Convolution algebras translation invariant operators

Citation for published version (APA):

Document status and date:
Published: 01/01/1995

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain.
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 02. Aug. 2021
Convolution algebras
translation invariant operators

by

S.J.L. van Eijndhoven
M.M.A. de Rijcke
Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507
Convolution algebras.
translation invariant operators

by
S.J.L. van Eijndhoven
M.M.A. de Rijcke

Summary

In this paper three commutative convolution algebras of functions on $\mathbb{R}$ of (locally) bounded variation are introduced. All three are represented as algebras of continuous, translation invariant linear operators on well-defined spaces of continuous functions and of Lebesgue measurable functions, respectively, in which the translation group is a locally equicontinuous $c_0$-group. As a result complete characterizations of the translation invariant operators for $L^1$-type spaces are derived.

December 1995
Introduction

Let $\mathcal{X}$ be a translation invariant subspace of Schwartz distribution space $\mathcal{D}'(\mathbb{R})$, and let $\mathcal{X}$ be endowed with a locally convex topology such that $\mathcal{X}$ is sequentially complete. Also, let the translation group $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathcal{X}$ be locally equicontinuous and strongly continuous for this topology. A continuous linear operator $L$ on $\mathcal{X}$ is said to be translation invariant if $\sigma_t L = L \sigma_t$ for all $t \in \mathbb{R}$. The collection of all translation invariant operators on $\mathcal{X}$ establishes an algebra. Note that it is not clear at this point whether this algebra is commutative.

From the paper [ER] we invoke that for all $\mu \in \text{bvc}(\mathbb{R})$ (cf. Definition 1.2) and all $f \in \mathcal{X}$ the linear operator $\sigma[\mu]$ on $\mathcal{X}$ defined by

$$\sigma[\mu]f = \int_{\mathbb{R}} \sigma_t f \, d\mu(t), \quad f \in \mathcal{X}$$

is translation invariant and continuous, where the integral is an $\mathcal{X}$-valued Riemann–Stieltjes integral. Now one may wonder whether the convolution algebra $\text{bvc}(\mathbb{R})$ represents all translation invariant operators on $\mathcal{X}$. In this paper we prove that for $\mathcal{X} = C(\mathbb{R}), C_c(\mathbb{R}), L_{\text{loc}}^p(\mathbb{R})$ and $L_{\text{comp}}^p(\mathbb{R})$ this is the case, but although a likely candidate for $\mathcal{X} = L_{\text{loc}}^p(\mathbb{R})$ or $L_{\text{comp}}^p(\mathbb{R})$, the problem whether $\text{bvc}(\mathbb{R})$ is the convolution algebra associated to these spaces is not yet solved.

The convolution algebra $\text{bvc}(\mathbb{R})$ is too small, in general, to describe all translation invariant operators. We introduce two larger convolution algebras, namely $\text{bv}_+(\mathbb{R})$ and $\text{bV}_{\text{loc},-}(\mathbb{R})$, and represent them as commutative operator algebras on spaces $\mathcal{X}$ satisfying additionally for $\text{bv}_+(\mathbb{R})$

$$(1) \quad \lim_{t \to -\infty} \sigma_t f = 0, \quad \forall f \in \mathcal{X}$$

and for $\text{bV}_{\text{loc},-}(\mathbb{R})$,

$$(2) \quad \forall f \in \mathcal{X} \exists a \in \mathbb{R} : \text{supp}(f) \subset [a, \infty).$$

The representation is by means of the same integral, yet considered as an $\mathcal{X}$-valued improper Riemann–Stieltjes integral. Examples of spaces of type (1) are $C_-(\mathbb{R})$ and $L_p(\mathbb{R})$ and examples of spaces of type (2) are $C_+(\mathbb{R})$ and $L_{\text{loc},+}^p(\mathbb{R}), 1 \leq p < \infty$, introduced in Sections 2 and 3.

In this paper we use freely, without further reference, the standard terminology of the theory of locally convex vector spaces and of the general one-parameter co-group theory. In this respect we deal with $F$-spaces (= Fréchet spaces) and strict $LF$-spaces (= strict inductive limits of $F$-spaces). Nice references are [Co], [Sch] and [Fl].

The present paper is divided into four sections. Section 1 is devoted merely to definitions and notations to be used. We introduce the space $C(\mathbb{R}, V)$ of continuous functions into a locally convex vector space $V$, and its subspaces $C_-(\mathbb{R}, V), C_c(\mathbb{R}, V), C_+(\mathbb{R}, V)$ and $C_{-,+}(\mathbb{R}, V)$. Also we discuss Riemann–Stieltjes integration for these spaces. Here the subspaces $\text{bvc}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bv}_{\text{loc},-}(\mathbb{R})$ and $\text{bV}_{\text{loc},-}(\mathbb{R})$ of
the space $bv_{loc}(\mathbb{R})$ consisting of all right continuous functions $\mu$ on $\mathbb{R}$ with bounded variation on compact intervals of $\mathbb{R}$, come into play.

In Section 2 we characterize the collection of all continuous, translation invariant, linear operators on the spaces $C(\mathbb{R})$ and $C_c(\mathbb{R})$, $C_{-}(\mathbb{R})$ and $C_{-,+}(\mathbb{R})$, and $C_{+}(\mathbb{R})$ where $V = C$ is omitted in the notation. They are represented by the convolution algebras $bv_c(\mathbb{R})$, $bv_+(\mathbb{R})$ and $bv_{loc,-}(\mathbb{R})$, respectively.

In Section 3 we introduce the spaces $L^p_{loc}(\mathbb{R})$, $L^p_{comp}(\mathbb{R})$, $L^p_{-}(\mathbb{R})$, $L^p_{+}(\mathbb{R})$ and $L^p_{loc,+}(\mathbb{R})$, we discuss their duality relations and show how they are related to the aforementioned convolution algebras.

In the last section we pay special attention to the case $p = 1$, describing completely all continuous, translation invariant, linear operators on these spaces.

1 Spaces of continuous functions

Let $V$ denote a sequentially complete locally convex topological vector space. The locally convex topology of $V$ is assumed to be generated by the indexed set of seminorms $\{p_\nu \mid \nu \in D\}$. In this section we introduce the space $C(\mathbb{R}, V)$ consisting of all continuous functions from $\mathbb{R}$ into $V$, and its subspaces $C_{+}(\mathbb{R}, V)$, $C_{-}(\mathbb{R}, V)$, $C_{-,+}(\mathbb{R}, V)$ and $C_{c}(\mathbb{R}, V)$.

The space $C(\mathbb{R}, V)$ is endowed with the compact open topology, i.e. the locally convex topology generated by the seminorms

$$p_{\nu,n}(f) = \max_{t \in [-n,n]} p_\nu(f(t)), \quad n \in \mathbb{N}, \nu \in D.$$  

Then $C(\mathbb{R}, V)$ is sequentially complete. Moreover, if $D$ is a countable set, both $V$ and $C(\mathbb{R}, V)$ are $F$-spaces (Fréchet spaces). We note that in [ER] the space $C(\mathbb{R}, V)$ is used as a means to describe general properties of one-parameter locally equicontinuous $c_0$-groups of continuous linear mappings on $V$.

**Definition 1.1.** Let $I \subseteq \mathbb{R}$ denote an interval. A function $\mu : I \to C$ is said to be of bounded variation if there exists $C > 0$ such that for all $n \in \mathbb{N}$ and all $\{t_0, \ldots, t_n\} \subset I$ with $t_0 < t_1 < \cdots < t_n$

$$\sum_{j=1}^{n} |\mu(t_j) - \mu(t_{j-1})| \leq C.$$  

By $\text{var}_I(\mu)$ we denote the total variation of $\mu$ on $I$, i.e. the infimum of the collection of all constants $C$ satisfying $(\ast)$. The vector space of all right continuous functions $\mu$ on $I$ with $\text{var}_I(\mu) < \infty$ is denoted by $bv(I)$.

**Definition 1.2.** The space $bv_c(\mathbb{R})$ consists of all $\mu \in bv(\mathbb{R})$ with the property that there is $T > 0$ depending on $\mu$, such that

$$\mu(t) = 0 \text{ for } t < -T,$$

$$\mu(t) = \mu(T) \text{ for } t > T.$$
Due to the sequential completeness of $V$ for all $\mu \in \operatorname{bv}(\mathbb{R})$ and $f \in C(\mathbb{R}, V)$ the $V$-valued (Riemann–Stieltjes) integral

$$J_{\mu}f := \int_{\mathbb{R}} f(\tau) d\mu(\tau)$$

can be defined properly. The mapping $J_{\mu}$ from $C(\mathbb{R}, V)$ into $V$, thus defined, is linear and continuous, since

$$p_{\nu}(J_{\mu}f) \leq \max_{t \in [-T,T]} p_{\nu}(f(t)) \cdot \operatorname{var}(\mu).$$

**Proposition 1.3.** Let $K : V_1 \to V_2$ be a continuous linear operator. Then the linear operator $L_K : C(\mathbb{R}, V_1) \to C(\mathbb{R}, V_2)$ defined by

$$(L_Kf)(t) := K(f)(t)$$

is continuous and satisfies

$$J_{\mu}L_K = KJ_{\mu}.$$

More details on the introduction of $J_{\mu}$ can be found in [ER].

**Definition 1.4.** The space $C_{\infty}(\mathbb{R}, V)$ consists of all $f \in C(\mathbb{R}, V)$ for which

$$\lim_{t \to -\infty} f(t) = 0,$$

i.e. $\forall \nu \in D : \lim_{t \to -\infty} p_{\nu}(f(t)) = 0$.

The locally convex topology for $C_{\infty}(\mathbb{R}, V)$ is the one generated by the seminorms

$$p_{\nu,n}(f) := \max_{t \geq -n} p_{\nu}(f(t)), \quad n \in \mathbb{N}, \nu \in D.$$

The space $C_{\infty}(\mathbb{R}, V)$ is sequentially complete. In correspondence, we introduce the subspace $\operatorname{bv}_{+}(\mathbb{R})$ of $\operatorname{bv}(\mathbb{R})$.

**Definition 1.5.** The space $\operatorname{bv}_{+}(\mathbb{R})$ consists of all $\mu \in \operatorname{bv}(\mathbb{R})$ for which there is $T > 0$ depending on $\mu$ such that $\mu(t) = 0$ for all $t < -T$.

Since the functions in $C_{\infty}(\mathbb{R}, V)$ are uniformly continuous on half-infinite intervals $[-T, \infty)$, for each $\mu \in \operatorname{bv}_{+}(\mathbb{R})$ and $f \in C_{\infty}(\mathbb{R}, V)$ the $V$-valued (improper Riemann–Stieltjes) integral

$$J_{\mu}f := \int_{\mathbb{R}} f(\tau) d\mu(\tau)$$

can be well-defined and $J_{\mu} : C_{\infty}(\mathbb{R}, V) \to V$ is continuous with
Definition 1.6. The space $C_+(\mathbb{R}, V)$ consists of all $f \in C(\mathbb{R}, V)$ with support bounded on the left, i.e. all $f \in C(\mathbb{R}, V)$ for which $T > 0$ exists such that $f(t) = 0$ for $t \leq -T$.

Let

$$C_{+, n}(\mathbb{R}, V) := \{ f \in C(\mathbb{R}, V) | \forall t \leq -n : f(t) = 0 \} .$$

Then $C_{+, n}(\mathbb{R}, V)$ is a closed subspace of $C(\mathbb{R}, V)$ and the collection $\{ C_{+, n}(\mathbb{R}, V) | n \in \mathbb{N} \}$ is a strict inductive system of sequentially complete locally convex spaces. We endow $C_+(\mathbb{R}, V)$ with the inductive limit topology generated by this strict inductive system and end up with a sequentially complete locally convex space.

Definition 1.7. The space $bV_{\text{loc,-}}(\mathbb{R})$ consists of all right continuous $\mu : \mathbb{R} \rightarrow C$ with the property that

$$\exists T > 0 : \mu(t) = \mu(T) , \quad t \geq T \quad (\ast \ast)$$

and

$$\mu|_{[-n, \infty)} \in bV([-n, \infty)), \quad n \in \mathbb{N} .$$

For each $\mu \in bV_{\text{loc,-}}(\mathbb{R})$ and $f \in C_+(\mathbb{R}, V)$ the $V$-valued (proper Riemann–Stieltjes) integral

$$\mathcal{J}_\mu f := \int_{\mathbb{R}} f(\tau) d\mu(\tau)$$

can be well-defined, and $\mathcal{J}_\mu : C_+(\mathbb{R}, V) \rightarrow V$ is continuous. To check the latter assertion, observe that for all $n \in \mathbb{N}$ and all $f \in C_{+, n}(\mathbb{R}, V)$

$$p_\nu(\mathcal{J}_\mu f) \leq \max_{t \in [-n, T]} p_\nu(f(t)) \cdot \text{var}_{[-n, \infty)}(\mu)$$

with $T$ chosen as in $(\ast \ast)$.

