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Wigner distribution moments measured as intensity moments in separable first-order optical systems

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Abstract

It is shown how all global Wigner distribution moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of separable first-order optical systems (generally anamorphic ones). The minimum number of such systems that are needed for the determination of these moments is derived.

1 Introduction

After the introduction of the Wigner distribution [1] (WD) for the description of coherent and partially coherent optical fields [2], it became an important tool for optical signal/image analysis and beam characterization [3–5]. The WD completely describes the complex amplitude of a coherent optical field (up to a constant phase factor) or the mutual coherence function of a partially coherent field. As the WD of a two-dimensional optical field is a function of four variables, it is difficult to analyze. Therefore, the optical field is often represented not by the WD itself, but by its global moments. Beam characterization based on the second-order moments is often represented not by the WD itself, but by its global coherence function [14–17] of the stochastic process

\[ \gamma(x, y; \tau) = \text{Exp} \{ j\omega \tau \} \]  

(1)

The function \( \gamma(x_1, x_2; y_1, y_2; \tau) \) is known as the mutual coherence function [14–17] of the stochastic process \( f(x, y; t) \). The mutual power spectrum [16, 17] or cross-spectral density function [18] \( \Gamma(x_1, x_2; y_1, y_2; \omega) \) is defined as the temporal Fourier transform of the mutual coherence function:

\[ \Gamma(x_1, x_2; y_1, y_2; \omega) = \int_{-\infty}^{\infty} \gamma(x_1, x_2; y_1, y_2; \tau) \exp(j\omega \tau) d\tau. \]  

(2)

For \( x_1 = x_2 = x, y_1 = y_2 = y \), the cross-spectral density function reduces to the (auto) power spectrum \( \Gamma(x, x; y, y; \omega) \), which represents the intensity distribution of the light for the temporal frequency \( \omega \). Since in the present discussion the explicit temporal-frequency dependence is of

measurements of only intensity distributions in an appropriate number of (generally anamorphic) separable first-order optical systems.
no importance, we shall, for the sake of convenience, omit the temporal-frequency variable $\omega$ from the formulas in the remainder of the paper.

The Wigner distribution of partially coherent light is defined in terms of the cross-spectral density function by

$$W(x, u; y, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x'; y + \frac{1}{2}y', y - \frac{1}{2}y') \times \exp[-j2\pi(ux' + vy')] \, dx' \, dy'. \quad (3)$$

A distribution function according to definition (3) was first introduced in optics by Walther [19, 20], who called it the generalized radianc. The WD $W(x, u; y, v)$ represents partially coherent light in a combined space-frequency domain, the so-called phase space, where $u$ is the spatial-frequency variable associated to the space variable $x$, and $v$ the spatial-frequency variable associated to the space variable $y$.

In this paper we consider the normalized moments of the WD, where the normalization is with respect to the total energy $E$ of the signal:

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, u; y, v) \, dx \, du \, dy \, dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x, x; y, y) \, dx \, dy. \quad (4)$$

These normalized moments $\mu_{pqrs}$ of the WD are thus defined by

$$\mu_{pqrs} E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, u; y, v) \times x^p \, u^q \, y^r \, v^s \, dx \, du \, dy \, dv \quad (p, q, r, s \geq 0)$$

$$= \frac{1}{(2\pi)^{q+s}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p \, y^r \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2} \right)^q \Gamma(x_1, y_2; x_2, y_1) \bigg|_{x_1 = x_2 = y_2 = y} \, dx \, dy. \quad (5)$$

Note that for $q = s = 0$ we have intensity moments, which can easily be measured:

$$\mu_{p0r0} E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, u; y, v) \times x^p \, y^r \, dx \, du \, dy \, dv \quad (p, r \geq 0)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p \, y^r \, \Gamma(x, x; y, y) \, dx \, dy. \quad (6)$$

The WD moments $\mu_{pqrs}$ provide valuable tools for the characterization of optical beams, see, for instance [21]. First-order moments yield the position of the beam ($\mu_{1000}$ and $\mu_{0100}$) and its direction ($\mu_{0101}$ and $\mu_{0011}$). Second-order moments give information about the spatial width of the beam (the shape $\mu_{2000}$ and $\mu_{0200}$ of the spatial ellipse and its orientation $\mu_{1010}$) and the angular width in which the beam is radiating (the shape $\mu_{2000}$ and $\mu_{0002}$ of the spatial-frequency ellipse and its orientation $\mu_{0101}$); moreover, they provide information about its curvature ($\mu_{1100}$ and $\mu_{0011}$) and its twist ($\mu_{1001}$ and $\mu_{0110}$). Many important beam characteristics, like the overall beam quality [12]

$$(\mu_{2000} \mu_{0200} - \mu_{1100}^2) + (\mu_{0002} \mu_{2000} - \mu_{0011}^2) + 2(\mu_{0101} \mu_{0110} - \mu_{1001} \mu_{1010}).$$

are based on second-order moments. Higher-order moments are used, for instance, to characterize the beam’s symmetry and its sharpness [21].

