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ON THE SIZE OF SYSTEMS OF SETS EVERY $t$ OF WHICH HAVE AN SDR, WITH AN APPLICATION TO THE WORST-CASE RATIO OF HEURISTICS FOR PACKING PROBLEMS*

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Abstract. Let $E_1, \cdots, E_m$ be subsets of a set $V$ of size $n$, such that each element of $V$ is in at most $k$ of the $E_i$ and such that each collection of $t$ sets from $E_1, \cdots, E_m$ has a system of distinct representatives (SDR). It is shown that $m/n \leq (k(k-1)^t - k)/(2(k-1)^t - k)$ if $t = 2r - 1$, and $m/n \leq (k(k-1)^t - 2)/(2(k-1)^t - 2)$ if $t = 2r$. Moreover it is shown that these upper bounds are the best possible. From these results the “worst-case ratio” of certain heuristics for the problem of finding a maximum collection of pairwise disjoint sets among a given collection of sets of size $k$ is derived.

Key words. packing, system of distinct representatives, worst-case ratio, heuristics

AMS(MOS) subject classifications. 05C65, 05A05, 90C27

1. Introduction. We prove the following theorem, where $m, n, k,$ and $t$ are positive integers, with $k \geq 3$.

THEOREM 1. Let $E_1, \cdots, E_m$ be subsets of the set $V$ of size $n$, such that we have the following:

1. Each element of $V$ is contained in at most $k$ of the sets $E_1, \cdots, E_m$;
2. Any collection of at most $t$ sets among $E_1, \cdots, E_m$ has a system of distinct representatives.

Then, we have the following:

2. (i) $m/n \leq (k(k-1)^t - k)/(2(k-1)^t - k)$ if $t = 2r - 1$;
(ii) $m/n \leq (k(k-1)^t - 2)/(2(k-1)^t - 2)$ if $t = 2r$.

Note that by the Kőnig–Hall Theorem, condition (1)(ii) can be replaced by the following:

3. For any $s \leq t$, any $s$ of the sets among $E_1, \cdots, E_m$ cover at least $s$ elements of $V$.

We give a proof of Theorem 1 in § 2. We also show that the bounds given in (2) are best possible in the following sense.

THEOREM 2. For any fixed $k$, $t$ (with $k \geq 3$), there exist $m, n$ and $E_1, \cdots, E_m \subseteq V$ (with $|V| = n$) satisfying (1) and having equality in the appropriate line of (2).

The proof of Theorem 2 is based on a construction using regular graphs of large girth (see § 3).

Finally, in § 4 we apply these results to derive the worst-case ratio of certain heuristic algorithms for the problem of finding a largest family of pairwise disjoint sets among a given family of sets of size $k$ (this problem is NP-complete for any $k \geq 3$).

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2. Proof of Theorem 1. To show Theorem 1, we first give a lemma. Let \( E_1, \cdots, E_m \) be a collection of finite nonempty sets, which we order so that \( |E_1|, \cdots, |E_h| \geq 2 \) and \( |E_{h+1}| = \cdots = |E_m| = 1 \), for some \( h \leq m \). We define a new collection as follows. Let

\[
W := E_{h+1} \cup \cdots \cup E_m.
\]

Let for each \( i = 1, \cdots, h \), \( X_i \) be a set of size \( |E_i| - 2 \), disjoint from \( E_1 \cup \cdots \cup E_m \) and so that if \( i \neq j \) then \( X_i \cap X_j = \emptyset \). Let \( X_1 \cup \cdots \cup X_h = \{ y_1, \cdots, y_q \} \). Then the derived collection of sets is formed by the following sets:

\[
(E_i \setminus W) \cup X_1, \cdots, (E_h \setminus W) \cup X_h, \{ y_1 \}, \cdots, \{ y_q \}.
\]

Furthermore, we define a collection \( E_1, \cdots, E_m \) to have the \( t \)-SDR-property if any \( t \) sets among \( E_1, \cdots, E_m \) have a system of distinct representatives.

**Lemma.** For \( t \geq 3 \), if \( E_1, \cdots, E_m \) has the \( t \)-SDR-property, then the derived collection \( (5) \) has the \( (t-2) \)-SDR-property.