Definition 1.8. The space $C_{-, +}(\mathbb{R}, V)$ consists of all $f \in C(\mathbb{R}, V)$ with the property that

$$\exists T > 0 : f(t) = 0 , \quad t \leq -T$$

and

$$\lim_{t \rightarrow -\infty} f(t) = 0 ,$$
\[ C_{-,+}(\mathbb{R}, V) = C_-(\mathbb{R}, V) \cap C_+(\mathbb{R}, V) . \]

Introduce the strict inductive system of sequentially complete locally convex vector spaces
\[ C_{-,+}(\mathbb{R}, V) := C_{-,+}(\mathbb{R}, V) . \]

Then
\[ C_{-,+}(\mathbb{R}, V) = \bigcup_{n=1}^{\infty} C_{-,+}(\mathbb{R}, V) \]

is endowed with the corresponding inductive limit topology and thus sequentially complete.

**Definition 1.9.** The space \( bV_{loc, -}(\mathbb{R}) \) consists of all right-continuous functions \( \mu : \mathbb{R} \to \mathcal{C} \) with the property that for all \( n \in \mathbb{N} \)
\( \mu|_{[-n, \infty)} \in \text{bv}([-n, \infty)) . \)

For each \( \mu \in bV_{loc, -}(\mathbb{R}) \) and \( f \in C_{-,+}(\mathbb{R}, V) \) the \( V \)-valued (improper Riemann–Stieltjes) integral
\[ J_{\mu} f := \int_{\mathbb{R}} f(\tau) d\mu(\tau) \]

can be well-defined and satisfies
\[ p_\nu(J_{\mu} f) \leq \max_{t \geq -n} p_\nu(f(t)) \cdot \text{var}_{[-n, \infty)}(\mu) . \]

Therefore \( J_{\mu} : C_{-,+}(\mathbb{R}, V) \to V \) is continuous for each \( \mu \in bV_{loc, -}(\mathbb{R}) \).

Finally, we introduce the subspace \( C_c(\mathbb{R}, V) \) of \( C(\mathbb{R}, V) \).

**Definition 1.10.** The space \( C_c(\mathbb{R}, V) \) consist of all \( f \in C(\mathbb{R}, V) \) for which there exists \( T > 0 \) such that
\[ f(t) = 0 , \quad |t| \geq T . \]

We see that
\[ C_c(\mathbb{R}, V) = \bigcup_{n=1}^{\infty} C_{c,n}(\mathbb{R}, V) , \]
where

\[ C_{c,n}(\mathbb{R}, V) := \{ f \in C(\mathbb{R}, V) \mid f(t) = 0, \forall t, |t| \geq n \} \, . \]

Being closed subspaces of \( C(\mathbb{R}, V) \), the collection \( \{ C_{c,n}(\mathbb{R}, V) \mid n \in \mathbb{N} \} \) is a strict inductive system of sequentially complete locally convex vector spaces. We endow the space \( C_c(\mathbb{R}, V) \) with the corresponding inductive limit topology and end up with a sequentially complete locally convex space.

**Definition 1.11.** The space \( bV_{\text{loc}}(\mathbb{R}) \) consists of all right continuous functions \( \mu : \mathbb{R} \to C \) such that

\[ \mu|_{[-n, n]} \in bv([-n, n]) \, , \quad n \in \mathbb{N} \, . \]

For each \( \mu \in bV_{\text{loc}}(\mathbb{R}) \) and \( f \in C_c(\mathbb{R}, V) \) the \( V \)-valued (Riemann–Stieltjes) integral

\[ J_\mu f := \int f(\tau) d\mu(\tau) \]

can be well-defined and \( J_\mu : C_c(\mathbb{R}, V) \to V \) is continuous because for all \( f \in C_{c,n}(\mathbb{R}, V) \)

\[ p_\nu(J_\mu f) \leq \max_{t \in [-n, n]} p_\nu(f(t)) \cdot \text{var}_{[-n, n]}(\mu) \, . \]

There is the following dense and continuous inclusion scheme

\[ C_c(\mathbb{R}, V) \hookrightarrow C_{-,+}(\mathbb{R}, V) \hookrightarrow C_{-,}(\mathbb{R}, V) \]

and correspondingly

\[ bV_{\text{loc}}(\mathbb{R}) \supseteq bV_{\text{loc,+}}(\mathbb{R}) \supseteq bV_{\text{loc,-}}(\mathbb{R}) \supseteq bV_c(\mathbb{R}) \, . \]

We remark that if \( V \) is an \( F \)-space, the spaces \( C(\mathbb{R}, V) \) and \( C_{-,+}(\mathbb{R}, V) \) are \( F \)-spaces and \( C_{+}(\mathbb{R}, V) \), \( C_{-}(\mathbb{R}, V) \) and \( C_c(\mathbb{R}, V) \) are strict \( LF \)-spaces.

There is an interesting subspace of \( bV_c(\mathbb{R}) \), namely \( bV_c^\infty(\mathbb{R}) := bV_c(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). We like to discuss it here in short. For \( \psi \in bV_c^\infty(\mathbb{R}) \), its derivative \( \psi' \) belongs to \( C_c^\infty(\mathbb{R}) \) and if \( \varphi \in C_c^\infty(\mathbb{R}) \), then \( \psi \) defined by

\[ \psi(t) := \int_{-\infty}^{t} \varphi(\tau) d\tau \, , \quad t \in \mathbb{R} \, , \]

is an element of \( bV_c^\infty(\mathbb{R}) \). For \( \psi \in bV_c^\infty(\mathbb{R}) \), \( \varphi = \psi' \),
\[ \mathcal{J}_\psi f = \int f(\tau) d\psi(\tau) = \int f(\tau) \varphi(\tau) d\tau. \]

If \( (\varphi_k) \) is a sequence in \( C^\infty_c(\mathbb{R}) \) with

\[ \text{supp}(\varphi_k) \subseteq [-\frac{1}{k}, \frac{1}{k}], \quad \varphi_k \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} \varphi_k(\tau) d\tau = 1, \]

then with \( \psi_k \in \text{bv}_c^\infty(\mathbb{R}) \) defined by

\[ \psi_k(t) := \int_{-\infty}^{t} \varphi_k(\tau) d\tau, \]

we have

\[ \mathcal{J}_{\psi_k} f \to f(0) \quad \text{in} \quad V \quad \text{as} \quad k \to \infty. \]

Indeed for all \( \nu \in D \)

\[ p_\nu(\mathcal{J}_{\psi_k} f - f(0)) \leq \max_{t \in [-\frac{1}{k}, \frac{1}{k}]} p_\nu(f(t) - f(0)). \]

The sequence \( (\psi_k) \) is called an approximate identity.

The remaining part of this section is devoted to the special case that \( V = C \) and so to the spaces \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-,+}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \), where the additional \( C \) in the notation is dropped.

**Theorem 1.12.**

I. For each \( \mu \in \text{bv}_c(\mathbb{R}), \text{bv}_{1oc,-}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bv}_{1oc,-}(\mathbb{R}) \) and \( \text{bv}_{1oc}(\mathbb{R}) \), the linear functional \( \mathcal{J}_\mu \),

\[ \mathcal{J}_\mu(f) = \int_{\mathbb{R}} f(\tau) d\mu(\tau) \]

is properly defined and continuous on \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-,+}(\mathbb{R}) \) and \( C_c(\mathbb{R}) \), respectively.

II. Let \( F \) be a continuous linear functional on \( C(\mathbb{R}), C_+(\mathbb{R}), C_-(\mathbb{R}), C_{-,+}(\mathbb{R}) \) or \( C_c(\mathbb{R}) \). Then \( F = \mathcal{J}_\mu \) for some \( \mu \in \text{bv}_c(\mathbb{R}), \text{bv}_{1oc,-}(\mathbb{R}), \text{bv}_+(\mathbb{R}), \text{bv}_{1oc,-}(\mathbb{R}) \) and \( \text{bv}_{1oc}(\mathbb{R}) \), respectively.

The correspondence \( F \leftrightarrow \mu \) is one-to-one for the spaces \( C(\mathbb{R}) \) and \( C_-(\mathbb{R}) \) and unique up to a constant for the other spaces.
Proof. The proof of part I has been indicated already in the context of a sequentially complete locally convex space $V$ instead of $C$.

The result stated in II has been proved for $C(\mathbb{R})$, e.g. in [So]. We outline the proof for $C_+(\mathbb{R}), C_- (\mathbb{R}), C_{-,+} (\mathbb{R})$ and $C_c(\mathbb{R})$, consecutively.

1. Let $F$ be a continuous linear functional on the strict LF-space $C_+(\mathbb{R})$. It means that for each $n \in \mathbb{N}$ the restriction $F_n$ of $F$ to $C_{+,n}(\mathbb{R})$ is continuous. The classical Riesz representation Theorem, cf. [Kr], yields $\hat{\mu}_n \in \text{bv}_c(\mathbb{R})$ with

$$F_n(f) = \int_{-n}^{\infty} f(\tau) d\hat{\mu}_n(\tau), \quad f \in C_{+,n}(\mathbb{R})$$

for each $n \in \mathbb{N}$. Since $F_{n+1}$ can be regarded as an extension of $F_n$, it follows that $\hat{\mu}_{n+1}$ and $\hat{\mu}_n$ differ only a constant on $[-n, \infty)$. Matching these constants we get $\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})$,

$$\mu(t) = \hat{\mu}_n(t), \quad t \in [-n, \infty), \; n \in \mathbb{N},$$

such that for all $n \in \mathbb{N}$ and $f \in C_{+,n}(\mathbb{R})$

$$\int_{-n}^{\infty} f(\tau) d\mu(\tau) = \int_{-n}^{\infty} f(\tau) d\hat{\mu}_n(\tau) = F_n(f).$$

This proves the first statement.

2. We characterize the continuous linear functionals on $C_{-}(\mathbb{R})$. To do so, first we characterize the continuous linear functionals on $C_{-,+},n(\mathbb{R}), \; n \in \mathbb{N}$.

For fixed $n \in \mathbb{N}$ let $G$ be a continuous linear functional on the Banach space $C_{-,+},n(\mathbb{R})$. Hence, there is $C > 0$ such that

$$|G(f)| \leq C \max_{t \geq -n} |f(t)|.$$ 

Applying Hahn–Banach extension Theorem, there is a continuous linear functional $G_{\text{ext}}$ on the Banach space $B([-n, \infty))$ of bounded functions on $[-n, \infty)$ with

$$\|G_{\text{ext}}\| = \|G\|$$

and $G_{\text{ext}}$ extending $G$. Then define

$$\hat{\mu}_G(t) := \begin{cases} G_{\text{ext}}(1_{[-n,t]}(\tau)) & t > -n \\ 0 & t \leq -n \end{cases}$$

where $1_{[-n,t]}$ is the characteristic function of the interval $[-n,t)$.

Since for $-n \leq t_0 < \ldots < t_m < \infty$

$$\sum_{j=1}^{m} |\hat{\mu}_G(t_j) - \hat{\mu}_G(t_{j-1})| = G_{\text{ext}} \left( \sum_{j=1}^{m} \alpha_j 1_{[t_{j-1},t_j]} \right) \leq \|G\|$$

9
for certain $\alpha_j$ with $|\alpha_j| = 1$, $\tilde{\mu}_G$ has bounded variation on $[-n, \infty)$. Also, for $f \in C_{-\infty, +, n}(\mathbb{R})$ and $\varepsilon > 0$ given, there are $-n \leq t_0 < t_1 < \ldots < t_m$ such that

$$
\sup_{t \geq -n} \left| f(t) - \sum_{j=1}^{m} f(t_j)1_{[t_{j-1}, t_j)}(t) \right| < \varepsilon ,
$$

so that

$$
|G(f) - \sum_{j=1}^{m} f(t_j)(\tilde{\mu}_G(t_j) - \tilde{\mu}_G(t_{j-1}))| < \varepsilon \|G\| .
$$

It follows that

$$
G(f) = \int_{-n}^{\infty} f(\tau)d\tilde{\mu}_G(\tau) ,
$$

and a posteriori that

$$
\|G\| = \text{var}_{[-n, \infty)}(\tilde{\mu}_G) .
$$

We observe that we can replace $\tilde{\mu}_G$ by the right-continuous function

$$
\mu_G(t) = \lim_{s \uparrow t} \tilde{\mu}_G(s)
$$

without changing anything.

We conclude that every continuous linear functional on the $B$-space $C_{-\infty, +, n}(\mathbb{R})$ is of the form $(***)$ for some $\mu_G \in \text{bv}([-n, \infty))$.

Now, let $F$ be a continuous linear functional on the $F$-space $C_{-\infty}(\mathbb{R})$. Then there is $n \in \mathbb{N}$ and $C > 0$ such that

$$
|F(f)| \leq C \cdot \max_{t \geq -n} |f(t)|
$$

for all $f \in C_{-\infty}(\mathbb{R})$.

