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FAMILIES OF LINEAR-QUADRATIC PROBLEMS
CONTINUITY PROPERTIES
by
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Eindhoven, The Netherlands
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Abstract. In this paper we investigate for given one-parameter families of linear time-invariant finite-dimensional systems the parameter dependence of the linear-quadratic optimal cost, optimal control inputs, optimal state-trajectories and optimal outputs. It is shown that results that have been obtained in the past in the context of the problem of 'cheap control' can in fact be generalized to a much broader class of parameter dependent cost-functionals, including cost-functionals in which for every parameter value the weighting matrix of the control inputs is singular. Essentially, only two assumptions on the parameter dependence of the cost-functionals are required in order to have continuity of the optimal cost and optimal control inputs with respect to the underlying parameter. One assumption is concerned with the continuity of the weighting matrices with respect to this parameter, the other with the monotonicity of the weighting matrices with respect to the parameter. Instrumental in our development is a characterization of the linear-quadratic optimal cost in terms of the so-called dissipation inequality. The results obtained are applied to the problem of 'cheap control' and to a problem of 'priority control'. The latter provides an example of a family of quadratic cost-functionals with a polynomial parameter dependence.
1. INTRODUCTION

It is a well-known fact that optimal control inputs and optimal state trajectories for the infinite horizon linear-quadratic time-invariant optimal control problem are in general distributions. The corresponding optimal cost may be obtained by calculating the maximal solution to the so-called dissipation inequality. These issues have been the subject of detailed studies as for example [1], [2], [3] and [4].

We will consider the finite-dimensional linear time-invariant system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]

(1.1)

together with the quadratic cost-functional

\[
J(x_0,u) = \int_0^\infty \|y(t)\|^2 \, dt.
\]

(1.2)

Here, it will be assumed that \(u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n\) and \(y(t) \in \mathbb{R}^p\). It will be a standing assumption in this paper that the pair \((A,B)\) is stabilizable with respect to \(C^- := \{s \in \mathbb{C} \mid \text{Re } s < 0\}\).

The linear-quadratic problem associated with the system \((A,B,C,D)\) is the problem of finding the infimal value of the cost-functional (1.2) with respect to an appropriate class of control inputs \(u\) and to calculate, if it exists, an optimal one. The problem is called regular if the mapping \(D\) appearing in (1.1b) is injective and singular if this is not the case. In the latter case in general the optimal controls will no longer be smooth functions (as in the regular case) but will be distributions.

Traditionally, one method of trying to capture the singular case has been to regularize the cost-functional (1.2). Instead of minimizing (1.2) one then minimizes its 'regularization'

\[
J_\varepsilon(x_0,u) = \int_0^\infty \|y(t)\|^2 + \varepsilon^2 \|u(t)\|^2 \, dt.
\]

(1.3)
Of course, minimizing (1.3) is the same as solving the linear-quadratic problem for the system \((A, B, (\gamma^C_0), (\gamma^D_\epsilon))\) which, for \(\epsilon > 0\), is a regular problem. For \(\epsilon = 0\) the original problem is recovered. In the past, it has been the subject of a considerable amount of papers to characterize the behavior of the optimal cost, the optimal control inputs and the optimal state-trajectories as the parameter \(\epsilon\) tends to zero ([5] to [17]).

In the present paper we shall consider the more general situation that the mappings \(C\) and \(D\) depend more or less arbitrarily on a real parameter \(\epsilon\) lying in some closed interval, say \([0,1]\), and study the continuity properties of the optimal cost, the optimal controls and the optimal state-trajectories as functions of this real parameter. More precisely, we shall consider the system (1.1a) together with the output equations

\[ y_\epsilon(t) = C(\epsilon)x(t) + D(\epsilon)u(t), \quad \epsilon \in [0,1] \]  

(1.4)

with associated cost-functionals

\[ J_\epsilon(x_0, u) = \int_0^\infty \|y_\epsilon(t)\|^2 \, dt, \quad \epsilon \in [0,1] \]  

(1.5)

and study the behavior of the optimal cost, optimal controls and optimal state-trajectories as functions of \(\epsilon\). In (1.4) for each \(\epsilon \in [0,1]\), \(C(\epsilon)\) and \(D(\epsilon)\) are assumed to be linear mappings from \(\mathbb{R}^n\) to \(\mathbb{R}^p\) and from \(\mathbb{R}^m\) to \(\mathbb{R}^p\) respectively.

We stress that in this set-up \(\epsilon\) no longer needs to have an interpretation in terms of 'small control weighting'. In fact, no assumptions will be made on the injectivity of the mappings \(D(\epsilon)\). Thus, our context will also capture for example the situation that for all \(\epsilon \in [0,1]\) the linear-quadratic problem for \((A, B, C(\epsilon), D(\epsilon))\) is singular.
2. A CLASS OF ADMISSIBLE INPUTS

In this section we shall specify the class of control inputs with respect to which we will perform the minimization of the cost-functionals (1.5). In the sequel, let \( \mathcal{D} \) denote the testfunction space of all smooth functions in \( \mathbb{R} \) with compact support. Let \( \mathcal{D}' \) be the distribution space of all continuous linear functionals on \( \mathcal{D} \) (for details on distributions, see [18]). Let \( \mathcal{D}'_+ \) denote the subspace of \( \mathcal{D}' \) of all distributions with support in \( \mathbb{R}^+ \). Let \( L^2_{2, \text{loc}}(\mathbb{R}^+) \) be the space of locally square-integrable functions on \( \mathbb{R}^+ \). This space may be identified with a subspace of \( \mathcal{D}'_+ \) by defining the functional value of \( \psi \in L^2_{2, \text{loc}}(\mathbb{R}^+) \) at \( \varphi \in \mathcal{D} \) to be

\[
<\psi, \varphi> := \int_{\mathbb{R}^+} \psi(t)\varphi(t)dt .
\]

