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Influence of initial conditions on decaying two-dimensional turbulence

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A numerical study of freely decaying two-dimensional turbulence is presented to show how the time evolution of characteristic flow quantities is influenced by the initial conditions. The numerical method adopted is a standard two-dimensional (2D) Fourier pseudospectral algorithm with Newtonian viscosity. Vortex statistics are extracted using a vortex census method. Several characteristic initial vorticity distributions analogous to those employed in previous laboratory experiments are considered. Some of the initial vorticity distributions have in common a dominant subset of vortices. Reliable statistics are obtained for each characteristic distribution by ensemble averaging. For the dominant subset, the time evolutions of the global enstrophy and the number density, respectively, are found to collapse confirming the self-similarity of 2D turbulence, one of the starting points for the scaling theory proposed by Carnevale et al. [Phys. Rev. Lett. 66, 2735 (1991)]. The relationship between the relevant scaling exponents as predicted by the scaling theory is not confirmed, however, and thus seems questionable within the considered parameter range. Furthermore, power-law exponents for both the number density and the global enstrophy are found to be affected by the initial number density and the initial vortex size distribution. Our results thus suggest that for experiments in shallow fluid layers, any agreement with a universal scaling exponent seems coincidental. © 2007 American Institute of Physics. [DOI: 10.1063/1.2716785]

I. INTRODUCTION

The atmosphere and oceans are essentially very thin layers of fluid. Motions in these systems are characterized by a vertical length scale ($\leq 10$ km) that is much smaller than the horizontal length scale ($\leq 10^3$ km). Besides geometrical confinement, background planetary rotation as well as vertical density stratification (due to temperature and salinity differences) provide the right environment to accommodate quasi-twodimensional (geophysical) flows.

One of the most striking features of two-dimensional (2D) turbulence is the development of coherent vortices from smaller-scale structures, a process often referred to as self-organization (first reported in Ref. 1). Kinetic energy, initially distributed over both larger and smaller eddies, eventually concentrates in large-scale structures by complicated vortex interactions, the merger of two nearby like-signed vortices being the fundamental mechanism of vortex growth in 2D turbulent flows.

Generally, the decay of a 2D turbulent flow from random initial conditions can be divided into three stages. The initial stage is characterized by self-organization of the fluid into a collection of coherent structures containing most of the surviving vorticity. The intermediate stage is governed primarily by two processes: (a) nearly conservative mutual advection of the vortices when they are well separated and (b) dissipative interaction of vortices when they become close. It has been shown in many numerical studies that the resulting evolution exhibits scaling behavior, meaning that aggregate measures of both the flow fields and the vortex population evolve algebraically in time. The final stage of decay is characterized by a single pair of counter-rotating vortices that decay by diffusion only (any nonlinearity has been depleted). In this study, prime interest goes to the intermediate stage for which several (temporal) scaling theories were proposed, e.g., by Batchelor and Carnevale et al. Throughout this paper, when mentioning scaling theory we mean the theory proposed by Carnevale et al., unless explicitly stated otherwise. Carnevale et al. assume conservation of both the kinetic energy $E$ and the average absolute vorticity extremum $\bar{\zeta}$, so that a length scale $L = \sqrt{E/\bar{\zeta}}$ and a time scale $T = \bar{\zeta}^{-1}$ can be defined. Moreover, assuming all enstrophy is contained by a self-similar vortex population, it is then derived on dimensional grounds that

$$\Omega(t) \propto T^{-3/2} \left[ \frac{t}{T} \right]^{-\xi}, \quad \rho(t) \propto L^{-2} \left[ \frac{t}{T} \right]^{-\xi},$$

$$a(t) \propto L^3 \left[ \frac{t}{T} \right]^{\xi/4}, \quad r(t) \propto L \left[ \frac{t}{T} \right]^{\xi/2},$$

where $\Omega$ represents the global enstrophy, $\rho$ is the number density of vortices, $a$ is a typical vortex radius, $r$ denotes the intervortex distance, and $\xi$ is a positive scaling exponent.

Inferences from the scaling theory were confirmed by Weiss and McWilliams by using a long-time integration of the fluid equations with hyperviscosity, and a dissipative point-vortex model. Bracco et al. recently performed very high-resolution pseudospectral simulations (also with hyperviscosity) of a random Gaussian initial vorticity field with a narrowband energy spectrum that confirmed many of the predictions proposed by Carnevale et al., as well as results obtained from numerical simulations by Weiss and McWilliams. Further support was deduced from laboratory...
In these experiments, electromagnetic forcing was shortly applied to a thin layer of electrolyte (salty water) in order to generate 2D turbulence. Tabeling et al. tacitly assumed that the decay of the resulting turbulence is independent of the initial vorticity distribution, which in their case consisted of an array of 10 by 10 vortices with alternating circulation.