So, the restriction $G$ of $F$ to $C_{-\infty, +, n+1}(\mathbb{R})$ is continuous, and $\mu_G \in \text{bv}([-n - 1, \infty))$ exists satisfying the representation $(***)$. The inequality $(***)$ indicates that $\mu_G$ must be constant on $(-n - 1, -n)$, say equal to $c_n$. Then, define $\mu \in \text{bv}_{+}(\mathbb{R})$ by

$$
\mu(t) := \begin{cases} 
0 & t < -n , \\
\mu_G(t) - c_n , & t \geq -n .
\end{cases}
$$

Now, for $f \in C_{-\infty}(\mathbb{R})$, there is $f_0 \in C_{-\infty, +, n+1}(\mathbb{R})$ such that $f(t) - f_0(t) = 0$ for $t \geq -n$. So,

$$
F(f) = F(f_0) = G(f_0) = \int_{\mathbb{R}} f_0(\tau)d\mu(\tau) = \int_{\mathbb{R}} f(\tau)d\mu(\tau) .
$$

3. Let $F$ be a continuous linear functional on the strict $LB$-space $C_{-\infty, +}(\mathbb{R})$. It means that for all $n \in \mathbb{N}$ the restriction $F_n$ of $F$ to $C_{-\infty, +, n}(\mathbb{R})$ is continuous. As we have seen in 2., for each $n \in \mathbb{N}$ there is $\tilde{\mu}_n \in \text{bv}([-n, \infty))$ such that
\[ F_n(f) = \int_{-n}^{\infty} f(\tau)d\mu_n(\tau), \quad f \in C_{-\infty,n}(\mathbb{R}). \]

It follows that \( \mu_{n+1}(t) - \mu_n(t) = c_n, t \in [-n, \infty), \) for some constants \( c_n. \) Matching these constants yields \( \mu_n \in \text{bv}([-n, \infty)) \) with \( \mu_{n+1}|_{[-n, \infty)} = \mu_n. \) Now, define \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) by

\[ \mu(t) = \mu_n(t), \quad t \in [-n, \infty), \ n \in \mathbb{N}. \]

Then

\[ F(f) = F_n(f) = \int_{-n}^{\infty} f(\tau)d\mu_n(\tau) = J_n(f) \]

for \( f \in C_{-\infty,n}(\mathbb{R}), \ n \in \mathbb{N}. \)

4. Let \( F \) be a continuous linear functional on \( C_c(\mathbb{R}). \) Since \( C_c(\mathbb{R}) \) is a strict \( LB \)-space with corresponding strict inductive system of \( B \)-spaces \( C_{c,n}(\mathbb{R}), \) the classical Riesz representation Theorem yields \( \mu_n \in \text{bv}([-n, n]) \) such that

\[ F(f) = \int_{-n}^{n} f(\tau)d\mu_n(\tau), \quad f \in C_{c,n}(\mathbb{R}) \]

for each \( n \in \mathbb{N}. \) Again we can assume that \( \mu_{n+1} \) extends \( \mu_n \) and define \( \mu \in \text{bv}_{\text{loc}}(\mathbb{R}) \) by

\[ \mu(t) = \mu_n(t), \quad t \in [-n, n], \ n \in \mathbb{N} \]

such that

\[ F(f) = \int_{\mathbb{R}} f(\tau)d\mu(\tau), \quad f \in C_c(\mathbb{R}). \]

We remark, but do not prove, that the space \( \text{bv}_{\text{loc}}(\mathbb{R}) \) and \( \text{bv}_{\text{loc},-}(\mathbb{R}) \) are \( F \)-spaces whereas the spaces \( \text{bv}_+(\mathbb{R}) \) and \( \text{bv}_c(\mathbb{R}) \) are strict \( LB \)-spaces, and \( \text{bv}_{\text{loc},-}(\mathbb{R}) \) is a strict \( LF \)-space.

2 Translation invariant operators on spaces of continuous functions

In this section we derive complete characterizations for translation invariant operators on the spaces \( C(\mathbb{R}), C_{-}(\mathbb{R}), C_{+}(\mathbb{R}), C_+(\mathbb{R}) \) and \( C_{-\infty,n}(\mathbb{R}) \) respectively.

Case 1: \( C(\mathbb{R}). \)
For each \( t \in \mathbb{R}, \) let \( \sigma_t : C(\mathbb{R}) \to C(\mathbb{R}) \) be the continuous linear operator defined by
\[(\sigma_tf)(s) := f(s + t), \quad s \in \mathbb{R} .\]

Then, the one-parameter family \((\sigma_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group on the Fréchet space \(C(\mathbb{R})\), i.e. for all \(f \in C(\mathbb{R})\)

\[
\lim_{t \to 0} \sigma_tf = f \text{ in } C(\mathbb{R}) .
\]

The \(C(\mathbb{R})\)-valued function \(T_\sigma f\), for \(f \in C(\mathbb{R})\) defined by

\[
(T_\sigma f)(t) = \sigma_tf, \quad t \in \mathbb{R} ,
\]

belongs to \(C(\mathbb{R}, C(\mathbb{R}))\). The operator \(T_\sigma\) from \(C(\mathbb{R})\) into \(C(\mathbb{R}, C(\mathbb{R}))\), thus defined, is linear and has a closed graph. \(C(\mathbb{R})\) and \(C(\mathbb{R}, C(\mathbb{R}))\) being \(F\)-spaces the Closed Graph Theorem applies so that \(T_\sigma\) is continuous (equivalently the \(c_0\)-group \((\sigma_t)_{t \in \mathbb{R}}\) is locally equicontinuous).

Now, for every \(\mu \in bvc(\mathbb{R})\) define the operator \(\sigma[\mu] := J_\mu T_\sigma\), i.e.

\[
\sigma[\mu]f = \int_{\mathbb{R}} \sigma_t f \, d\mu(t), \quad f \in C(\mathbb{R}) ;
\]

\(\sigma[\mu]\) maps \(C(\mathbb{R})\) into \(C(\mathbb{R})\) with \(\sigma_t \sigma[\mu] = \sigma[\mu] \sigma_t, t \in \mathbb{R};\) so \(\sigma[\mu]\) is translation invariant. Also, if \(L: C(\mathbb{R}) \to C(\mathbb{R})\) is continuous, linear and translation invariant, then by Theorem 1.11, \(\mu \in bvc(\mathbb{R})\) exists such that

\[
(Lg)(0) = J_\mu(g), \quad g \in C(\mathbb{R})
\]

and so

\[
(Lf)(t) = (L \sigma_t f)(0) = J_\mu(\sigma_t f) = (\sigma[\mu] f)(t) .
\]

As a consequence, for \(\mu_1, \mu_2 \in bvc(\mathbb{R})\), there is \(\mu \in bvc(\mathbb{R})\) such that

\[
\sigma[\mu] = \sigma[\mu_1] \sigma[\mu_2] .
\]

In fact, \(\mu = \mu_1 * \mu_2\) with \(\mu_1 * \mu_2 = \mu_2 * \mu_1\) and

\[
(\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau) d\mu_2(\tau) .
\]

The commutative convolution ring \(bvc(\mathbb{R})\) has been subject of study in [So], [ES] and [Rij]. It plays the central role in the study of mean periodic functions, cf. [Schw] and [Ka].

**Case 2: \(C_\infty(\mathbb{R})\).**

For each \(t \in \mathbb{R}\) the translation operator \(\sigma_t\) maps \(C_\infty(\mathbb{R})\) into \(C_\infty(\mathbb{R})\) continuously. The one-parameter family \((\sigma_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group on \(C_\infty(\mathbb{R})\), additionally satisfying
\[ \lim_{t \to \infty} \sigma_t f = 0, \quad f \in C_{-\infty}(\mathbb{R}) \, . \]

In this case, \( T_\sigma f \in C_{-\infty}(\mathbb{R}, C_{-\infty}(\mathbb{R})) \), \( f \in C_{-\infty}(\mathbb{R}) \) and \( T_\sigma \) is continuous as a linear mapping from the \( F \)-space \( C_{-\infty}(\mathbb{R}) \) into the \( F \)-space \( C_{-\infty}(\mathbb{R}, C_{-\infty}(\mathbb{R})) \). For all \( \mu \in \text{bv}^+_c(\mathbb{R}) \) we put

\[ \sigma[\mu] := J_\mu T_\sigma \, , \]

with \( J_\mu \) as in Definition 1.4. Then \( \sigma[\mu] : C_{-\infty}(\mathbb{R}) \to C_{-\infty}(\mathbb{R}) \) is continuous and translation invariant. Since \( \text{bv}^+_c(\mathbb{R}) \) represents the dual of \( C_{-\infty}(\mathbb{R}) \) (cf. Theorem 1.11) arguments similar as for the space \( C(\mathbb{R}) \) show that

\[ \{ \sigma[\mu] \mid \mu \in \text{bv}^+_c(\mathbb{R}) \} \]

is the collection of all continuous, linear, translation invariant operators from \( C_{-\infty}(\mathbb{R}) \) into \( C_{-\infty}(\mathbb{R}) \). So, \( \text{bv}^+_c(\mathbb{R}) \) is a commutative convolution ring with convolution defined by

\[ \sigma[\mu_1 * \mu_2] := \sigma[\mu_1] \sigma[\mu_2] \]

with \( (\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau) d\mu_2(\tau) = (\mu_2 * \mu_1)(t) \).

**Case 3: \( C^+_c(\mathbb{R}) \).**

Thirdly, we consider the \( LF \)-space \( C^+_c(\mathbb{R}) \). Again it can be checked that the translation group \((\sigma_t)_{t \in \mathbb{R}}\) is a locally equicontinuous \( c_0 \)-group on \( C^+_c(\mathbb{R}) \), so that \( T_\sigma \) maps \( C^+_c(\mathbb{R}) \) into \( C(\mathbb{R}, C^+_c(\mathbb{R})) \). Consequently, for \( \mu \in \text{bv}^c_\mathbb{c}(\mathbb{R}) \) the operator \( \sigma[\mu] = J_\mu T_\sigma \) is continuous and translation invariant from \( C^+_c(\mathbb{R}) \) into \( C^+_c(\mathbb{R}) \). However, not all translation invariant operators can be presented this way.

By Definition 1.6 every \( \mu \in \text{bv}^\text{loc, c}_{-\infty}(\mathbb{R}) \) induces a translation invariant continuous linear mapping \( \sigma[\mu] \), defined by

\[ \sigma[\mu] := J_\mu T_\sigma \, , \]

from \( C^+_c(\mathbb{R}) \) into \( C(\mathbb{R}) \). Explicitly, we have

\[ (\sigma[\mu] f)(t) = \int_{\mathbb{R}} f(t + \tau) d\mu(\tau) \]

for all \( f \in C^+_c(\mathbb{R}) \). From this we see that for all \( f \in C^+_{+,n}(\mathbb{R}) \) and \( \mu \in \text{bv}^\text{loc, c}_{-\infty}(\mathbb{R}) \) with \( \text{supp}(\mu) \subseteq (-\infty, m) \),

\[ \sigma[\mu] f \in C^+_{+,m+n}(\mathbb{R}) \]

and for \( t \in [-n - m, k] \)
\[ |(\sigma[\mu])f(t)| = \left| \int_{-(n+t)}^{m} f(t + \tau)d\mu(\tau) \right| \leq \max_{\tau \in [-n,m+k]} |f(t)| \cdot \var_{[-n-k,m]}(\mu). \]

We conclude that each \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) defines a continuous, linear, translation invariant operator \( \sigma[\mu] \) on \( C_+(\mathbb{R}) \).

If, conversely, \( \mathcal{L} \) is a continuous, linear, translation invariant operator from \( C_+(\mathbb{R}) \) into \( C_+(\mathbb{R}) \), then \( f \mapsto (\mathcal{L}f)(0) \) is a continuous linear functional on \( C_+(\mathbb{R}) \) and, by Theorem 1.11, \( \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \) exists such that

\[ (\mathcal{L}f)(t) = (\mathcal{L}(\sigma_t f))(0) = \int_{\mathbb{R}} (\sigma_t f)(\tau)d\mu(\tau). \]

Summarizing, the collection \( \{\sigma[\mu] | \mu \in \text{bv}_{\text{loc},-}(\mathbb{R})\} \) consists of precisely all continuous, linear, translation invariant operators from \( C_+(\mathbb{R}) \) into \( C_+(\mathbb{R}) \).

Consequently, \( \text{bv}_{\text{loc},-}(\mathbb{R}) \) is a commutative convolution algebra with convolution defined by

\[ \sigma[\mu_1 \ast \mu_2] := \sigma[\mu_1]\sigma[\mu_2], \]

or, explicitly,

\[ (\mu_1 \ast \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t - \tau)d\mu_2(\tau), \quad t \in \mathbb{R}. \]

The way of characterizing the translation invariant operators for the spaces \( C_c(\mathbb{R}) \) and \( C_{-,+}(\mathbb{R}) \) is of a different nature. We consider the space \( C_c(\mathbb{R}) \), first.

**Case 4: \( C_c(\mathbb{R}) \).**

The family \( \{\sigma_t\}_{t \in \mathbb{R}} \) establishes a locally equicontinuous \( c_0 \)-group on \( C_c(\mathbb{R}) \). So, for \( f \in C_c(\mathbb{R}) \),

\[ T_\sigma f : t \mapsto \sigma_t f \in C(\mathbb{R}, C_c(\mathbb{R})) \]

and \( T_\sigma : C_c(\mathbb{R}) \to C(\mathbb{R}, C_c(\mathbb{R})) \) is continuous. It follows that for each \( \mu \in \text{bv}_c(\mathbb{R}) \), the mapping \( \sigma[\mu] = J_\mu T_\sigma \) is linear continuous and translation invariant from \( C_c(\mathbb{R}) \) into \( C_c(\mathbb{R}) \).