**Proof.** Suppose \( (5) \) does not have the \( (t-2) \)-SDR-property. Then there exists a collection \( \Pi \) of \( p \) sets among \( (5) \) covering at most \( p - 1 \) elements, for some \( p \leq t - 2 \). Assume we have chosen \( p \) minimal. This immediately implies the following:

\[
\begin{align*}
(6) &\quad |\cup \Pi| = p - 1; \\
(\text{ii}) &\quad \text{Each element in } \cup \Pi \text{ is covered by at least two sets in } \Pi.
\end{align*}
\]

From (6)(ii) we directly have for any \( i = 1, \cdots, h \) and \( x \in X_i \):

\[
E_i \setminus W \text{ and } \cup X_i \in \Pi.
\]

Without loss of generality, all sets \( (E_1 \setminus W) \cup X_1, \cdots, (E_h \setminus W) \cup X_h \) belong to \( \Pi \) (as we can delete all sets \( E_j \) from \( E_1, \cdots, E_h \) for which \( (E_j \setminus W) \cup X_j \notin \Pi \)), and without loss of generality, \( (E_1 \cup \cdots \cup E_h) \cap W = E_{h+1} \cup \cdots \cup E_m \).

Note the following:

\[
|X_1 \cup \cdots \cup X_h| = \sum_{i=1}^{h} (|E_i| - 2), \quad p = h + q,
\]

\[
|\bigcup_{i=1}^{h} (E_i \setminus W)| = |\cup \Pi| - q = (p - 1) - q = h - 1.
\]

So,

\[
|\bigcup_{i=1}^{m} E_i| = \left| \bigcup_{i=1}^{h} (E_i \cap W) \right| + \left| \bigcup_{i=1}^{h} (E_i \setminus W) \right| = (m - h) + (h - 1) = m - 1.
\]

Moreover, by (6)(ii), \( \sum_{i=1}^{h} |E_i \setminus W| \geq 2 \cdot |\bigcup_{i=1}^{h} (E_i \setminus W)| \), and hence

\[
\begin{align*}
m = h + &\left| \bigcup_{i=1}^{h} (E_i \cap W) \right| \leq h + \sum_{i=1}^{h} |E_i \cap W| = h + \sum_{i=1}^{h} |E_i| - \sum_{i=1}^{h} |E_i \setminus W| \\
\leq h + &\sum_{i=1}^{h} |E_i| - 2 \cdot \left| \bigcup_{i=1}^{h} (E_i \setminus W) \right| = h + 2h + \sum_{i=1}^{h} (|E_i| - 2) - 2(h - 1) \\
= h + 2h + q - 2(h - 1) = h + q + 2 = p + 2 \leq t.
\end{align*}
\]

Inequalities (9) and (10) contradict the fact that \( E_1, \cdots, E_m \) has the \( t \)-SDR-property. \( \square \)
Proof of Theorem 1. We prove Theorem 1 by induction on $t$.

Case 1. $t = 1$. Then we have that each of $E_1, \cdots, E_m$ is nonempty, and hence $m \leq \sum_{i=1}^m |E_i| \leq kn$, by (1)(i).

Case 2. $t = 2$. Then we have that each of $E_1, \cdots, E_m$ is nonempty, and that no two of the singletons among $E_1, \cdots, E_m$ are the same. Without loss of generality, let $E_{h+1}, \cdots, E_m$ be the singletons among $E_1, \cdots, E_m$. Then $m - h \leq n$, and

$$m + h = 2h + (m - h) \leq \sum_{i=1}^h |E_i| + \sum_{i=h+1}^m |E_i| = \sum_{i=1}^m |E_i| \leq kn$$

(by (1)(i)). Hence $2m = (m - h) + (m + h) \leq (k + 1)n$, and (2) follows.

Case 3. $t \geq 3$. Then consider the derived collection $E'_1, \cdots, E'_m$, on $V' := \cup_{i=1}^m E_i$ as in (5). Note that $m' = h + q$ and $n' := |V'| = n - |W'| + q$. Denote the right-hand side term in (2) by $\varphi(k, t)$.

