Convolution algebras translation invariant operators

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by

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M.M.A. de Rijcke
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Summary

In this paper three commutative convolution algebras of functions on \( \mathbb{R} \) of (locally) bounded variation are introduced. All three are represented as algebras of continuous, translation invariant linear operators on well-defined spaces of continuous functions and of Lebesgue measurable functions, respectively, in which the translation group is a locally equicontinuous \( c_0 \)-group. As a result complete characterizations of the translation invariant operators for \( L^1 \)-type spaces are derived.

December 1995
Introduction

Let $\mathcal{X}$ be a translation invariant subspace of Schwartz distribution space $\mathcal{D}'(\mathbb{R})$, and let $\mathcal{X}$ be endowed with a locally convex topology such that $\mathcal{X}$ is sequentially complete. Also, let the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{X}$ be locally equicontinuous and strongly continuous for this topology. A continuous linear operator $\mathcal{L}$ on $\mathcal{X}$ is said to be translation invariant if $\sigma_t \mathcal{L} = \mathcal{L} \sigma_t$ for all $t \in \mathbb{R}$. The collection of all translation invariant operators on $\mathcal{X}$ establishes an algebra. Note that it is not clear at this point whether this algebra is commutative.

From the paper [ER] we invoke that for all $\mu \in \text{bvc}(\mathbb{R})$ (cf. Definition 1.2) and all $f \in \mathcal{X}$ the linear operator $\sigma[\mu]$ on $\mathcal{X}$ defined by

$$\sigma[\mu] f = \int_{\mathbb{R}} \sigma_t f \, d\mu(t), \quad f \in \mathcal{X}$$

is translation invariant and continuous, where the integral is an $\mathcal{X}$-valued Riemann–Stieltjes integral. Now one may wonder whether the convolution algebra $\text{bvc}(\mathbb{R})$ represents all translation invariant operators on $\mathcal{X}$. In this paper we prove that for $\mathcal{X} = C(\mathbb{R}), C_c(\mathbb{R}), L^p_{\text{loc}}(\mathbb{R})$ and $L^p_{\text{comp}}(\mathbb{R})$ this is the case, but although a likely candidate for $\mathcal{X} = L^p_{\text{loc}}(\mathbb{R})$ or $L^p_{\text{comp}}(\mathbb{R})$, the problem whether $\text{bvc}(\mathbb{R})$ is the convolution algebra associated to these spaces is not yet solved.

The convolution algebra $\text{bvc}(\mathbb{R})$ is too small, in general, to describe all translation invariant operators. We introduce two larger convolution algebras, namely $\text{bv}_+(\mathbb{R})$ and $\text{bV}_{\text{loc},-(\mathbb{R})}$, and represent them as commutative operator algebras on spaces $\mathcal{X}$ satisfying additionally for $\text{bv}_+(\mathbb{R})$

$$(1) \quad \lim_{t \to -\infty} \sigma_t f = 0, \quad \forall f \in \mathcal{X}$$

and for $\text{bV}_{\text{loc},-(\mathbb{R})}$,

$$(2) \quad \forall f \in \mathcal{X}, \exists a \in \mathbb{R} : \text{supp}(f) \subset [a, \infty).$$

The representation is by means of the same integral, yet considered as an $\mathcal{X}$-valued improper Riemann–Stieltjes integral. Examples of spaces of type (1) are $C_{-}(\mathbb{R})$ and $L^p_{-}(\mathbb{R})$ and examples of spaces of type (2) are $C_{+}(\mathbb{R})$ and $L^p_{\text{loc},+}(\mathbb{R})$, $1 \leq p < \infty$, introduced in Sections 2 and 3.

In this paper we use freely, without further reference, the standard terminology of the theory of locally convex vector spaces and of the general one-parameter co-group theory. In this respect we deal with $F$-spaces (= Fréchet spaces) and strict $LF$-spaces (= strict inductive limits of $F$-spaces). Nice references are [Co], [Sch] and [Fl].

The present paper is divided into four sections.
Section 1 is devoted merely to definitions and notations to be used. We introduce the space $C(\mathbb{R}, V)$ of continuous functions into a locally convex vector space $V$, and its subspaces $C_{-}(\mathbb{R}, V), C_c(\mathbb{R}, V), C_{+}(\mathbb{R}, V)$ and $C_{-,+}(\mathbb{R}, V)$. Also we discuss Riemann–Stieltjes integration for these spaces. Here the subspaces $\text{bvc}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bV}_{\text{loc},-(\mathbb{R})}$ and $\text{bV}_{\text{loc},-(\mathbb{R})}$ of
the space \( bV_{\text{loc}}(\mathbb{R}) \) consisting of all right continuous functions \( \mu \) on \( \mathbb{R} \) with bounded variation on compact intervals of \( \mathbb{R} \), come into play.

In Section 2 we characterize the collection of all continuous, translation invariant, linear operators on the spaces \( C(\mathbb{R}) \) and \( C_{c}(\mathbb{R}) \), \( C_{-}(\mathbb{R}) \) and \( C_{-,+}(\mathbb{R}) \), and \( C_{+}(\mathbb{R}) \) where \( V = C \) is omitted in the notation. They are represented by the convolution algebras \( bV_{c}(\mathbb{R}) \), \( bV_{+}(\mathbb{R}) \) and \( bV_{\text{loc},-}(\mathbb{R}) \), respectively.

In Section 3 we introduce the spaces \( L^{p}_{\text{loc}}(\mathbb{R}) \), \( L^{p}_{\text{comp}}(\mathbb{R}) \), \( L^{p}_{-}(\mathbb{R}) \), \( L^{p}_{+}(\mathbb{R}) \) and \( L^{p}_{\text{loc},+}(\mathbb{R}) \), we discuss their duality relations and show how they are related to the aforementioned convolution algebras.

In the last section we pay special attention to the case \( p = 1 \), describing completely all continuous, translation invariant, linear operators on these spaces.

1 Spaces of continuous functions

Let \( V \) denote a sequentially complete locally convex topological vector space. The locally convex topology of \( V \) is assumed to be generated by the indexed set of seminorms \( \{ p_{\nu} \mid \nu \in D \} \). In this section we introduce the space \( C(\mathbb{R}, V) \) consisting of all continuous functions from \( \mathbb{R} \) into \( V \), and its subspaces \( C_{+}(\mathbb{R}, V) \), \( C_{-}(\mathbb{R}, V) \), \( C_{-,+}(\mathbb{R}, V) \) and \( C_{c}(\mathbb{R}, V) \).

The space \( C(\mathbb{R}, V) \) is endowed with the compact open topology, i.e. the locally convex topology generated by the seminorms

\[
p_{\nu,n}(f) = \max_{t \in [-n,n]} p_{\nu}(f(t)), \quad n \in \mathbb{N}, \ \nu \in D.
\]

Then \( C(\mathbb{R}, V) \) is sequentially complete. Moreover, if \( D \) is a countable set, both \( V \) and \( C(\mathbb{R}, V) \) are \( F \)-spaces (Fréchet spaces). We note that in [ER] the space \( C(\mathbb{R}, V) \) is used as a means to describe general properties of one-parameter locally equicontinuous \( c_{0} \)-groups of continuous linear mappings on \( V \).

**Definition 1.1.** Let \( I \subseteq \mathbb{R} \) denote an interval. A function \( \mu : I \to \mathbb{C} \) is said to be of bounded variation if there exists \( C > 0 \) such that for all \( n \in \mathbb{N} \) and all \( \{t_{0}, \ldots, t_{n}\} \subseteq I \) with \( t_{0} < t_{1} < \cdots < t_{n} \)

\[
\sum_{j=1}^{n} |\mu(t_{j}) - \mu(t_{j-1})| \leq C. \tag{*}
\]

By \( \text{var}_{I}(\mu) \) we denote the total variation of \( \mu \) on \( I \), i.e. the infimum of the collection of all constants \( C \) satisfying (\(*\)). The vector space of all right continuous functions \( \mu \) on \( I \) with \( \text{var}_{I}(\mu) < \infty \) is denoted by \( bV(I) \).

**Definition 1.2.** The space \( bV_{c}(\mathbb{R}) \) consists of all \( \mu \in bV(\mathbb{R}) \) with the property that there is \( T > 0 \) depending on \( \mu \), such that

\[
\mu(t) = 0 \text{ for } t < -T, \quad \mu(t) = \mu(T) \text{ for } t > T.
\]
Due to the sequential completeness of $V$ for all $\mu \in \text{bv}_c(\mathbb{R})$ and $f \in C(\mathbb{R}, V)$ the $V$-valued (Riemann–Stieltjes) integral

$$\mathcal{J}_\mu f := \int_{\mathbb{R}} f(\tau) d\mu(\tau)$$

can be defined properly. The mapping $\mathcal{J}_\mu$ from $C(\mathbb{R}, V)$ into $V$, thus defined, is linear and continuous, since

$$p_\nu(\mathcal{J}_\mu f) \leq \max_{t \in [-T, T]} p_\nu(f(t)) \cdot \text{var}_{\mathbb{R}}(\mu) .$$

**Proposition 1.3.** Let $\mathcal{K} : V_1 \to V_2$ be a continuous linear operator. Then the linear operator $\mathcal{L}_\mathcal{K} : C(\mathbb{R}, V_1) \to C(\mathbb{R}, V_2)$ defined by

$$(\mathcal{L}_\mathcal{K} f)(t) := \mathcal{K}(f)(t)$$

is continuous and satisfies

$$\mathcal{J}_\mu \mathcal{L}_\mathcal{K} = \mathcal{K} \mathcal{J}_\mu .$$

More details on the introduction of $\mathcal{J}_\mu$ can be found in [ER].

**Definition 1.4.** The space $C_-(\mathbb{R}, V)$ consists of all $f \in C(\mathbb{R}, V)$ for which

$$\lim_{t \to \infty} f(t) = 0,$$

i.e., $\forall \nu \in D : \lim_{t \to \infty} p_\nu(f(t)) = 0$.

The locally convex topology for $C_-(\mathbb{R}, V)$ is the one generated by the seminorms

$$p_\nu^{-n}(f) := \max_{t \in [-n, n]} p_\nu(f(t)), \quad n \in \mathbb{N}, \nu \in D .$$

The space $C_-(\mathbb{R}, V)$ is sequentially complete. In correspondence, we introduce the subspace $\text{bv}_+(\mathbb{R})$ of $\text{bv}(\mathbb{R})$.

**Definition 1.5.** The space $\text{bv}_+(\mathbb{R})$ consists of all $\mu \in \text{bv}(\mathbb{R})$ for which there is $T > 0$ depending on $\mu$ such that $\mu(t) = 0$ for all $t < -T$.

Since the functions in $C_-(\mathbb{R}, V)$ are uniformly continuous on half-infinite intervals $[-T, \infty)$, for each $\mu \in \text{bv}_+(\mathbb{R})$ and $f \in C_-(\mathbb{R}, V)$ the $V$-valued (improper Riemann–Stieltjes) integral

$$\mathcal{J}_\mu f := \int_{\mathbb{R}} f(\tau) d\mu(\tau)$$

can be well-defined and $\mathcal{J}_\mu : C_-(\mathbb{R}, V) \to V$ is continuous with
\[ p_\nu(\mathcal{J}_\mu f) \leq \max_{t \geq -T} p_\nu(f(t)) \cdot \text{var}_R(\mu). \]

**Definition 1.6.** The space \( C_+([R, V]) \) consists of all \( f \in C([R, V]) \) with support bounded on the left, i.e. all \( f \in C([R, V]) \) for which \( T > 0 \) exists such that \( f(t) = 0 \) for \( t \leq -T \).

Let

\[ C_{+,n}([R, V]) := \{ f \in C([R, V]) \mid \forall t \leq -n : f(t) = 0 \}. \]

Then \( C_{+,n}([R, V]) \) is a closed subspace of \( C([R, V]) \) and the collection \( \{ C_{+,n}([R, V]) \mid n \in \mathbb{N} \} \) is a strict inductive system of sequentially complete locally convex spaces. We endow \( C_+([R, V]) \) with the inductive limit topology generated by this strict inductive system and end up with a sequentially complete locally convex space.

**Definition 1.7.** The space \( bV_{\text{loc.-}}([R]) \) consists of all right continuous \( \mu : [R] \rightarrow \mathcal{C} \) with the property that

\[ \exists T > 0 : \mu(t) = \mu(T), \quad t \geq T \]

and

\[ \mu|_{[-n, \infty)} \in bV([-n, \infty)), \quad n \in \mathbb{N}. \]

For each \( \mu \in bV_{\text{loc.-}}([R]) \) and \( f \in C_+([R, V]) \) the \( V \)-valued (proper Riemann–Stieltjes) integral

\[ \mathcal{J}_\mu f := \int_{[R]} f(\tau) d\mu(\tau) \]

can be well-defined, and \( \mathcal{J}_\mu : C_+([R, V]) \rightarrow V \) is continuous. To check the latter assertion, observe that for all \( n \in \mathbb{N} \) and all \( f \in C_{+,n}([R, V]) \)

\[ p_\nu(\mathcal{J}_\mu f) \leq \max_{t \in [-n, T]} p_\nu(f(t)) \cdot \text{var}_{[-n, \infty]}(\mu) \]

with \( T \) chosen as in \((**))\).