Distributions in \( L^2_{2, \text{loc}}(\mathbb{R}^+) \) will be called regular distributions. Linear combinations of the Dirac distribution \( \delta \) and its higher order derivatives will be called impulsive distributions. If \( m \in \mathbb{N} \) then \( \mathcal{D}'_+^m \) and \( L^2_{2, \text{loc}}(\mathbb{R}^+) \) will denote the spaces of \( m \)-vectors with components in \( \mathcal{D}'_+ \) and \( L^2_{2, \text{loc}}(\mathbb{R}^+) \) respectively. A distributional control input \( u \in \mathcal{D}'_+^m \) will be called admissible if it is in the space \( \mathcal{U}_{\text{dist}} \) of impulsive-regular distributions defined by

\[
\mathcal{U}_{\text{dist}} := \{ u \in \mathcal{D}'_+^m \mid u = u_1 + u_2 \text{ with } u_1 \text{ impulsive and } u_2 \text{ regular} \} .
\]

Given (1.1a), let

\[
K(t) := e^{At}B1_{\mathbb{R}^+}(t) \quad \text{and} \quad d^+(t) := e^{At}1_{\mathbb{R}^+}(t)
\]

(here \( \mathbb{R}^+ = [0, \infty) \)). If \( u \in \mathcal{D}'_+^m \) then we define the corresponding state-trajectory of (1.1a) by

\[
x(x_0, u) := d^+x_0 + K * u ,
\]

(2.1)

where ' * ' stands for convolution of distributions in \( \mathcal{D}'_+ \). We note that \( x(x_0, u) \) is in \( \mathcal{D}'_+^m \). In fact, if \( u \in \mathcal{U}_{\text{dist}} \) then \( x(x_0, u) \) is impulsive-regular, i.e. \( x(x_0, u) = x_1 + x_2 \) with \( x_1 \) and \( x_2 \) \( n \)-vectors of impulse and regular distributions respectively (see, for example, [19, prop. 2.5]). Now define
Given an initial condition $x_0 \in \mathbb{R}^n$ the subclass of $U_{\text{dist}}$ consisting of all admissible inputs that yield stable state-trajectories is defined by

$$U^\text{stab}(x_0) := \{ u \in U_{\text{dist}} \mid x(x_0,u)(\infty) = 0 \}.$$ 

Of course, if $x_0 \in \mathbb{R}^n$ and $u \in U_{\text{dist}}$ then the corresponding outputs $y_\varepsilon(x_0,u) := C(\varepsilon)x(x_0,u) + D(\varepsilon)u$ will also be impulsive-regular distributions. Thus, in general the integrals (1.5) are not well-defined. Let $L^p_2(\mathbb{R}^2)$ denote the space of all $p$-vectors whose components are square-integrable over $\mathbb{R}^2$. If $u \in U_{\text{dist}}$ happens to be such that $y_\varepsilon(x_0,u) \notin L^p_2(\mathbb{R}^2)$ then we formally define

$$J_\varepsilon(x_0,u) := + \infty.$$ 

Incorporating this formal convention, we shall now define the optimal cost for the infinite-horizon linear-quadratic problem with stability for the system $(A,B,C(\varepsilon),D(\varepsilon))$ by

$$J^*_\varepsilon(x_0) := \inf \{ J_\varepsilon(x_0,u) \mid u \in U^\text{stab}(x_0) \}.$$ 

Given $x_0 \in \mathbb{R}^n$, a control $u^*_\varepsilon \in U^\text{stab}(x_0)$ is called optimal for this problem if

$$J_\varepsilon(x_0,u^*_\varepsilon) = J^*_\varepsilon(x_0).$$

### 3. CONTINUITY OF THE OPTIMAL COST

In this section we shall show that under fairly mild conditions on the functions $\varepsilon \mapsto C(\varepsilon)$ and $\varepsilon \mapsto D(\varepsilon)$ the function $\varepsilon \mapsto J^*_\varepsilon(x_0)$ is continuous in $\varepsilon = 0$ for every $x_0 \in \mathbb{R}^n$. For the time being, take a fixed $\varepsilon \in [0,1]$. Recall that we assumed $(A,B)$ stabilizable. This implies that for all $x_0$ $U^\text{stab}(x_0)$ is non-empty and $J^*_\varepsilon(x_0) < \infty$.

Using an argument similar to the one in [3, theorem 6.12] it can be shown that in fact
\[ J^*_\varepsilon(x_0) = \inf \{ J(x_0, u) \mid u \in U_{\text{dist}}^{\text{stab}}(x_0) \cap L^m_{2, \text{loc}}(\mathbb{R}^+) \}, \]

i.e. the optimal cost can be obtained as the infimum over all regular inputs in \( U_{\text{dist}}^{\text{stab}}(x_0) \). Also, it is well-known (see [1]) that the optimal cost depends quadratically on \( x_0 \), i.e. there exists a nonnegative semi-definite symmetric matrix \( P^*_\varepsilon \in \mathbb{R}^{n \times n} \) such that

\[ J^*_\varepsilon(x_0) = x_0^T P^*_\varepsilon x_0. \quad (3.1) \]

Combining these facts it becomes possible to characterize the matrix \( P^*_\varepsilon \) in terms of the so-called dissipation inequality ([1]). Given \((A, B, C(\varepsilon), D(\varepsilon))\), define a map \( F_\varepsilon : \mathbb{R}^{n \times n} \to \mathbb{R}^{(n+m) \times (n+m)} \) by

\[ F_\varepsilon(P) := \begin{bmatrix} A^T P + PA + C^T(\varepsilon)C(\varepsilon) & PB + C^T(\varepsilon)D(\varepsilon) \\ B^T P + D^T(\varepsilon)C(\varepsilon) & D^T(\varepsilon)D(\varepsilon) \end{bmatrix}. \quad (3.2) \]