For the converse, let \( \mathcal{K} : C_c(\mathbb{R}) \to C_c(\mathbb{R}) \) be continuous, linear and translation invariant. Then by Theorem 1.11 there is \( \bar{\mu} \in \text{bv}_{\text{loc}}(\mathbb{R}) \) such that

\[ (\mathcal{K}f)(0) = \int_{\mathbb{R}} f(\tau)d\bar{\mu}(\tau), \quad f \in C_c(\mathbb{R}), \]

and we conclude that

\[ (\mathcal{K}f)(t) = \int_{\mathbb{R}} f(t + \tau)d\bar{\mu}(\tau). \]
Since $K$ is continuous there is $m \in \mathbb{N}$ such that for all $f \in C_{c,1}(\mathbb{R})$,

$$Kf \in C_{c,m}(\mathbb{R})$$

and

$$\max_{t \in [-m,m]} |(Kf)(t)| \leq C \cdot \max_{t \in [-1,1]} |f(t)| .$$

So, for all $t \in \mathbb{R}$ with $|t| \geq m$ and all $f \in C_{+,1}(\mathbb{R})$

$$\int_{\mathbb{R}} f(t + \tau)d\mu(\tau) = 0 .$$

Since $\text{supp}(\sigma_f) \subseteq [-1-t,1-t]$, we conclude that

$$\mu(t) = \mu(m-1), \quad t > m - 1$$

$$\mu(t) = \mu(1-m), \quad t < 1 - m .$$

If we put

$$\mu(t) = \mu(t) - \mu(1-m), \quad t \in \mathbb{R} ,$$

then $\mu \in \text{bv}_c(\mathbb{R})$ and $K = \sigma[\mu]$.

Summarizing, the collection $\{\sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R})\}$ consists of precisely all continuous, linear, translation invariant operators on $C_c(\mathbb{R})$.

**Case 5: $C_{-,+}(\mathbb{R})$.**

Finally we consider $C_{-,+}(\mathbb{R})$. Also for this space the translation group $(\sigma_t)_{t \in \mathbb{R}}$ is a locally equicontinuous $c_0$-group. So, for each $\mu \in \text{bv}_c(\mathbb{R})$ the operator $\sigma[\mu]$ maps $C_{-,+}(\mathbb{R})$ into $C_{-,+}(\mathbb{R})$ continuously. We go a step further. For each $f \in C_{-,+}(\mathbb{R})$ the function $\tilde{f}$ on $\mathbb{R}$ defined by

$$\tilde{f}(t) := f(-t), \quad t \in \mathbb{R} ,$$

belongs to $C_{-}(\mathbb{R})$. Then, as we have seen, for all $\mu \in \text{bv}_+(\mathbb{R})$, the function $\sigma[\mu]\tilde{f}$ belongs to $C_{-}(\mathbb{R})$.

Let $f \in C_{-,+}(\mathbb{R})$, $f \neq 0$, with $f(t) = 0$ for $t \leq -n$, and $\mu \in \text{bv}_+(\mathbb{R})$ with $\mu(s) = 0$ for $s \leq -m$. Then

$$(\sigma[\mu]\tilde{f})(t) = \int_{\mathbb{R}} \tilde{f}(t + \tau)d\mu(\tau) = \begin{cases} 0 & t \geq n + m \\ \int_{n+m}^t \tilde{f}(t + \tau)d\mu(\tau) & t < n + m \end{cases} .$$
so that

\[ \text{supp}(\sigma[\mu]f) \subset (-\infty, n + m] \]

and

\[ \max_{t \leq (n+m)} |(\sigma[\mu]f)(t)| \leq \max_{t \geq -n} |f(t)| \cdot \var_F(\mu). \]

Let \( \varepsilon > 0 \) and take \( a > 0 \) so large that

\[ \var_{[a, \infty)}(\mu) < \varepsilon \cdot (\max_{t \geq -n} |f(t)|)^{-1}. \]

Then

\[ \left| \int_{-m}^{\infty} f(t + \tau)d\mu(\tau) \right| \leq \max_{\tau \in [-m,a]} |\hat{f}(t + \tau)| \cdot \var_F(\mu) + \varepsilon. \]

Since \( \lim_{t \to -\infty} \hat{f}(t) = 0 \) we conclude that

\[ \lim_{t \to -\infty} (\sigma[\mu]f)(t) = 0. \]

Thus we proved that \((\sigma[\mu]f)^\vee \in C_{-\infty,n+m}(\mathbb{R})\) with

\[ \max_{t \geq -(n+m)} |(\sigma[\mu]f)^\vee(t)| \leq \max_{t \geq -n} |f(t)| \cdot \var_F(\mu). \]

Consequently, for each \( \mu \in \text{bv}_+(\mathbb{R}) \) we can define properly the continuous, linear, translation invariant operator \( \tilde{\sigma}[\mu] \) from \( C_{-\infty,+}(\mathbb{R}) \) into \( C_{-\infty,+}(\mathbb{R}) \) by

\[ \tilde{\sigma}[\mu]f := (\sigma[\mu]f)^\vee, \quad f \in C_{-\infty,+}(\mathbb{R}). \]

We have

\[ (\tilde{\sigma}[\mu]f)(t) = \int_{\mathbb{R}} f(t - \tau)d\mu(\tau), \quad t \in \mathbb{R}. \]

Next, we shall prove that each continuous, translation invariant linear operator from \( C_{-\infty,+}(\mathbb{R}) \) into \( C_{-\infty,+}(\mathbb{R}) \) arises from an element of \( \text{bv}_+(\mathbb{R}) \) in the above described way. So, let \( \mathcal{L} \) be such an operator. Then the usual argument and Theorem 1.11 shows existence of \( \tilde{\mu} \in \text{bv}_{1\text{oc},+}(\mathbb{R}) \) such that for all \( t \in \mathbb{R} \) and all \( f \in C_{-\infty,+}(\mathbb{R}) \)
\[(\mathcal{L}f)(t) = \int_{\mathbb{R}} (\sigma_t f)(\tau) d\tilde{\mu}(\tau).\]

There exists \(m \in \mathbb{N}\) such that for all \(f \in C_{-\infty,0}(\mathbb{R})\), \(\mathcal{L}f \in C_{-\infty,m}(\mathbb{R})\) and

\[
\max_{t \geq -m} \left| (\mathcal{L}f)(t) \right| \leq C \max_{t \geq 0} \left| f(t) \right|.
\]

(*)

So, for all \(f \in C_{-\infty,0}(\mathbb{R})\) and all \(t \leq -m\)

\[
\int_{-t}^{\infty} f(t + \tau) d\tilde{\mu}(\tau) = 0.
\]

Consequently, we can assume that \(\tilde{\mu}(t) = 0\) for \(t \geq m\).

Define for \(t \in \mathbb{R}\),

\[\tilde{\mu}_t(\tau) := \tilde{\mu}(\tau - t), \quad \tau \in \mathbb{R}.\]

Then by (*) for all \(t > -m\)

\[
\left| \int_{0}^{m+t} f(\tau) d\tilde{\mu}_t(\tau) \right| = \left| \int_{-t}^{m} f(t + \tau) d\tilde{\mu}(\tau) \right| \leq C \cdot \max_{s \geq 0} \left| f(s) \right|,
\]

and so for all \(t > -m\)

\[\text{var}_{[0,m+t]}(\tilde{\mu}_t) \leq C.\]

We conclude that for all \(t > -m\)

\[\text{var}_{[-t,m]}(\tilde{\mu}) \leq C,
\]

i.e. \(\tilde{\mu} \in \text{bv}(\mathbb{R})\). Now, define \(\mu \in \text{bv}_{+}(\mathbb{R})\) by

\[\mu(t) := -\tilde{\mu}(-t), \quad t \in \mathbb{R},\]

then

\[(\mathcal{L}f)(t) = \int_{\mathbb{R}} f(t + \tau) d\tilde{\mu}(\tau) = \int_{\mathbb{R}} f(t - \tau) d\mu(\tau) = (\tilde{\sigma}[\mu]f)(t)
\]

for all \(f \in C_{-\infty,0}(\mathbb{R})\) and \(t \in \mathbb{R}\).

We conclude that the collection \(\{\tilde{\sigma}[\mu] \mid \mu \in \text{bv}_{+}(\mathbb{R})\}\) consists of precisely all continuous, translation invariant, linear operators from \(C_{-\infty,0}(\mathbb{R})\) into \(C_{-\infty,0}(\mathbb{R})\).

Summarizing, we obtained the following result.
Theorem 2.1.

I. The collection of all continuous, translation invariant, linear operators on \( C(\mathbb{R}) \), and \( C_c(\mathbb{R}) \) is precisely the collection

\[
\{ \sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}) \}
\]

respectively.

II. The collection of all continuous, translation invariant linear operators on \( C_+(\mathbb{R}) \) is precisely the collection

\[
\{ \sigma[\mu] \mid \mu \in \text{bv}_{\text{loc},-}(\mathbb{R}) \}.
\]

III. The collection of all continuous, translation invariant linear operators on \( C_{-,+}(\mathbb{R}) \) is precisely the collection

\[
\{ \sigma[\mu] \mid \mu \in \text{bv}_+(\mathbb{R}) \}.
\]

IV. The collection of all continuous translation-invariant linear operators on \( C_{-}(\mathbb{R}) \) is precisely the collection

\[
\{ \sigma[\mu] \mid \mu \in \text{bv}_+(\mathbb{R}) \}.
\]

3 Translation invariant operators on \( L^p \)-type spaces

For \( 1 \leq p < \infty \), \( L^p(\mathbb{R}) \) is the Banach space of (equivalence classes of) Lebesgue measurable functions \( f \) on \( \mathbb{R} \) for which \( |f|^p \) is integrable with associated norm

\[
\|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p}.
\]

The Banach space \( L^\infty(\mathbb{R}) \) consists (up to equivalence) of all essentially bounded Lebesgue measurable functions \( f \) on \( \mathbb{R} \) with associated norm

\[
\|f\|_\infty = \text{esssup}_{t \in \mathbb{R}} |f(t)|.
\]

For \( A \subset \mathbb{R} \), we denote by \( 1_A \) the characteristic function of the set \( A \). Consecutively we shall define the \( F \)-spaces and strict \( LF \)-spaces \( L^p_{\text{loc}}(\mathbb{R}) \), \( L^p_{\text{comp}}(\mathbb{R}) \), \( L^p_{-}(\mathbb{R}) \), \( L^p_{+}(\mathbb{R}) \) and \( L^p_{\text{loc},+}(\mathbb{R}) \), which have a similar behaviour with respect to the translation group \( (\sigma_t)_{t \in \mathbb{R}} \) as the spaces \( C(\mathbb{R}) \), \( C_c(\mathbb{R}) \), \( C_{-}(\mathbb{R}) \), \( C_{-}+(\mathbb{R}) \) and \( C_+(\mathbb{R}) \), respectively.

The space \( L^p_{\text{loc}}(\mathbb{R}) \), \( 1 \leq p \leq \infty \), consists of all Lebesgue measurable functions \( f \) on \( \mathbb{R} \) for which \( f \cdot 1_A \) belongs to \( L^p(\mathbb{R}) \) for all bounded Borel subsets \( A \) of \( \mathbb{R} \). The locally convex topology on the vector space \( L^p_{\text{loc}}(\mathbb{R}) \) is the one generated by the countable system of seminorms \( \{ s_n^p \}_{n \in \mathbb{N}} \),
Thus $L^p_{\text{loc}}(\mathbb{R})$ is a Fréchet space.

The space $L^p_{\text{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$, is the subspace of $L^p(\mathbb{R})$ consisting of all $f \in L^p(\mathbb{R})$ with bounded support $\text{supp}(f)$. Define

$$L^p_{\text{comp},n}(\mathbb{R}) := \{ f \in L^p(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, n] \} .$$

Then

$$L^p_{\text{comp}}(\mathbb{R}) = \bigcup_{n=1}^{\infty} L^p_{\text{comp},n}(\mathbb{R})$$

and $L^p_{\text{comp}}(\mathbb{R})$ is a strict $LB$-space induced by the strict inductive system of Banach spaces $\{L^p_{\text{comp},n}(\mathbb{R})\}_{n \in \mathbb{N}}$.

The space $L^p_{-\infty}(\mathbb{R})$, $1 \leq p \leq \infty$, is the subspace of $L^p_{\text{loc}}(\mathbb{R})$ consisting of all $f \in L^p_{\text{loc}}(\mathbb{R})$ such that $f \cdot 1_{[-n, \infty)} \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$. Introducing the seminorms

$$r^p_n(f) = \| f \cdot 1_{[-n, \infty)} \|_p , \quad f \in L^p_{-\infty}(\mathbb{R}) ,$$

$L^p_{-\infty}(\mathbb{R})$ becomes an $F$-space.