**Definition 1.8.** The space \( C_{-,+}([R, V]) \) consists of all \( f \in C([R, V]) \) with the property that

\[ \exists T > 0 : f(t) = 0, \quad t \leq -T \]

and

\[ \lim_{t \to \infty} f(t) = 0, \]
\[ C_{-,-,+}(\mathbb{R}, V) = C_{-}(\mathbb{R}, V) \cap C_{+}(\mathbb{R}, V) . \]

Introduce the strict inductive system of sequentially complete locally convex vector spaces

\[ C_{-,-,+}(\mathbb{R}, V) := \bigcap_{n=1}^{\infty} C_{-,-,+}(\mathbb{R}, V) . \]

Then

\[ C_{-,-,+}(\mathbb{R}, V) = \bigcup_{n=1}^{\infty} C_{-,-,+}(\mathbb{R}, V) \]

is endowed with the corresponding inductive limit topology and thus sequentially complete.

**Definition 1.9.** The space \( \text{bv}_{\text{loc, } \rightarrow}(\mathbb{R}) \) consists of all right-continuous functions \( \mu : \mathbb{R} \rightarrow \mathcal{C} \) with the property that for all \( n \in \mathbb{N} \)

\[ \mu|_{[-n, \infty)} \in \text{bv}([-n, \infty)) . \]

For each \( \mu \in \text{bv}_{\text{loc, } \rightarrow}(\mathbb{R}) \) and \( f \in C_{-,-,+}(\mathbb{R}, V) \) the \( V \)-valued (improper Riemann–Stieltjes) integral

\[ \mathcal{J}_\mu f := \int_{\mathbb{R}} f(\tau) d\mu(\tau) \]

can be well-defined and satisfies

\[ p_\nu(\mathcal{J}_\mu f) \leq \max_{t \geq -n} p_\nu(f(t)) \cdot \text{var}_{[-n, \infty]}(\mu) . \]

Therefore \( \mathcal{J}_\mu : C_{-,-,+}(\mathbb{R}, V) \rightarrow V \) is continuous for each \( \mu \in \text{bv}_{\text{loc, } \rightarrow}(\mathbb{R}) \).

Finally, we introduce the subspace \( C_c(\mathbb{R}, V) \) of \( C(\mathbb{R}, V) \).

**Definition 1.10.** The space \( C_c(\mathbb{R}, V) \) consist of all \( f \in C(\mathbb{R}, V) \) for which there exists \( T > 0 \) such that

\[ f(t) = 0 , \quad |t| \geq T . \]

We see that

\[ C_c(\mathbb{R}, V) = \bigcup_{n=1}^{\infty} C_{c,n}(\mathbb{R}, V) , \]
where

\[ C_{c,n}(\mathbb{R}, V) := \{ f \in C(\mathbb{R}, V) \mid f(t) = 0, \forall t, |t| \geq n \}. \]

Being closed subspaces of \( C(\mathbb{R}, V) \), the collection \( \{ C_{c,n}(\mathbb{R}, V) \mid n \in \mathbb{N} \} \) is a strict inductive system of sequentially complete locally convex vector spaces. We endow the space \( C_c(\mathbb{R}, V) \) with the corresponding inductive limit topology and end up with a sequentially complete locally convex space.

**Definition 1.11.** The space \( bV_{loc}(\mathbb{R}) \) consists of all right continuous functions \( \mu : \mathbb{R} \to C \) such that

\[ \mu|_{[-n,n]} \in bv([-n,n]), \quad n \in \mathbb{N}. \]

For each \( \mu \in bV_{loc}(\mathbb{R}) \) and \( f \in C_c(\mathbb{R}, V) \) the \( V \)-valued (Riemann–Stieltjes) integral

\[ J_{\mu}f := \int_{\mathbb{R}} f(\tau) d\mu(\tau) \]

can be well-defined and \( J_{\mu} : C_c(\mathbb{R}, V) \to V \) is continuous because for all \( f \in C_{c,n}(\mathbb{R}, V) \)

\[ p_\nu(J_{\mu}f) \leq \max_{t \in [-n,n]} p_\nu(f(t)) \cdot \text{var}_{[-n,n]}(\mu). \]

There is the following dense and continuous inclusion scheme

\[ C_c(\mathbb{R}, V) \hookrightarrow C_{-,+}(\mathbb{R}, V) \hookrightarrow C_{-+}(\mathbb{R}, V) \hookrightarrow C(\mathbb{R}, V), \]

and correspondingly

\[ bV_{loc}(\mathbb{R}) \supseteq bV_{loc,-}(\mathbb{R}) \supseteq bV_{+,+}(\mathbb{R}) \supseteq bV_{+}(\mathbb{R}) \supseteq bV_{c}(\mathbb{R}). \]

We remark that if \( V \) is an \( F \)-space, the spaces \( C(\mathbb{R}, V) \) and \( C_{-}(\mathbb{R}, V) \) are \( F \)-spaces and \( C_{+}(\mathbb{R}, V), C_{-+,+}(\mathbb{R}, V) \) and \( C_c(\mathbb{R}, V) \) are strict \( LF \)-spaces.

There is an interesting subspace of \( bV_{c}(\mathbb{R}) \), namely \( bV_{c}^{\infty}(\mathbb{R}) := bV_{c}(\mathbb{R}) \cap C^{\infty}(\mathbb{R}) \). We like to discuss it here in short. For \( \psi \in bV_{c}^{\infty}(\mathbb{R}) \), its derivative \( \psi' \) belongs to \( C_{c}^{\infty}(\mathbb{R}) \) and if \( \varphi \in C_{c}^{\infty}(\mathbb{R}) \), then \( \psi \) defined by

\[ \psi(t) := \int_{-\infty}^{t} \varphi(\tau) d\tau, \quad t \in \mathbb{R}, \]

is an element of \( bV_{c}^{\infty}(\mathbb{R}) \). For \( \psi \in bV_{c}^{\infty}(\mathbb{R}) \), \( \varphi = \psi' \),
If \( (\varphi_k) \) is a sequence in \( C^\infty_c(\mathbb{R}) \) with

\[
\text{supp}(\varphi_k) \subseteq [-\frac{1}{k}, \frac{1}{k}], \quad \varphi_k \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} \varphi_k(\tau) d\tau = 1,
\]

then with \( \psi_k \in \text{bv}^\infty_c(\mathbb{R}) \) defined by

\[
\psi_k(t) := \int_{-\infty}^{t} \varphi_k(\tau) d\tau,
\]

we have

\[
\mathcal{J}_{\psi_k} f \to f(0) \quad \text{in} \quad V \quad \text{as} \quad k \to \infty.
\]

Indeed for all \( \nu \in D \)

\[
p_{\nu}(\mathcal{J}_{\psi_k} f - f(0)) \leq \max_{t \in [-\frac{1}{k}, \frac{1}{k}]} p_{\nu}(f(t) - f(0)) .
\]

The sequence \( (\psi_k) \) is called an approximate identity.

The remaining part of this section is devoted to the special case that \( V = C \) and so to the spaces \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-+}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \), where the additional \( C \) in the notation is dropped.

**Theorem 1.12.**

I. For each \( \mu \in \text{bv}_c(\mathbb{R}), \text{bv}_{loc,-}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bv}_{loc,-}(\mathbb{R}) \) and \( \text{bv}_{loc}(\mathbb{R}) \), the linear functional

\[
\mathcal{J}_\mu(f) = \int_{\mathbb{R}} f(\tau) d\mu(\tau)
\]

is properly defined and continuous on \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-+}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \), respectively.

II. Let \( F \) be a continuous linear functional on \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-+}(\mathbb{R}) \) or \( C_c(\mathbb{R}) \). Then \( F = \mathcal{J}_\mu \) for some \( \mu \in \text{bv}_c(\mathbb{R}), \text{bv}_{loc,-}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bv}_{loc,-}(\mathbb{R}) \) and \( \text{bv}_{loc}(\mathbb{R}) \), respectively.

The correspondence \( F \leftrightarrow \mu \) is one-to-one for the spaces \( C(\mathbb{R}) \) and \( C_-(\mathbb{R}) \) and unique up to a constant for the other spaces.
Proof. The proof of part I has been indicated already in the context of a sequentially complete locally convex space $V$ instead of $C$.

The result stated in II has been proved for $C(\mathbb{R})$, e.g. in [So]. We outline the proof for $C_+(\mathbb{R})$, $C_-(\mathbb{R})$, $C_{-,+}(\mathbb{R})$ and $C_c(\mathbb{R})$, consecutively.

1. Let $F$ be a continuous linear functional on the strict LF-space $C_+(\mathbb{R})$. It means that for each $n \in \mathbb{N}$ the restriction $F_n$ of $F$ to $C_{+,n}(\mathbb{R})$ is continuous. The classical Riesz representation Theorem, cf. [Kr], yields

$$
F_n(f) = \int_{-n}^{\infty} f(\tau) d\tilde{\mu}_n(\tau), \quad f \in C_{+,n}(\mathbb{R})
$$

for each $n \in \mathbb{N}$. Since $F_{n+1}$ can be regarded as an extension of $F_n$, it follows that $\tilde{\mu}_{n+1}$ and $\tilde{\mu}_n$ differ only a constant on $[-n, \infty)$. Matching these constants we get $\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})$,

$$
\mu(t) = \tilde{\mu}_n(t), \quad t \in [-n, \infty), \quad n \in \mathbb{N},
$$

such that for all $n \in \mathbb{N}$ and $f \in C_{+,n}(\mathbb{R})$

$$
\int_{-n}^{\infty} f(\tau) d\mu(\tau) = \int_{-n}^{\infty} f(\tau) d\mu_n(\tau) = F_n(f).
$$

This proves the first statement.

2. We characterize the continuous linear functionals on $C_-(\mathbb{R})$. To do so, first we characterize the continuous linear functionals on $C_{-,+,n}(\mathbb{R})$, $n \in \mathbb{N}$.

For fixed $n \in \mathbb{N}$ let $G$ be a continuous linear functional on the Banach space $C_{-,+,n}(\mathbb{R})$. Hence, there is $C > 0$ such that

$$
|G(f)| \leq C \max_{t \geq -n} |f(t)|.
$$

Applying Hahn–Banach extension Theorem, there is a continuous linear functional $G_{\text{ext}}$ on the Banach space $B([-n, \infty))$ of bounded functions on $[-n, \infty)$ with

$$
\|G_{\text{ext}}\| = \|G\|
$$

and $G_{\text{ext}}$ extending $G$. Then define

$$
\tilde{\mu}_{G}(t) := \begin{cases} 
G_{\text{ext}}(1_{[-n,t)}) & t > -n \\
0 & t \leq -n
\end{cases}
$$

where $1_{[-n,t)}$ is the characteristic function of the interval $[-n, t)$.

Since for $-n \leq t_0 < \ldots < t_m < \infty$

$$
\sum_{j=1}^{m} |\tilde{\mu}_{G}(t_j) - \tilde{\mu}_{G}(t_{j-1})| = G_{\text{ext}} \left( \sum_{j=1}^{m} \alpha_j 1_{[t_{j-1}, t_j)} \right) \leq \|G\|
$$

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for certain $\alpha_j$ with $|\alpha_j| = 1$, $\tilde{\mu}_G$ has bounded variation on $[-n, \infty)$. Also, for $f \in C_{-\omega,+n}(\mathbb{R})$ and $\varepsilon > 0$ given, there are $-n \leq t_0 < t_1 < \ldots < t_m$ such that

$$\sup_{t \geq -n} |(f - \sum_{j=1}^{m} f(t_j)1_{[t_{j-1},t_j)})(t)| < \varepsilon,$$

so that

$$|G(f) - \sum_{j=1}^{m} f(t_j)(\tilde{\mu}_G(t_j) - \tilde{\mu}_G(t_{j-1}))| < \varepsilon\|G\|.$$ 

It follows that

$$G(f) = \int_{-n}^{\infty} f(\tau)d\tilde{\mu}_G(\tau) \quad (***)$$

and a posteriori that

$$\|G\| = \text{var}_{[-n,\infty)}(\tilde{\mu}_G).$$

We observe that we can replace $\tilde{\mu}_G$ by the right-continuous function

$$\mu_G(t) = \lim_{s \uparrow t} \tilde{\mu}_G(s)$$

without changing anything. We conclude that every continuous linear functional on the $B$-space $C_{-\omega,+n}(\mathbb{R})$ is of the form (***) for some $\mu_G \in \text{bv}([-n, \infty))$.

Now, let $F$ be a continuous linear functional on the $F$-space $C_{-\omega}(\mathbb{R})$. Then there is $n \in \mathbb{N}$ and $C > 0$ such that

$$|F(f)| \leq C \cdot \max_{t \geq -n} |f(t)| \quad (***)$$

for all $f \in C_{-\omega}(\mathbb{R})$.