A matrix \( P \) is said to satisfy the dissipation inequality for the system \((A, B, C(\varepsilon), D(\varepsilon))\) if \( F_\varepsilon(P) \geq 0 \). It was proven in [1] that \( P^*_\varepsilon \) can be characterized as the maximal element in the set of solutions of the dissipation inequality (see also [4] for a quick proof):

**Lemma 3.1.** \( F_\varepsilon(P^*_\varepsilon) \geq 0 \). Moreover, if \( P \) is a nonnegative semi-definite symmetric \( n \times n \) matrix such that \( F_\varepsilon(P) \geq 0 \) then \( P \leq P^*_\varepsilon \). \( \square \)

We shall now impose the following assumptions concerning the dependency of the mappings \( C(\varepsilon) \) and \( D(\varepsilon) \) on the parameter \( \varepsilon \). In our first assumption \( C(\varepsilon) \) and \( D(\varepsilon) \) are interpreted as functions from \([0,1]\) to \( \mathbb{R}^{p \times n} \) and \( \mathbb{R}^{p \times m} \) respectively:

\[(A.1) \quad \varepsilon \mapsto C(\varepsilon) \text{ and } \varepsilon \mapsto D(\varepsilon) \text{ are continuous in } 0.\]

Our second assumption deals with the monotonicity in \( \varepsilon \) of the nonnegative semi-definite symmetric matrices

\[ Q(\varepsilon) := (C(\varepsilon) D(\varepsilon))^T C(\varepsilon) D(\varepsilon)). \quad (3.3) \]

In terms of these the cost-functionals (1.5) can be written as
\[
J(x_0,u) = \int_0^\infty (x(t)^T u(t)^T) Q(\varepsilon) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \, dt.
\]

We shall assume that in a neighbourhood of 0 \( Q(\varepsilon) \) is a monotonically nondecreasing function:

(A.2) There exists a \( \delta > 0 \) such that for all \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \delta \) we have \( Q(\varepsilon_1) \leq Q(\varepsilon_2) \).

It turns out that under the above two assumptions the optimal cost \( J^*(x_0) \) is continuous in \( x_0 \):\

**Theorem 3.2.** Assume that (A.1) and (A.2) hold. Let \( x_0 \in \mathbb{R}^n \). Then

\[
\lim_{\varepsilon \to 0} J^*(x_0) = J^*_0(x_0).
\]

**Proof.** By (3.1), \( J^*(x_0) = x_0^T P^*_\varepsilon x_0 \) for all \( \varepsilon \in [0,1] \). We claim that

\[
\lim_{\varepsilon \to 0} P^*_\varepsilon = \bar{P}
\]

exists. Indeed, as a consequence of the assumption (A.2) the function \( \varepsilon \mapsto P^*_\varepsilon \) is monotonically nondecreasing on \( [0,\delta] \). Since also \( P^*_\varepsilon \geq 0 \) for all \( \varepsilon \), this proves our claim. Next, we note that by the assumption (A.1) the function

\[
(P,\varepsilon) \mapsto F(\varepsilon)
\]

is continuous in \((P,0)\) for every \( P \in \mathbb{R}^{n \times n} \) and thus in particular in \((\bar{P},0)\). We may therefore conclude that

\[
\lim_{\varepsilon \to 0} F(\varepsilon) = F(0,\bar{P})
\]

Since \( F(\varepsilon) \geq 0 \) for all \( \varepsilon \), this yields \( F(0,\bar{P}) \geq 0 \), i.e. \( \bar{P} \) satisfies the dissipation inequality for the system \((A, B, C(0), D(0))\). By lemma 3.1 we therefore have \( \bar{P} \preceq P^*_0 \). On the other hand, again by the assumption (A.2) we find \( P^*_0 \preceq P^*_\varepsilon \) for all \( \varepsilon \in [0,\delta] \) and thus also \( P^*_0 \preceq \bar{P} \). We conclude that \( \bar{P} = P^*_0 \).

**Remark 3.3.** An argument similar to the above was used in [4] to obtain the corresponding result for the special case that \( C(\varepsilon) = C(0) \) and...
D(\epsilon) = (D_{\epsilon}) (see also [5] and [9]). Note that for the latter case the assumptions (A.1) and (A.2) are trivially satisfied.

**REMARK 3.4.** Instead of considering the linear-quadratic problem with stability, one could also consider the version of this problem in which we do not require stability of the optimal state trajectories (see also [2]). Again consider the system (1.1a) with cost-functionals (1.5). Instead of minimizing over \( U_{\text{dist}}^{\text{stab}}(x_0) \), now minimize over the (larger) class \( U_{\text{dist}} \) and define the associated cost by

\[
\tilde{J}_{\epsilon}^*(x_0) := \inf \{ J_{\epsilon}(x_0, u) \mid u \in U_{\text{dist}} \}.
\]

It can be shown that under the assumptions (A.1) and (A.2) the function \( \epsilon \mapsto \tilde{J}_{\epsilon}^*(x_0) \) does not need to be continuous in 0 for all \( x_0 \), i.e. the analogue of theorem 3.2 does not hold for the linear-quadratic problem without stability. In order to obtain a class of counterexamples we shall briefly recall the notion of invariant zero ([20], [21], [22]). Given a system \((A, B, C, D)\), let \( V^* \) denote the associated output-nulling subspace and let \( R^* \) denote the controllable output-nulling subspace. It is well-known that there exists a mapping \( F: \mathbb{R}^n \to \mathbb{R}^m \) such that \((A+BF)V^* \subseteq V^* \) and \((C+DF)V^* = \{0\} \). Moreover, any such \( F \) automatically satisfies \((A+BF)R^* \subseteq R^* \) and it turns out that the spectrum \( \sigma(A+BF \mid V^*/R^*) \) is independent of these \( F \)'s. This fixed spectrum is denoted by \( \sigma^*(A,B,C,D) \) and is called the set of invariant zeros of \((A,B,C,D)\). Now, let \( J^*(x_0) \) denote the optimal cost for the linear-quadratic problem for \((A,B,C,D)\) with stability and let \( \tilde{J}^*(x_0) \) denote the optimal cost for the problem without stability. Obviously, in general \( \tilde{J}^*(x_0) \leq J^*(x_0) \). However, it follows from the results in [2] that

\[
\tilde{J}^*(x_0) = J^*(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n
\]

if and only if

\[
\sigma^*(A,B,C,D) \subseteq \tilde{\mathbf{C}}^{-} := \{ s \in \mathbf{C} \mid \text{Re } s > 0 \}.
\]