As by the lemma above, $E'_1, \cdots, E'_m$ has the $(t - 2)$-SDR-property, and as trivially each element of $V'$ is in at most $k$ of the sets $E'_1, \cdots, E'_m$, we have by induction that $m' \leq \varphi(k, t - 2)n'$. That is,

$$h + q \leq \varphi(k, t - 2)(n - |W'| + q).$$

Writing the terms in different order, we have

$$\varphi(k, t - 2) |W'| + h - (\varphi(k, t - 2) - 1)q \leq \varphi(k, t - 2)n.$$

Moreover, as $E_1, \cdots, E_m$ cover any element at most $k$ times:

$$|W'| + 2h + q = |W'| + 2h + \sum_{i=1}^h (|E_i| - 2) = |W'| + \sum_{i=1}^h |E_i| = \sum_{i=1}^m |E_i| \leq kn.$$

Hence,

$$m = h + |W'|$$

$$= \frac{1}{2\varphi(k, t - 2) - 1} (\varphi(k, t - 2) |W'| + h - (\varphi(k, t - 2) - 1)q) + \frac{\varphi(k, t - 2) - 1}{2\varphi(k, t - 2) - 1} (|W'| + 2h + q)$$

$$\leq \frac{1}{2\varphi(k, t - 2) - 1} \varphi(k, t - 2)n + \frac{\varphi(k, t - 2) - 1}{2\varphi(k, t - 2) - 1} kn$$

$$= \frac{(k + 1)\varphi(k, t - 2) - k}{2\varphi(k, t - 2) - 1} n = \varphi(k, t)n.$$

The last equality follows directly by substituting the corresponding right-hand side of (2). □

3. Proof of Theorem 2. To prove Theorem 2 we use a result of Erdős and Sachs [1]:

(16) For every $k$ and $\gamma$ there exists a $k$-regular graph of girth $\gamma$.

As a consequence of (16) we have the following:

(17) For every $k$, $s$, and $\gamma$ there exists a bipartite graph of girth at least $\gamma$, with color classes $U$ and $W$, say, such that each vertex in $U$ has degree $k$, and each vertex in $W$ has degree $s$. 
(To see that (17) follows from (16), let $H$ be a $2ks$-regular graph of girth $\gamma$. Consider any Eulerian orientation of the edges of $H$ (i.e., one for which all indegrees and outdegrees equal $ks$). Split each vertex $v$ into $k + s$ vertices $v_1, \ldots, v_k, w_1, \ldots, w_s$, and divide the arcs entering $v$ equally over $v_1, \ldots, v_k$ and divide the arcs leaving $v$ equally over $w_1, \ldots, w_s$. Forgetting the orientations, we obtain a bipartite graph with the required properties.)

Now choose $k$, $t$. Let $r := \lfloor \frac{1}{2} t \rfloor$. Consider the tree $T$, with vertices $1, 2, \ldots, (k-1) + (k-1)^2 + \cdots + (k-1)^{r-1}$, so that for $i < j$, vertices $i$ and $j$ are connected by an edge, if and only if $(k-1)i \leq j \leq (k-1)i + (k-2)$. So each vertex has degree $k$, except for vertex 1, which has degree $k-1$, and for the vertices $1 + (k-1) + \cdots + (k-1)^{r-2} + 1$, $1 + (k-1) + \cdots + (k-1)^{r-1}$, which have degree one.

First let $t$ be even. Let $G$ be a $(k-1)^t$-regular graph of girth $t + (\text{cf. (16)}).$ Let $G$ have $p$ vertices: $v_1, \ldots, v_p$. Consider $p$ copies $T_1, \ldots, T_p$ of $T$ (denoting the copy of vertex $i$ in $T_j$ by $i_j$). For each $j = 1, \ldots, p$, partition the set of $(k-1)^t$ edges of $G$ incident to $v_j$ (arbitrarily) into $(k-1)^{t-1}$ classes of size $k-1$, and connect them to the $(k-1)^{t-1}$ vertices $i_j$ in $T_j$ of degree one. So the final graph $H = (W, F)$ has all degrees equal to $k$, except for the vertices $1, \ldots, 1_p$, which have degree $k-1$. Let $E_1, \ldots, E_m$ be the collection $F \cup \{\{1\}, \ldots, \{1_p\}\}$. This collection clearly satisfies (1)(i), and direct counting shows equality in (2)(ii). To see that the collection satisfies (1)(ii), let $E_1, \ldots, E_s$ form a subcollection with $|E_1 \cup \cdots \cup E_t| < s$ and $s$ as small as possible. Suppose $s \leq t$. As $E_1, \ldots, E_s$ must form a connected hypergraph, it contains at most one singleton (since any path between $i$ and $j$ in $H$ contains at least $t-1$ edges). So assume $E_1, \ldots, E_s$ are edges of $H$. Then they do not contain any circuit (as each $T_i$ is a tree and $G$ has girth $t + 1$). So $|E_1 \cup \cdots \cup E_t| \geq s$, a contradiction.