So, the restriction $G$ of $F$ to $C_{-\omega,+n+1}(\mathbb{R})$ is continuous, and $\mu_G \in \text{bv}([-n-1, \infty))$ exists satisfying the representation (**). The inequality (***) indicates that $\mu_G$ must be constant on $(-n-1, -n)$, say equal to $c_n$. Then, define $\mu \in \text{bv}_{+}(\mathbb{R})$ by

$$\mu(t) = \begin{cases} 0 & t < -n, \\ \mu_G(t) - c_n & t \geq -n. \end{cases}$$

Now, for $f \in C_{-\omega}(\mathbb{R})$, there is $f_0 \in C_{-\omega,+n+1}(\mathbb{R})$ such that $f(t) - f_0(t) = 0$ for $t \geq -n$. So,

$$F(f) = F(f_0) = G(f_0) = \int_{\mathbb{R}} f_0(\tau)d\mu(\tau) = \int_{\mathbb{R}} f(\tau)d\mu(\tau).$$

3. Let $F$ be a continuous linear functional on the strict $LB$-space $C_{-\omega,+}(\mathbb{R})$. It means that for all $n \in \mathbb{N}$ the restriction $F_n$ of $F$ to $C_{-\omega,+n}(\mathbb{R})$ is continuous. As we have seen in 2., for each $n \in \mathbb{N}$ there is $\tilde{\mu}_n \in \text{bv}([-n, \infty))$ such that
\[ F_n(f) = \int_{-n}^{\infty} f(\tau) d\mu_n(\tau), \quad f \in C_{-\infty, +n}(\mathbb{R}). \]

It follows that \( \bar{\mu}_{n+1}(t) - \bar{\mu}_n(t) = \epsilon_n, \) for some constants \( \epsilon_n. \) Matching these constants yields \( \mu_n \in \text{bv}([-n, \infty)) \) with \( \mu_{n+1}([-n, \infty)) = \mu_n. \) Now, define \( \mu \in \text{bv}_{\text{loc}, -}(\mathbb{R}) \) by

\[
\mu(t) = \mu_n(t), \quad t \in [-n, \infty), \quad n \in \mathbb{N}.
\]

Then

\[
F(f) = F_n(f) = \int_{-n}^{\infty} f(\tau) d\mu_n(\tau) = J_\mu(f)
\]

for \( f \in C_{-\infty, +n}(\mathbb{R}), \) \( n \in \mathbb{N}. \)

4. Let \( F \) be a continuous linear functional on \( C_c(\mathbb{R}). \) Since \( C_c(\mathbb{R}) \) is a strict \( LB \)-space with corresponding strict inductive system of \( B \)-spaces \( C_{c,n}(\mathbb{R}), \) the classical Riesz representation Theorem yields \( \mu_n \in \text{bv}([-n, n]) \) such that

\[
F(f) = \int_{-n}^{n} f(\tau) d\mu_n(\tau), \quad f \in C_{c,n}(\mathbb{R})
\]

for each \( n \in \mathbb{N}. \) Again we can assume that \( \mu_{n+1} \) extends \( \mu_n \) and define \( \mu \in \text{bv}_{\text{loc}}(\mathbb{R}) \) by

\[
\mu(t) = \mu_n(t), \quad t \in [-n, n], \quad n \in \mathbb{N}
\]

such that

\[
F(f) = \int_{\mathbb{R}} f(\tau) d\mu(\tau), \quad f \in C_c(\mathbb{R})
\]

We remark, but do not prove, that the space \( \text{bv}_{\text{loc}}(\mathbb{R}) \) and \( \text{bv}_{\text{loc}, -}(\mathbb{R}) \) are \( F \)-spaces whereas the spaces \( \text{bv}_+(\mathbb{R}) \) and \( \text{bv}_c(\mathbb{R}) \) are strict \( LB \)-spaces, and \( \text{bv}_{\text{loc}, -}(\mathbb{R}) \) is a strict \( LF \)-space.

### 2 Translation invariant operators on spaces of continuous functions

In this section we derive complete characterizations for translation invariant operators on the spaces \( C(\mathbb{R}), C_{-\infty}(\mathbb{R}), C_{+}(\mathbb{R}), C_c(\mathbb{R}) \) and \( C_{-\infty, +}(\mathbb{R}) \) respectively.

**Case 1:** \( C(\mathbb{R}). \)

For each \( t \in \mathbb{R}, \) let \( \sigma_t : C(\mathbb{R}) \to C(\mathbb{R}) \) be the continuous linear operator defined by
\[(\sigma_t f)(s) := f(s + t) , \quad s \in \mathbb{R} .\]

Then, the one-parameter family \((\sigma_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group on the Fréchet space \(C(\mathbb{R})\), i.e. for all \(f \in C(\mathbb{R})\)

\[
\lim_{t \to 0} \sigma_t f = f \quad \text{in} \quad C(\mathbb{R}) .
\]

The \(C(\mathbb{R})\)-valued function \(T_\sigma f\), for \(f \in C(\mathbb{R})\) defined by

\[
(T_\sigma f)(t) = \sigma_t f , \quad t \in \mathbb{R} ,
\]

belongs to \(C(\mathbb{R}, C(\mathbb{R}))\). The operator \(T_\sigma\) from \(C(\mathbb{R})\) into \(C(\mathbb{R}, C(\mathbb{R}))\), thus defined, is linear and has a closed graph. \(C(\mathbb{R})\) and \(C(\mathbb{R}, C(\mathbb{R}))\) being \(F\)-spaces the Closed Graph Theorem applies so that \(T_\sigma\) is continuous (equivalently the \(c_0\)-group \((\sigma_t)_{t \in \mathbb{R}}\) is locally equicontinuous).

Now, for every \(\mu \in \text{bv}_c(\mathbb{R})\) define the operator \(\sigma[\mu] := J_\mu T_\sigma\), i.e.

\[
\sigma[\mu] f = \int \sigma_t f \, d\mu(t) , \quad f \in C(\mathbb{R}) ;
\]

\(\sigma[\mu]\) maps \(C(\mathbb{R})\) into \(C(\mathbb{R})\) with \(\sigma_t \sigma[\mu] = \sigma[\mu] \sigma_t , t \in \mathbb{R}\); so \(\sigma[\mu]\) is translation invariant.

Also, if \(L : C(\mathbb{R}) \to C(\mathbb{R})\) is continuous, linear and translation invariant, then by Theorem 1.11, \(\mu \in \text{bv}_c(\mathbb{R})\) exists such that

\[
(Lg)(0) = J_\mu (g) , \quad g \in C(\mathbb{R})
\]

and so

\[
(Lf)(t) = (L\sigma_t f)(0) = J_\mu (\sigma_t f) = (\sigma[\mu] f)(t) .
\]

As a consequence, for \(\mu_1, \mu_2 \in \text{bv}_c(\mathbb{R})\), there is \(\mu \in \text{bv}_c(\mathbb{R})\) such that

\[
\sigma[\mu] = \sigma[\mu_1] \sigma[\mu_2] .
\]

In fact, \(\mu = \mu_1 \ast \mu_2\) with \(\mu_1 \ast \mu_2 = \mu_2 \ast \mu_1\) and

\[
(\mu_1 \ast \mu_2)(t) = \int \mu_1(t - \tau) d\mu_2(\tau) .
\]

The commutative convolution ring \(\text{bv}_c(\mathbb{R})\) has been subject of study in [So], [ES] and [Rij]. It plays the central role in the study of mean periodic functions, cf. [Schw] and [Ka].

**Case 2:** \(C_{-\infty}(\mathbb{R})\).

For each \(t \in \mathbb{R}\) the translation operator \(\sigma_t\) maps \(C_{-\infty}(\mathbb{R})\) into \(C_{-\infty}(\mathbb{R})\) continuously. The one-parameter family \((\sigma_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group on \(C_{-\infty}(\mathbb{R})\), additionally satisfying
\[
\lim_{t \to \infty} \sigma f = 0, \quad f \in C_\infty(\mathbb{R}).
\]

In this case, \( T_\sigma f \in C_\infty(\mathbb{R}, C_\infty(\mathbb{R})) \), \( f \in C_\infty(\mathbb{R}) \) and \( T_\sigma \) is continuous as a linear mapping from the \( F \)-space \( C_\infty(\mathbb{R}) \) into the \( F \)-space \( C_\infty(\mathbb{R}, C_\infty(\mathbb{R})) \). For all \( \mu \in \text{bv}_+(\mathbb{R}) \) we put

\[
\sigma[\mu] := \mathcal{J}_\mu T_\sigma,
\]

with \( \mathcal{J}_\mu \) as in Definition 1.4. Then \( \sigma[\mu] : C_\infty(\mathbb{R}) \to C_\infty(\mathbb{R}) \) is continuous and translation invariant. Since \( \text{bv}_+(\mathbb{R}) \) represents the dual of \( C_\infty(\mathbb{R}) \) (cf. Theorem 1.11) arguments similar as for the space \( C(\mathbb{R}) \) show that

\[
\{ \sigma[\mu] : \mu \in \text{bv}_+(\mathbb{R}) \}
\]

is the collection of all continuous, linear, translation invariant operators from \( C_\infty(\mathbb{R}) \) into \( C_\infty(\mathbb{R}) \). So, \( \text{bv}_+(\mathbb{R}) \) is a commutative convolution ring with convolution defined by

\[
\sigma[\mu_1 * \mu_2] := \sigma[\mu_1] \sigma[\mu_2]
\]

with \((\mu_1 * \mu_2)(t) = \int \mu_1(t - \tau) d\mu_2(\tau) = (\mu_2 * \mu_1)(t)\).

Case 3: \( C_+(\mathbb{R}) \).

Thirdly, we consider the \( LF \)-space \( C_+(\mathbb{R}) \). Again it can be checked that the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) is a locally equicontinuous \( c_0 \)-group on \( C_+(\mathbb{R}) \), so that \( T_\sigma \) maps \( C_+(\mathbb{R}) \) into \( C(\mathbb{R}, C_+(\mathbb{R})) \). Consequently, for \( \mu \in \text{bv}_c(\mathbb{R}) \) the operator \( \sigma[\mu] = \mathcal{J}_\mu T_\sigma \) is continuous and translation invariant from \( C_+(\mathbb{R}) \) into \( C_+(\mathbb{R}) \). However, not all translation invariant operators can be presented this way.

By Definition 1.6 every \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) induces a translation invariant continuous linear mapping \( \sigma[\mu] \), defined by

\[
\sigma[\mu] := \mathcal{J}_\mu T_\sigma,
\]

from \( C_+(\mathbb{R}) \) into \( C(\mathbb{R}) \). Explicitly, we have

\[
(\sigma[\mu]f)(t) = \int f(t + \tau) d\mu(\tau)
\]

for all \( f \in C_+(\mathbb{R}) \). From this we see that for all \( f \in C_{+,n}(\mathbb{R}) \) and \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) with \( \text{supp}(\mu) \subseteq (-\infty, m) \),

\[
\sigma[\mu]f \in C_{+,m+n}(\mathbb{R})
\]

and for \( t \in [-n - m, k] \)
\[(\sigma[\mu]f)(t) = \int_{-(n+t)}^{m} f(t + \tau) d\mu(\tau) \leq \max_{t \in [-n,m+k]} |f(t)| \cdot \text{var}_{[-n-k,m]}(\mu).\]

We conclude that each \(\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})\) defines a continuous, linear, translation invariant operator \(\sigma[\mu]\) on \(C_{+}(\mathbb{R})\).

If, conversely, \(\mathcal{L}\) is a continuous, linear, translation invariant operator from \(C_{+}(\mathbb{R})\) into \(C_{+}(\mathbb{R})\), then \(f \mapsto (\mathcal{L}f)(0)\) is a continuous linear functional on \(C_{+}(\mathbb{R})\) and, by Theorem 1.11, \(\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})\) exists such that

\[(\mathcal{L}f)(t) = (\mathcal{L}(\sigma_{t}f))(0) = \int_{\mathbb{R}} (\sigma_{t}f)(\tau) d\mu(\tau).\]

Summarizing, the collection \(\{\sigma[\mu] \mid \mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})\}\) consists of precisely all continuous, linear, translation invariant operators from \(C_{+}(\mathbb{R})\) into \(C_{+}(\mathbb{R})\).

Consequently, \(\text{bv}_{\text{loc,-}}(\mathbb{R})\) is a commutative convolution algebra with convolution defined by

\[\sigma[\mu_{1} \ast \mu_{2}] := \sigma[\mu_{1}][\sigma[\mu_{2}] , \]

or, explicitly,

\[(\mu_{1} \ast \mu_{2})(t) = \int_{\mathbb{R}} \mu_{1}(t - \tau)d\mu_{2}(\tau) , \quad t \in \mathbb{R} .\]

The way of characterizing the translation invariant operators for the spaces \(\text{Cc}(\mathbb{R})\) and \(\text{C}_{-+}(\mathbb{R})\) is of a different nature. We consider the space \(\text{Cc}(\mathbb{R})\), first.

**Case 4: \(\text{Cc}(\mathbb{R})\).**

The family \(\{\sigma_{t}\}_{t \in \mathbb{R}}\) establishes a locally equicontinuous \(c_{0}\)-group on \(\text{Cc}(\mathbb{R})\). So, for \(f \in \text{Cc}(\mathbb{R})\),

\[T_{\sigma}f : t \mapsto \sigma_{t}f \in C(\mathbb{R}, \text{Cc}(\mathbb{R}))\]

and \(T_{\sigma} : \text{Cc}(\mathbb{R}) \to C(\mathbb{R}, \text{Cc}(\mathbb{R}))\) is continuous. It follows that for each \(\mu \in \text{bv}_{c}(\mathbb{R})\), the mapping \(\sigma[\mu] = J_{\mu} T_{\sigma}\) is linear continuous and translation invariant from \(\text{Cc}(\mathbb{R})\) into \(\text{Cc}(\mathbb{R})\).