The space $L^p_{+\infty}(\mathbb{R})$ is the subspace of $L^p(\mathbb{R})$ consisting of all $f \in L^p(\mathbb{R})$ with $\text{supp}(f) \subseteq [-n, \infty)$. Introducing the closed subspaces $L^p_{+\infty}(\mathbb{R})$ of $L^p(\mathbb{R})$ by

$$L^p_{+,n}(\mathbb{R}) = \{ f \in L^p(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, \infty) \} ,$$

then $\{ L^p_{+,n}(\mathbb{R}) \mid n \in \mathbb{N} \}$ is a strict inductive system of $B$-spaces with corresponding strict $LB$-space

$$L^p_{-\infty}(\mathbb{R}) := \bigcup_{n=1}^{\infty} L^p_{-\infty,n}(\mathbb{R}).$$

Introduce, similarly, the closed subspace $L^p_{\text{loc},+.n}(\mathbb{R})$ of $L^p_{\text{loc}}(\mathbb{R})$ by

$$L^p_{\text{loc},+.n}(\mathbb{R}) = \{ f \in L^p_{\text{loc}}(\mathbb{R}) \mid \text{supp}(f) \subseteq [-n, \infty) \} .$$

Then $\{ L^p_{\text{loc},+.n}(\mathbb{R}) \mid n \in \mathbb{N} \}$ is a strict inductive system of $F$-spaces generating the strict $LF$-space

$$L^p_{\text{loc},+(\mathbb{R}) := \bigcup_{n=1}^{\infty} L^p_{\text{loc},+.n}(\mathbb{R}).$$
The next step is to examine the duality relations for the above introduced spaces.

For two Lebesgue measurable function \( f \) and \( g \) on \( \mathbb{R} \) for which \( f \cdot g \in L^1(\mathbb{R}) \) we introduce the notation

\[
\langle f, g \rangle = \int_{\mathbb{R}} f(t)g(t)dt.
\]

Let \( 1 \leq p < \infty \) and \( 1 < q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). We observe that for \( f \in L^q(\mathbb{R}) \), \( L_{\text{comp}}^q(\mathbb{R}) \), or \( L_{\text{loc}}^q(\mathbb{R}) \) and \( g \in L^p(\mathbb{R}) \), \( L_{\text{comp}}^p(\mathbb{R}) \) or \( L_{\text{loc}}^p(\mathbb{R}) \) the product \( f \cdot g \in L^1(\mathbb{R}) \).

The \( B \)-space \( L^q(\mathbb{R}) \) represents the dual of \( L^p(\mathbb{R}) \) in the sense that each continuous linear functional \( F \) on \( L^p(\mathbb{R}) \) is of the form

\[
F(g) = \langle f, g \rangle, \quad g \in L^p(\mathbb{R}),
\]

for some \( f \in L^q(\mathbb{R}) \) with \( ||F||_q = ||f||_q \).

This is a classical result, see [DS].

A linear functional \( F \) on \( L_{\text{loc}}^p(\mathbb{R}) \) is continuous iff there are \( n \in \mathbb{N} \) and \( C > 0 \) such that

\[
|F(g)| \leq C s^n(g).
\]

So, each \( f \in L_{\text{comp}}^q(\mathbb{R}) \) yields the continuous linear functional \( g \mapsto \langle f, g \rangle \) on \( L_{\text{loc}}^p(\mathbb{R}) \). And if \( F \) is continuous on \( L_{\text{loc}}^p(\mathbb{R}) \), then for sufficiently large \( n \)

\[
F(g) = F(g \cdot 1_{[-n,n]}), \quad g \in L_{\text{loc}}^p(\mathbb{R}).
\]

So, there exists \( f \in L_{\text{comp},n}^q(\mathbb{R}) \) such that

\[
F(g) = \langle f, g \rangle.
\]

We see that \( L_{\text{comp}}^q(\mathbb{R}) \) represent the dual of \( L_{\text{loc}}^p(\mathbb{R}) \).

A linear functional \( F \) on \( L_{\text{comp}}^p(\mathbb{R}) \) is continuous iff \( F|_{L_{\text{comp},n}^p(\mathbb{R})} \) is continuous for each \( n \in \mathbb{N} \).

So, for each \( f \in L_{\text{loc}}^q(\mathbb{R}) \), \( g \mapsto \langle f, g \rangle \) defines a continuous linear functional on \( L_{\text{comp}}^p(\mathbb{R}) \). And if \( F \) is continuous, then there is a sequence \( (f_n)_{n \in \mathbb{N}} \) with \( f_n \in L_{\text{comp},n}^q(\mathbb{R}) \) such that

\[
F(g) = \langle f_n, g \rangle, \quad g \in L_{\text{comp},n}^p(\mathbb{R}), \quad n \in \mathbb{N}.
\]

Hence, \( f_n = f_{n+1}|_{[-n,n]} \) and \( f \) on \( \mathbb{R} \) defined by

\[
f(t) = f_n(t), \quad t \in [-n,n]
\]
belongs to $L^q_{\text{loc}}(\mathbb{R})$ with

$$F(g) = \langle f, g \rangle, \quad g \in L^p_{\text{comp}}(\mathbb{R}).$$

A linear functional $F$ on $L^p_\infty(\mathbb{R})$ is continuous if and only if there are $n \in \mathbb{N}$ and $C > 0$ such that

$$|F(g)| \leq C \cdot r^n_0(g), \quad g \in L^p_\infty(\mathbb{R}).$$

So, for $f \in L^q_+(\mathbb{R})$, $g \mapsto \langle f, g \rangle$ is a continuous linear functional on $L^p_\infty(\mathbb{R})$. And if $F$ is continuous, $F|_{L^p_{\text{comp}}}(\mathbb{R})$ is continuous for $n$ sufficiently large with

$$F(g) = F|_{L^p_{\text{comp}}}(\mathbb{R})(g \cdot 1_{[-n,\infty)}) \quad g \in L^p_\infty(\mathbb{R});$$

so there exists $f \in L^q_{+,n}(\mathbb{R})$ such that $F(g) = \langle f, g \rangle$. We conclude that $L^q_{+,n}(\mathbb{R})$ represents the dual of $L^p_\infty(\mathbb{R})$.

A linear functional $F$ on $L^p_+(\mathbb{R})$ is continuous iff its restriction $F|_{L^p_{+,n}(\mathbb{R})}$ is continuous for each $n \in \mathbb{N}$. So, for each $f \in L^q_+(\mathbb{R})$ the linear functional $g \mapsto \langle f, g \rangle$ is continuous on $L^p_{+,n}(\mathbb{R})$. Reasoning as for $L^p_{\text{comp}}(\mathbb{R})$ we come to the conclusion that $L^p_{+,n}(\mathbb{R})$ represents the dual of $L^p_+(\mathbb{R})$.

A linear functional $F$ on $L^p_{\text{loc},+}(\mathbb{R})$ is continuous iff for each $n \in \mathbb{N}$ the restriction $F|_{L^p_{\text{loc},+,n}(\mathbb{R})}$ is continuous. So, each $f \in L^q_{\text{loc},+}(\mathbb{R})$ yields a continuous linear functional $g \mapsto \langle f, g \rangle$ on $L^p_{\text{loc},+}(\mathbb{R})$, where $\tilde{f}(t) = f(-t)$, $t \in \mathbb{R}$.

Now, suppose the linear functional $F$ on $L^p_{\text{loc},+}(\mathbb{R})$ is continuous. Let $F_n$ denote the restriction of $F$ to $L^p_{\text{loc},+,n}(\mathbb{R})$. Then $F_n$ extends to a continuous linear functional on $L^p_{\text{loc}}(\mathbb{R})$. So $f_n \in L^p_{\text{comp}}(\mathbb{R})$ exists such that

$$F_n(g) = \langle f_n, g \rangle = \int_{-n}^{\infty} f_n(t)g(t)dt$$

for all $g \in L^p_{\text{loc},+,n}(\mathbb{R})$. Since $F_{n+1}$ extends $F_n$, it follows that

$$f_n = f_{n+1}|_{[-n,\infty)}.$$

Define $f$ on $\mathbb{R}$ by

$$f(t) = f_n(t), \quad t \geq -n, \ n \in \mathbb{N}.$$

Then $f \in L^q_{\text{loc},+}(\mathbb{R})$ and $F(g) = \langle f, g \rangle$, $g \in L^p_{\text{loc},+}(\mathbb{R})$.

We conclude that $L^q_{\text{loc},+}(\mathbb{R})$ represents the dual of $L^p_{\text{loc},+}(\mathbb{R})$ as described.

Now, for $V$ one of the spaces $L^p_{\text{loc}}(\mathbb{R})$, $L^p_{\text{comp}}(\mathbb{R})$, $L^p_\infty(\mathbb{R})$, $L^p_+(\mathbb{R})$ and $L^p_{\text{loc},+}(\mathbb{R})$, where $p \in [1, \infty)$, the translation group $(\sigma_t)_{t \in \mathbb{R}}$, defined formally by
is a locally equicontinuous \(c_0\)-group on \(V\). (We note that it is sufficient to prove that it is a \(c_0\)-group, since each \(c_0\)-group on a Fréchet space or on a strict LF-space is locally equicontinuous! Cf. [ER] and [Ko].) Therefore, the associated trace operator \(T_\sigma\) from \(V\) into \(C(\mathbb{R}, V)\),

\[(T_\sigma g)(t) = \sigma_t g, \quad t \in \mathbb{R},\]

is continuous. So, for each \(\mu \in \text{bv}_c(\mathbb{R})\) the operator

\[\sigma[\mu] := J_\mu T_\sigma\]

is continuous on \(V\) and satisfies \(\sigma_t \sigma[\mu] = \sigma[\mu] \sigma_t\), i.e. \(\sigma[\mu]\) is a continuous translation invariant operator on \(V\). We recall that for each \(g \in V\), \(\sigma[\mu]g\) is defined by the \(V\)-valued Riemann–Stieltjes integral

\[\sigma[\mu]g = \int \sigma_t g \ d\mu(\tau).\]

This way the convolution algebra \(\text{bv}_c(\mathbb{R})\) is associated to each of the spaces \(L^p_{\text{loc}}(\mathbb{R})\) and \(L^p_{\text{comp}}(\mathbb{R})\), \(p \in [1, \infty)\).

Next, we show that to the spaces \(L^p_{\text{loc}}(\mathbb{R})\) and \(L^p_{\text{comp}}(\mathbb{R})\), \(p \in [1, \infty)\), the larger convolution algebra \(\text{bv}_+(\mathbb{R})\) can be associated. Indeed, for \(g \in L^p_{\text{loc}}(\mathbb{R})\) and each \(n \in \mathbb{N}\) we have

\[\lim_{t \to \infty} \int_{-n}^{\infty} |(\sigma_t f)(\tau)|^p d\tau = 0.\]

It follows that \(T_\sigma\) maps \(L^p_{\text{loc}}(\mathbb{R})\) into \(C_{-}(\mathbb{R}, L^p_{\text{loc}}(\mathbb{R}))\) continuously. Therefore, for all \(\mu \in \text{bv}_+(\mathbb{R})\) the linear operator \(J_\mu\) from \(C_{-}(\mathbb{R}, L^p_{\text{loc}}(\mathbb{R}))\) into \(L^p_{\text{loc}}(\mathbb{R})\) defined by the (improper Riemann–Stieltjes) \(L^p_{\text{loc}}(\mathbb{R})\)-valued integral,

\[J_\mu f = \int_{\mathbb{R}} f(\tau) d\mu(\tau),\]

is continuous, and so

\[\sigma[\mu] := J_\mu T_\sigma\]

is a continuous translation invariant operator on \(L^p_{\text{loc}}(\mathbb{R})\).

If \(1 < p \leq \infty\), then \(L^q(\mathbb{R})\) represents the dual of \(L^p(\mathbb{R})\) where \(1 \leq q \leq \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\). For each \(\mu \in \text{bv}_+(\mathbb{R})\) the dual mapping \(\sigma[\mu]'\) of \(\sigma[\mu]\) : \(L^q(\mathbb{R}) \to L^q(\mathbb{R})\) is a continuous linear translation invariant operator on \(L^q(\mathbb{R})\). There is the relation
\[ \langle \sigma[\mu] f, g \rangle = \int_{\mathbb{R}} \langle \sigma_{-\tau} f, g \rangle d\mu(\tau) = \int_{\mathbb{R}} \langle \sigma_{\tau} f, g \rangle d\mu(\tau), \]

where \( f \in L^p_+(\mathbb{R}) \), \( g \in L^\infty_+(\mathbb{R}) \) and \( \mu \in \text{bv}_+(\mathbb{R}) \). We see that on each of the spaces \( L^p_+(\mathbb{R}) \) with \( 1 < p \leq \infty \) the operator \( \tilde{\sigma}[\mu] \), \( \mu \in \text{bv}_+(\mathbb{R}) \) can be defined weakly.

We shall prove that for \( 1 \leq p < \infty \), the operator \( \tilde{\sigma}[\mu] \) can be defined strongly.