Now, take any system \((A,B,C,D)\) with \( D = 0 \), \((A,B)\) stabilizable and \((C,A)\) detectable, such that at least one invariant zero lies in \( \mathbf{c}^+ := \{ s \in \mathbf{C} \mid \text{Re } s > 0 \} \). Define \( C(\epsilon) := (C_{\epsilon}) \) and \( D(\epsilon) := (D_{\epsilon}) \) (with \( 1 \) the \( m \times m \) identity matrix. Obviously, for \( \epsilon > 0 \) the system \((A,B,C(\epsilon),D(\epsilon))\)
defines a regular linear-quadratic problem. This problem is also standard (i.e., $C^T(c)D(c) = 0$) and consequently (see [23]), since $(A,B)$ is stabilizable and $(A,C(c))$ is detectable, for $c > 0$ we have $J^*(x_0) = J^*_c(x_0)$ for all $x_0 \in \mathbb{R}^n$. However, caused by the presence of an invariant zero in $\mathfrak{C}^+$, there exists an $x_0 \in \mathbb{R}^n$ such that the strict inequality $J^*_0(x_0) < J^*_0(x_0)$ holds. For this $x_0$ we must have $J^*_c(x_0) + J^*_0(x_0)$ (by theorem 3.2) and consequently $J^*_c(x_0) \neq J^*_0(x_0)$ ($c > 0$).

4. CONTINUITY OF OPTIMAL CONTROLS AND STATE-TRAJECTORIES

This section is concerned with the continuity properties of the optimal control inputs and optimal state-trajectories. We shall prove that if these exist then considered as functions of $c$ they are continuous in $c = 0$, provided that (A.1) and (A.2) hold and provided that the system $(A,B,C(0),D(0))$ is left-invertible.

Let us first consider the linear-quadratic problem with stability for a given system $(A,B,C,D)$ and recall some facts on the existence and uniqueness of optimal controls for this problem. The existence of optimal controls is known to be closely connected to the position of the set of invariant zeros $\sigma^*(A,B,C,D)$ (see remark 3.4) with respect to the imaginary axis $\mathfrak{C}^0 := \{ s \in \mathfrak{C} \mid \text{Re } s = 0 \}$, while the uniqueness depends on the invertibility properties of the system. The system $(A,B,C,D)$ is called left-invertible (right-invertible) if its transfer matrix $C(\mathfrak{C}s-A)^{-1}B + D$ is injective (surjective) as a matrix over the field of real rational functions. $(A,B,C,D)$ is called invertible if it is both left- and right-invertible.

The following results were proven in [2]:

PROPOSITION 4.1. Consider the linear-quadratic problem with stability for the system $(A,B,C,D)$. For $x_0 \in \mathbb{R}^n$, let $J^*(x_0)$ denote the optimal cost. Then the following holds:

(i) For all $x_0 \in \mathbb{R}^n$ there exists an optimal control $u^*_x$ if and only if $\sigma^*(A,B,C,D) \cap \mathfrak{C}^0 = \emptyset$. 

(ii) For all \( x_0 \in \mathbb{R}^n \) there exists exactly one optimal control \( u^*_0 \) if and only if \( \sigma^*(A,B,C,D) \cap \mathfrak{K}^0 = \emptyset \) and \( (A,B,C,D) \) is left-invertible.

(iii) \( J^*(x_0) = 0 \) for all \( x_0 \in \mathbb{R}^n \) if and only if \( \sigma^*(A,B,C,D) \subset \mathfrak{K}^0 \).

Thus, returning to our original problem, in order to be able to speak about optimal controls \( u^*_0 \) for the linear-quadratic problem for \((A,B,C(\varepsilon),D(\varepsilon))\) we shall assume that for \( \varepsilon \) sufficiently small these systems do not have invariant zeros on the imaginary axis:

(A.3) There exists \( \eta > 0 \) such that for all \( 0 \leq \varepsilon \leq \eta \) we have \( \sigma^*(A,B,C(\varepsilon),D(\varepsilon)) \cap \mathfrak{K}^0 = \emptyset \).

Next we shall label the following invertibility condition on the 'boundary system' \((A,B,C(0),D(0))\):

(A.4) The system \((A,B,C(0),D(0))\) is left-invertible.

By proposition 4.1, (A.3) and (A.4) will guarantee that the linear-quadratic problem for \((A,B,C(0),D(0))\) has a unique solution for every \( x_0 \).

A fortiori however, (A.1) and (A.4) imply that for \( \varepsilon > 0 \) sufficiently small \((A,B,C(\varepsilon),D(\varepsilon))\) is also left-invertible. Consequently, we can make the following observation:

**Lemma 4.2.** Assume that (A.1) holds. Then the following statements are equivalent:

(i) \( \exists \eta > 0 \) such that for all \( 0 \leq \varepsilon \leq \eta \) for all \( x_0 \in \mathbb{R}^n \) the linear-quadratic problem for \((A,B,C(\varepsilon),D(\varepsilon))\) has a unique optimal control input \( u^*_\varepsilon, x_0 \).

(ii) (A.3) and (A.4) hold.