3 Separable first-order optical systems

It is well-known that the input-output relationship between the WD $W_{\text{in}}(x, u; y, v)$ at the input plane and the WD $W_{\text{out}}(x, u; y, v)$ at the output plane of a separable first-order optical system reads [3–5]

$$W_{\text{out}}(x, u; y, v) = W_{\text{in}}(dx - bu_x, -cy_x + a_x u; \, dy - by_v, -c_y v + a_y v). \quad (7)$$

The coefficients $a_x, b_x, c_x, d_x$ and $a_y, b_y, c_y, d_y$ are the matrix entries of the symplectic ray transformation matrix [7] that relates the position $x, y$ and direction $u, v$ of an optical ray in the input and the output plane of the first-order optical system:

$$\begin{bmatrix}
    x_{\text{out}} \\
    y_{\text{out}} \\
    u_{\text{out}} \\
    v_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
    a_x & 0 & b_x & 0 \\
    0 & a_y & 0 & b_y \\
    c_x & 0 & d_x & 0 \\
    0 & c_y & 0 & d_y
\end{bmatrix}
\begin{bmatrix}
    x_{\text{in}} \\
    y_{\text{in}} \\
    u_{\text{in}} \\
    v_{\text{in}}
\end{bmatrix}. \quad (8)$$

For separable systems, symplecticity reads simply $a_x d_x = b_x c_x = 1$ and $d_x d_y - b_y c_y = 1$. Note that in a first-order optical system, with such a symplectic ray transformation matrix, the total energy $E$, see Eq. (4), is invariant.

As examples of first-order optical systems we mention in particular

- a section of free space in the paraxial approximation, or ‘parabolic’ system [22] (with $a = d = 1$, $c = 0$, and $b$ proportional to the propagation distance $z$,
- a fractional Fourier transform system [23], or ‘elliptic’ system [22] (with $a = d = \cos \alpha$ and $b = -c = \sin \alpha$), and
- a ‘hyperbolic’ system [22] (with $a = d = \cosh \alpha$ and $b = c = \sinh \alpha$).

These three systems are characterized by one parameter. Other one-parameter first-order optical systems are
• a thin lens (with \(a = d = 1, b = 0\), and \(c\) inverse proportional to the focal distance) and

• an ideal magnifier (with \(a = m, d = 1/m, b = c = 0\)).

The latter systems however – like all systems for which the input and output planes are conjugate planes – cannot be used to determine the moments, as we will see later, because they have the property \(b = 0\).

The normalized moments \(\mu_{pqrs}^\text{out}\) of the output WD \(W_{\text{out}}(x, u; y, v)\) and \(\mu_{pqrs}^\text{in}\) of the input WD \(W_{\text{in}}(x, u; y, v)\) as

\[
\mu_{pqrs}^\text{out} E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_{pqrs}^\text{in} W_{\text{out}}(x, u; y, v) \times x^p y^q u^r v^s dx dy du dv
\]

For the first-order moments, the following two equations have the property

\[
\mu_{pqrs}^\text{out} = \mu_{pqrs}^\text{in}
\]

The latter systems however – like all systems for which the other moments can be chosen freely, as long as it leads to four independent equations. In the case of free space propagation, we simple choose two different values of the propagation distance \(z\), corresponding to two different values of \(b_4\) and \(b_5\) (with \(a_3 = a_4 = 1\), of course).