Previous studies, however, have shown that isolated vortex interactions strongly depend on parameters such as vortex size, vorticity amplitude, separation distance of the vortex centers, and background shear induced by the surrounding vortices. One may therefore wonder whether the initial vortex distribution affects the time evolution of decaying 2D turbulence, and whether the apparent agreement with the scaling theory obtained in the above-mentioned numerical and experimental studies is only coincidental. Previous investigators already pointed out the necessity to examine the effect of initial conditions (ICs) on decaying 2D turbulence as it might explain the different values obtained for the scaling exponent $\xi$ as well as the different relationships between the power-law exponents of the various vortex quantities, see Tables I and II. Nevertheless, much attention has not been given to this topic so far, except for the numerical work by Benzi, Santangelo, and colleagues. These studies emphasize that the distribution of vortex properties strongly depends on the ICs: for smooth ICs, the dynamics of 2D decaying turbulence produces both coherent vortices and vorticity sheets, the former dominating the large-scale flow while the latter dominating the small-scale flow; for nonsmooth ICs, on the other hand, a self-similar coherent vortex population results, the decay of which is merely dominated by vortex merger, and which can be quantitatively described by a Hamiltonian point-vortex model in which enstrophy dissipation is represented by a simple merger mechanism.

In the present study, a pseudospectral code is used to investigate the influence of ICs on the temporal evolution of characteristic (vortex) quantities, such as the number density, the average vortex area, etc. While most of the numerical studies on 2D turbulence use ICs based on well-defined energy spectra, our ICs are based on the spatial vorticity distribution of an array of Gaussian vortices. Our ICs were inspired by the narrowband ICs used by Tabeling et al. in their laboratory experiments. Such artificial flow fields develop in much the same manner as would a random realization from a narrowband, horizontally isotropic, kinematic energy spectrum. We have defined different ICs by varying the vortex size, the vorticity amplitude, and the (average) vortex density. A vortex census technique similar to that of McWilliams has been used to extract meaningful vortex statistics.

The initial Reynolds number, here defined as the inverse of the nondimensional kinematic viscosity, was varied from $R_{\infty}=3450$, close to Reynolds numbers encountered in the fluid layer experiments in order to put reported experimental results to the proof, to $R_{\infty}=34500$, almost twice as large as in previous (similar) numerical simulations in order to determine the positive scaling exponent $\xi$. As this study is almost entirely restricted to the use of Newtonian viscosity, it may appear insignificant in relation with infinite Reynolds number scaling theory. However, Chasnov convincingly

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\rho$</th>
<th>$\gamma$</th>
<th>$a$</th>
<th>$\zeta_{\text{ext}}$</th>
<th>$\Omega/E$</th>
<th>$R_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carnevale et al.</td>
<td>0.75</td>
<td>0.375</td>
<td>0.19</td>
<td>0</td>
<td>0.375</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Weiss and McWilliams</td>
<td>0.72±0.03</td>
<td>0.36</td>
<td>0.23</td>
<td>0.10</td>
<td>0.46</td>
<td>$2.9 \times 10^8$</td>
</tr>
<tr>
<td>Dritschel</td>
<td>0.29±0.03</td>
<td>$\cdots$</td>
<td>0.051±0.012</td>
<td>$\cdots$</td>
<td>0.347±0.03</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>Clercx and Nielsen</td>
<td>1.03±0.10</td>
<td>$\cdots$</td>
<td>0.25±0.03</td>
<td>0.38±0.04</td>
<td>0.98±0.05</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Bracco et al.</td>
<td>0.76±0.03</td>
<td>0.50±0.05</td>
<td>0.24±0.03</td>
<td>0.09±0.02</td>
<td>0.38±0.02</td>
<td>$1.0 \times 10^{12}$</td>
</tr>
</tbody>
</table>

*Weiss and McWilliams (Ref. 2) define $R_{\infty}=1/\nu_2$ with hyperviscosity $\nu_2=3.5 \times 10^{-7}$.*

*Only exponents related to the highest Reynolds number are reported here.*

*Bracco et al. (Ref. 8) do not report this numerical value. We have derived the numerical value from Fig. 2(b) in Ref. 8.*

*Bracco et al. (Ref. 8) report a hyperviscosity $\nu_2=9.53 \times 10^{-11}$. The Reynolds number is given by $R_{\infty}=1/\nu_2$.**
demonstrated that for sufficiently large initial Reynolds number, the small amount of energy that dissipates is of no apparent consequence, and a nearly inviscid enstrophy decay law is found with an approximate exponent of −0.80. Table I shows that the latter value for the enstrophy decay law is in clear contrast with the values obtained with pseudospectral simulations using hyperviscosity or contour dynamics simulations.4

This paper is organized as follows. In Secs. II and III, a short description of the numerical method and the vortex census technique is given. Section IV presents the time evolution of the vorticity field for several ICs. The time evolution of common global quantities, like energy and enstrophy, and the time evolution of various vortex quantities is discussed in Secs. V and VI, respectively. To conclude, a discussion of the obtained results is presented in Sec. VII.