Next let $t$ be odd. Let $G$ be a bipartite graph, of girth at least $t + 1$, so that in one color class $U$ each vertex has degree $(k-1)^t$ and in the other color class $W$ each vertex has degree $k$. Let $U = \{u_1, \ldots, u_p\}$. Consider again $p$ copies $T_1, \ldots, T_p$ of $T$, as above. For $j = 1, \ldots, p$, partition the set of $(k-1)^t$ edges of $G$ incident to $u_j$ (arbitrarily) into $(k-1)^{t-1}$ classes of size $k-1$, and connect them to the $(k-1)^{t-1}$ vertices $i_j$ in $T_j$ of degree one. Again, the final graph $H = (W, F)$ has all degrees equal to $k$, except for the vertices $1, \ldots, 1_p$ that have degree $k-1$. Let $E_1, \ldots, E_m$ be the collection $F \cup \{\{1\}, \ldots, \{1_p\}\}$. Similarly, as above, we show that this collection satisfies (1) and has equality in (2)(i).

4. Application to the worst-case ratio of heuristics. The problem of finding a largest collection of pairwise disjoint sets among a given collection $X_1, \ldots, X_q$ of $k$-sets is NP-complete, for any $k \geq 3$. Call any collection of pairwise disjoint sets a packing.

For any fixed $s$, we can apply the following heuristic algorithm $H_s$. Start with the empty packing. If we have found a packing $Y_1, \ldots, Y_n$ from $X_1, \ldots, X_q$, we could select $p \leq s$ sets among $Y_1, \ldots, Y_n$, and replace them by $p + 1$ sets from $X_1, \ldots, X_q$, so that the arising collection is a packing with $n + 1$ sets. Repeating this, the algorithm terminates with a collection $Y_1, \ldots, Y_n$ so that

\begin{equation}
(18) \quad \text{For each } p \leq s, \text{ the union of any } p + 1 \text{ pairwise disjoint sets among } X_1, \ldots, X_q \text{ intersects at least } p + 1 \text{ sets among } Y_1, \ldots, Y_n.
\end{equation}

This defines heuristic $H_s$, which is, for any fixed $s$, a polynomial-time algorithm—however it clearly need not lead to a largest packing. We might ask how far the packing found with $H_s$ is from the largest packing.

To this end, consider a largest packing $Z_1, \ldots, Z_m$ from $X_1, \ldots, X_q$. We claim that $m/n$ satisfies the bounds given in (2), taking $t := s + 1$, and that these bounds are best possible. That is, the “worst-case ratio” of the heuristic is given in (2).
Indeed, let

\[ V := \{ Y_1, \cdots, Y_n \} \quad \text{and} \quad E_i := \{ Y_j \mid Y_j \cap Z_i \neq \emptyset \} \quad \text{for} \quad i = 1, \cdots, m. \]

Then by (18), \( E_1, \cdots, E_m \) satisfy (1), and hence we obtain the bounds given in (2).

In turn, it is not difficult to see that for any collection \( E_1, \cdots, E_m \) of sets of size at most \( k \), containing any point at most \( k \) times, we can assume they are of form (19) for certain packings \( Y_1, \cdots, Y_n \) and \( Z_1, \cdots, Z_m \) of \( k \)-sets. Thus starting with \( E_1, \cdots, E_m \) as described in § 3 above, making these \( Y_1, \cdots, Y_n, Z_1, \cdots, Z_m \), and taking \( \{ X_1, \cdots, X_q \} := \{ Y_1, \cdots, Y_n, Z_1, \cdots, Z_m \} \), we obtain a system of sets attaining the worst-case ratio. (That is because we may assume that \( H \) selects the sets \( Y_1, \cdots, Y_n \) in the first \( n \) iterations.)

Note that we may assume even that the sets \( Y_1, \cdots, Y_n, Z_1, \cdots, Z_m \) form the collection of all cliques of size \( k \) in a graph. Hence, we cannot obtain a better worst-case ratio by restricting the collections of sets to collections of \( k \)-cliques.

REFERENCE