The first-order optical systems can always be chosen such that they are anamorphic, \(a_3 = a_5 = a_1, b_3 = b_5 = b_1, \text{etc.} (i = 1, 2)\), with identical behaviour in the \(x\) and \(y\) direction.

\[
\begin{align*}
\mu_{pqrs}^\text{out} & = a_1^2 \mu_{pqrs}^\text{in} + 2a_1 b_1 \mu_{pqrs}^{1001} + b_1^2 \mu_{pqrs}^{2000}, \\
\mu_{pqrs}^{1001} & = a_3 a_5 \mu_{pqrs}^{1001} + a_4 b_5 \mu_{pqrs}^{10001} + a_3 b_4 \mu_{pqrs}^{1001} + b_3 b_5 \mu_{pqrs}^{10010} + b_3 b_5 \mu_{pqrs}^{10010}. \\
\mu_{pqrs}^{2000} & = a_2^2 \mu_{pqrs}^{0020} + 2a_2 b_2 \mu_{pqrs}^{0010} + b_2^2 \mu_{pqrs}^{0002}. \\
\end{align*}
\]

For the 3+4+3=10 second-order moments, the following two equations are relevant:

\[
\begin{align*}
\mu_{pqrs}^{0020} & = a_2^2 \mu_{pqrs}^{0020} + 2a_2 b_2 \mu_{pqrs}^{1001} + b_2^2 \mu_{pqrs}^{2000}, \\
\mu_{pqrs}^{1001} & = a_3 a_5 \mu_{pqrs}^{1001} + a_4 b_5 \mu_{pqrs}^{10001} + a_3 b_4 \mu_{pqrs}^{1001} + b_3 b_5 \mu_{pqrs}^{10010} + b_3 b_5 \mu_{pqrs}^{10010}. \\
\end{align*}
\]

Note that the two first-order optical systems can always be chosen such that they are isotropic, \(a_3 = a_5 = a_1, b_3 = b_5 = b_1, \text{etc.} (i = 1, 2)\), with identical behaviour in the \(x\) and \(y\) direction.

\[
\begin{align*}
\mu_{pqrs}^{0020} & = a_2^2 \mu_{pqrs}^{0020} + 2a_2 b_2 \mu_{pqrs}^{0010} + b_2^2 \mu_{pqrs}^{0002}. \\
\end{align*}
\]

4 Relations between input and output moments

4.1 First-order moments

For the first-order moments, the following two equations are relevant:

\[
\begin{align*}
\mu_{pqrs}^{1000} & = a_3 \mu_{pqrs}^{1000} + b_3 \mu_{pqrs}^{1000}, \\
\mu_{pqrs}^{0010} & = a_3 \mu_{pqrs}^{0010} + b_3 \mu_{pqrs}^{0001},
\end{align*}
\]

which equations correspond to Eq. (10) with \(pqrs = 1000\) and \(pqrs = 0010\), respectively, and the four input moments \(\mu_{1000}, \mu_{0100}, \mu_{0010}, \text{and} \mu_{0001}\) can be determined by measuring the intensity moments \(\mu_{1000}\) and \(\mu_{0010}\) in the output planes of two systems with different values of \(a\) and \(b\), respectively.

In the case of a fractional Fourier transform system we can choose, for instance \([24, 25]\), the fractional angles \(\alpha_x = \alpha_y = 0\) (leading to \(a_3 = a_5 = 1, b_3 = b_5 = 0\)) and \(\alpha_x = \alpha_y = \frac{1}{4}\pi\) (leading to \(a_3 = a_5 = 0\) and \(b_3 = b_5 = 1\)), but any other choice could be made as well, as long as it leads to four independent equations. In the case of free space propagation, we simple choose two different values of the propagation distance \(z\), corresponding to two different values of \(b_4\) and \(b_5\) (with \(a_3 = a_4 = 1\), of course).

Note that the two first-order optical systems can always be chosen such that they are anamorphic, \(a_3 = a_5 = a_1, b_3 = b_5 = b_1, \text{etc.} (i = 1, 2)\), with identical behaviour in the \(x\) and \(y\) direction.

4.2 Second-order moments

For the 3+4+3=10 second-order moments, the following equations are relevant:

\[
\begin{align*}
\mu_{pqrs}^{0020} & = a_2^2 \mu_{pqrs}^{0020} + 2a_2 b_2 \mu_{pqrs}^{1001} + b_2^2 \mu_{pqrs}^{2000}, \\
\mu_{pqrs}^{1001} & = a_3 a_5 \mu_{pqrs}^{1001} + a_4 b_5 \mu_{pqrs}^{10001} + a_3 b_4 \mu_{pqrs}^{1001} + b_3 b_5 \mu_{pqrs}^{10010} + b_3 b_5 \mu_{pqrs}^{10010}. \\
\mu_{pqrs}^{2000} & = a_2^2 \mu_{pqrs}^{0020} + 2a_2 b_2 \mu_{pqrs}^{0010} + b_2^2 \mu_{pqrs}^{0002}. \\
\end{align*}
\]
We conclude that all ten second-order moments can be determined from the knowledge of the output intensities of four first-order optical systems, where one of them has to be anamorphic. In the case of fractional Fourier transform systems we could choose, for instance [24, 25], the fractional angles \( \alpha_x = \alpha_y = 0 \) (leading to \( a_x = a_y = 1 \) and \( b_x = b_y = 0 \)), \( \alpha_x = \alpha_y = \frac{\pi}{4} \) (leading to \( a_x = a_y = b_x = b_y = \frac{\sqrt{2}}{2} \)), \( \alpha_x = a_y = \frac{\pi}{2} \) (leading to \( a_x = a_y = 0 \) and \( b_x = b_y = 1 \)), and the anamorphic combination \( \alpha_x = \frac{\pi}{4} \) and \( \alpha_y = 0 \) (leading to \( a_x = b_x = 0 \) and \( a_y = b_y = 1 \)). If we decide to determine the moments using free space propagation, we should be aware of the fact that an anamorphic free space system cannot be realized by mere free space, but can only be simulated by using a proper arrangement of cylindrical lenses.

Of course, optical schemes to determine all ten second-order moments have been described before, see, for instance [8,9, 11–13], but the way to determine these moments as presented in this paper is based on a general scheme that can also be used for the determination of arbitrary higher-order moments.

### 4.3 Higher-order moments

For higher-order moments we can proceed analogously. For the 4+6+6+4=20 third-order moments, the following equations are relevant:

\[
\begin{align*}
\mu_{3000}^{\text{out}} & = a_x^3 \mu_{3000} + 3a_x^2 b_x \mu_{2100} + 3a_x b_x^2 \mu_{1200} \\
\mu_{2010}^{\text{out}} & = a_x^2 a_y \mu_{2010} + a_x^2 b_y \mu_{2011} + 2a_x b_x a_y \mu_{1110} \\
\mu_{1020}^{\text{out}} & = a_x a_y^2 \mu_{1020} + 2a_x b_x b_y \mu_{1011} + a_x b_x^2 \mu_{1002} \\
\mu_{0030}^{\text{out}} & = a_x b_x^3 \mu_{0030} + 3a_x^2 b_x \mu_{0121} + 3a_x b_x^2 \mu_{0012} + b_x^3 \mu_{0003}.
\end{align*}
\]

Note again that these equations correspond to Eq. (10) with \( pqr = 3000, pqr = 2010, pqr = 1020, \) and \( pqr = 0030, \) respectively. The 20 third-order moments can be determined from the knowledge of the output intensities of six first-order optical systems, where two of them have to be anamorphic. Let us consider in more detail how the third-order moments could be determined.

- The four input moments \( \mu_{3000}, \mu_{2100}, \mu_{1200}, \) and \( \mu_{0030} \) can be determined by measuring the intensity moment \( \mu_{3000,i}^{\text{out}} \) in the output planes of four systems with different values of \( a_x \) and \( b_x \), see Eq. (16). Likewise, with the transversal coordinate \( x \) replaced by \( y \), the four input moments \( \mu_{0030}, \mu_{0201}, \mu_{0012}, \) and \( \mu_{0003} \) can be determined by measuring the intensity moment \( \mu_{0030,i}^{\text{out}} \) in the output planes of four systems with different values of \( a_y \) and \( b_y \), see Eq. (19). Note that we can choose \( a_x = a_y = a_i \) and \( b_x = b_y = b_i \) \((i = 1,2,3,4)\) for these four different systems, in which case we are obviously using isotropic systems. This then leads to the set of four equations

\[
\begin{align*}
a_i^3 \mu_{3000} + 3a_i^2 b_i \mu_{2100} + 3a_i b_i^2 \mu_{1200} \\
+ b_i^3 \mu_{0030} & = \mu_{3000,i}^{\text{out}} \quad (i = 1,2,3,4)
\end{align*}
\]

based on Eq. (16) and a similar set of four equations

\[
\begin{align*}
a_i^3 \mu_{0030} + 3a_i^2 b_i \mu_{0201} + 3a_i b_i^2 \mu_{0012} \\
+ b_i^3 \mu_{0003} & = \mu_{0030,i}^{\text{out}} \quad (i = 1,2,3,4)
\end{align*}
\]

based on Eq. (19). Possible system choices are, for instance: four sections of free space, with \( a_i = 1 \) and \( b_i \) proportional to the four different propagation distances \( z_i \) \((i = 1,2,3,4)\); or four isotropic fractional Fourier transform systems with \( a_i = \cos \alpha_i \) and \( b_i = \sin \alpha_i \), and \( a_i \) \((i = 1,2,3,4)\) four different fractional angles.