II. NUMERICAL METHOD

The numerical simulations were carried out using a standard 2D Fourier pseudospectral method with de-aliasing that solves the 2D vorticity equation,

\[
\frac{\partial \zeta}{\partial t} = - \left( \frac{\partial \zeta}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \zeta}{\partial y} \frac{\partial \psi}{\partial x} \right) + (-1)^{p+1} \nu_p \nabla^2 \zeta
\]  

(2)

and

\[
\zeta = - \nabla^2 \psi,
\]  

(3)

on a square periodic domain with side \( l \). Here, \( \zeta \) represents vorticity, \( \psi \) the stream function, \( \nu_p \) the nondimensional kinematic viscosity, where \( p=1 \) for Newtonian viscosity (also referred to as \( \nu \)) and \( p=2 \) for hyperviscosity, and \( \nabla^2 \) the Laplacian operator. The computational grid consists of \( N^2 \) grid points, which corresponds to a maximum of 341 active Fourier modes in each direction. Time integration is performed by a fully corrected Adams–Bashforth third-order scheme. For further details on the numerical code, the reader is referred to Refs. 23 and 24.

The ICs of the numerical simulations presented in this paper are based on the initial vorticity fields employed in previous experiments.9,21 Doing so, we restrict ourselves to vortex populations with a relatively narrow size distribution, so that the vortex properties can be described meaningfully by average quantities.20 A predefined number of Gaussian vortices (with the initial vortex radius \( R \) defined as the radius at which \( \zeta = \gamma' / \pi \), and \( \gamma \) the vortex circulation) was placed on a numerical grid. The following four ICs were investigated:

(I) 64 identical vortices, all with radius \( R \) and vorticity extremum \( \zeta_{\text{ext}} \).

(II) 64 vortices with equal vorticity extremum \( \zeta_{\text{ext}} \), half of them with radius \( R \), the other half with radius \( 0.5R \).

(III) 64 vortices with equal radius \( R \), half of them with vorticity extremum \( \zeta_{\text{ext}} \), the other half with vorticity extremum \( 0.5\zeta_{\text{ext}} \).

(IV) 32 identical vortices, all with radius \( R \) and vorticity extremum \( \zeta_{\text{ext}} \).

Note that all cases share a common subset of vortices, namely the 32 vortices of case IV. Cases I–III can thus be considered as extensions of case IV in the sense that an additional subset of vortices with different properties has been added to perturb the evolution of the common subset.

Following the approach of Clercx and Nielsen,21 12 initial vorticity fields were generated for each IC. Each initial vorticity field differed due to the random placement of the center and the random choice of the sign of vorticity of each vortex. Figure 1 (first column) shows an example of an initial vorticity field for each IC. The results of the 12 runs were ensemble-averaged (denoted by \( \langle \cdot \rangle \) to remove noise and to obtain reliable statistics. Initially, all runs have by definition zero total circulation and equal kinetic energy, \( E_0 \). As a result, the vorticity extremes differ, namely \( \langle \zeta_{\text{ext}} \rangle = 61,82,78,88 \) for IC I, II, III, and IV, respectively.

In order to ensure numerical stability, the time step for each IC was chosen such that at \( t=0 \), the Courant–Friedrichs–Lewy number, defined as \( \text{CFL} = \max \left[ \frac{\max(u_{ij})}{\Delta t / \Delta x}, \frac{\max(v_{ij})}{\Delta t / \Delta y} \right] \),24 was approximately 0.1. The total number of time steps of the numerical simulation was chosen such that all numerical simulations reached the final stage of decay mentioned in the Introduction. Assuming \( \mathcal{U} = (2E_0/\zeta^2)^{1/2} \) and \( \mathcal{L} = \pi/2 \), respectively, are representative velocity and length scales, the initial Reynolds number \( Re_0 = \mathcal{U} \mathcal{L} / \nu = (E_0/2)^{1/2} / \nu \) was either 3450 or 34500 in the numerical simulations with Newtonian viscosity.

III. VORTEX CENSUS

A vortex census technique was used to determine characteristic vortex properties such as the average radius and the average vorticity extremum. Generally, two approaches are possible: a physical approach in which flow structures that approximately conform to the idealized shape of an isolated coherent vortex are identified,20 or a more mathematical approach based on wavelet theory.25 The latter approach is based on ad hoc parameters whose optimal value may vary from flow field to flow field, and may accidentally count filamentary debris as a vortex. Therefore, the physical approach formed the preferred basis of the vortex census technique described below.

According to Weiss,26 one can define a function \( Q = \text{tr}[(\nabla \psi)^2] \), which is proportional to the rate of strain squared, \( S^2 \), minus the vorticity squared, \( \zeta^2 \), i.e., \( Q = 1/2 (S^2 - \zeta^2) \). [Note that with the incompressibility condition \( \nabla \cdot \psi = 0 \), it follows that \( Q = -2 \text{det} (\nabla \psi) \).] The Weiss function \( Q \) appears to be a convenient measure to distinguish between the vortex core and the halo of filamentary features, which surrounds the cores of continuous vortices. Regions with positive \( Q \), i.e., \( |S| > |\zeta| \), are characterized by structures that are dominated by strain, such as vorticity filaments, whereas regions with negative \( Q \), i.e., \( |S| < |\zeta| \), are related to structures that are dominated by rotation, such as coherent vortices. Motivated by the fact that for axisymmetric vortices \( Q \) changes sign where the azimuthal velocity reaches a maxi-
mum, we associate each region of negative $Q$ with the vortex core.