For the converse, let \(K : \text{Cc}(\mathbb{R}) \to \text{Cc}(\mathbb{R})\) be continuous, linear and translation invariant. Then by Theorem 1.11 there is \(\tilde{\mu} \in \text{bv}_{\text{loc}}(\mathbb{R})\) such that

\[(Kf)(0) = \int_{\mathbb{R}} f(\tau)d\tilde{\mu}(\tau) , \quad f \in \text{Cc}(\mathbb{R}) ,\]

and we conclude that

\[(Kf)(t) = \int_{\mathbb{R}} f(t + \tau)d\tilde{\mu}(\tau) .\]
Since $K$ is continuous there is $m \in \mathbb{N}$ such that for all $f \in C_{c,1}(\mathbb{R})$,

$$Kf \in C_{c,m}(\mathbb{R})$$

and

$$\max_{t \in [-m,m]} |(Kf)(t)| \leq C \cdot \max_{t \in [-1,1]} |f(t)| .$$

So, for all $t \in \mathbb{R}$ with $|t| \geq m$ and all $f \in C_{+,1}(\mathbb{R})$

$$\int_{\mathbb{R}} f(t + \tau) d\mu(\tau) = 0 .$$

Since supp$(\sigma_t f) \subseteq [-1 - t, 1 - t]$, we conclude that

$$\hat{\mu}(t) = \hat{\mu}(m - 1), \quad t > m - 1$$

$$\hat{\mu}(t) = \hat{\mu}(1 - m), \quad t < 1 - m .$$

If we put

$$\mu(t) = \hat{\mu}(t) - \hat{\mu}(1 - m), \quad t \in \mathbb{R} ,$$

then $\mu \in \text{bv}_c(\mathbb{R})$ and $K = \sigma[\mu]$.

Summarizing, the collection $\{\sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R})\}$ consists of precisely all continuous, linear, translation invariant operators on $C_c(\mathbb{R})$.

**Case 5: $C_{-,+}(\mathbb{R})$.**

Finally we consider $C_{-,+}(\mathbb{R})$. Also for this space the translation group $(\sigma_t)_{t \in \mathbb{R}}$ is a locally equicontinuous $C_0$-group. So, for each $\mu \in \text{bv}_c(\mathbb{R})$ the operator $\sigma[\mu]$ maps $C_{-,+}(\mathbb{R})$ into $C_{-,+}(\mathbb{R})$ continuously. We go a step further. For each $f \in C_{-,+}(\mathbb{R})$ the function $\hat{f}$ on $\mathbb{R}$ defined by

$$\hat{f}(t) := f(-t), \quad t \in \mathbb{R} ,$$

belongs to $C_{-,+}(\mathbb{R})$. Then, as we have seen, for all $\mu \in \text{bv}_+(\mathbb{R})$, the function $\sigma[\mu]\hat{f}$ belongs to $C_{-,+}(\mathbb{R})$.

Let $f \in C_{-,+}(\mathbb{R})$, $f \neq 0$, with $f(t) = 0$ for $t \leq -n$, and $\mu \in \text{bv}_+(\mathbb{R})$ with $\mu(s) = 0$ for $s \leq -m$. Then

$$(\sigma[\mu]\hat{f})(t) = \int_{\mathbb{R}} \hat{f}(t + \tau) d\mu(\tau) = \begin{cases} 0 & t \geq n + m \\ \int_{-m}^{n-t} \hat{f}(t + \tau) d\mu(\tau) & t < n + m \end{cases} ,$$

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so that

\[ \text{supp}(\sigma[\mu]f) \subset (-\infty, n + m) \]

and

\[ \max_{t \leq (n+m)} |(\sigma[\mu]f)(t)| \leq \max_{t \geq -n} |f(t)| \cdot \var R(\mu). \]

Let \( \varepsilon > 0 \) and take \( a > 0 \) so large that

\[ \var_{[a,\infty)}(\mu) < \varepsilon \cdot (\max_{t \geq -n} |f(t)|)^{-1}. \]

Then

\[ \left| \int_{-m}^{\infty} \hat{f}(t + \tau) d\mu(\tau) \right| \leq \max_{\tau \in [-m,a]} |\hat{f}(t + \tau)| \cdot \var R(\mu) + \varepsilon. \]

Since \( \lim_{t \to -\infty} \hat{f}(t) = 0 \) we conclude that

\[ \lim_{t \to -\infty} (\sigma[\mu]f)(t) = 0. \]

Thus we proved that \( (\sigma[\mu]f)^\vee \in C_{-n+m}(\mathbb{R}) \) with

\[ \max_{t \geq -(n+m)} |(\sigma[\mu]f)^\vee(t)| \leq \max_{t \geq -n} |f(t)| \cdot \var R(\mu). \]

Consequently, for each \( \mu \in \text{bv}_+(\mathbb{R}) \) we can define properly the continuous, linear, translation invariant operator \( \sigma[\mu] \) from \( C_{-n+m}(\mathbb{R}) \) into \( C_{-n,+}(\mathbb{R}) \) by

\[ \sigma[\mu]f := (\sigma[\mu]f)^\vee, \quad f \in C_{-n,+}(\mathbb{R}). \]

We have

\[ (\sigma[\mu]f)(t) = \int_{\mathbb{R}} f(t - \tau) d\mu(\tau), \quad t \in \mathbb{R}. \]

Next, we shall prove that each continuous, translation invariant linear operator from \( C_{-n,+}(\mathbb{R}) \) into \( C_{-n,+}(\mathbb{R}) \) arises from an element of \( \text{bv}_+(\mathbb{R}) \) in the above described way. So, let \( \mathcal{L} \) be such an operator. Then the usual argument and Theorem 1.11 shows existence of \( \tilde{\mu} \in \text{bv}_{\text{loc},+}(\mathbb{R}) \) such that for all \( t \in \mathbb{R} \) and all \( f \in C_{-n,+}(\mathbb{R}) \)
\[(Lf)(t) = \int_R (\sigma f)(\tau) d\tilde{\mu}(\tau).\]

There exists \(m \in \mathbb{N}\) such that for all \(f \in C_{-\cdot,0}(\mathbb{R})\), \(Lf \in C_{-\cdot,m}(\mathbb{R})\) and

\[
\max_{t \geq -m} |(Lf)(t)| \leq C \max_{t \geq 0} |f(t)|. 
\]

(\#)

So, for all \(f \in C_{-\cdot,0}(\mathbb{R})\) and all \(t \leq -m\)

\[
\int_{-t}^{\infty} f(t + \tau) d\tilde{\mu}(\tau) = 0.
\]

Consequently, we can assume that \(\tilde{\mu}(t) = 0\) for \(t \geq m\).

Define for \(t \in \mathbb{R}\),

\[
\tilde{\mu}_t(\tau) := \tilde{\mu}(\tau - t), \quad \tau \in \mathbb{R}.
\]

Then by (\#) for all \(t > -m\)

\[
\left| \int_0^{m+t} f(\tau) d\tilde{\mu}_t(\tau) \right| = \left| \int_{-t}^{m} f(t + \tau) d\tilde{\mu}(-\tau) \right| \leq C \cdot \max_{s \geq 0} |f(s)|,
\]

and so for all \(t > -m\)

\[
\operatorname{var}_{[0,m+t]}(\tilde{\mu}_t) \leq C.
\]

We conclude that for all \(t > -m\)

\[
\operatorname{var}_{[-t,m]}(\tilde{\mu}) \leq C,
\]

i.e. \(\tilde{\mu} \in \operatorname{bv}(\mathbb{R})\). Now, define \(\mu \in \operatorname{bv_+}(\mathbb{R})\) by

\[
\mu(t) := -\tilde{\mu}(-t), \quad t \in \mathbb{R},
\]

then

\[
(Lf)(t) = \int_R f(t + \tau) d\tilde{\mu}(\tau) = \int_R f(t - \tau) d\mu(\tau) = (\tilde{\sigma}[\mu] f)(t)
\]

for all \(f \in C_{-\cdot,+}(\mathbb{R})\) and \(t \in \mathbb{R}\).

We conclude that the collection \(\{\tilde{\sigma}[\mu] \mid \mu \in \operatorname{bv_+}(\mathbb{R})\}\) consists of precisely all continuous, translation invariant, linear operators from \(C_{-\cdot,+}(\mathbb{R})\) into \(C_{-\cdot,+}(\mathbb{R})\).

Summarizing, we obtained the following result.
Theorem 2.1.

I. The collection of all continuous, translation invariant, linear operators on \( C(\mathbb{R}) \), and \( C_c(\mathbb{R}) \) is precisely the collection

\[ \{ \sigma[\mu] | \mu \in \text{bv}_c(\mathbb{R}) \} \]

respectively.

II. The collection of all continuous, translation invariant linear operators on \( C_+(\mathbb{R}) \) is precisely the collection

\[ \{ \sigma[\mu] | \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \} \, . \]

III. The collection of all continuous, translation invariant linear operators on \( C_{-+}(\mathbb{R}) \) is precisely the collection

\[ \{ \sigma[\mu] | \mu \in \text{bv}_+(\mathbb{R}) \} \, . \]

IV. The collection of all continuous translation-invariant linear operators on \( C_-(\mathbb{R}) \) is precisely the collection

\[ \{ \sigma[\mu] | \mu \in \text{bv}_+(\mathbb{R}) \} \]

3 Translation invariant operators on \( L^p \)-type spaces

For \( 1 \leq p < \infty \), \( L^p(\mathbb{R}) \) is the Banach space of (equivalence classes of) Lebesgue measurable functions \( f \) on \( \mathbb{R} \) for which \( |f|^p \) is integrable with associated norm

\[ \|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p} . \]

The Banach space \( L^\infty(\mathbb{R}) \) consists (up to equivalence) of all essentially bounded Lebesgue measurable functions \( f \) on \( \mathbb{R} \) with associated norm

\[ \|f\|_\infty = \text{esssup}_{t \in \mathbb{R}} |f(t)| \, . \]

For \( A \subseteq \mathbb{R} \), we denote by \( 1_A \) the characteristic function of the set \( A \). Consecutively we shall define the \( F \)-spaces and strict \( LF \)-spaces \( L^p_{\text{loc}}(\mathbb{R}) \), \( L^p_{\text{comp}}(\mathbb{R}) \), \( L^p_-(\mathbb{R}) \), \( L^p_+(\mathbb{R}) \) and \( L^p_{\text{loc},+}(\mathbb{R}) \), which have a similar behaviour with respect to the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) as the spaces \( C(\mathbb{R}) \), \( C_c(\mathbb{R}) \), \( C_{-+}(\mathbb{R}) \), \( C_{--}(\mathbb{R}) \) and \( C_+(\mathbb{R}) \), respectively.

The space \( L^p_{\text{loc}}(\mathbb{R}) \), \( 1 \leq p \leq \infty \), consists of all Lebesgue measurable functions \( f \) on \( \mathbb{R} \) for which \( f \cdot 1_A \) belongs to \( L^p(\mathbb{R}) \) for all bounded Borel subsets \( A \) of \( \mathbb{R} \). The locally convex topology on the vector space \( L^p_{\text{loc}}(\mathbb{R}) \) is the one generated by the countable system of seminorms \( \{ s_n^p \} \subseteq \mathbb{N} \),
Thus \( L^p_{\text{loc}}(\mathbb{R}) \) is a Fréchet space.

The space \( L^p_{\text{comp}}(\mathbb{R}), 1 \leq p \leq \infty \) is the subspace of \( L^p(\mathbb{R}) \) consisting of all \( f \in L^p(\mathbb{R}) \) with bounded support \( \text{supp}(f) \). Define

\[
L^p_{\text{comp},n}(\mathbb{R}) := \{ f \in L^p(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, n] \}.
\]

Then

\[
L^p_{\text{comp}}(\mathbb{R}) = \bigcup_{n=1}^{\infty} L^p_{\text{comp},n}(\mathbb{R})
\]

and \( L^p_{\text{comp}}(\mathbb{R}) \) is a strict \( LB \)-space induced by the strict inductive system of Banach spaces \( \{ L^p_{\text{comp},n}(\mathbb{R}) \}_{n \in \mathbb{N}} \).

The space \( L^p_{+,n}(\mathbb{R}), 1 \leq p \leq \infty \), is the subspace of \( L^p_{\text{loc}}(\mathbb{R}) \) consisting of all \( f \in L^p_{\text{loc}}(\mathbb{R}) \) such that \( f \cdot 1_{[-n, \infty)} \in L^p(\mathbb{R}) \) for all \( n \in \mathbb{N} \). Introducing the seminorms

\[
r^p_{n}(f) = \| f \cdot 1_{[-n, \infty)} \|_p, \quad f \in L^p_{+,n}(\mathbb{R}),
\]

\( L^p_{+,n}(\mathbb{R}) \) becomes an \( F \)-space.