- Using the same four isotropic systems as above, the two input moments \( \mu_{2010} \) and \( \mu_{0102} \), together with the two moment combinations \( \mu_{2011} + 2\mu_{1110} \) and \( 2\mu_{1011} + \mu_{0210} \), follow from measuring the intensity moment \( \mu_{2010,i}^{\text{out}} \) \((i = 1,2,3,4)\) in the output planes of these four systems, see Eq. (17), while the two input moments \( \mu_{1020} \) and \( \mu_{0102} \), together with the two moment combinations \( 2\mu_{1011} + \mu_{0120} \) and \( \mu_{1002} + 2\mu_{0111} \), follow from measuring the intensity moment \( \mu_{1020,i}^{\text{out}} \) \((i = 1,2,3,4)\), see Eq. (18). This leads to the set of four equations

\[
\begin{align*}
a_i^3 \mu_{2010} + a_i b_i (2\mu_{1011} + \mu_{0210}) \\
+ b_i^3 \mu_{0102} & = \mu_{2010,i}^{\text{out}} \quad (i = 1,2,3,4)
\end{align*}
\]

based on Eq. (17) and a similar set of four equations

\[
\begin{align*}
a_i^3 \mu_{1020} + a_i^2 b_i (2\mu_{1011} + \mu_{0120}) \\
+ a_i b_i^3 (2\mu_{1002} + \mu_{0111}) + b_i^3 \mu_{0102} & = \mu_{1020,i}^{\text{out}} \quad (i = 1,2,3,4)
\end{align*}
\]

based on Eq. (18).

- Twelve of the 20 input moments (together with four moment combinations) can thus be determined by using four isotropic systems. To determine the remaining eight moments, we need four more equations based on Eqs. (17) and (18), for which we have to use two
more systems (labeled $i = 5$ and $i = 6$), which should now be anamorphic. Among the many possibilities, an easy choice would be a system with $a_x = b_y = 0$, $b_x \neq 0, a_y \neq 0$, leading to
\[
b^2 a_y \mu_{0120} = \mu_{2010,5}^{\text{out}},
\]
and a system with $b_x = a_y = 0, a_x \neq 0, b_y \neq 0$, leading to
\[
a^2 b_y \mu_{2001} = \mu_{2010,6}^{\text{out}},
\]
\[
a^2 b_y \mu_{1002} = \mu_{1020,6}^{\text{out}}.
\]

The former system may be an anamorphic fractional Fourier transform system with fractional angles $a_x = \frac{1}{2}\pi$ and $a_y = 0$ (and hence $a_x = b_y = 0$ and $b_x = a_y = 1$), while the latter may be an anamorphic fractional Fourier transform system with $a_x = 0$ and $a_y = \frac{1}{2}\pi$ (and hence $b_x = a_y = 0$ and $b_y = a_x = 1$).

Altogether we have thus constructed 20 equations for the 20 third-order moments, using a total of six first-order systems: four isotropic systems where we measure the 16 output intensity moments $\mu_{3000,i}, \mu_{0030,i}, \mu_{2010,i}, \mu_{1020,i}$ ($i = 1, 2, 3, 4$), and two anamorphic systems where we measure the four output intensity moments $\mu_{2010,i}^{\text{out}}$ and $\mu_{1020,i}^{\text{out}}$ ($i = 5, 6$).

For the $5+8+9+8+5=35$ fourth-order moments, the relevant equations follow from Eq. (10) with $pqrs = 4000$, $pqrs = 3010$, $pqrs = 2020$, $pqrs = 1030$, and $pqrs = 0040$, respectively. The 35 fourth-order moments can be determined from the knowledge of the output intensities of nine first-order optical systems spectra, where four of them have to be anamorphic. Constructing a measuring scheme along the lines described above for the second-order case and the third-order case, is rather straightforward.

To find the number of $n$th-order moments $N_n$, and the total number of first-order optical systems $N_f$ (with $N_a$ the number of anamorphic ones) that we need to determine these $N_n$ moments, use can be made of the following triangle, which can easily be extended to higher order:

<table>
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<tr>
<th>$n$</th>
<th>number of $n$th order moments</th>
<th>$N_n$</th>
<th>$N_{f,n}$</th>
<th>$N_{a,n}$</th>
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<td>0</td>
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<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
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<td>2</td>
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<td>4</td>
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