Let \( f \in L^p_+(\mathbb{R}) \) and let \( \mu \in \text{bv}_+(\mathbb{R}) \) with \( \mu(t) = 0 \) for \( t \leq -m \), where \( n, m \in \mathbb{N} \). For all \( t \geq -m \) we have \( \sigma_{-t} f \in L^p_{+,n+m}(\mathbb{R}) \) and

\[ \|\sigma_{-t} f\|_p = \|f\|_p. \]

For each \( A \geq -m \) the integral expression

\[ \int_{-m}^A \sigma_{-\tau} f d\mu(\tau) \]

defines an element of \( L^p_+(\mathbb{R}) \), where the integral converges in \( L^p(\mathbb{R}) \). Further, for \( A_2 > A_1 > -m \)

\[ \| \int_{A_1}^{A_2} \sigma_{-\tau} f d\mu(\tau) \|_p \leq \int_{A_1}^{A_2} \|\sigma_{-\tau} f\|_p |d\mu(\tau)| \leq \|f\|_p \cdot \text{var}_{[A_1, A_2]}(\mu). \]

Since \( L^p_{+,n+m}(\mathbb{R}) \) is a Banach space, the limit

\[ \int_{-m}^{\infty} \sigma_{-\tau} f d\mu(\tau) = \lim_{A \to \infty} \int_{-m}^{A} \sigma_{-\tau} f d\mu(\tau) \]

exists in \( L^p_{+,n+m}(\mathbb{R}) \) (observe that \( \lim_{A \to \infty} \text{var}_{[A, \infty)}(\mu) = 0 \)) and

\[ \| \int_{-m}^{\infty} (\sigma_{-\tau} f) d\mu(\tau) \|_p \leq \|f\|_p \cdot \text{var}_{[-m, \infty)}(\mu). \]

Thus we can conclude that for each \( \mu \in \text{bv}_+(\mathbb{R}) \) the linear operator \( \tilde{\sigma}[\mu] \) on \( L^p_+(\mathbb{R}) \), \( 1 \leq p < \infty \), defined by the (improper Riemann–Stieltjes) integral

\[ \tilde{\sigma}[\mu] f = \int_{\mathbb{R}} \sigma_{-\tau} f d\mu(\tau) \]
is continuous and translation invariant. In particular, $\sigma[\mu]$ maps $L^p_{+,n}(\mathbb{R})$ into $L^p_{+,n+m}(\mathbb{R})$ continuously, where $m \in \mathbb{N}$ satisfies $\mu(t) = 0$, $t < -m$.

Finally, we discuss translation invariant operators on the spaces $L^p_{\text{loc,+}}(\mathbb{R})$, $1 \leq p < \infty$.

For $g \in L^p_{\text{loc,+},n}(\mathbb{R})$ and $t \leq k$, $\sigma_t g \in L^p_{\text{loc,+},n+k}(\mathbb{R})$ with for each $m \in \mathbb{N}$

$$\int_{-n-k}^{m} |(\sigma_t g)(s)|^p ds \leq \int_{-n}^{m+k} |g(s)|^p ds.$$  (***)

So, for $\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})$ with $\mu(t) = \mu(k)$, $t \geq k$, the inequality (**) indicates that for every $A \leq k$ the proper $L^p$-valued Riemann–Stieltjes integral

$$\int_{A}^{k} \sigma_t g d\mu(\tau)$$

converges and defines an element of $L^p_{\text{loc,+},n+k}(\mathbb{R})$. Since the left hand side of (**) vanishes for $t \leq -m - n$, the limit

$$\int_{-\infty}^{k} \sigma_t g d\mu(\tau) = \lim_{A \to -\infty} \int_{A}^{k} \sigma_t g d\mu(\tau)$$

exists in $L^p_{\text{loc,+},n+k}(\mathbb{R})$ and satisfies for each $m \in \mathbb{N}$

$$\left( \int_{-n-k}^{m} \left( \int_{-\infty}^{k} \sigma_t g d\mu(\tau) \right)(s)^p ds \right)^{1/p} \leq \left( \int_{-n}^{m+k} |g(t)|^p dt \right)^{1/p} \text{var}_{[-m-n,k]}(\mu).$$

So, for each $\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})$ the operator $\sigma[\mu]$,

$$\sigma[\mu] g = \int_{\mathbb{R}} \sigma_t g d\mu(\tau)$$

is a continuous linear translation invariant operator on $L^p_{\text{loc,+}}(\mathbb{R})$. In particular, $\sigma[\mu]$ maps $L^p_{\text{loc,+},n}(\mathbb{R})$ into $L^p_{\text{loc,+},n+k}(\mathbb{R})$, where $k \in \mathbb{N}$ is such that $\mu(t) = \mu(k)$ for all $t \geq k$.

Summarizing, we obtained the following results

**Theorem 3.1.**

I. For $p \in [1,\infty)$, $L^p_{\text{loc}}(\mathbb{R})$ and $L^p_{\text{comp}}(\mathbb{R})$ are modules over the commutative convolution algebra $\text{bv}_{c}(\mathbb{R})$, where in both cases multiplication is defined by

$$\mu \circ g := \sigma[\mu] g = \int_{\mathbb{R}} \sigma_t g d\mu(t),$$

24
for \( g \in L^p_{\text{loc}}(\mathbb{R}) \) or \( g \in L^p_{\text{comp}}(\mathbb{R}) \) and \( \mu \in \text{bv}_c(\mathbb{R}) \).

II. For \( p \in [1, \infty) \), \( L^p_{\text{loc}}(\mathbb{R}) \) and \( L^p_{\text{loc}}(\mathbb{R}) \) are modules over the commutative convolution algebra \( \text{bv}_+(\mathbb{R}) \), where multiplication is defined by

\[
\mu \circ g := \sigma[\mu]g = \int \sigma g \, d\mu(t), \quad g \in L^p_{\text{loc}}(\mathbb{R}),
\]

and

\[
\mu \circ g := \sigma[\mu]g = \int \sigma g \, d\mu(t), \quad g \in L^p_{\text{loc}}(\mathbb{R}).
\]

III. For \( p \in [1, \infty) \), \( L^p_{\text{loc, c}}(\mathbb{R}) \) is a module over the commutative convolution algebra \( \text{bv}_{\text{loc,-}}(\mathbb{R}) \), where multiplication is defined by

\[
\mu \circ g := \sigma[\mu]g = \int \sigma g \, d\mu(t).
\]

The translation group \((\sigma_t)_{t \in \mathbb{R}}\) is not \( \sigma_0 \)-group on either of the spaces \( L^\infty_{\text{loc}}(\mathbb{R}) \), \( L^\infty_{\text{comp}}(\mathbb{R}) \), \( L^\infty(\mathbb{R}) \), \( L^\infty_{\text{loc}}(\mathbb{R}) \) and \( L^\infty_{\text{loc, c}}(\mathbb{R}) \). In this connection and for later use we mention the following results.

**Proposition 3.2.**

I. Let \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \) with trace \( T_\sigma f : t \mapsto \sigma_t f \). Then \( T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \) if and only if \( f \in C(\mathbb{R}) \).

**Proof.** Since \( C(\mathbb{R}) \subset L^\infty_{\text{loc}}(\mathbb{R}) \), sufficiency is evident. Now, suppose \( T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \). For \((\psi_k)\) an approximate identity in \( \text{bv}_c^\infty(\mathbb{R}) \), see Section 1, p. 7,

\[
J_{\psi_k}T_\sigma f \to (T_\sigma f)(0) = f \quad \text{in} \ L^\infty_{\text{loc}}(\mathbb{R})
\]

as \( k \to \infty \). Since

\[
J_{\psi_k}T_\sigma f = \int J_{\psi_k}^{-1}(\sigma f) \, d\tau,
\]

we see that

\[
J_{\psi_k}T_\sigma f \in C^\infty(\mathbb{R}), \quad k \in \mathbb{N}
\]

and so \( f \in C(\mathbb{R}) \).

**Proposition 3.2.** (continued)
II. Let \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{comp}}(\mathbb{R})) \) if and only if \( f \in C_e(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty_{\text{comp}}(\mathbb{R}) \).

III. Let \( f \in L^\infty_+(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R})) \) if and only if \( f \in C_+(\mathbb{R}) \).

IV. Let \( f \in L^\infty_+(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R})) \) if and only if \( f \in C_+(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty_+(\mathbb{R}) \).

V. Let \( f \in L^\infty_{\text{loc},+}(\mathbb{R}) \). Then \( T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{loc},+}(\mathbb{R})) \) if and only if \( f \in C_+(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty_{\text{loc},+}(\mathbb{R}) \).

**Proof.** Let \( V \) be one of the spaces \( L^\infty_{\text{comp}}(\mathbb{R}), L^\infty_+(\mathbb{R}), L^\infty_-(\mathbb{R}) \) and \( L^\infty_{\text{loc},+}(\mathbb{R}) \). Then \( V \hookrightarrow L^\infty_{\text{loc}}(\mathbb{R}) \) and so for \( f \in V \), \( T_\sigma f \in C(\mathbb{R}, V) \) if and only if \( f \in C(\mathbb{R}) \cap V \). For II. in addition

\[
\lim_{t \to \infty} \left( \max_{r \geq -n} |f(t+r)| \right) = 0 ,
\]

if and only if \( f \in C_-(\mathbb{R}) \). \( \square \)

## 4 Complete characterizations of the translation invariant operators on \( L_1 \)-type spaces

In this section we deduce a complete characterization of the linear, continuous translation invariant operators for four spaces of \( L_1 \)-type. The first two characterizations can be found in [Eij] also (Theorem 9 and Theorem 12). We have included them here for sake of completeness.

**Case 1** \( L^1_{\text{comp}}(\mathbb{R}) \) and \( L^\infty_{\text{loc}}(\mathbb{R}) \).

We recall that \( L^\infty_{\text{loc}}(\mathbb{R}) \) represents the dual of \( L^1_{\text{comp}}(\mathbb{R}) \). For each \( \mu \in \text{bv}_e(\mathbb{R}) \) the linear operator \( \sigma[\mu] : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) is defined as the dual of \( \hat{\sigma}[\mu] : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R}) \),

\[
\hat{\sigma}[\mu] g = \int_{\mathbb{R}} \sigma_{-r} g \, d\mu(r) ,
\]

as introduced in the previous section. It means that for all \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \) and \( g \in L^1_{\text{comp}}(\mathbb{R}) \)

\[
\langle \hat{\sigma}[\mu] g, f \rangle = \langle g, \sigma[\mu] f \rangle .
\]

The Closed Graph Theorem for \( F \)-spaces guarantees that \( \sigma[\mu] \) is continuous on \( L^\infty_{\text{loc}}(\mathbb{R}) \). Also, \( \sigma[\mu] \) is translation invariant.

Now, let \( K : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be a continuous translation invariant operator. Let, as in Proposition 1.3, \( L_K \) be defined by

\[
(L_K u)(t) := K(u(t)) , \quad t \in \mathbb{R}, u \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) .
\]
For \( f \in C(\mathbb{R}) \) we have \( T_K f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \) and so \( T_K f = \mathcal{L}_K T_K f \in C(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \). By Proposition 3.2, we conclude that \( K f \in C(\mathbb{R}) \). So \( C(\mathbb{R}) \) is an invariant subspace of \( K \) and \( K|_{C(\mathbb{R})} : C(\mathbb{R}) \to C(\mathbb{R}) \) is continuous. Then, by Theorem 2.1, there is \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( K f = \sigma[\mu] f \) for all \( f \in C(\mathbb{R}) \). Further for all \( g \in L^1_{\text{comp}}(\mathbb{R}) \) and \( f \in C(\mathbb{R}) \)

\[
(g, K f) = \langle \sigma[\mu] g, f \rangle
\]
due to the \( L_1 \)-convergence of the \( L^1_{\text{comp}} \)-valued integral

\[
\int_{\mathbb{R}} \sigma_{-r} g \, d\mu(r).
\]

We arrive at the following Theorem.

**Theorem 4.1.** Let \( K : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there exists \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( K|_{C(\mathbb{R})} = \sigma[\mu] \). Let \( K^* : L^\infty_{\text{loc}}(\mathbb{R})^* \to L^\infty_{\text{loc}}(\mathbb{R})^* \) denote the dual of \( K \). If \( K^*|_{L^1_{\text{comp}}(\mathbb{R})} \) maps \( L^1_{\text{comp}}(\mathbb{R}) \) into \( L^1_{\text{comp}}(\mathbb{R}) \), then \( K = \sigma[\mu] \) and \( K^*|_{L^2_{\text{comp}}(\mathbb{R})} = \hat{\sigma}[\mu] \).

**Proof.** The first part of the Theorem has been established above. For the second part observe that \( K = (K^*)_\ast \) where \( K_\ast = K^*|_{L^1_{\text{comp}}(\mathbb{R})} \) identifying \( L^1_{\text{comp}}(\mathbb{R}) \) with a subspace of \( L^\infty_{\text{loc}}(\mathbb{R})^* \), whence for all \( f \in C(\mathbb{R}) \) and \( g \in L^1_{\text{comp}}(\mathbb{R}) \)

\[
(g, K f) = \langle \sigma[\mu] g, f \rangle = \langle \sigma[\mu] g, f \rangle
\]

\( \square \)

**Theorem 4.2.** Let \( S : L^1_{\text{comp}}(\mathbb{R}) \to L^1_{\text{comp}}(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there exists \( \mu \in \text{bv}_c(\mathbb{R}) \) such that \( S = \sigma[\mu] \). So, the collection \( \{ \sigma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}) \} \) consists of precisely all continuous, translation invariant, linear operators on \( L^1_{\text{comp}}(\mathbb{R}) \).