The space \( L^p_{+,n}(\mathbb{R}) \) is the subspace of \( L^p(\mathbb{R}) \) consisting of all \( f \in L^p(\mathbb{R}) \) with \( \text{supp}(f) \subseteq [-n, \infty) \). Introducing the closed subspaces \( L^p_{+,n}(\mathbb{R}) \) of \( L^p(\mathbb{R}) \) by

\[
L^p_{+,n}(\mathbb{R}) = \{ f \in L^p(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, \infty) \},
\]

then \( \{ L^p_{+,n}(\mathbb{R}) \mid n \in \mathbb{N} \} \) is a strict inductive system of \( B \)-spaces with corresponding strict \( LB \)-space

\[
L^p_{+,}(\mathbb{R}) := \bigcup_{n=1}^{\infty} L^p_{+,n}(\mathbb{R}) .
\]

Introduce, similarly, the closed subspace \( L^p_{\text{loc},+,n}(\mathbb{R}) \) of \( L^p_{\text{loc}}(\mathbb{R}) \) by

\[
L^p_{\text{loc},+,n}(\mathbb{R}) = \{ f \in L^p_{\text{loc}}(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, \infty) \}.
\]

Then \( \{ L^p_{\text{loc},+,n}(\mathbb{R}) \mid n \in \mathbb{N} \} \) is a strict inductive system of \( F \)-spaces generating the strict \( LF \)-space

\[
L^p_{\text{loc},+}(\mathbb{R}) := \bigcup_{n=1}^{\infty} L^p_{\text{loc},+,n}(\mathbb{R}).
\]
The next step is to examine the duality relations for the above introduced spaces.

For two Lebesgue measurable functions $f$ and $g$ on $\mathbb{R}$ for which $f \cdot g \in L^1(\mathbb{R})$ we introduce the notation

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)g(t)dt .$$

Let $1 \leq p < \infty$ and $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We observe that for $f \in L^q(\mathbb{R})$, $L^q_{\text{comp}}(\mathbb{R})$, or $L^q_{\text{loc}}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $L^p_{\text{loc}}(\mathbb{R})$ or $L^p_{\text{loc}}(\mathbb{R})$ the product $f \cdot g \in L^1(\mathbb{R})$.

The $B$-space $L^q(\mathbb{R})$ represents the dual of $L^p(\mathbb{R})$ in the sense that each continuous linear functional $F$ on $L^p(\mathbb{R})$ is of the form

$$F(g) = \langle f, g \rangle , \quad g \in L^p(\mathbb{R}) ,$$

for some $f \in L^q(\mathbb{R})$ with $\|F\|_q = \|f\|_q$.

This is a classical result, see [DS].

A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous iff there are $n \in \mathbb{N}$ and $C > 0$ such that

$$\|F(g)\| \leq C s_n^p(g) .$$

So, each $f \in L^q_{\text{comp}}(\mathbb{R})$ yields the continuous linear functional $g \mapsto \langle f, g \rangle$ on $L^p_{\text{loc}}(\mathbb{R})$. And if $F$ is continuous on $L^p_{\text{loc}}(\mathbb{R})$, then for sufficiently large $n$

$$F(g) = F(g \cdot 1_{[-n,n]}) , \quad g \in L^p_{\text{loc}}(\mathbb{R}) .$$

So, there exists $f \in L^q_{\text{comp},n}(\mathbb{R})$ such that

$$F(g) = \langle f, g \rangle .$$

We see that $L^q_{\text{comp}}(\mathbb{R})$ represent the dual of $L^p_{\text{loc}}(\mathbb{R})$.

A linear functional $F$ on $L^p_{\text{comp}}(\mathbb{R})$ is continuous iff $F|_{L^p_{\text{comp},n}(\mathbb{R})}$ is continuous for each $n \in \mathbb{N}$.

So, for each $f \in L^q_{\text{loc}}(\mathbb{R})$, $g \mapsto \langle f, g \rangle$ defines a continuous linear functional on $L^p_{\text{comp}}(\mathbb{R})$. And if $F$ is continuous, then there is a sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n \in L^q_{\text{comp},n}(\mathbb{R})$ such that

$$F(g) = \langle f_n, g \rangle , \quad g \in L^p_{\text{comp},n}(\mathbb{R}) , \quad n \in \mathbb{N} .$$

Hence, $f_n = f_{n+1}|_{[-n,n]}$ and $f$ on $\mathbb{R}$ defined by

$$f(t) = f_n(t) , \quad t \in [-n,n]$$

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belongs to $L^p_{\text{loc}}(\mathbb{R})$ with

$$F(g) = \langle f, g \rangle, \quad g \in L^p_{\text{comp}}(\mathbb{R}).$$

A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous if and only if there are $n \in \mathbb{N}$ and $C > 0$ such that

$$|F(g)| \leq C \cdot r^n_p(g), \quad g \in L^p_{\text{loc}}(\mathbb{R}).$$

So, for $f \in L^q_{+}(\mathbb{R})$, $g \mapsto \langle f, g \rangle$ is a continuous linear functional on $L^p_{\text{loc}}(\mathbb{R})$. And if $F$ is continuous, $F|_{L^p_{\text{loc}, n}(\mathbb{R})}$ is continuous for $n$ sufficiently large with

$$F(g) = F|_{L^p_{\text{loc}, n}(\mathbb{R})}(g \cdot 1_{[-n, \infty)}) \quad g \in L^p_{\text{loc}}(\mathbb{R});$$

so there exists $f \in L^q_{+}(\mathbb{R})$ such that $F(g) = \langle f, g \rangle$. We conclude that $L^q_{+}(\mathbb{R})$ represents the dual of $L^p_{\text{loc}}(\mathbb{R})$.

A linear functional $F$ on $L^p_{\text{loc}}(\mathbb{R})$ is continuous iff its restriction $F|_{L^p_{\text{loc}, n}(\mathbb{R})}$ is continuous for each $n \in \mathbb{N}$. So, for each $f \in L^q_{+}(\mathbb{R})$ the linear functional $g \mapsto \langle f, g \rangle$ is continuous on $L^p_{\text{loc}}(\mathbb{R})$. Reasoning as for $L^p_{\text{comp}}(\mathbb{R})$, we come to the conclusion that $L^q_{+}(\mathbb{R})$ represents the dual of $L^p_{\text{loc}}(\mathbb{R})$.

A linear functional $F$ on $L^p_{\text{loc},+}(\mathbb{R})$ is continuous iff each $f \in L^q_{\text{loc},+}(\mathbb{R})$ yields a continuous linear functional $g \mapsto \langle f, g \rangle$ on $L^p_{\text{loc},+}(\mathbb{R})$, where $f(t) = f(-t)$, $t \in \mathbb{R}$.

Now, suppose the linear functional $F$ on $L^p_{\text{loc},+}(\mathbb{R})$ is continuous. Let $F_n$ denote the restriction of $F$ to $L^p_{\text{loc},+}(\mathbb{R})$. Then $F_n$ extends to a continuous linear functional on $L^p_{\text{loc}}(\mathbb{R})$. So $f_n \in L^p_{\text{comp}}(\mathbb{R})$ exists such that

$$F_n(g) = \langle f_n, g \rangle = \int_{-n}^{\infty} f_n(t) g(t) \, dt$$

for all $g \in L^p_{\text{loc},+}(\mathbb{R})$. Since $F_{n+1}$ extends $F_n$, it follows that

$$f_n = f_{n+1} 1_{[-n, \infty)}.$$

Define $f$ on $\mathbb{R}$ by

$$f(t) = f_n(t), \quad t \geq -n, \quad n \in \mathbb{N}.$$

Then $f \in L^q_{\text{loc},+}(\mathbb{R})$ and $F(g) = \langle f, g \rangle$, $g \in L^p_{\text{loc},+}(\mathbb{R})$.

We conclude that $L^q_{\text{loc},+}(\mathbb{R})$ represents the dual of $L^p_{\text{loc},+}(\mathbb{R})$ as described.

Now, for $V$ one of the spaces $L^p_{\text{loc}}(\mathbb{R})$, $L^p_{\text{comp}}(\mathbb{R})$, $L^p_{\text{loc},+}(\mathbb{R})$, $L^p_{+}(\mathbb{R})$ and $L^p_{\text{loc},+}(\mathbb{R})$, where $p \in [1, \infty)$, the translation group $(\sigma_t)_t \in \mathbb{R}$, defined formally by

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\[(\sigma_t g)(s) := g(t + s), \quad s \in \mathbb{R}, \ t \in \mathbb{R},\]

is a locally equicontinuous \(c_0\)-group on \(V\). (We note that it is sufficient to prove that it is a \(c_0\)-
group, since each \(c_0\)-group on a Fréchet space or on a strict \(L^F\)-space is locally equicontinuous! Cf. [ER] and [Ko].) Therefore, the associated trace operator \(T_\sigma\) from \(V\) into \(C(\mathbb{R}, V)\),

\[(T_\sigma g)(t) = \sigma_t g, \quad t \in \mathbb{R},\]

is continuous. So, for each \(\mu \in \text{bv}_c(\mathbb{R})\) the operator

\[\sigma[\mu] := J_\mu T_\sigma\]

is continuous on \(V\) and satisfies \(\sigma_t \sigma[\mu] = \sigma[\mu] \sigma_t\), i.e. \(\sigma[\mu]\) is a continuous translation invariant operator on \(V\). We recall that for each \(g \in V\), \(\sigma[\mu]g\) is defined by the \(V\)-valued Riemann–Stieltjes integral

\[\sigma[\mu]g = \int \sigma_t g \ d\mu(\tau).\]

This way the convolution algebra \(\text{bv}_c(\mathbb{R})\) is associated to each of the spaces \(L^p_{\text{loc}}(\mathbb{R})\) and \(L^p_{\text{comp}}(\mathbb{R})\), \(p \in [1, \infty)\).

Next, we show that to the spaces \(L^p_{\text{loc}}(\mathbb{R})\) and \(L^p_{+}(\mathbb{R})\), \(p \in [1, \infty)\), the larger convolution algebra \(\text{bv}^+(\mathbb{R})\) can be associated.

Indeed, for \(g \in L^p_{\text{loc}}(\mathbb{R})\) and each \(n \in \mathbb{N}\) we have

\[\lim_{t \to \infty} \int_{-n}^n |(\sigma_t f)(\tau)|^p \, d\tau = 0.\]

It follows that \(T_\sigma\) maps \(L^p_{\text{loc}}(\mathbb{R})\) into \(C_{-}(\mathbb{R}, L^p_{\text{comp}}(\mathbb{R}))\) continuously. Therefore, for all \(\mu \in \text{bv}^+(\mathbb{R})\) the linear operator \(J_\mu\) from \(C_{-}(\mathbb{R}, L^p_{\text{comp}}(\mathbb{R}))\) into \(L^p_{\text{comp}}(\mathbb{R})\) defined by the (improper Riemann–Stieltjes) \(L^p_{\text{comp}}(\mathbb{R})\)-valued integral,

\[J_\mu f = \int_{\mathbb{R}} f(\tau) \, d\mu(\tau),\]

is continuous, and so

\[\sigma[\mu] := J_\mu T_\sigma\]

is a continuous translation invariant operator on \(L^p_{\text{loc}}(\mathbb{R})\).

If \(1 < p \leq \infty\), then \(L^q_{\text{loc}}(\mathbb{R})\) represents the dual of \(L^p_{\text{loc}}(\mathbb{R})\) where \(1 \leq q < \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\). For each \(\mu \in \text{bv}^+(\mathbb{R})\) the dual mapping \(\sigma[\mu]'\) of \(\sigma[\mu]: L^p_{\text{loc}}(\mathbb{R}) \to L^q_{\text{loc}}(\mathbb{R})\) is a continuous linear translation invariant operator on \(L^p_{\text{loc}}(\mathbb{R})\). There is the relation
\[
\langle \sigma[\mu]' f, g \rangle = \int_{\mathbb{R}} \langle \sigma_{-\tau} f, g \rangle d\mu(\tau) = \int_{\mathbb{R}} \langle \sigma_{\tau} f, g \rangle d\nu(\tau),
\]

where \( f \in L^p_+(\mathbb{R}), \ g \in L^p_-\)(\mathbb{R}) and \( \mu \in \text{bv}_+ (\mathbb{R}) \). We see that on each of the spaces \( L^p_+ (\mathbb{R}) \) with \( 1 < p \leq \infty \) the operator \( \sigma[\mu] \), \( \mu \in \text{bv}_+ (\mathbb{R}) \) can be defined weakly.

We shall prove that for \( 1 \leq p < \infty \), the operator \( \sigma[\mu] \) can be defined strongly.