Before stating our main result, a few words on the type of convergence we shall consider are at order. A sequence of distributions \( \psi_n \in D'_+ \) is said to converge to \( \psi \in D'_+ \) in distributional sense if \( \langle \psi_n, \varphi \rangle \to \langle \psi, \varphi \rangle \) (\( n \to \infty \)) for all \( \varphi \in D \). In the obvious way, a sequence \( u_n \in D'_{+m} \) is said to converge to \( u \in D'_{+m} \) in distributional sense if this convergence holds componentwise. Finally, a sequence \( y_n \in L^p_{2}(\mathbb{R}^+) \) is said to converge to \( y \in L^p_{2}(\mathbb{R}^+) \) strongly if convergence holds componentwise in the topology of \( L^p_{2}(\mathbb{R}^+) \).
Now consider the system (1.1a) together with the cost-functionals (1.5). For \( x_0 \in \mathbb{R}^n \) let

\[
x^*_{\varepsilon, x_0} := x(x_0, u^*_{\varepsilon, x_0}) \quad \text{and} \quad y^*_{\varepsilon, x_0} := C(\varepsilon)x^*_{\varepsilon, x_0} + D(\varepsilon)u^*_{\varepsilon, x_0}
\]

be the optimal state-trajectory and optimal output for \( (A,B,C(\varepsilon),D(\varepsilon)) \).

Our main result is the following:

**Theorem 4.3.** Assume that (A.1), (A.2), (A.3) and (A.4) hold. Then for all \( x_0 \in \mathbb{R}^n \) we have \( u^*_{\varepsilon, x_0} \to u^*_{0, x_0} \) and \( x^*_{\varepsilon, x_0} \to x^*_{0, x_0} \) \((\varepsilon \to 0)\) in distributional sense. Moreover, \( y^*_{\varepsilon, x_0} \to y^*_{0, x_0} \) \((\varepsilon \to 0)\) strongly.

In order to establish a proof of this theorem we shall proceed as follows. Let \( P_0^+ \) be the nonnegative semi-definite symmetric matrix defining the optimal cost for \( (A,B,C(0),D(0)) \) via (3.1). Again, let \( F_0(P;\varepsilon) \) be given by (3.2). By lemma 3.1, \( F_0(P^+;0) \geq 0 \). Since \( F_0(P^+;0) \) is also symmetric it admits a factorization

\[
F_0(P^+;0) = (C_+ \ D_+)^T(C_+ \ D_+)
\]

(see also [3]). Assume that in (4.1) the matrix \( (C_+ \ D_+) \) has linearly independent rows. This can always be achieved by taking the number of rows equal to the rank of \( F_0(P^+;0) \). Thus (4.1) assigns to \( (A,B,C(0),D(0)) \) a uniquely defined system \( (A,B,C_+,D_+) \). Consider now the linear-quadratic problem for \( (A,B,C_+,D_+) \). Given \( x_0 \in \mathbb{R}^n \) and \( u \in U_{\text{dist}} \) let

\[
J_+(x_0,u) := \int_0^\infty \|C_+ x(x_0,u) + D_+ u\|^2 \, dt
\]

if \( C_+ x(x_0,u) + D_+ u \) is square-integrable and \( J_+(x_0,u) := +\infty \) otherwise.

Let the corresponding optimal cost be given by

\[
J_+^*(x_0) := \inf \{ J_+(x_0,u) \mid u \in U_{\text{dist}}^{\text{stab}}(x_0) \}.
\]

The following important lemma relates the optimal costs for the linear-quadratic problems associated with the systems \( (A,B,C(0),D(0)) \) and \( (A,C_+,D_+) \):
LEMMA 4.4. Let $x_0 \in \mathbb{R}^n$ and $u \in U_{\text{dist}}$. Then we have $J^+(x_0, u) < \infty$ if and only if $J^0(x_0, u) < \infty$. For every $u \in U_{\text{stab}}^\text{dist}(x_0)$ we have

$$J^+(x_0, u) = J^0(x_0, u) - J^\lambda_0(x_0).$$

(4.2)

PROOF. A proof of this can be found in [3, lemma 6.21].

As an immediate consequence of the above we see that for every $x_0 \in \mathbb{R}^n$ the linear-quadratic problem for $(A, B, C(0), D(0))$ has a unique solution $u^+_0, x_0$ if and only if for every $x_0 \in \mathbb{R}^n$ the problem for $(A, B, C^+, D^+)$ has a unique solution $u^+_+, x_0$ and that in this case $u^+_0, x_0 = u^+_+, x_0$ for every $x_0$. This fact will be instrumental in our further development. In the sequel, a system $(A, B, C, D)$ will be called minimum-phase if its invariant zeros are contained in the open left-halfplane, i.e. if $\sigma^*(A, B, C, D) \subset \mathbb{C}^-$ (see also [24]).

LEMMA 4.5. $(A, B, C(0), D(0))$ has no invariant zeros on the imaginary axis if and only if $(A, B, C^+, D^+)$ is minimum-phase.

PROOF. Taking infima on both sides in (4.2) we find that $J^\lambda_+(x_0) = 0$ for all $x_0$. It thus follows from proposition 4.1 (iii) that $\sigma^*(A, B, C^+, D^+) \subset \mathbb{C}^-$. Consequently, it suffices to show that $\sigma^*(A, B, C(0), D(0)) \cap \mathbb{C}^0 = \emptyset$ if and only if $\sigma^*(A, B, C^+, D^+) \cap \mathbb{C}^0 = \emptyset$. However, this follows immediately by combining proposition 4.1 (i) and lemma 4.4 upon noting that the existence for all $x_0$ of an optimal control for $(A, B, C(0), D(0))$ is equivalent to the existence for all $x_0$ of an optimal control for $(A, B, C^+, D^+)$. 

REMARK 4.6. The fact that the system $(A, B, C(0), D(0))$ has no invariant zeros on the imaginary axis if and only if the same holds for the transformed system $(A, B, C^+, D^+)$ is in fact a manifestation of the fact that the invariant zeros on the imaginary axis are fixed under the transformation $(A, B, C(0), D(0)) \mapsto (A, B, C^+, D^+)$ defined by (4.1). This has also been noted in [25] in the (dual) context of the singular filtering problem.