Our feature education is based on the following condition: the Weiss function field has a single sign ($Q<0$) in a simply connected region about a single local minimum of significant amplitude (typically smaller than $Q_{\text{min}}/10$ with $Q_{\text{min}}$ the global minimum Weiss function). Furthermore, the size of the region must be at least $30l^2/N^2/10^{-4}$ and the shape of the region must fulfill the restriction $I/I_c<6$, where $I$ represents the inertia of the region and $I_c$ the inertia of an equally sized circular region—each point of the region is assumed to have the same “weight.” (The vortex census technique was found to be extremely insensitive to any of these parameters.) Doing so, only the three selection criteria mentioned above were needed to filter vortex structures, which significantly simplifies the interpretation of the vortex census results. Extensive testing resulted in an optimized set of parameters with which striking agreement was found between the time evolution of the vortex density as calculated by the automated vortex census technique and that determined by counting the number of vortices by hand.

IV. TIME EVOLUTION OF THE FLOW

In the following analysis, we will first focus on the results for $Re_0=34500$. Later, in Sec. VII, we will deal with the case $Re_0=3450$. Figure 1 shows typical time evolutions of the vorticity field for each of the four ICs defined in Sec. II. The top row of panels corresponds to case I, the second row to case II, etc. The first column of Fig. 1 shows characteristic initial vorticity distributions for cases I, II, III, and IV, respectively. Time is rescaled as $\tau=\frac{t}{T}$, where $t$ represents the numerical time, and $T=(\xi_{\text{ext}})^{-1}$ is the average eddy turnover time associated with the subset of vortices common to all ICs, i.e., a Gaussian vortex with radius $R$ and vorticity extremum $\xi_{\text{ext}}$ (see Sec. II). The proposed time scaling was found to be similar to a scaling based on the initial global enstrophy, viz., $\left(\frac{\langle \Omega_0 \rangle}{l^2}\right)^{1/2}$.

Consider first the characteristic time evolution of the vorticity field for case IV (bottom row of Fig. 1). After a short time ($\tau=20$), some of the initially circular vortices have been strained-out due to the strain field locally induced by the neighboring vortices. First mergers are fulfilled around $\tau=60$. As time proceeds, self-organization of the flow continues and the vortex density rapidly decreases while the average vortex size increases ($\tau=170$). It is essentially this stage of the decay process that should display the scaling behavior mentioned in the Introduction. Around $\tau=600$, only a few vortices remain, and the subsequent self-organization is a lengthy process. Eventually, a slowly propagating dipole remains that is slowly dissipated by viscous effects, the advection becomes regular, and the evolution ceases to be sig-

![Image of time evolution of vorticity field for cases I–IV](image-url)
significantly nonlinear. This final stage of the decay process is almost reached at $\tau=1550$.

The time evolution of the vorticity field for case I (first row of panels in Fig. 1) differs from that of case IV. In case I, the vortex interactions are characterized by significantly stronger vortex distortion, readily visible from the more elongated vortex sheets at $\tau=60$. Furthermore, the number of vortices is observed to decrease at a somewhat higher rate, so that the final stage is reached earlier.

The characteristic time evolutions of the vorticity fields associated with cases II and III reveal that the small vortices initially present in case II retain their axisymmetric shape, whereas the vortices with low peak vorticity initially present in case III seem to be strained out fairly easily. Around $\tau=600$, the small vortices (case II) and the vortices with low peak vorticity (case III) are no longer present (this was verified for the depicted time evolutions by manually tracking each vortex in time). They have either merged with vortices from the common subset, or have been strained out by surrounding vortices. The subsequent time evolution of the remaining vortices strongly resembles the final decay observed for case IV.

V. GLOBAL QUANTITIES

Figure 2(a) shows the normalized kinetic energy of the flow $\langle E \rangle / E_0$ as a function of time $\tau$. The solid, dotted, dash-dotted, and dashed lines correspond to cases I, II, III, and IV, respectively. This representation of the lines is maintained throughout the remainder of this paper. The kinetic energy is virtually constant for all ICs, so that we may consider the kinetic energy as a conserved quantity.

Figure 2(b) shows the time evolution of the normalized global enstrophy $\langle \Omega \rangle / \langle \Omega_0 \rangle$ (normalization is required since all runs are initiated with different initial enstrophy $\Omega_0$). The rate of decrease of $\Omega$ is a qualitative measure of the rate at which self-organization evolves as it illustrates the rate at which the average length scale $[\propto (\Omega/E)^{-1/2}]$ increases. Figure 2(b) shows clear power-law behavior during the intermediate stage of the evolution for all cases, reflecting the self-similar behavior of 2D turbulent flows. The power-law exponents for $\Omega$ (Table III)—determined for the intermediate stage of each IC, as will be explained in Sec. VI—are consistent with previous investigations$^{11,22}$ with regard to Reynolds number dependence, but differ slightly from case to case.

The time evolution of the global vorticity extremum, $\langle \xi_{\text{ext0}} \rangle$, is depicted in Fig. 2(c).$^{30}$ Clearly, conservation of the global vorticity extremum as assumed by Carnevale et al.$^7$ is strongly violated due to the limited Reynolds number in our simulations. Nevertheless, it is striking that $\langle \xi_{\text{ext0}} \rangle$ displays power-law behavior for $\tau>200$, and that the corresponding decay rates are in agreement for all cases (see Table III).