Let \( f \in L^p_+(\mathbb{R}) \) and let \( \mu \in \text{bv}_+ (\mathbb{R}) \) with \( \mu(t) = 0 \) for \( t \leq -m \), where \( n, m \in \mathbb{N} \). For all \( t \geq -m \) we have \( \sigma_{-t} f \in L^p_+, n+m (\mathbb{R}) \) and
\[
\| \sigma_{-t} f \|_p = \| f \|_p.
\]

For each \( A \geq -m \) the integral expression
\[
\int_{-m}^{A} \sigma_{-\tau} f d\mu(\tau)
\]
defines an element of \( L^p_+ (\mathbb{R}) \), where the integral converges in \( L^p (\mathbb{R}) \). Further, for \( A_2 > A_1 > -m \)
\[
\| \int_{A_1}^{A_2} \sigma_{-\tau} f d\mu(\tau) \|_p \leq \int_{A_1}^{A_2} \| \sigma_{-\tau} f \|_p |d\mu(\tau)| \leq \| f \|_p \cdot \text{var}_{[A_1, A_2]}(\mu).
\]

Since \( L^p_+, n+m (\mathbb{R}) \) is a Banach space, the limit
\[
\int_{-m}^{\infty} \sigma_{-\tau} f d\mu(\tau) = \lim_{A \to \infty} \int_{-m}^{A} \sigma_{-\tau} f d\mu(\tau)
\]
exists in \( L^p_+, n+m (\mathbb{R}) \) (observe that \( \lim_{A \to \infty} \text{var}_{[A, \infty)} (\mu) = 0 \)) and
\[
\| \int_{-m}^{\infty} (\sigma_{-\tau} f) d\mu(\tau) \|_p \leq \| f \|_p \cdot \text{var}_{[-m, \infty)} (\mu).
\]

Thus we can conclude that for each \( \mu \in \text{bv}_+ (\mathbb{R}) \) the linear operator \( \sigma[\mu] \) on \( L^p_+ (\mathbb{R}) \), \( 1 \leq p < \infty \), defined by the (improper Riemann–Stieljes) integral
\[
\sigma[\mu] f = \int_{\mathbb{R}} \sigma_{-\tau} f d\mu(\tau)
\]
is continuous and translation invariant. In particular, $\sigma[\mu]$ maps $L^p_{+,n}(\mathbb{R})$ into $L^p_{+,n+m}(\mathbb{R})$ continuously, where $m \in \mathbb{N}$ satisfies $\mu(t) = 0$, $t < -m$.

Finally, we discuss translation invariant operators on the spaces $L^p_{\text{loc},+}(\mathbb{R})$, $1 \leq p < \infty$.

For $g \in L^p_{\text{loc},+}(\mathbb{R})$ and $t \leq k$, $\sigma_t g \in L^p_{\text{loc},+}(\mathbb{R})$ with for each $m \in \mathbb{N}$

$$\int_{-n-k}^{m} |(\sigma_t g)(s)|^p ds \leq \int_{-n}^{m+k} |g(s)|^p ds. \quad (***)$$

So, for $\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})$ with $\mu(t) = \mu(k)$, $t \geq k$, the inequality $(**)$ indicates that for every $A \leq k$ the proper $L^p$-valued Riemann–Stieltjes integral

$$\int_{A}^{k} \sigma_{\tau} g \, d\mu(\tau)$$

converges and defines an element of $L^p_{\text{loc},+}(\mathbb{R})$. Since the left hand side of $(**)$ vanishes for $t \leq -m - n$, the limit

$$\int_{-\infty}^{k} \sigma_{\tau} g \, d\mu(\tau) = \lim_{A \to -\infty} \int_{A}^{k} \sigma_{\tau} g \, d\mu(\tau)$$

exists in $L^p_{\text{loc},+}(\mathbb{R})$ and satisfies for each $m \in \mathbb{N}$

$$\left( \int_{-n-k}^{m} (\int_{-\infty}^{k} \sigma_{\tau} g \, d\mu(\tau)) |(s)|^p ds \right)^{1/p} \leq \left( \int_{-n}^{m+k} |g(t)|^p dt \right)^{1/p} \text{var}[_{m-n,k}]_{(\mu)}. $$

So, for each $\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})$ the operator $\sigma[\mu]$,

$$\sigma[\mu]g = \int_{\mathbb{R}} \sigma_{\tau} g \, d\mu(\tau)$$

is a continuous linear translation invariant operator on $L^p_{\text{loc},+}(\mathbb{R})$. In particular, $\sigma[\mu]$ maps $L^p_{\text{loc},+}(\mathbb{R})$ into $L^p_{\text{loc},+}(\mathbb{R})$, where $k \in \mathbb{N}$ is such that $\mu(t) = \mu(k)$ for all $t \geq k$.

Summarizing, we obtained the following results

**Theorem 3.1.**

I. For $p \in [1, \infty)$, $L^p_{\text{loc}}(\mathbb{R})$ and $L^p_{\text{comp}}(\mathbb{R})$ are modules over the commutative convolution algebra $\text{bv}_{c}(\mathbb{R})$, where in both cases multiplication is defined by

$$\mu \circ g := \sigma[\mu]g = \int_{\mathbb{R}} \sigma_{t} g \, d\mu(t),$$
for \( g \in L^p_{\text{loc}}(\mathbb{R}) \) or \( g \in L^p_{\text{comp}}(\mathbb{R}) \) and \( \mu \in \text{bv}_c(\mathbb{R}) \).

II. For \( p \in [1, \infty) \), \( L^p_{\text{loc}}(\mathbb{R}) \) and \( L^p_{\text{loc}+}(\mathbb{R}) \) are modules over the commutative convolution algebra \( \text{bv}_+(\mathbb{R}) \), where multiplication is defined by

\[
\mu \circ g := \sigma [\mu] g = \int_{\mathbb{R}} \sigma_1 g \, d\mu(t), \quad g \in L^p_{\text{loc}+}(\mathbb{R}),
\]

and

\[
\mu \circ g := \sigma [\mu] g = \int_{\mathbb{R}} \sigma_{-1} g \, d\mu(t), \quad g \in L^p_{\text{loc}}(\mathbb{R}).
\]

III. For \( p \in [1, \infty) \), \( L^p_{\text{loc},+}(\mathbb{R}) \) is a module over the commutative convolution algebra \( \text{bv}_{\text{loc},-}(\mathbb{R}) \), where multiplication is defined by

\[
\mu \circ g := \sigma [\mu] g = \int_{\mathbb{R}} \sigma_1 g \, d\mu(t).
\]

The translation group \((\sigma_t)_{t \in \mathbb{R}}\) is not a \(\sigma^\circ\)-group on either of the spaces \( L^\infty_{\text{loc}}(\mathbb{R}) \), \( L^\infty_{\text{comp}}(\mathbb{R}) \), \( L^\infty_+(\mathbb{R}) \), \( L^\infty_{\text{loc},+}(\mathbb{R}) \) and \( L^\infty_{\text{loc},-}(\mathbb{R}) \). In this connection and for later use we mention the following results.

**Proposition 3.2.**

I. Let \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \) with trace \( T_{\sigma} f : t \mapsto \sigma_t f \). Then \( T_{\sigma} f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \) if and only if \( f \in C(\mathbb{R}) \).

**Proof.** Since \( C(\mathbb{R}) \subset L^\infty_{\text{loc}}(\mathbb{R}) \), sufficiency is evident. Now, suppose \( T_{\sigma} f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \). For \((\psi_k)\) an approximate identity in \( \text{bv}_c^{\infty}(\mathbb{R}) \), see Section1, p.7,

\[
J_{\psi_k} T_{\sigma} f \to (T_{\sigma} f)(0) = f \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}),
\]

as \( k \to \infty \). Since

\[
J_{\psi_k} T_{\sigma} f = \int_{\mathbb{R}} \psi'_k(\tau) \sigma_\tau f \, d\tau,
\]

we see that

\[
J_{\psi_k} T_{\sigma} f \in C^\infty(\mathbb{R}), \quad k \in \mathbb{N},
\]

and so \( f \in C(\mathbb{R}) \). \( \square \)

**Proposition 3.2.** (continued)
II. Let \( f \in L_{\text{comp}}^{\infty}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L_{\text{comp}}^{\infty}(\mathbb{R})) \) if and only if \( f \in C_\varepsilon(\mathbb{R}) = C(\mathbb{R}) \cap L_{\text{comp}}^{\infty}(\mathbb{R}) \).

III. Let \( f \in L_{\text{loc}}^{\infty}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L_{\text{loc}}^{\infty}(\mathbb{R})) \) if and only if \( f \in C_\varepsilon(\mathbb{R}) \).

IV. Let \( f \in L_{\text{+loc}}^{\infty}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L_{\text{+loc}}^{\infty}(\mathbb{R})) \) if and only if \( f \in C_\varepsilon(\mathbb{R}) = C(\mathbb{R}) \cap L_{\text{loc}}^{\infty}(\mathbb{R}) \).

V. Let \( f \in L_{\text{c,loc}}^{\infty}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L_{\text{c,loc}}^{\infty}(\mathbb{R})) \) if and only if \( f \in C_\varepsilon(\mathbb{R}) = C(\mathbb{R}) \cap L_{\text{c,loc}}^{\infty}(\mathbb{R}) \).

**Proof.** Let \( V \) be one of the spaces \( L_{\text{comp}}^{\infty}(\mathbb{R}) \), \( L_{\text{loc}}^{\infty}(\mathbb{R}) \), \( L_{\text{+loc}}^{\infty}(\mathbb{R}) \) and \( L_{\text{c,loc}}^{\infty}(\mathbb{R}) \). Then \( V \in L_{\text{loc}}^{\infty}(\mathbb{R}) \) and so for \( f \in V \), \( T_\sigma f \in C(\mathbb{R}, V) \) if and only if \( f \in C(\mathbb{R}) \cap V \). For II. in addition

\[
\lim_{t \to \infty} \left( \max_{r \geq -n} |f(t + r)| \right) = 0 \text{ ,}
\]

if and only if \( f \in C_\varepsilon(\mathbb{R}) \).

\[\square\]

4 Complete characterizations of the translation invariant operators on \( L_1 \)-type spaces

In this section we deduce a complete characterization of the linear, continuous translation invariant operators for four spaces of \( L_1 \)-type. The first two characterizations can be found in [Eij] also (Theorem 9 and Theorem 12). We have included them here for sake of completeness.

**Case 1** \( L_{\text{comp}}^1(\mathbb{R}) \) and \( L_{\text{loc}}^{\infty}(\mathbb{R}) \).

We recall that \( L_{\text{loc}}^{\infty}(\mathbb{R}) \) represents the dual of \( L_{\text{comp}}^1(\mathbb{R}) \). For each \( \mu \in \text{bv}_c(\mathbb{R}) \) the linear operator \( \sigma[\mu] : L_{\text{loc}}^{\infty}(\mathbb{R}) \to L_{\text{loc}}^{\infty}(\mathbb{R}) \) is defined as the dual of \( \hat{\sigma}[\mu] : L_{\text{comp}}^1(\mathbb{R}) \to L_{\text{comp}}^1(\mathbb{R}) \),

\[\hat{\sigma}[\mu] g = \int_{\mathbb{R}} \sigma_+ g \ d\mu(\tau) ,\]

as introduced in the previous section. It means that for all \( f \in L_{\text{loc}}^{\infty}(\mathbb{R}) \) and \( g \in L_{\text{comp}}^1(\mathbb{R}) \)

\[\langle \hat{\sigma}[\mu] g, f \rangle = \langle g, \sigma[\mu] f \rangle .\]

The Closed Graph Theorem for \( F \)-spaces guarantees that \( \sigma[\mu] \) is continuous on \( L_{\text{loc}}^{\infty}(\mathbb{R}) \). Also, \( \sigma[\mu] \) is translation invariant.

Now, let \( K : L_{\text{loc}}^{\infty}(\mathbb{R}) \to L_{\text{loc}}^{\infty}(\mathbb{R}) \) be a continuous translation invariant operator. Let, as in Proposition 1.3, \( L_K \) be defined by

\[(L_K u)(t) := K(u(t)) , \quad t \in \mathbb{R} , u \in C(\mathbb{R}, L_{\text{loc}}^{\infty}(\mathbb{R})) .\]
For \( f \in C(\mathbb{R}) \) we have \( T_f \in C(\mathbb{R}, L^\infty_0(\mathbb{R})) \) and so \( T_f K f = L_K T_f f \in C(\mathbb{R}, L^\infty_0(\mathbb{R})) \). By Proposition 3.2, we conclude that \( K f \in C(\mathbb{R}) \). So \( C(\mathbb{R}) \) is an invariant subspace of \( K \) and \( K|_C(\mathbb{R}) : C(\mathbb{R}) \to C(\mathbb{R}) \) is continuous. Then, by Theorem 2.1, there is \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( K f = \sigma[\mu] f \) for all \( f \in C(\mathbb{R}) \). Further for all \( g \in L^1_{\text{comp}}(\mathbb{R}) \) and \( f \in C(\mathbb{R}) \)

\[
\langle g, K f \rangle = \langle \sigma[\mu] g, f \rangle
\]
due to the \( L_1 \)-convergence of the \( L^1_{\text{comp}} \)-valued integral

\[
\int_\mathbb{R} \sigma_{-\tau} g \, d\mu(\tau).
\]

We arrive at the following Theorem.

**Theorem 4.1.** Let \( K : L^\infty_0(\mathbb{R}) \to L^\infty_0(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there exists \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( K|_C(\mathbb{R}) = \sigma[\mu] \). Let \( K^* : L^\infty_0(\mathbb{R})^* \to L^\infty_0(\mathbb{R})^* \) denote the dual of \( K \). If \( K^*|_{L^1_{\text{comp}}(\mathbb{R})} \) maps \( L^1_{\text{comp}}(\mathbb{R}) \) into \( L^1_{\text{comp}}(\mathbb{R}) \), then \( K = \sigma[\mu] \) and \( K^*|_{L^1_{\text{comp}}(\mathbb{R})} = \sigma[\mu] \).