**Proof.** Apply the preceding Theorem to the dual operator \( S^\ast \).

**Remark.** Sometimes there is a priori knowledge of \( K \).

Let \( \mathcal{D}'(\mathbb{R}) \) denote the space of Schwartz distributions corresponding to the test space \( \mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) \). Then \( L^\infty_{\text{loc}}(\mathbb{R}) \) can be considered as a subspace of \( \mathcal{D}'(\mathbb{R}) \) identifying \( f \in L^\infty_{\text{loc}}(\mathbb{R}) \) and the distribution

\[
\varphi \mapsto \langle f, \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}) .
\]

Let \( \mathcal{E}'(\mathbb{R}) \) denote the subspace of \( \mathcal{D}'(\mathbb{R}) \) consisting of all \( \Phi \in \mathcal{D}'(\mathbb{R}) \) with compact support. According to [So] each functional in \( \mathcal{E}'(\mathbb{R}) \) can be written as

\[
\Phi(\varphi) = \int_{\mathbb{R}} (p \frac{d}{dt}) \varphi(\tau) \, d\mu(\tau)
\]

for some \( \mu \in \text{bv}_c(\mathbb{R}) \) and some polynomial \( p \). Now, for \( \Psi \in \mathcal{D}'(\mathbb{R}) \) and \( \Phi \in \mathcal{E}'(\mathbb{R}) \) we define \( \Phi \ast \Psi \) by

\[\text{27}\]
\[(\Phi \ast \Psi)(\varphi) := \Psi(p(\frac{d}{dt})\sigma[\mu]\varphi)\]

with \(\Phi\) of the form (\(\ast\)). Then \(\Phi \ast \Psi \in \mathcal{D}'(\mathbb{R})\) and the linear mapping \(\Phi \mapsto \Phi \ast \Psi\) is translation invariant on \(\mathcal{D}'(\mathbb{R})\). If we know that \(\mathcal{K} = \Phi \ast f, f \in L^\infty_{\text{loc}}(\mathbb{R})\) and that \(\mathcal{K}\) maps \(L^\infty_{\text{loc}}(\mathbb{R})\) into \(L^\infty_{\text{loc}}(\mathbb{R})\), then \(\mathcal{K}\) is continuous, linear and translation invariant due to the Closed Graph Theorem. So, there is \(\mu_1 \in \text{bv}_c(\mathbb{R})\) such that for all \(f \in C(\mathbb{R})\) and \(\varphi \in \mathcal{D}(\mathbb{R})\)

\[\langle \varphi, \mathcal{K} f \rangle = \langle p(\frac{d}{dt})\sigma[\mu] \varphi, f \rangle\]

and

\[\langle \varphi, \mathcal{K} f \rangle = \langle \varphi, \sigma[\mu_1] f \rangle = \langle \sigma[\mu_1] \varphi, f \rangle.\]

Consequently,

\[\sigma[\mu_1] = p(\frac{d}{dt})\sigma[\mu] \text{ and } \mathcal{K} = \sigma[\mu_1].\]

We are left with the following question.

Does there exist a continuous translation invariant operator \(\mathcal{K} \neq 0\) from \(L^\infty_{\text{loc}}(\mathbb{R})\) into \(L^\infty_{\text{loc}}(\mathbb{R})\) such that \(\mathcal{K} f = 0\) for all \(f \in C(\mathbb{R})\)?

**Case II** \(L^1_{\text{loc}}(\mathbb{R})\) and \(L^\infty_{\text{comp}}(\mathbb{R})\).

Also in this case we define \(\sigma[\mu]\) for \(\mu \in \text{bv}_c(\mathbb{R})\) as a translation invariant operator by duality, i.e. for each \(g \in L^1_{\text{loc}}(\mathbb{R})\) and \(f \in L^\infty_{\text{comp}}(\mathbb{R})\)

\[\langle \sigma[\mu] g, f \rangle = \langle g, \sigma[\mu] f \rangle\]

Then \(\sigma[\mu]\) is continuous due to the Closed Graph Theorem for strict \(LB\)-spaces, and, clearly, \(\sigma[\mu]\) is translation invariant.

Now, let \(\mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R})\) be linear, translation invariant and continuous. Since for \(f \in C_c(\mathbb{R})\),

\[T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{comp}}(\mathbb{R}))\]

it follows that

\[T_\sigma \mathcal{K} f = \mathcal{L}_\mathcal{K} T_\sigma f \in C(\mathbb{R}, L^\infty_{\text{comp}}(\mathbb{R}))\]

and, by Proposition 3.2, \(\mathcal{K} f \in C_c(\mathbb{R})\). So, as in Case I, we derive that \(\mathcal{K}\) maps \(C_c(\mathbb{R})\) into \(C_c(\mathbb{R})\) continuously and consequently there is \(\mu \in \text{bv}_c(\mathbb{R})\), according to Theorem 2.1, such that
For all \( f \in C_c(\mathbb{R}) \) and \( g \in L^1_{\text{comp}}(\mathbb{R}) \) we thus have
\[
\langle g, Kf \rangle = \langle \hat{\sigma}[\mu]f, g \rangle.
\]

**Theorem 4.3.** Let \( \mathcal{K} : L^\infty_{\text{comp}}(\mathbb{R}) \to L^\infty_{\text{comp}}(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in bvc_{c}(\mathbb{R}) \) such that \( \mathcal{K}|_{C_c(\mathbb{R})} = \sigma[\mu] \). Let \( \mathcal{K}^* : L^\infty_{\text{comp}}(\mathbb{R})^* \to L^\infty_{\text{comp}}(\mathbb{R})^* \) denote the dual operator of \( \mathcal{K} \). If \( \mathcal{K}^*|_{L^1_{\text{loc}}(\mathbb{R})} \) maps \( L^1_{\text{loc}}(\mathbb{R}) \) into \( L^1_{\text{loc}}(\mathbb{R}) \), then \( \mathcal{K} = \sigma[\mu] \) and \( \mathcal{K}^*|_{L^1_{\text{loc}}(\mathbb{R})} = \hat{\sigma}[\mu] \).

**Theorem 4.4.** Let \( S : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}) \) be a continuous translation invariant linear operator. Then there is \( \mu \in bvc_{c}(\mathbb{R}) \) such that \( S = \sigma[\mu] \). The collection
\[
\{ \sigma[\mu] \mid \mu \in bvc_{c}(\mathbb{R}) \}
\]
consists of precisely all continuous, translation invariant, linear operators from \( L^1_{\text{loc}}(\mathbb{R}) \) into \( L^1_{\text{loc}}(\mathbb{R}) \).

The distribution space \( \mathcal{E}'(\mathbb{R}) \) is a commutative convolution ring (without zero divisors): indeed for
\[
\Phi_1(\psi) = \int_\mathbb{R} (p_1(\frac{d}{dt})\psi)(\tau)d\mu(\tau), \quad j = 1, 2, ,
\]
\[
\Phi_1 \ast \Phi_2 = \Phi_1 \circ p_2(\frac{d}{dt})\sigma[\mu_2] = \Phi_2 \circ p_1(\frac{d}{dt})\sigma[\mu_1].
\]
Identifying \( L^\infty_{\text{comp}}(\mathbb{R}) \) as a subspace of \( \mathcal{E}'(\mathbb{R}) \) in the usual way, it can be checked readily, as in Case I, that if \( \mathcal{K}f = \Phi \ast f \in L^\infty_{\text{comp}}(\mathbb{R}) \) for some \( \Phi \in \mathcal{E}'(\mathbb{R}) \) and for all \( f \in L^\infty_{\text{comp}}(\mathbb{R}) \), then \( \mathcal{K} \) is continuous and \( \mathcal{K} = \sigma[\mu] \) for some \( \mu \in bvc_{c}(\mathbb{R}) \).

**Case III** \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^\infty_{\text{loc}}(\mathbb{R}) \).

In the previous section we defined \( \hat{\sigma}[\mu], \mu \in bvc_{c}(\mathbb{R}) \), on \( L^1_{\text{loc}}(\mathbb{R}) \) by
\[
\hat{\sigma}[\mu]g = \int_\mathbb{R} \sigma_{-\tau}g d\mu(\tau), \quad g \in L^1_{\text{loc}}(\mathbb{R}).
\]

We introduce the linear operators \( \sigma[\mu], \mu \in bvc_{c}(\mathbb{R}) \), on \( L^\infty_{\text{loc}}(\mathbb{R}) \) by using the duality of \( L^1_{\text{loc}}(\mathbb{R}) \) and \( L^\infty_{\text{loc}}(\mathbb{R}) \):
\[
\forall f \in L^\infty_{\text{loc}}(\mathbb{R}) \forall g \in L^1_{\text{loc}}(\mathbb{R}) : \langle \hat{\sigma}[\mu]g, f \rangle = \langle g, \sigma[\mu]f \rangle.
\]
Then \( \sigma[\mu] : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) is continuous and translation invariant.

Let \( \mathcal{K} : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}) \) be continuous, translation invariant and linear. By Proposition 3.2, for \( f \in C_-(\mathbb{R}), T_0f \in C_-(\mathbb{R}, L^\infty_{\text{loc}}(\mathbb{R})) \) and consequently
\[ T_\sigma K f = L \sigma T_\sigma f \in C_{-+}(\mathbb{R}, L^\infty_+(\mathbb{R})) . \]

Then, the same Proposition yields \( K f \in C_{-+}(\mathbb{R}) \). According to Theorem 2.1, there exists \( \mu \in \text{bv}_+(\mathbb{R}) \) such that

\[ K f = \sigma[\mu] f , \]

and

\[ \langle g, K f \rangle = \langle \delta[\mu] g, f \rangle \]

for all \( g \in L^1_+(\mathbb{R}) \) and \( f \in C_{-+}(\mathbb{R}) \).

**Theorem 4.5.** Let \( K : L^\infty_+(\mathbb{R}) \to L^\infty_+(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in \text{bv}_+(\mathbb{R}) \) such that \( K[\mu] = \sigma[\mu] \). If \( K^* \mid_{L^1_+(\mathbb{R})} \) maps \( L^1_+(\mathbb{R}) \) into \( L^1_+(\mathbb{R}) \), then \( K = \sigma[\mu] \) and \( K^* \mid_{L^1_+(\mathbb{R})} = \delta[\mu] \) for some \( \mu \in \text{bv}_+(\mathbb{R}) \).

**Theorem 4.6.** Let \( S : L^1_+(\mathbb{R}) \to L^1_+(\mathbb{R}) \) be a continuous, translation invariant, linear operator. Then there is \( \mu \in \text{bv}_+(\mathbb{R}) \) such that \( S = \delta[\mu] \). The collection

\[ \{ \delta[\mu] \mid \mu \in \text{bv}_+(\mathbb{R}) \} \]

consists of precisely all continuous, translation, invariant, linear operators from \( L^1_+(\mathbb{R}) \) into \( L^1_+(\mathbb{R}) \).

**Case IV** \( L^\infty_{-+}(\mathbb{R}) \) and \( L^\infty_+(\mathbb{R}) \).

For \( \mu \in \text{bv}_c(\mathbb{R}) \), in the previous section, we defined the linear operator \( \sigma[\mu] \) on \( L^\infty_{-+}(\mathbb{R}) \) by the \( L^\infty_{-+}(\mathbb{R}) \)-valued (improper) Riemann-Stieltjes integral

\[ \sigma[\mu]g = \int_{\mathbb{R}} \sigma_+ g \ d\mu(\tau) , \quad g \in L^1_{-+}(\mathbb{R}) . \]

Since \( \sigma[\mu] \) is continuous we can define \( \delta[\mu] \) on \( L^\infty_+(\mathbb{R}) \) by duality:

\[ \forall f \in L^\infty_+(\mathbb{R}) \forall g \in L^\infty_{-+}(\mathbb{R}) : \langle \sigma[\mu] g, f \rangle = \langle g, \delta[\mu] f \rangle . \]

Then \( \delta[\mu] \) is translation invariant and, by the Closed Graph Theorem for strict LB-spaces, continuous on \( L^\infty_+(\mathbb{R}) \).

Now, let \( K : L^\infty_{-+}(\mathbb{R}) \to L^\infty_+(\mathbb{R}) \) be continuous and translation invariant. For \( f \in C_{-+}(\mathbb{R}) \subseteq C_+ (\mathbb{R}), T - \sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R})) \) and

\[ T_\sigma K f = L \sigma T_\sigma f \in C(\mathbb{R}, L^\infty_+(\mathbb{R})) . \]
So by Proposition 3.2, $Kf \in C(\mathbb{R}) \cap L_+^\infty(\mathbb{R})$. Besides $K\sigma_t f = \sigma_t Kf$, the continuity of $K$ yields

$$\lim_{t \to \infty} \sigma_t Kf = 0 \quad \text{in} \quad L_+^\infty(\mathbb{R}),$$

so that $Kf \in C_{-+}(\mathbb{R})$. We see that $K|_{C_{-+}(\mathbb{R})} : C_{-+}(\mathbb{R}) \to C_{-+}(\mathbb{R})$ is continuous and translation invariant. So, by Theorem 2.1 we conclude existence of $\mu \in \text{bv}_+(\mathbb{R})$ such that

$$K|_{C_{-+}(\mathbb{R})} = \delta[\mu].$$

Further, for all $f \in C_{-+}(\mathbb{R})$ and $g \in L_+^1(\mathbb{R})$

$$(g, Kf) = (\sigma[\mu]g, f).$$

As in the cases I, II and III we arrive at the following characterizations.

**Theorem 4.7.** Let $K : L_+^\infty(\mathbb{R}) \to L_+^\infty(\mathbb{R})$ be a continuous, translation invariant linear operator. Then there is $\mu \in \text{bv}_+(\mathbb{R})$ such that $K|_{C_{-+}(\mathbb{R})} = \delta[\mu]$.