Our next result relates the invertibility properties of the original system and its transform:
LEMMA 4.7. Assume that \((A,B,C(O),D(O))\) has no invariant zeros on the imaginary axis. Then \((A,B,C(O),D(O))\) is left-invertible if and only if \((A,B,C_+,D_+)\) is invertible.

PROOF. It follows immediately from proposition 4.1 (iii) and the fact that \(J^b(x_0) = 0\) for all \(x_0\) that \((A,B,C_+,D_+)\) is right-invertible. Thus it suffices to show that \((A,B,C(O),D(0))\) is left-invertible if and only if \((A,B,C_+,D_+)\) is left-invertible. Again, this follows by combining proposition 4.1 (ii), lemma 4.4 and lemma 4.5 upon noting that the existence for all \(x_0\) of a unique optimal control for \((A,B,C(O),D(0))\) is equivalent to the existence for all \(x_0\) of a unique optimal control for \((A,B,C_+,D_+)\).

REMARK 4.8. The above lemma is in fact also true without the premise that \((A,B,C(O),D(O))\) has no invariant zeros on the imaginary axis. A proof of this claim can be given using the geometric characterization of left-invertibility in terms of controlled invariant subspaces (cf. [22, ex. 4.4]). Since we do not use this stronger result here, the proof is omitted.

Before proving our main theorem we still need one more preliminary result. In the following, a left-invertible system \((A,B,C,D)\) will be called strongly detectable if for any \(x_0 \in \mathbb{R}^n\) and any \(u \in U_{\text{dist}}\) such that \(Cx(x_0,u) + Du = 0\) we have \(x(x_0,u)(\cdot) = 0\) (cf. (2.1) and 2.2).

The latter definition generalizes the one in [3, def. 6.7], where a system is called strongly detectable if the above property holds for regular inputs \(u\). With our definition the following extension of [3, th. 6.8] holds.

LEMMA 4.9. Assume that \((A,B,C,D)\) is left-invertible. Then \((A,B,C,D)\) is strongly detectable if and only if it is minimum-phase.

PROOF. Obviously, if \((A,B,C,D)\) is strongly detectable in the sense of our definition, it is also strongly detectable in the sense of [3, def. 6.7]. Thus, by applying [3, th. 6.8] we find that \((A,B,C,D)\) is minimum-phase.

Conversely, assume that \((A,B,C,D)\) is minimum-phase and let \(u \in U_{\text{dist}}\) and \(x_0 \in \mathbb{R}^n\) be such that \(Cx(x_0,u) + Du = 0\). The distribution \(x(x_0,u)\) can be written uniquely as \(x(x_0,u) = x_1 + x_2\) with \(x_1\) impulsive and \(x_2\) regular.
We want to show that \( \lim_{t \to \infty} x_2(t) = 0 \). Now, since \( u \in U_{\text{dist}} \), it has a unique representation \( u = u_1 + u_2 \) with \( u_1 \) impulsive and \( u_2 \) regular. Define \( x(0^+) := \lim_{t \to 0^+} x_2(t) \). It is well-known that \( x_2 = x(x(0^+),u_2) \) (see e.g. [19, prop. 2.5]), i.e. \( x_2 \) is the state-trajectory emanating from initial condition \( x(0^+) \) and regular control \( u_2 \). Since obviously \( C x_2 + D u_2 = 0 \) it therefore follows from [3, th. 6.7] that \( x_2(t) \to 0 \) (\( t \to \infty \)). 

**PROOF OF THEOREM 4.2.** In this proof, take an arbitrary but fixed \( x_0 \) and denote \( u^*_{\epsilon,x_0}, x^*_{\epsilon,x_0} \) and \( y^*_{\epsilon,x_0} \) by \( u^*_{\epsilon}, x^*_{\epsilon} \) and \( y^*_{\epsilon} \). Let \( \delta > 0 \) be sufficiently small such that \( Q(\epsilon_1) \leq Q(\epsilon_2) \) for all \( 0 \leq \epsilon_1 \leq \epsilon_2 \leq \delta \) (cf. (3.3)) and such that for all \( \epsilon \in (0,\delta] \) the linear-quadratic problem for \( (A,B,C(\epsilon),D(\epsilon)) \) has a unique solution \( u^*_\epsilon \) (cf. lemma 4.2). Define \( \bar{y}_{\epsilon} := C_{\epsilon} x^*_{\epsilon} + D_{\epsilon} u^*_\epsilon \). By lemma 4.4 and the monotonic assumption (A.2), for all \( \epsilon \in [0,\delta] \) we have

\[
\int_0^\infty \|y^*_{\epsilon}\|^2 \, dt = J_+(x_0,u^*_\epsilon) = J_0(x_0,u^*_\epsilon) - J_0^*(x_0) = 0
\]

\[
= \int_0^\infty \|C(0)x^*_{\epsilon} + D(0)u^*_{\epsilon}\|^2 \, dt - J_0^*(x_0) \leq
\]

\[
\leq \int_0^\infty \|C(\epsilon)x^*_{\epsilon} + D(\epsilon)u^*_{\epsilon}\|^2 \, dt - J_0^*(x_0) =
\]

\[
= J_+(x_0) - J_0^*(x_0)
\]

Consequently, by theorem 3.2, \( \bar{y}_{\epsilon} \to 0 \) (\( \epsilon \to 0 \)) strongly in \( L^2_2(\mathbb{R}^+) \) (here \( r \) is equal to the number of rows of \( (C_+, \, D_+) \)). A fortiori this implies that \( \bar{y}_{\epsilon} \to 0 \) (\( \epsilon \to 0 \)) in distributional sense. Now define

\[
G_+(t) := C_+ e^{At} B \int_\mathbb{R^+} h_+(t) + D_+
\]

\[
H_+(t) := C_+ e^{At} \int_\mathbb{R^+} h_+(t)
\]