Concerning the initial stage of the evolution, Fig. 2(b) shows that the initial period of adjustment is shortest in case I and longest in case IV. The evolutions of $\langle \Omega \rangle / \langle \Omega_0 \rangle$ for cases II and III almost collapse, and their decrease sets in somewhere in between that of cases I and IV. The initial period of adjustment is primarily determined by the frequency of close encounters and the efficiency of the mergers that take place. For identical vortices, it follows directly that a higher initial vortex density implies a larger merger frequency if the average vortex advection speed [proportional to $\dot{\langle E \rangle}$ (Ref. 7)] remains unchanged, such as the case here, since $E$ is conserved [see Fig. 2(a)]. A larger merger frequency results in a shorter initial period of adjustment, which is directly confirmed by comparing the time evolution of
TABLE III. Power-law exponents for the power-law decay of the global enstrophy \(\langle \Omega \rangle\), the vortex enstrophy \(\langle \Omega \rangle / \langle \Omega_0 \rangle\) for cases I and IV [see Fig. 2(b)]. For distinct vortices (cases II and III), such a simple argument is no longer valid since one must then also take into account the different vortex properties. Previous studies have shown that the inelastic interaction between unequal vortices is generally less efficient than that between equal vortices. Assuming that this result holds for interactions involving more than two vortices, one may expect that \(\langle \Omega \rangle / \langle \Omega_0 \rangle\) decreases later in cases II and III than in case I. Figure 2(b) supports this inference.

VI. VORTEX STATISTICS

Figure 3(a) shows the time evolution of the normalized vortex density, \(\langle \rho \rangle / \langle \rho_0 \rangle\), for each of the cases described in Sec. II. The time evolution of \(\langle \rho \rangle / \langle \rho_0 \rangle\) is equal to that of \(\langle \langle \Omega \rangle / \langle \Omega_0 \rangle \rangle\) [Fig. 2(b)] with respect to the sequence of the lines for all cases except for case III, in which \(\langle \rho \rangle / \langle \rho_0 \rangle\) evolves as in case I for \(\tau \leq 500\). The latter may be explained (at least partly) by vortex stripping, i.e., vortex breaking into filamentary structures due to a sufficiently strong external straining field. Such a phenomenon is more relevant for vortices with low peak vorticity and so it contributes to an accelerated decrease of \(\langle \rho \rangle / \langle \rho_0 \rangle\) in case III. This explanation is supported by the observations given in Sec. IV, which show that the weak vortices in case III are easily strained out and are no longer observed for \(\tau \approx 600\).

Figure 3(b) shows the temporal growth of the (normalized) average vortex area \(\langle A \rangle / \langle A_0 \rangle\), where the overbar indicates an average over the entire vortex population at a given time. The growth rate of the average vortex area was found to be significantly larger than the growth of a single member of the common subset (i.e., a Gaussian vortex in isolation with initial radius \(R\) and initial vorticity extremum \(\zeta_{ext}\) indicating that the vortex growth observed in cases I–IV is for the larger part due to vortex mergers. We expect the sequence of lines of the time evolution of \(\langle A \rangle / \langle A_0 \rangle\) to be identical to the sequence of lines observed for the inverse global enstrophy—the latter being proportional to a squared average length scale (Sec. V). However, comparison of Figs. 2(b) and 3(b) reveals that this is only true for the final stage of the decay (\(\tau > 700\)).

In contrast with \(\langle A \rangle / \langle A_0 \rangle\), Fig. 3(c) reveals that the decrease of the normalized total vortex area \(\langle \Lambda_{tot} \rangle / \langle A_{tot,0} \rangle\) displays clear power-law behavior for the time interval for which the enstrophy also displays power-law behavior [Fig. 2(b)]. Since the average vortex area and the total vortex area are related by \(\Lambda = \Lambda_{tot} / \rho\), it seems obvious that the less apparent power-law behavior observed for the average vortex area is caused by averaging over a vortex population with too diverse vortex properties (e.g., the vortex area).

Before discussing in detail the power-law behavior of the vortex quantities to see to which extent our results agree with the scaling theory, it is important to show whether the assumptions of the scaling theory are fulfilled, viz., (a) that all enstrophy is contained by the coherent structures and (b) that averages of powers of quantities scale the same as powers of averages, e.g., \(\bar{A}^{\mu}(t) = c_\mu \langle A \rangle^\mu\), where \(c_\mu\) is independent of time.\(^2\)

In order to check the first assumption, we have plotted in Fig. 4 the time evolution of the normalized vortex enstrophy \(\langle \Omega \rangle / \langle \Omega_0 \rangle\), where \(\Omega_0\) is defined as the total enstrophy contained by the coherent vortices extracted by the vortex census. Clearly, the vortex enstrophy behaves in exactly the same way as does the global enstrophy [Fig. 2(b)]. We report that during the entire evolution, the ratio \(\langle \Omega \rangle / \langle \Omega_0 \rangle\) fluctuates about 0.8 with deviations no greater than 0.05. This means that the coherent vortices contain approximately 80% of the global enstrophy during the entire decay, based on which we consider the first assumption sufficiently satisfied.