**Proof.** The first part of the Theorem has been established above. For the second part observe that \( K = (K_*)^* \) where \( K_* = K^*|_{L^1_{\text{comp}}(\mathbb{R})} \) identifying \( L^1_{\text{comp}}(\mathbb{R}) \) with a subspace of \( L^\infty_0(\mathbb{R})^* \), whence for all \( f \in C(\mathbb{R}) \) and \( g \in L^1_{\text{comp}}(\mathbb{R}) \)

\[
\langle K_* g, f \rangle = \langle g, \sigma[\mu] f \rangle = \langle \sigma[\mu] g, f \rangle.
\]

**Theorem 4.2.** Let \( S : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( S = \sigma[\mu] \). So, the collection \( \{ \sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}) \} \) consists of precisely all continuous, translation invariant, linear operators on \( L^1_{\text{comp}}(\mathbb{R}) \).

**Proof.** Apply the preceding Theorem to the dual operator \( S^* \).

**Remark.** Sometimes there is a priori knowledge of \( K \).

Let \( D'(\mathbb{R}) \) denote the space of Schwartz distributions corresponding to the test space \( D(\mathbb{R}) = C_0^\infty(\mathbb{R}) \). Then \( L^\infty_0(\mathbb{R}) \) can be considered as a subspace of \( D'(\mathbb{R}) \) identifying \( f \in L^\infty_0(\mathbb{R}) \) and the distribution

\[
\varphi \mapsto \langle f, \varphi \rangle, \quad \varphi \in D(\mathbb{R}).
\]

Let \( E'(\mathbb{R}) \) denote the subspace of \( D'(\mathbb{R}) \) consisting of all \( \Phi \in D'(\mathbb{R}) \) with compact support. According to [So] each functional in \( E'(\mathbb{R}) \) can be written as

\[
\Phi(\varphi) = \int_{\mathbb{R}} (p(\frac{d}{dt})\varphi)(\tau) d\mu(\tau) \tag{*}
\]

for some \( \mu \in \text{bv}_c(\mathbb{R}) \) and some polynomial \( p \). Now, for \( \Psi \in D'(\mathbb{R}) \) and \( \Phi \in E'(\mathbb{R}) \) we define \( \Phi \ast \Psi \) by
\[(\Phi \ast \Psi)(\varphi) := \Psi(\frac{d}{dx})\sigma[\mu]\varphi\]

with \(\Phi\) of the form \((\ast)\). Then \(\Phi \ast \Psi \in \mathcal{D}'(IR)\) and the linear mapping \(\Phi \mapsto \Phi \ast \Psi\) is translation invariant on \(\mathcal{D}'(IR)\). If we know that \(Kf = \Phi * f, \, f \in L_{\text{loc}}^\infty(IR)\) and that \(K\) maps \(L_{\text{loc}}^\infty(IR)\) into \(L_{\text{loc}}^\infty(IR)\), then \(K\) is continuous, linear and translation invariant due to the Closed Graph Theorem. So, there is \(\mu_1 \in \text{bv}_c(IR)\) such that for all \(f \in C(IR)\) and \(\varphi \in \mathcal{D}(IR)\)

\[\langle \varphi, Kf \rangle = \langle p(\frac{d}{dx})\sigma[\mu]\varphi, f \rangle\]

and

\[\langle \varphi, Kf \rangle = \langle \varphi, \sigma[\mu_1]f \rangle = \langle \sigma[\mu_1]\varphi, f \rangle .\]

Consequently,

\[\sigma[\mu_1] = p(\frac{d}{dx})\sigma[\mu] \text{ and } K = \sigma[\mu_1].\]

We are left with the following question.

Does there exist a continuous translation invariant operator \(K \neq 0\) from \(L_{\text{loc}}^\infty(IR)\) into \(L_{\text{loc}}^\infty(IR)\) such that \(Kf = 0\) for all \(f \in C(IR)\)?

Case II \(L_{\text{loc}}^1(IR)\) and \(L_{\text{comp}}^\infty(IR)\).

Also in this case we define \(\sigma[\mu]\) for \(\mu \in \text{bv}_c(IR)\) as a translation invariant operator by duality, i.e. for each \(g \in L_{\text{loc}}^1(IR)\) and \(f \in L_{\text{comp}}^\infty(IR)\)

\[\langle \sigma[\mu]g, f \rangle = \langle g, \sigma[\mu]f \rangle .\]

Then \(\sigma[\mu]\) is continuous due to the Closed Graph Theorem for strict \(LB\)-spaces, and, clearly, \(\sigma[\mu]\) is translation invariant.

Now, let \(K : L_{\text{comp}}^\infty(IR) \rightarrow L_{\text{comp}}^\infty(IR)\) be linear, translation invariant and continuous. Since for \(f \in C_c(IR)\),

\[T_\sigma f \in C(IR, L_{\text{comp}}^\infty(IR)) ,\]

it follows that

\[T_\sigma Kf = LKT_\sigma f \in C(IR, L_{\text{comp}}^\infty(IR))\]

and, by Proposition 3.2, \(Kf \in C_c(IR)\). So, as in Case I, we derive that \(K\) maps \(C_c(IR)\) into \(C_c(IR)\) continuously and consequently there is \(\mu \in \text{bv}_c(IR)\), according to Theorem 2.1, such that

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\( K|_{C_c(\mathbb{R})} = \sigma[\mu] \).

For all \( f \in C_c(\mathbb{R}) \) and \( g \in L^1_{\text{comp}}(\mathbb{R}) \) we thus have

\[ \langle g, K f \rangle = \langle \delta[\mu] f, g \rangle . \]

**Theorem 4.3.** Let \( K : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( K|_{C_c(\mathbb{R})} = \sigma[\mu] \). Let \( K^* : L^\infty_{\text{comp}}(\mathbb{R})^* \to L^\infty_{\text{comp}}(\mathbb{R})^* \) denote the dual operator of \( K \). If \( K^*|_{L^1_{\text{loc}}(\mathbb{R})} \) maps \( L^1_{\text{loc}}(\mathbb{R}) \) into \( L^1_{\text{loc}}(\mathbb{R}) \), then \( K = \sigma[\mu] \) and \( K^*|_{L^1_{\text{loc}}(\mathbb{R})} = \delta[\mu] \). □

**Theorem 4.4.** Let \( S : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}) \) be a continuous translation invariant linear operator. Then there is \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( S = \sigma[\mu] \). The collection

\[ \{ \sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}) \} \]

consists of precisely all continuous, translation invariant, linear operators from \( L^1_{\text{loc}}(\mathbb{R}) \) into \( L^1_{\text{loc}}(\mathbb{R}) \). □

The distribution space \( \mathcal{E}'(\mathbb{R}) \) is a commutative convolution ring (without zero divisors): indeed for

\[ \Phi_j(\psi) = \int_{\mathbb{R}} (p_j(\frac{d}{dt})\psi)(\tau) d\mu(\tau), \quad j = 1, 2, \ldots, \]

\[ \Phi_1 \ast \Phi_2 = \Phi_1 \circ p_2(\frac{d}{dt})^* \sigma[\mu_2] = \Phi_2 \circ p_1(\frac{d}{dt}) \sigma[\mu_1] . \]

Identifying \( L^\infty_{\text{comp}}(\mathbb{R}) \) as a subspace of \( \mathcal{E}'(\mathbb{R}) \) in the usual way, it can be checked readily, as in Case I, that if \( K f = \Phi \ast f \in L^\infty_{\text{comp}}(\mathbb{R}) \) for some \( \Phi \in \mathcal{E}'(\mathbb{R}) \) and for all \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \), then \( K \) is continuous and \( K = \sigma[\mu] \) for some \( \mu \in \text{bv}_c(\mathbb{R}) \).

**Case III** \( L^1(\mathbb{R}) \) and \( L^\infty(\mathbb{R}) \).

In the previous section we defined \( \delta[\mu], \mu \in \text{bv}_c(\mathbb{R}) \), on \( L^1(\mathbb{R}) \) by

\[ \delta[\mu] g = \int_{\mathbb{R}} \sigma_{-\tau} g \ d\mu(\tau), \quad g \in L^1(\mathbb{R}) . \]

We introduce the linear operators \( \sigma[\mu], \mu \in \text{bv}_c(\mathbb{R}) \), on \( L^\infty(\mathbb{R}) \) by using the duality of \( L^1(\mathbb{R}) \) and \( L^\infty(\mathbb{R}) \):

\[ \forall f \in L^\infty(\mathbb{R}) \forall g \in L^1(\mathbb{R}) : \langle \delta[\mu] g, f \rangle = \langle g, \sigma[\mu] f \rangle . \]

Then \( \sigma[\mu] : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \) is continuous and translation invariant.

Let \( K : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \) be continuous, translation invariant and linear. By Proposition 3.2, for \( f \in C_-(\mathbb{R}), T_\sigma f \in C_-(\mathbb{R}, L^\infty(\mathbb{R})) \) and consequently

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Then, the same Proposition yields $\mathcal{K}f \in C_+(\mathbb{R})$. According to Theorem 2.1, there exists $\mu \in \text{bv}_+(\mathbb{R})$ such that 

$$\mathcal{K}f = \sigma[\mu]f,$$

and

$$\langle g, \mathcal{K}f \rangle = \langle \delta[\mu] g, f \rangle$$

for all $g \in L^1_+(\mathbb{R})$ and $f \in C_+(\mathbb{R})$.

**Theorem 4.5.** Let $\mathcal{K} : L^\infty_+(\mathbb{R}) \to L^\infty_+(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then there is $\mu \in \text{bv}_+(\mathbb{R})$ such that $\mathcal{K}|_{C_+(\mathbb{R})} = \sigma[\mu]$. If $\mathcal{K}^*|_{L^1_+(\mathbb{R})}$ maps $L^1_+(\mathbb{R})$ into $L^1_+(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}^*|_{L^1_+(\mathbb{R})} = \delta[\mu]$ for some $\mu \in \text{bv}_+(\mathbb{R})$. \hfill \Box

**Theorem 4.6.** Let $\mathcal{S} : L^1_+(\mathbb{R}) \to L^1_+(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then there is $\mu \in \text{bv}_+(\mathbb{R})$ such that $\mathcal{S} = \delta[\mu]$. The collection

$$\{ \delta[\mu] \mid \mu \in \text{bv}_+(\mathbb{R}) \}$$

consists of precisely all continuous, translation, invariant, linear operators from $L^1_+(\mathbb{R})$ into $L^1_+(\mathbb{R})$. \hfill \Box

**Case IV** $L^\infty_+(\mathbb{R})$ and $L^\infty_+(\mathbb{R})$.

For $\mu \in \text{bv}_+(\mathbb{R})$, in the previous section, we defined the linear operator $\sigma[\mu]$ on $L^1_-(\mathbb{R})$ by the $L^1_-(\mathbb{R})$-valued (improper) Riemann-Stieltjes integral

$$\sigma[\mu]g = \int g \, d\mu(\tau), \quad g \in L^1_-(\mathbb{R}).$$

Since $\sigma[\mu]$ is continuous we can define $\delta[\mu]$ on $L^\infty_+(\mathbb{R})$ by duality:

$$\forall f \in L^\infty_+(\mathbb{R}) \forall g \in L^1_-(\mathbb{R}) : \langle \sigma[\mu]g, f \rangle = \langle g, \delta[\mu]f \rangle.$$ 

Then $\delta[\mu]$ is translation invariant and, by the Closed Graph Theorem for strict LB-spaces, continuous on $L^\infty_+(\mathbb{R})$.

Now, let $\mathcal{K} : L^\infty_+(\mathbb{R}) \to L^\infty_+(\mathbb{R})$ be continuous and translation invariant. For $f \in C_+, T = \sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R}))$ and

$$T_\sigma \mathcal{K}f = L_\mathcal{K}T_\sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R})).$$
So by Proposition 3.2, $Kf \in C(\mathbb{R}) \cap L^\infty_+(\mathbb{R})$. Besides $K\sigma f = \sigma Kf$, the continuity of $K$ yields

$$\lim_{t \to \infty} \sigma_t Kf = 0 \quad \text{in } L^\infty_+(\mathbb{R}),$$

so that $Kf \in C_{-\to+}(\mathbb{R})$. We see that $K|_{C_{-\to+}(\mathbb{R})} : C_{-\to+}(\mathbb{R}) \to C_{-\to+}(\mathbb{R})$ is continuous and translation invariant. So, by Theorem 2.1 we conclude existence of $\mu \in \text{bv}_+(\mathbb{R})$ such that

$$K|_{C_{-\to+}(\mathbb{R})} = \delta[\mu].$$

Further, for all $f \in C_{-\to+}(\mathbb{R})$ and $g \in L^1_-(\mathbb{R})$

$$\langle g, Kf \rangle = \langle \sigma[\mu]g, f \rangle.$$

As in the cases I, II and III we arrive at the following characterizations.