If $K^*|_{L_+^1(\mathbb{R})}$ maps $L_+^1(\mathbb{R})$ into $L_+^1(\mathbb{R})$, then $K = \delta[\mu]$ and $K^*|_{L_+^1(\mathbb{R})} = \sigma[\mu]$. □

**Theorem 4.8.** Let $S : L_+^1(\mathbb{R}) \to L_+^1(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then there is $\mu \in \text{bv}_c(\mathbb{R})$ such that $S = \sigma[\mu]$.

The collection

$$\{\sigma[\mu] \mid \mu \in \text{bv}_+(\mathbb{R})\}$$

consists of precisely all continuous, translation invariant linear operators on $L_+^1(\mathbb{R})$. □

**Case V** $L_+^1_{\text{loc},+}(\mathbb{R})$ and $L_+^\infty_{\text{loc},+}(\mathbb{R})$.

For $\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})$ we defined the linear operator $\sigma[\mu]$ on $L_+^1_{\text{loc},+}(\mathbb{R})$ by

$$\sigma[\mu]g = \int_{\mathbb{R}} \sigma g \mu(t), \quad g \in L_+^1_{\text{loc},+}(\mathbb{R}),$$

as an $L_+^1_{\text{loc},+}(\mathbb{R})$-valued (improper) Riemann–Stieltjes integral. Thus defined, $\sigma[\mu]$ is a continuous translation invariant, linear operator on $L_+^1_{\text{loc},+}(\mathbb{R})$. Applying the duality of $L_+^1_{\text{loc},+}(\mathbb{R})$ and $L_+^1_{\text{loc},+}(\mathbb{R})$, we introduce $\sigma[\mu]$ on $L_+^\infty_{\text{loc},+}(\mathbb{R})$,

$$\forall f \in L_+^\infty_{\text{loc},+}(\mathbb{R}) \forall g \in L_+^1_{\text{loc},+}(\mathbb{R}) : (\sigma[\mu]g, \tilde{f}) = (g, (\sigma[\mu]f)^\gamma).$$

The Closed Graph Theorem for strict LF-spaces guarantees that $\sigma[\mu]$ is continuous on $L_+^\infty_{\text{loc},+}(\mathbb{R})$, also $\sigma[\mu]$ is translation invariant.

Let $K : L_+^\infty_{\text{loc},+}(\mathbb{R}) \to L_+^\infty_{\text{loc},+}(\mathbb{R})$ be a continuous, translation invariant, linear operator. Then with $L_K$ on $C(\mathbb{R}, L_+^\infty_{\text{loc},+}(\mathbb{R}))$ defined by
for $f \in C_+ (\mathbb{R})$,

$$T_\sigma K f = L_\sigma T_\sigma f \in C (\mathbb{R}, L^\infty_{loc+} (\mathbb{R})) .$$

By Proposition 3.1, $K f \in C_+ (\mathbb{R})$ for all $f \in C_+ (\mathbb{R})$, and $K |_{C_+ (\mathbb{R})}$ is continuous and translation invariant. According to Theorem 2.1 there exists $\mu \in bv_{loc,-} (\mathbb{R})$ such that

$$K |_{C_+ (\mathbb{R})} = \sigma [\mu] ,$$

and for all $g \in L^1_{loc,+} (\mathbb{R})$ and $f \in C_+ (\mathbb{R})$,

$$\langle g , (K f)^\vee \rangle = \langle \hat{g} , K f \rangle = \langle \hat{\sigma [\mu]} \hat{g} , f \rangle = \langle \sigma [\mu] g , \hat{f} \rangle = \langle K^* g , \hat{f} \rangle .$$

**Theorem 4.9.** Let $K : L^\infty_{loc,+} (\mathbb{R}) \to L^\infty_{loc,+} (\mathbb{R})$ be a continuous, translation invariant, linear operator with dual $K^* : L^\infty_{loc,+} (\mathbb{R})^* \to L^\infty_{loc,+} (\mathbb{R})^*$. Then there is $\mu \in bv_{loc,-} (\mathbb{R})$ such that $K |_{C_+ (\mathbb{R})} = \sigma [\mu]$.

If $K^* (L^1_{loc,+} (\mathbb{R})) \subseteq L^1_{loc,+} (\mathbb{R})$, then $K = \sigma [\mu]$ and $K^* |_{L^1_{loc,+} (\mathbb{R})} = \sigma [\mu]$.

(Here we identified $L^1_{loc,+} (\mathbb{R})$ as a closed subspace of $L^\infty_{loc,+} (\mathbb{R})^*$ in the indicated way.)

**Theorem 4.10.** Let $S : L^1_{loc,+} (\mathbb{R}) \to L^1_{loc,+} (\mathbb{R})$ be a continuous, translation invariant, linear operator. Then there is $\mu \in bv_{loc,-} (\mathbb{R})$ such that $S = \sigma [\mu]$.

The collection

$$\{ \sigma [\mu] | \mu \in bv_{loc,-} (\mathbb{R}) \}$$

consists of precisely all continuous, linear, translation invariant operators on $L^1_{loc,+} (\mathbb{R})$.

Let $D'_+ (\mathbb{R})$ denote the subspace of $D' (\mathbb{R})$ consisting of all $F \in D' (\mathbb{R})$ with support $\text{supp} (F)$ contained in some half-infinite interval $[a, \infty)$, $a$ depending on $F$, i.e. $F (\varphi) = 0$ for all $\varphi \in D (\mathbb{R})$ with $\text{supp} (\varphi) \subset (-\infty, a]$.

**Lemma 4.11.** For each $F \in D'_+ (\mathbb{R})$ and $\varphi \in D (\mathbb{R})$ the function $C_F \varphi$ defined by

$$(C_F \varphi) (t) = F (\sigma_t \varphi) , \quad t \in \mathbb{R} ,$$

belongs to $C^\infty (\mathbb{R})$ with $\text{supp} (C_F \varphi) \subset (-\infty, b - a]$ if $\text{supp} (F) \subset [a, \infty)$ and $\text{supp} (\varphi) \subset (-\infty, b]$, $a, b \in \mathbb{R}$.

**Proof.** Since $(\sigma_t)_{t \in \mathbb{R}}$ is a $c_0$-group with continuous infinitesimal generator $\frac{d}{dt}$ on $D (\mathbb{R})$, for each continuous linear functional $F \in D' (\mathbb{R})$, the function $C_F \varphi \in C^\infty (\mathbb{R})$.

Now, suppose $\text{supp} (F) \subset [a, \infty)$ and $\text{supp} (\varphi) \subset (-\infty, b]$. Then $\text{supp} (\sigma_t \varphi) \subset (-\infty, b - t]$ and so for all $t$ with $t > b - a$, $\text{supp} (\sigma_t \varphi) \subset (-\infty, a]$ which implies $F (\sigma_t \varphi) = 0$ for $t > b - a$.

Now, let $g \in L^\infty_{loc,+} (\mathbb{R})$. Then we define $F * g \in D' (\mathbb{R})$ by

32
$$(F * g)(\varphi) := \int_R g(t)F(\sigma_t \varphi)dt, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

This integral is for all $F \in \mathcal{D}'_+(\mathbb{R})$, $g \in L^\infty_{\text{loc,+}}(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$ a proper Riemann-Stieltjes integral. Indeed, assume $\text{supp}(F) \subseteq [a, \infty)$, $\text{supp} \varphi \subseteq (-\infty, b]$ and $\text{supp}(g) \subseteq [c, \infty)$. Then

$$F(\sigma_t \varphi) = 0 \text{ for all } t \text{ with } t > b - a$$

and

$$\int_R g(t)F(\sigma_t \varphi)dt = \begin{cases} \int_c^b g(t)F(\sigma_t \varphi)dt & b > c + a \\ 0 & b \leq c + a \end{cases}.$$

We conclude additionally that $g * F \in \mathcal{D}'_+(\mathbb{R})$ with $\text{supp}(g * F) \subseteq [c + a, \infty)$.

Now, $L^\infty_{\text{loc,+}}(\mathbb{R})$ can be considered a subspace of $\mathcal{D}'_+(\mathbb{R})$ by identifying each $g \in L^\infty_{\text{loc,+}}(\mathbb{R})$ and $F g \in \mathcal{D}'_+(\mathbb{R})$ defined by

$$F g(\varphi) = \langle \varphi, g \rangle = \int \varphi(t)g(t)dt.$$ 

**Theorem 4.12.** Let $F \in \mathcal{D}'_+(\mathbb{R})$. Assume for all $g \in L^\infty_{\text{loc,+}}(\mathbb{R})$

$$F * g \in L^\infty_{\text{loc,+}}(\mathbb{R}) \text{ (under the identification mentioned).}$$

Then there is $\mu \in \text{bv}_{\text{loc,-}}(\mathbb{R})$ such that

$$\forall g \in L^\infty_{\text{loc,+}}(\mathbb{R}): F * g = \sigma[\mu]g.$$

**Proof.**

(1) Define the linear operator $\mathcal{L}$ from $L^\infty_{\text{loc,+}}(\mathbb{R})$ into $L^\infty_{\text{loc,+}}(\mathbb{R})$ by

$$\mathcal{L} g = F * g.$$ 

Then $\forall t \in \mathbb{R}: \mathcal{L}\sigma_t = \sigma_t \mathcal{L}$. Also, $\mathcal{L}$ is (sequentially) closed and therefore continuous. Indeed, let $g_n \to g$ and $\mathcal{L} g_n \to h$ ($n \to \infty$) in $L^\infty_{\text{loc,+}}(\mathbb{R})$. Then for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$\lim_{n \to \infty} \int_R g_n(t)F(\sigma_t \varphi)dt = \int_R g(t)F(\sigma_t \varphi)dt$$ 

due to uniform convergence, and also

$$\lim_{n \to \infty} (F * g_n)(\varphi) = \langle \varphi, h \rangle.$$ 

33
It follows that \((F * g)(\varphi) = \langle \varphi, h \rangle, \varphi \in \mathcal{D}(\mathbb{R})\), which means \(F * g = h\).

(2) \(\mathcal{L}\) being continuous, Theorem 4.9 yields there is \(\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})\) such that for all \(g \in C_+(\mathbb{R})\)

\[
\mathcal{L}g = \sigma[\mu]g .
\]

This means that for all \(g \in C_+(\mathbb{R}), \varphi \in \mathcal{D}(\mathbb{R})\)

\[
\int_{\mathbb{R}} g(t)F(\sigma_t\varphi)dt = \int_{\mathbb{R}} \varphi(\tau)\left(\int_{\mathbb{R}} (\sigma_t g)(\tau)d\mu(t)\right)d\tau = \int_{\mathbb{R}} \varphi(\tau)\left(\int_{\mathbb{R}} (\sigma_{-t} g)(\tau)d\hat{\mu}(t)\right)d\tau
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(\tau + t)g(\tau)d\hat{\mu}(t)d\tau = \int_{\mathbb{R}} g(t)F_\mu(\sigma_t\varphi)dt .
\]

So \(F = F_\mu\) with \(F_\mu(\varphi) = \int_{\mathbb{R}} \varphi(\tau)d\hat{\mu}(\tau)\) and, consequently, for all \(g \in L_{\text{loc},+}^\infty(\mathbb{R})\)

\[
\mathcal{L}g = F * g = F_\mu * g = \sigma[\mu]g .
\]

\[\square\]

**Corollary 4.13.** Let \(\mathcal{L} : L_{\text{loc},+}^\infty(\mathbb{R}) \rightarrow L_{\text{loc},+}^\infty(\mathbb{R})\) be continuous and translation invariant. Assume there is \(F \in \mathcal{D}_+(\mathbb{R})\) such that \(\mathcal{L}g = F * g\) for all \(g \in L_{\text{loc},+}^\infty(\mathbb{R})\). There there is \(\mu \in \text{bv}_{\text{loc},-}(\mathbb{R})\) such that \(\mathcal{L} = \sigma[\mu]\).

For self-containedness of the paper we have presented the above set up. However, Schwartz in [Schw2] proved that \(\mathcal{D}_+(\mathbb{R})\) is a convolution algebra where convolution generalizes naturally the ordinary convolution between functions with half-infinite support. The condition on \(\mathcal{L}\) presented in the above Corollary means that \(\mathcal{L}\) can be extended to the whole of \(\mathcal{D}_+(\mathbb{R})\) and \(F = \mathcal{L}\delta_0\) with \(\delta_0\) the point evaluation at \(t = 0\).

Replacing \(L_{\text{loc},+}^\infty(\mathbb{R})\) by \(L_+^\infty(\mathbb{R})\) in Theorem 4.12 and Corollary 4.13 we come to a similar assertion if also \(\text{bv}_{\text{loc},-}(\mathbb{R})\) is replaced by \(\text{bv}_-(\mathbb{R}) = \{\hat{\mu} \mid \mu \in \text{bv}_+(\mathbb{R})\}\).

**Theorem 4.14.** Let \(F \in \mathcal{D}_+(\mathbb{R})\). Assume \(F * g \in L_+^\infty(\mathbb{R})\) for all \(g \in L_+^\infty(\mathbb{R})\). Then there is \(\mu \in \text{bv}_+(\mathbb{R})\) such that

\[
F * g = \sigma[\mu]g , \quad g \in L_+^\infty(\mathbb{R}) .
\]

The above result was proved in [WC] also; the proof is based on quite different arguments.
References