The second assumption of the scaling theory requires a self-similar evolution of the probability distribution function (pdf) describing the vortex population. This pdf depends on the vortex size, shape, and vorticity amplitude. If the pdf only depends on the vortex size, then one can show that a self-similar evolution of the pdf implies a time-independent
distribution \( p(x) \) with \( x = A(t)/\bar{A}(t) \). Following Weiss and McWilliams,\(^7\) we have investigated the time evolution of \( p(x) \). Careful analysis of the entire time evolution of \( p(x) \) allows us to identify time intervals \( \Delta \tau \) for which the distribution function maintains its shape. These intervals are reported in Table III. For each case, we show in Fig. 5 the characteristic shape of the distribution function \( p(x) \) during the time interval \( \Delta \tau \). Each distribution function \( p(x) \) is obtained by combining the distributions of the 12 independent realizations of a given case and for a given time, and normalizing such that the integral \( \int p(x)dx = 1 \). In addition, the distributions are averaged over a short period of time (about 150 scaled time units) to suppress the sampling variability.

Since the above-mentioned assumptions of the scaling theory are to a large extent fulfilled in our numerical simulations, we have investigated the power-law behavior during the time interval \( \Delta \tau \) for quantities \( \langle \rho \rangle \) and \( \langle A_{\text{tot}} \rangle \). The corresponding power-law exponents are given in Table III. (The power-law exponents reported for \( \langle \Omega \rangle \) and \( \langle \Omega_v \rangle \) were derived in exactly the same way.) The error estimates result solely from the uncertainty in defining the time interval \( \Delta \tau \). We feel the need to emphasize that it is very important to select the correct time interval in order to compare the resulting power-law exponents with the scaling theory, and it will be evident from Figs. 3(a) and 3(b) that an incorrectly chosen time interval will lead to considerably different power-law exponents.

Table III reveals that the relationship between the power-law exponents of \( \langle \rho \rangle \rangle /\langle \rho_0 \rangle \rangle \) and \( \langle A_{\text{tot}} \rangle /\langle A_{\text{tot},0} \rangle \rangle \) varies significantly for cases I–IV. We believe that this variation is caused by the interactions between the common subset and the added subset, the initial number density, and the initial pdf [reflected by the distribution of vortex size \( p(x) \)]. With regard to the initial number density, note that while the initial distributions of vortex size are identical in cases I and IV, the distributions are seen to evolve during a short phase of initialization to very distinct (time-independent) distributions [compare Figs. 5(a) and 5(d)] characterized by different power-law behavior. As cases I and IV only differ by the initial number density, \( \rho_0 \), it appears to be an important parameter that directly affects the power-law behavior. Furthermore, with regard to the initial pdf, it is important to note that Benzi et al.\(^{14}\) showed that the statistical distribution of vortex size is sufficient to fully describe scaling behavior in a vortex system, so that one may expect different relationships between the power-law exponents when altering the distribution of vortex size while maintaining the initial number density (as in cases I–III).
It is interesting to quantify the effect of the perturbing subset of vortices on the common subset of vortices. The qualitatively observed behavior of the weaker vortices in case III suggests that they do not really impede the vortices of the common subset, so that in this particular case one may expect the evolution of the common subset of vortices to be very similar to that observed in case IV. A similar expectation goes for case II, since the smaller vortices are no longer observed from $\tau_9=600$. Indeed, our expectations are largely confirmed by Fig. 6, which shows the same quantities as Fig. 3, but now for the common subset only (cases II and III). The power-law exponents associated with the common subset are found to match well with case IV, see Table III. For the common subsets in cases II–IV, the relationship between the power-law exponents $\rho$ and $A_{\text{tot}}$ is approximately $1:0.5$, which is in good agreement with the scaling theory.

VII. DISCUSSION

In the preceding sections, we have demonstrated how small differences in narrowband ICs lead to significant differences in the temporal behavior of the global enstrophy, the total vortex enstrophy, the vortex number density, the average vortex size, and the total vortex area. These differences are mainly ascribed to the differences in the initial vortex number density and the initial vortex size distribution. Moreover, although the evolution of the vortex number density, the average vortex area, and the total vortex area of the entire vortex population differs from case to case (Fig. 3), a remarkable collapse is observed for cases II–IV when the analysis is restricted to the common subset of vortices (Fig. 6). This shows that important properties that characterize the common subset of vortices are not significantly altered by adding an equal number of weaker vortices (with “weaker” referring to either their size or their vorticity amplitude). In connection with the above-mentioned material, we note that Bracco et al. already observed two distinct populations of vortices, viz., small vortices with weak amplitude and large vortices with strong amplitude, and concluded that the scaling theory only applies to the latter population.

It has been argued that pseudospectral simulations at present (fixed) spatial resolution cannot properly simulate inviscid dynamics since vortex edge erosion by numerical dissipation effectively removes all small scales, not all of them being unimportant. According to Dritschel, the resulting vortex properties and time trends are dominated by the effects of numerical dissipation. In order to avoid this problem, Dritschel used contour dynamics to simulate the evolution of 200 patches of equal vorticity but disparate size, the results of which strongly contradict the scaling theory. The study of Dritschel suggests that 2D turbulence is strongly influenced by the constant generation of small-scale vortices by the various merger processes that take place. Our results show that a subset of weak vortices affects the average properties of the total set of vortices. The subset of weak vortices does not, however, significantly influence the dynamics of the subset of dominant vortices.

