Solution to Problem 60-8: Another sorting problem

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the integrand (of Eq. 3) is asymptotic to

$$\frac{\sqrt{3/\pi}}{x} \left\{ 12^{1/6} \Gamma\left(\frac{7}{6}\right) \alpha + \Gamma\left(\frac{5}{6}\right) 12^{-1/6} \right\}.$$

Thus for arbitrary α ,

$$F'(x) \sim c \log x.$$

In order for $\lim_{x \rightarrow \infty} F'(x)$ to exist,

$$\alpha = -\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{7}{6})} \cdot 12^{-1/3},$$

and then

$$F'(\infty) = \frac{2\Gamma(\frac{5}{6})}{\sqrt[6]{2^5 \cdot 3^4}} \int_0^\infty \{I_{-1/6}(y) - I_{1/6}(y)\} \frac{e^{-y} dy}{\sqrt{y}}.$$

From a table of Laplace transforms,

$$t^{\mu-1/2} K_{\nu+1/2}(at) \supset \sqrt{\frac{\pi}{2a}} \Gamma(\mu - \nu) \Gamma(\mu + \nu + 1) s^{-\mu} P_{\nu}^{-\mu}(p/a).$$

where $s = \sqrt{p^2 - a^2}$.

Since

$$K_{\nu}(x) = \frac{\pi}{2 \sin \nu \pi} \{I_{-\nu}(x) - I_{\nu}(x)\},$$

$$F'(\infty) = \left(\frac{2}{3}\right)^{2/3} \frac{\sqrt{\pi} \Gamma(\frac{5}{6})}{2 \cos \pi/6} P_{-1/3}(1) = \left(\frac{2}{3}\right)^{2/3} \sqrt{\frac{\pi}{3}} \Gamma\left(\frac{5}{6}\right).$$

Also solved by the proposer and incompletely by William Squire (Southwest Research Institute, San Antonio, Texas).

Editorial note: Even though the expressions for α and L in the different solutions are dissimilar, it is easy to show that they are equivalent.

Problem 60-8, Another Sorting Problem, by J. H. VAN LINT (Technische Hogeschool, Te Eindhoven, Nederland).

Consider all sequences of length M consisting of p_1 1's, p_2 2's, $\dots$ p_k k 's ($k \geq 2$) whose maximal monotonic non-decreasing subsequences of contiguous numbers are of length n where n satisfies the inequalities

$$(1) \quad n > M - p_i \quad i = 1, 2, \dots, k.$$

Determine the number $N(p_1, p_2, \dots, p_k; n)$ of sequences with this property.

Editorial note: This problem is similar to that of Brock's Optimum Sorting Procedure, Problem 59-3. However, in the latter problem, contiguity is not required.

Solution by the proposer. As a consequence of (1) the subsequence must contain every number $1, 2, \dots, k$ at least once and hence it starts with the number 1 and ends with the number k .

First consider $n \leq M - 2$.

If the monotonic subsequence of length n is at the beginning (at the end) of the sequence there are $M - n$ elements left, of which $M - n - 1$ can have arbitrary values, because $M - n < p_i$ for all i , while the first (last) of these elements can have all values except k (except 1). If we choose these elements the monotonic subsequence is fully determined. Hence there are $2 \cdot k^{M-n-1} \cdot (k - 1)$ sequences of this type.

Now consider the case where the monotonic subsequence is preceded by elements and followed by $M - n - l$ elements. Here l can have the values $1, 2, \dots, M - n - 1$. The last of the preceding elements can have all values except 1. The first of the elements following the subsequence can have all values except k . The other $M - n - 2$ preceding and following elements can have arbitrary values. (We again use (1)). After choosing these elements the subsequence is fully determined. We can still choose l .

So there are $(M - n - 1) \cdot (k - 1)^2 \cdot k^{M-n-2}$ sequences of this type. Adding the two results we find:

$$\begin{aligned} (2) \quad N &= 2 \cdot k^{M-n-1} (k - 1) + (M - n - 1) \cdot (k - 1)^2 \cdot k^{M-n-2} \\ &= (k - 1) \cdot k^{M-n-2} \cdot \{2k + (k - 1) (M - n - 1)\} \end{aligned}$$

We see that N is the same for all $p_1, p_2, \dots, p_k$ as long as (1) is satisfied and of course $p_1 + p_2 + \dots + p_k = M$.

For $k = 2$ we find for (2): $N = 2^{M-n-2} (M - n + 3)$.

We still have 2 cases to discuss: $n = M$ and $n = M - 1$. Both are trivial. For $n = M$ we have $N = 1$ and for $n = M - 1$ the above holds but we only have the first term i.e., $N = 2(k - 1)$.