**Theorem 4.7.** Let $K : L^\infty_+(\mathbb{R}) \to L^\infty_+(\mathbb{R})$ be a continuous, translation invariant linear operator. Then there is $\mu \in \text{bv}_+(\mathbb{R})$ such that $K|_{C_{-\to+}(\mathbb{R})} = \delta[\mu]$. If $K^*|_{L^1_-(\mathbb{R})}$ maps $L^1_-(\mathbb{R})$ into $L^1_-(\mathbb{R})$, then $K = \delta[\mu]$ and $K^*|_{L^1_-(\mathbb{R})} = \sigma[\mu]$. □

**Theorem 4.8.** Let $S : L^1_-(\mathbb{R}) \to L^1_-(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then there is $\mu \in \text{bv}_c(\mathbb{R})$ such that $S = \sigma[\mu]$. The collection

$$\{ \sigma[\mu] \mid \mu \in \text{bv}_+(\mathbb{R}) \}$$

consists of precisely all continuous, translation invariant linear operators on $L^1_-(\mathbb{R})$. □

**Case V** $L^1_{\text{loc,+}}(\mathbb{R})$ and $L^\infty_{\text{loc,+}}(\mathbb{R})$.

For $\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})$ we defined the linear operator $\sigma[\mu]$ on $L^1_{\text{loc,+}}(\mathbb{R})$ by

$$\sigma[\mu]g = \int_{\mathbb{R}} \sigma g \mu(t), \quad g \in L^1_{\text{loc,+}}(\mathbb{R}),$$

as an $L^1_{\text{loc,+}}(\mathbb{R})$-valued (improper) Riemann–Stieltjes integral. Thus defined, $\sigma[\mu]$ is a continuous translation invariant, linear operator on $L^1_{\text{loc,+}}(\mathbb{R})$. Applying the duality of $L^1_{\text{loc,+}}(\mathbb{R})$ and $L^1_{\text{loc,+}}(\mathbb{R})$, we introduce $\sigma[\mu]$ on $L^\infty_{\text{loc,+}}(\mathbb{R})$,

$$\forall f \in L^\infty_{\text{loc,+}}(\mathbb{R}) \forall g \in L^1_{\text{loc,+}}(\mathbb{R}) : \langle \sigma[\mu]g, f \rangle = \langle g, (\sigma[\mu]f)^\vee \rangle.$$

The Closed Graph Theorem for strict $LF$-spaces guarantees that $\sigma[\mu]$ is continuous on $L^\infty_{\text{loc,+}}(\mathbb{R})$, also $\sigma[\mu]$ is translation invariant.

Let $K : L^\infty_{\text{loc,+}}(\mathbb{R}) \to L^\infty_{\text{loc,+}}(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then with $\mathcal{L}_K$ on $C(\mathbb{R}, L^\infty_{\text{loc,+}}(\mathbb{R}))$ defined by
for \( f \in C_+(\mathbb{R}) \),
\[
T_\sigma Kf = L_X T_\sigma f \in C(\mathbb{R}, L_{\text{loc},+}^\infty(\mathbb{R}))
\]

By Proposition 3.1, \( Kf \in C_+(\mathbb{R}) \) for all \( f \in C_+(\mathbb{R}) \), and \( K|_{C_+(\mathbb{R})} \) is continuous and translation invariant. According to Theorem 2.1 there exists \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) such that
\[
K|_{C_+(\mathbb{R})} = \sigma[\mu]
\]
and for all \( g \in L_{\text{loc},+}^1(\mathbb{R}) \) and \( f \in C_+(\mathbb{R}) \),
\[
\langle g, (Kf)' \rangle = \langle \hat{g}, Kf \rangle = \langle \check{\sigma[\mu]} \hat{g}, f \rangle = \langle \sigma[\mu]g, \check{f} \rangle = \langle K^*g, \check{f} \rangle.
\]

**Theorem 4.9.** Let \( K : L_{\text{loc},+}^\infty(\mathbb{R}) \to L_{\text{loc},+}^\infty(\mathbb{R}) \) be a continuous, translation invariant, linear operator with dual \( K^* : L_{\text{loc},+}^\infty(\mathbb{R})^* \to L_{\text{loc},+}^\infty(\mathbb{R})^* \). Then there is \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) such that
\[
K|_{C_+(\mathbb{R})} = \sigma[\mu].
\]
If \( K^*(L_{\text{loc},+}^1(\mathbb{R})) \subseteq L_{\text{loc},+}^1(\mathbb{R}) \), then \( K = \sigma[\mu] \) and \( K^*|_{L_{\text{loc},+}^1(\mathbb{R})} = \sigma[\mu]. \)
(Here we identified \( L_{\text{loc},+}^1(\mathbb{R}) \) as a closed subspace of \( L_{\text{loc},+}^\infty(\mathbb{R})^* \) in the indicated way.)

**Theorem 4.10.** Let \( S : L_{\text{loc},+}^1(\mathbb{R}) \to L_{\text{loc},+}^1(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) such that
\[
S = \sigma[\mu].
\]
The collection
\[
\{ \sigma[\mu] \mid \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \}
\]
consists of precisely all continuous, linear, translation invariant operators on \( L_{\text{loc},+}^1(\mathbb{R}) \).

Let \( \mathcal{D}'(\mathbb{R}) \) denote the subspace of \( \mathcal{D}'(\mathbb{R}) \) consisting of all \( F \in \mathcal{D}'(\mathbb{R}) \) with support \( \text{supp}(F) \) contained in some half-infinite interval \([a, \infty)\), \( a \) depending on \( F \), i.e. \( F(\varphi) = 0 \) for all \( \varphi \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp}(\varphi) \subset (-\infty, a] \).

**Lemma 4.11.** For each \( F \in \mathcal{D}'(\mathbb{R}) \) and \( \varphi \in \mathcal{D}(\mathbb{R}) \) the function \( C_F \varphi \) defined by
\[
(C_F \varphi)(t) = F(\sigma_t \varphi), \quad t \in \mathbb{R},
\]
belongs to \( C^\infty(\mathbb{R}) \) with \( \text{supp}(C_F \varphi) \subset (-\infty, b - a] \) if \( \text{supp}(F) \subset [a, \infty) \) and \( \text{supp}(\varphi) \subset (-\infty, b] \), \( a, b \in \mathbb{R} \).

**Proof.** Since \( (\sigma_t)_{t \in \mathbb{R}} \) is a \( c_0 \)-group with continuous infinitesimal generator \( \frac{d}{dt} \) on \( \mathcal{D}(\mathbb{R}) \), for each continuous linear functional \( F \in \mathcal{D}'(\mathbb{R}) \), the function \( C_F \varphi \in C^\infty(\mathbb{R}) \).
Now, suppose \( \text{supp}(F) \subset [a, \infty) \) and \( \text{supp}(\varphi) \subset (-\infty, b] \). Then \( \text{supp}(\sigma_t \varphi) \subset (-\infty, b - t] \) and so for all \( t \) with \( t > b - a \), \( \text{supp}(\sigma_t \varphi) \subset (-\infty, a] \) which implies \( F(\sigma_t \varphi) = 0 \) for \( t > b - a \). □

Now, let \( g \in L_{\text{loc},+}^\infty(\mathbb{R}) \). Then we define \( F * g \in \mathcal{D}'(\mathbb{R}) \) by
\[(F * g)(\varphi) := \int_R g(t)F(\sigma_t \varphi)dt, \quad \varphi \in \mathcal{D}(\mathbb{R}).\]

This integral is for all \(F \in \mathcal{D}'(\mathbb{R}), g \in L^\infty_{\text{loc,+}}(\mathbb{R}),\) and \(\varphi \in \mathcal{D}(\mathbb{R})\) a proper Riemann-Stieltjes integral. Indeed, assume \(\text{supp}(F) \subset [a, \infty), \) supp \(\varphi \subset (-\infty, b]\) and \(\text{supp}(g) \subset [c, \infty).\) Then

\[F(\sigma_t \varphi) = 0 \text{ for all } t \text{ with } t > b - a\]

and

\[
\int_R g(t)F(\sigma_t \varphi)dt = \begin{cases} 
\int_a^b g(t)F(\sigma_t \varphi)dt & b > c + a \\
\int_c^a 0 & b \leq c + a.
\end{cases}
\]

We conclude additionally that \(g * F \in \mathcal{D}'(\mathbb{R})\) with \(\text{supp}(g * F) \subset [c + a, \infty).\)

Now, \(L^\infty_{\text{loc,+}}(\mathbb{R})\) can be considered a subspace of \(\mathcal{D}'(\mathbb{R})\) by identifying each \(g \in L^\infty_{\text{loc,+}}(\mathbb{R})\) and \(F * g \in \mathcal{D}'(\mathbb{R})\) defined by

\[F_g(\varphi) = (\varphi, g) = \int \varphi(t)g(t)dt.\]

**Theorem 4.12.** Let \(F \in \mathcal{D}'(\mathbb{R}).\) Assume for all \(g \in L^\infty_{\text{loc,+}}(\mathbb{R})\)

\[F * g \in L^\infty_{\text{loc,+}}(\mathbb{R})\] (under the identification mentioned).

Then there is \(\mu \in b\nu_{\text{loc,-}}(\mathbb{R})\) such that

\[\forall g \in L^\infty_{\text{loc,+}}(\mathbb{R}) : F * g = \sigma[\mu]g.\]

**Proof.**

(1) Define the linear operator \(\mathcal{L}\) from \(L^\infty_{\text{loc,+}}(\mathbb{R})\) into \(L^\infty_{\text{loc,+}}(\mathbb{R})\) by

\[\mathcal{L}g = F * g.\]

Then \(\forall t \in \mathbb{R} : \mathcal{L}\sigma_t = \sigma_t \mathcal{L}.\) Also, \(\mathcal{L}\) is (sequentially) closed and therefore continuous. Indeed, let \(g_n \to g\) and \(\mathcal{L}g_n \to h (n \to \infty)\) in \(L^\infty_{\text{loc,+}}(\mathbb{R}).\) Then for all \(\varphi \in \mathcal{D}(\mathbb{R})\)

\[
\lim_{n \to \infty} \int_R g_n(t)F(\sigma_t \varphi)dt = \int_R g(t)F(\sigma_t \varphi)dt
\]

due to uniform convergence, and also

\[
\lim_{n \to \infty} (F * g_n)(\varphi) = (\varphi, h).
\]

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It follows that \((F \ast g)(\varphi) = \langle \varphi, h \rangle, \varphi \in \mathcal{D}(\mathbb{R})\), which means \(F \ast g = h\).

(2) \(\mathcal{L}\) being continuous, Theorem 4.9 yields there is \(\mu \in \text{bv}_{\text{loc}}(\mathbb{R})\) such that for all \(g \in C_+(\mathbb{R})\)

\[
\mathcal{L} g = \sigma[\mu] g .
\]

This means that for all \(g \in C_+(\mathbb{R}), \varphi \in \mathcal{D}(\mathbb{R})\)

\[
\int_{\mathbb{R}} g(t) F(\sigma_t \varphi) dt = \int_{\mathbb{R}} \varphi(t) \left(\int_{\mathbb{R}} (\sigma_t g)(\tau) d\mu(\tau)\right) dt = \int_{\mathbb{R}} \varphi(t) \left(\int_{\mathbb{R}} (\sigma_{-t} g)(\tau) d\hat{\mu}(\tau)\right) dt
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t+\tau) g(\tau) d\hat{\mu}(\tau) d\tau = \int_{\mathbb{R}} g(t) F_\mu(\sigma_t \varphi) dt .
\]

So \(F = F_\mu\) with \(F_\mu(\varphi) = \int_{\mathbb{R}} \varphi(t) d\hat{\mu}(\tau)\) and, consequently, for all \(g \in L_{\text{loc}}^\infty(\mathbb{R})\)

\[
\mathcal{L} g = F \ast g = F_\mu \ast g = \sigma[\mu] g .
\]

\[\square\]

**Corollary 4.13.** Let \(\mathcal{L} : L_{\text{loc}}^\infty(\mathbb{R}) \rightarrow L_{\text{loc}}^\infty(\mathbb{R})\) be continuous and translation invariant. Assume there is \(F \in \mathcal{D}'(\mathbb{R})\) such that \(\mathcal{L} g = F \ast g\) for all \(g \in L_{\text{loc}}^\infty(\mathbb{R})\). There is \(\mu \in \text{bv}_{\text{loc}}(\mathbb{R})\) such that \(\mathcal{L} = \sigma[\mu]\).

For self-containedness of the paper we have presented the above set up. However, Schwartz in [Schw2] proved that \(\mathcal{D}'(\mathbb{R})\) is a convolution algebra where convolution generalizes naturally the ordinary convolution between functions with half-infinite support. The condition on \(\mathcal{L}\) presented in the above Corollary means that \(\mathcal{L}\) can be extended to the whole of \(\mathcal{D}'(\mathbb{R})\) and \(F = \mathcal{L} \delta_0\) with \(\delta_0\) the point evaluation at \(t = 0\).

Replacing \(L_{\text{loc}}^\infty(\mathbb{R})\) by \(L_{\text{loc}}^\infty(\mathbb{R})\) in Theorem 4.12 and Corollary 4.13 we come to a similar assertion if also \(\text{bv}_{\text{loc}}(\mathbb{R})\) is replaced by \(\text{bv}_-(\mathbb{R}) = \{\hat{\mu} | \mu \in \text{bv}_+(\mathbb{R})\}\).

**Theorem 4.14.** Let \(F \in \mathcal{D}'(\mathbb{R})\). Assume \(F \ast g \in L_{\text{loc}}^\infty(\mathbb{R})\) for all \(g \in L_{\text{loc}}^\infty(\mathbb{R})\). Then there is \(\mu \in \text{bv}_+(\mathbb{R})\) such that

\[
F \ast g = \delta[\mu] g , \quad g \in L_{\text{loc}}^\infty(\mathbb{R}) .
\]

The above result was proved in [WC] also; the proof is based on quite different arguments.
References