We have also presented an extensive analysis of the
power-law behavior of quantities \( \langle \Omega \rangle \), \( \langle \xi_{\text{ext}} \rangle \), \( \langle \rho \rangle \), and \( \langle A_{\text{tot}} \rangle \) during the intermediate stage of the decay. Our power-law exponents (Table III) are consistent with previous investigations, but their relationship differs from the relationship predicted by the scaling theory. This different relationship may be partly ascribed to finite Reynolds number effects in our simulations, which most important of all lead to a nonconserved average vortex extremum. Assuming a diffusive effect \( \xi_{\text{ext}} \propto t^{-7} \), Weiss and McWilliams have proposed a finite Reynolds number correction for the scaling theory, viz.,

\[
\Omega(t) \propto T^{-2} \left[ \frac{t}{T} \right]^{-\frac{1}{2}-\gamma} , \quad \rho(t) \propto L^{-2} \left[ \frac{t}{T} \right]^{-\xi} ,
\]

\[
a(t) \propto L \left[ \frac{t}{T} \right]^{\frac{1}{2}+\frac{1}{2}\gamma} , \quad r(t) \propto L \left[ \frac{t}{T} \right]^{\frac{1}{2}} .
\]

Using (4), we have predicted the corrected exponents for \( \langle \Omega \rangle \) and \( \langle A_{\text{tot}} \rangle \), see Table III. The relationship between the exponents of \( \rho \) and the corrected exponents of \( \langle \Omega \rangle \) clearly improves, but the relationship between the exponents of \( \rho \) and the corrected exponents of \( A_{\text{tot}} \) becomes worse.

One may argue that the lack of clear power-law behavior is due to the small initial number of vortices. However, we have performed three simulations (\( \text{Re}_0 = 34500 \)) with a larger initial number of (identical) vortices while maintaining the spatial resolution of the vortices in case I, viz., 256 vortices with resolution \( l/N = 4/1024 \). The statistics of these runs collapse with the statistics obtained for case I, see Fig. 7.

Recall from the Introduction that the choice of the ICs was inspired by experiments on an electromagnetically forced fluid layer by Tabeling et al., who claim to have confirmed the scaling theory. However, analysis of all runs of cases I–IV for \( \text{Re}_0 = 34500 \) (similar to the initial Reynolds number in the above-mentioned laboratory experiment) yields results very similar to those obtained for \( \text{Re}_0 = 34500 \). (The decay rates are obviously larger due to the stronger dissipation, see Table IV.) Given the dependence of the investigated quantities on the ICs for the investigated range of Reynolds numbers, the results of fluid layer experiments are expected to depend on the initial vortex number density and the initial vortex size distribution. If besides that we take into consideration the strong dependence on the fluid layer thickness, it seems that any agree-
ment between the laboratory experiments in shallow fluid layers and the scaling theory is rather coincidental.

To check to what extent our data are contaminated by viscous diffusion, we have computed the time evolution for all runs of cases I–IV using hyperviscosity ($\nu_2 = 3.5 \times 10^{-6}$) instead of Newtonian viscosity (Table V). The relationship between the corresponding power-law exponents improves in favor of the scaling theory. In particular, the decay of $\langle \Omega \rangle$ is strongly reduced due to the reduced dissipation at the small scales. Despite the use of hyperviscosity, however, the relationship between the global enstrophy and the total vortex area still differs from the relationship predicted by the scaling theory. Even with the finite Reynolds number correction, the scaling theory is ruled out in all of our cases with only one exception. If we compare the corrected exponents of $\langle \Omega \rangle$ and $\langle A_{v0} \rangle$ with their uncorrected counterparts, it follows that in case II with hyperviscosity the difference falls within one standard deviation. The scaling theory is therefore not ruled out in this particular case.

Finally, one may wonder what happens if the common subset of vortices is perturbed by a subset of smaller vortices with higher vorticity amplitude, such that the circulation of each vortex is equal for each subset. Such an IC is to some extent comparable to numerical simulations with no-slip boundary conditions (although there, a continuous production of strong small-scale vortices is expected). Such simulations are characterized by significantly slower decay rates due to the small, strong vortices, generated through boundary layers that roll up, which interact with vortices in the inner domain. Therefore, a fifth IC (case V) has been simulated: 64 Gaussian vortices with equal circulation, half of them with radius $R$ and vorticity extremum $\zeta_{ext}$ (the common subset), the other half with radius $\frac{1}{2}R$ and vorticity extremum $2\zeta_{ext}$. Doing so, the effective intervortex distance is increased compared to case I, while the circulation of each vortex remains unchanged. Since the inelastic interaction between unequal vortices is generally less efficient than that between equal vortices (see Refs. 4, 12, and 13), it is expected that the time

