to build on their knowledge in less standard problems, such as they may encounter in future practice. In this way we think that their understanding of the basic concepts is improved, perhaps at the expense of some technical skills or a strictly formal way of mathematical reasoning.

With respect to the time schedule there are some complaints, mainly about the second term. Here they find that at some places there is too little time to master a topic. Partly this is not specific to a computer course, as in general in the second term topics become more difficult and one has to assume former knowledge, because time is insufficient to start from first principles over and over again. For example, with the calculation of null space and image space of a linear transformation it is necessary that one has mastered the treatment of the fundamental spaces of a matrix in the first term. Yet, with the computer, less time is needed for calculations, so it looks as if topics can be treated in a shorter period. Here one has to keep in mind that some minimal time is needed for a new topic to settle down and one has to be careful in filling in the saved time with new topics.

The tutorial sessions function very well. Students work there with enthusiasm, especially on problems where they can use PC-MATLAB. This demonstrates, once more, how the actual use of a computer can motivate and stimulate the teaching process. In some measure this positive effect would be lost if students did not have to generate the relevant computer output themselves; i.e. if one provided them *a priori* with a copy of it and left only the interpretation to them. Furthermore in this way there might be too much presuggestion for a strategy of solution. Therefore we would advise this approach only if the selection of a strategy plays no role or if one has

an insufficient number of computers available. However, with the actual use of the computer, one has to watch out for the effect that some students tend to concentrate on details of operating the program and use this as an excuse to neglect the mathematics behind it.

It is appreciated that working with the computer takes place in couples. In this way minor (technical) problems can be solved by the students together, while the number of real questions, to be handled by the tutor, remains manageable. Yet, there is another important advantage in this approach: it forces students to discuss explicitly the strategy of solving the problem or the results obtained by the computer. In this way they get some practise in using the language of mathematics, while formulating their ideas. For this reason we would prefer working in pairs, even if there are enough PCs available to let students work individually.

A lot of care is necessary in selecting and formulating the exercises and problems for the tutorial sessions, because otherwise the number of questions might become too large to be handled by a single tutor. Particularly in situations where students are introduced, with the help of the computer, to unknown techniques or concepts, one has to find the right balance between questions that are formulated only in a broad sense, leaving room for creativity and experimentation, and very detailed questions, guiding students carefully through every step they have to take. We feel that using the computer here is especially effective if the new topics are extensions of already known results, otherwise this way of learning might be too time-consuming. An example is the use of power methods to determine eigenvalues and eigenvectors: the basic principle of the method is already known but further refinements of the technique, like the scaled or the inverse power method, are new.

At the end of each term there is a written examination. Here, mainly for practical reasons of organization, the students have no computer at their disposal. If necessary we supplement questions with a copy of computer output that might play a role in solving the problem. In fact, to test the choice of a strategy for solving the problem, we sometimes supply redundant information, but here the possibilities are restricted. Though this approach works reasonably, we feel there is a discrepancy with the way in which we expect students to work during the tutorials. Some students are also aware of this and they tend to concentrate on types of questions they expect in the exam. In this way there is a danger that they fix on recipes or techniques, a trend we wanted to stop. To prevent this we are considering, for next year, an examination with the use of a computer, as in this way it is possible to have an exam that is more in line with the tutorials and with our aims.

The overall results of the exams are satisfactory. The score in the first term is slightly better than in the second term, but this is expected as topics in the second term are more complex. However students seem to be somewhat careless in the argumentation and formulation of the solutions at an exam. This may be caused by the fact that during the tutorials emphasis is on the discussion of the strategy and the results obtained, more than on the formulation of it. We could improve on this by letting the students hand in written reports on the exercises they make. However at the moment the extra time that is needed for this is not available.

To conclude, one may reflect on the general effects of the use of PCs in a linear algebra course like ours. It is difficult to compare directly the results of our course with those of a 'traditional' course on the basis of their respective exams, as they differ too much. Furthermore, in this way only short-term effects would be measured, while effects in the long term, e.g. the way in which one can use

mathematical skills in practice, are more important. At the moment we have no objective measurements of these long-term effects. However we have the impression that our students are better equipped with respect to this, and so we think that, as users of applied mathematics, they are better off.

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