<table>
<thead>
<tr>
<th>Case</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \Omega \rangle$</td>
<td>0.375</td>
<td>1.17±0.05</td>
<td>1.05±0.05</td>
<td>1.11±0.05</td>
</tr>
<tr>
<td>$\langle \zeta_{ext} \rangle$</td>
<td>0.375</td>
<td>1.24±0.05</td>
<td>1.25±0.05</td>
<td>1.19±0.05</td>
</tr>
<tr>
<td>$\langle \zeta_{ext} \rangle / \sqrt{\langle E \rangle}$</td>
<td>0</td>
<td>0.45±0.03</td>
<td>0.44±0.03</td>
<td>0.39±0.03</td>
</tr>
<tr>
<td>$\langle \rho \rangle$</td>
<td>0.75</td>
<td>1.27±0.05</td>
<td>0.95±0.05</td>
<td>1.09±0.05</td>
</tr>
<tr>
<td>$\langle A_{v0} \rangle$</td>
<td>0.375</td>
<td>0.30±0.04</td>
<td>0.25±0.04</td>
<td>0.20±0.04</td>
</tr>
<tr>
<td>$\langle \Omega \rangle$ corrected</td>
<td>0.375</td>
<td>1.09±0.05</td>
<td>1.02±0.05</td>
<td>1.02±0.05</td>
</tr>
<tr>
<td>$\langle A_{v0} \rangle$ corrected</td>
<td>0.375</td>
<td>0.19±0.05</td>
<td>−0.07±0.05</td>
<td>0.16±0.05</td>
</tr>
</tbody>
</table>

TABLE V. Power-law exponents for the power-law decay of the global enstrophy $\langle \Omega \rangle$, the vortex enstrophy $\langle \Omega_v \rangle$, the average absolute vorticity extremum $\langle \zeta_{ext} \rangle$, the average number density $\langle \rho \rangle$, and the total vortex area $\langle A_{v0} \rangle$ for various ICs. "Corrected" refers to the finite Reynolds number correction (4) that was used to predict the exponents for $\langle \Omega \rangle$ and $\langle A_{v0} \rangle$ based on the exponents found for $\langle \rho \rangle$. Standard deviations show the quality of the fit based on the estimated time interval $\Delta \tau$. Initial Reynolds number $Re_0 = 3450$ for all numerical simulations.

<table>
<thead>
<tr>
<th>Case</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \Omega \rangle$</td>
<td>0.375</td>
<td>0.61±0.05</td>
<td>0.49±0.05</td>
<td>0.42±0.05</td>
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<tr>
<td>$\langle \zeta_{ext} \rangle$</td>
<td>0.375</td>
<td>0.57±0.05</td>
<td>0.45±0.05</td>
<td>0.36±0.05</td>
</tr>
<tr>
<td>$\langle \zeta_{ext} \rangle / \sqrt{\langle E \rangle}$</td>
<td>0</td>
<td>0.08±0.03</td>
<td>0.08±0.03</td>
<td>0.10±0.03</td>
</tr>
<tr>
<td>$\langle \rho \rangle$</td>
<td>0.75</td>
<td>0.69±0.05</td>
<td>0.73±0.05</td>
<td>0.66±0.05</td>
</tr>
<tr>
<td>$\langle A_{v0} \rangle$</td>
<td>0.375</td>
<td>0.37±0.04</td>
<td>0.30±0.04</td>
<td>0.20±0.04</td>
</tr>
<tr>
<td>$\langle \Omega \rangle$ corrected</td>
<td>0.375</td>
<td>0.43±0.05</td>
<td>0.45±0.05</td>
<td>0.43±0.05</td>
</tr>
<tr>
<td>$\langle A_{v0} \rangle$ corrected</td>
<td>0.375</td>
<td>0.27±0.05</td>
<td>0.29±0.05</td>
<td>0.23±0.05</td>
</tr>
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</table>
would like to know whether the decay of case V sets in between that of cases I and IV. Of course, one will deal with this matter.

The present vortex census technique. Future investigations

tices. This requires vortex tracking, not yet implemented in

common

with the

evolution of the flow quantities discussed in Secs. V and VI is delayed in case V compared to case I. Indeed, Fig. 8 (obtained in exactly the same manner as in the other cases) shows that the decrease of both \(\langle \Omega \rangle /\langle \Omega_t \rangle\) and \(\langle \rho \rangle /\langle \rho_0 \rangle\) in case V sets in between that of cases I and IV. Of course, one would like to know whether the decay of \(\langle \rho \rangle /\langle \rho_0 \rangle\) associated with the common subset of vortices coincides with that of case IV. Unfortunately, one cannot use simple criteria to discriminate the small, strong vortices from the large, weak vortices. This requires vortex tracking, not yet implemented in the present vortex census technique. Future investigations will deal with this matter.

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30 Increases in the overall extreme vorticity, e.g., the spikes at $\tau = 50$, result from a minor error in one of the subroutines of the numerical code. We have corrected the subroutine in question and recalculated the time evolution of a single initial vorticity field in order to verify that the false subroutine did not affect the time evolution of the vorticity field nor any of the other statistics, as well as to verify that the corrected subroutine reproduced the evolution of the vorticity extremum as shown in Fig. 2(c) without any spikes. Since this was the case, there was no need to recalculate the full time evolution of all the 60 (viz., 5 ICs consisting of 12 runs each) individual initial vorticity fields and the corresponding ensemble averages.