By definition of \( \bar{y}_{\epsilon} \) the following convolution relation holds:

\[
\bar{y}_\epsilon = G_+ \ast u^*_{\epsilon} + H_+ x_0
\]
Since, by (A.3), (A.4) and lemma 4.6, the system \((A,B,C_+,D_+)*\) is invertible, the convolution operator with kernel \(G_+\) has an inverse with kernel \(G_+^{-1}\). \((G_+^{-1}\) is an impulsive regular distribution: it can be characterized as the inverse Laplace transform of \([C_+(1s-A)^{-1}B+D_+]^{-1}\). It follows that

\[
u_\varepsilon^* = G_+^{-1} * (\bar{y}_\varepsilon - H_+x_0)\]

and therefore, since \(\bar{y}_\varepsilon \to 0\) as a distribution,

\[
u_\varepsilon^* \to -G_+^{-1} * H_+x_0 =: \bar{u} \quad (\varepsilon \downarrow 0) \tag{4.3}\]

in distributional sense. Note that, since \(G_+^{-1}\) is impulsive-regular, \(\bar{u}\) is admissible, i.e. \(\bar{u} \in U_{\text{dist}}\). Also, it follows from (4.3) that \(\bar{u}\) and \(x_0\) yield zero output for the system \((A,B,C_+,D_+)\), i.e. \(C_+x(x_0,\bar{u}) + D_+ \bar{u} = 0\), or equivalently \(J_+(x_0,\bar{u}) = 0\).

Now, we contend that in fact \(\bar{u}\) is optimal for the linear-quadratic problem for \((A,B,C_+,D_+)\). In view of the foregoing, to show this it suffices to prove that \(\bar{u} \in U_{\text{stab}}(x_0)\). Indeed, since \((A,B,C_+,D_+)\) is left-invertible and minimum-phase by lemma 4.9 it is strongly detectable. Thus, since \(\bar{u}\) and \(x_0\) yield zero output, we have \(x(x_0,\bar{u}) = 0\) or, equivalently, \(\bar{u} \in U_{\text{stab}}(x_0)\).

Obviously, by lemma 4.4, \(\bar{u}\) is also optimal for \((A,B,C(0),D(0))\) and must therefore be equal to the unique optimal control \(u_0^*\). This proves that \(u_\varepsilon^* \to u_0^* \quad (\varepsilon \downarrow 0)\) in distributional sense.

To show that \(x_\varepsilon^* \to x_0^*\), note that \(x_\varepsilon^* = d^+x_0 + K * u_\varepsilon^*\). The latter converges to \(d^+x_0 + K * u_0^* = x_0^*\).

Finally, we will show that \(y_\varepsilon^* \to y_0^* \quad (\varepsilon \downarrow 0)\) strongly. To show this, consider the inner product space \(L_2^p(\mathbb{R}^+)\) with inner product

\[
<y_1,y_2> := \int_0^\infty y_1^T(t)y_2(t)dt.\]

We will first show that \(y_\varepsilon^* \to y_0^*\) weakly in this space, i.e. \(\langle y_\varepsilon^*,y \rangle \to \langle y_0^*,y \rangle\) for all \(y \in L_2^p(\mathbb{R}^+)\). By theorem 3.2 we have

\[
\int_0^\infty \|y_\varepsilon^*\|^2 dt \to \int_0^\infty \|y_0^*\|^2 dt.
\]
Thus, there is a $\delta > 0$ such that the set $\{y_\varepsilon^* \mid \varepsilon \in [0,\delta]\}$ is bounded in $L^2_2(\mathbb{R}^+)$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence in $[0,\delta]$ such that $\varepsilon_n \to 0$ ($n \to \infty$). The sequence $(y_{\varepsilon_n})_{n \in \mathbb{N}}$ is then bounded in $L^2_2(\mathbb{R}^+)$ and therefore has a subsequence that converges weakly to, say, $\bar{y} \in L^2_2(\mathbb{R}^+)$. On the other hand, $y_\varepsilon^* = C(\varepsilon)x_\varepsilon^* + D(\varepsilon)u_\varepsilon^*$ which converges in distributional sense to $C(0)x_0^* + D(0)u_0^* = y_0^*$ by the foregoing. Thus $\bar{y} = y_0^*$ from which it is easy to see that $y_\varepsilon^* \to y_0^*$ ($\varepsilon \to 0$) weakly. The proof is now completed by noting that

$$\int_0^\infty \|y_\varepsilon^* - y_0^*\|^2 \, dt = \int_0^\infty \|y_\varepsilon^*\|^2 \, dt - 2<y_\varepsilon^*,y_0^*> + \int_0^\infty \|y_0^*\|^2 \, dt .$$

The right-hand side of this equality converges to 0 as $\varepsilon \to 0$.

5. APPLICATIONS: CHEAP CONTROL AND PRIORITY CONTROL

In the present section we will apply the very general result of theorem 4.2 to some important special cases.

First we shall consider the special case that is commonly referred to as the problem of 'cheap control' ([5] to [17]). Consider the system (1.1) together with the cost-functionals (1.2) and (1.3). As before, assume that $(A,B)$ is stabilizable with respect to $\mathcal{U}^-$. As noted in the introduction the above falls within our context by taking $C(c) := \left(\begin{smallmatrix} c_0 \\ 0 \end{smallmatrix}\right)$ and $D(c) := \left(\begin{smallmatrix} D_0 \\ 0 \end{smallmatrix}\right)$. Obviously in this case (A.1) and (A.2) are satisfied. Condition (A.4) requires that $(A,B,C,D)$ is left-invertible. In the following, let

$$<\ker C \mid A> := \cap_{i=1}^n \ker CA^{i-1}$$

be the unobservable subspace of $(C,A)$. The next lemma states that (A.3) is equivalent to the requirement that $(A,B,C,D)$ has no invariant zeros on the imaginary axis and that $(C,A)$ has no 'unobservable poles' on the imaginary axis:

**Lemma 5.1.** (A.3) is satisfied for $(A,B,(\begin{smallmatrix} c_0 \\ 0 \end{smallmatrix}), (\begin{smallmatrix} D_0 \\ 0 \end{smallmatrix}))$ if and only if $\sigma^*(A,B,C,D) \cap \mathcal{I}^0 = \emptyset$ and $\sigma(A \mid \ker C \mid A>) \cap \mathcal{I}^0 = \emptyset$. 
PROOF. In this proof denote \( \sigma^*(A,B,(C)_{\varepsilon}^{(D)_{\varepsilon}}) \) by \( \sigma_{\varepsilon}^* \). It suffices to show that \( \sigma_{\varepsilon}^* = \sigma(A \mid <\ker C \mid A>) \) for all \( \varepsilon > 0 \). Let \( V_{\varepsilon}^* \) denote the output-nulling subspace of the system \((A,B,(C)_{\varepsilon}^{(D)_{\varepsilon}})\). We claim that for \( \varepsilon > 0 \)
\[ V_{\varepsilon}^* = <\ker C \mid A> \]. Indeed, let \( x_0 \in V_{\varepsilon}^* \). There is a regular input \( u \) such that
\[ (C_{\varepsilon}x_0,u) + (D_{\varepsilon})u = 0. \]
This implies that \( u = 0 \) and that \( Cx_0,0,0) = 0 \), whence \( x_0 \in <\ker C \mid A> \). The converse is also immediate. Finally, since
\[ \sigma_{\varepsilon}^* = \sigma(A+BF_{\varepsilon} \mid V_{\varepsilon}^*) \] for every \( F_{\varepsilon} \) such that \((A+BF_{\varepsilon})V_{\varepsilon}^* \subseteq V_{\varepsilon}^* \), we conclude that
\[ \sigma_{\varepsilon}^* = \sigma(A \mid <\ker C \mid A>) \] (just take \( F_{\varepsilon} = 0 \) upon noting that \(<\ker C \mid A> \) is \( A \)-invariant). \( \square \)

Now, given \( x_0 \in \mathbb{R}^n \), let \( u^*,x_0^*,x_0^*,x_0^* \) and \( y^*,x_0^*,y^*,x_0^* \) be optimal for the cost-functional \((1.2)\) and \( u^*,x_0^*,y^*,x_0^*,x_0^* \) and \( y^*,x_0^*,x_0^* \) for the cost-functional \((1.3)\). We have the following corollary:

**Corollary 5.2.** There exists \( \delta > 0 \) such that for all \( \varepsilon \in [0,\delta] \) and \( x_0 \in \mathbb{R}^n \)
\[ u^*,x_0^* \] exists and is unique if and only if \((A,B,C,D)\) is left-invertible,
\[ \sigma_{\varepsilon}^*(A,B,C,D) \cap C^0 = \emptyset \text{ and } \sigma(A \mid <\ker C \mid A>) \cap C^0 = \emptyset. \]

In that case we have \( u^* \to u_0^*,x_0^* \) and \( y^* \to y_0^*,x_0^* \) \((\varepsilon \to 0)\) in distributional sense and \( y^* \to y_0^*,x_0^* \) \((\varepsilon \to 0)\) strongly.

We note that in the above for \( \varepsilon > 0 \) the linear-quadratic problem for
\((A,B,(C)_{\varepsilon}^{(D)_{\varepsilon}})\) is regular and that therefore \( u^*,x_0^* \) and \( x^*,x_0^* \) are regular for all \( x_0 \). For \( \varepsilon = 0 \) the problem might become singular. Also note that a sufficient condition for \( \sigma(A \mid <\ker C \mid A>) \cap C^0 = \emptyset \) to hold is that \((C,A)\) is detectable.

Finally, we will discuss the following application. Again consider the system \((1.1a)\). Instead of the output equation \((1.1b)\) however, consider the situation that we have \( N+1 \) output equations:

\[
\begin{align*}
  y^0(t) &= C_0 x(t) + D_0 u(t), \\
  y^1(t) &= C_1 x(t) + D_1 u(t), \\
  &\vdots \\
  y^N(t) &= C_N x(t) + D_N u(t).
\end{align*}
\]

Here, we assume that \( y^k(t) \) takes its values in, say, \( \mathbb{R}^{p_k} \). For \( \varepsilon > 0 \), define the following cost-functional:
Keeping in mind that $\varepsilon$ is a small positive real number, the cost-functional (5.2) could be interpreted as assigning to the various outputs specified in (5.3) weightings of different 'orders'. In practical control problems this kind of cost-functional could be used in systems with several to-be-controlled outputs to reflect the fact that there are different orders of priority in controlling these outputs ('priority control'). We would like to characterize the behavior of the optimal closed loop system as $\varepsilon$ becomes small. First note that the above situation is captured in our context by taking

$$J_{\varepsilon}(x_0,u) := \int_0^\infty \|y^0(t)\|^2 + \varepsilon \|y^1(t)\|^2 + \ldots + \varepsilon^N \|y^N(t)\|^2 \, dt.$$  

(5.2)

Again, (A.1) and (A.2) are satisfied. Condition (A.4) is equivalent to left-invertibility of $(A,B,C_0,D_0)$. Denoting

$$C(\varepsilon) := \begin{pmatrix} C_0 \\ \varepsilon C_1 \\ \vdots \\ \varepsilon^N C_N \end{pmatrix}, \quad D(\varepsilon) := \begin{pmatrix} D_0 \\ \varepsilon D_1 \\ \vdots \\ \varepsilon^N D_N \end{pmatrix},$$

(5.3)

we have the following lemma:

**Lemma 5.3.** Let $C(\varepsilon)$ and $D(\varepsilon)$ be given by (5.3). Assume that $(A,B,C_0,D_0)$ is left-invertible. Then (A.3) is satisfied if and only if

$$\sigma^*(A,B,C_0,D_0) \cap \mathcal{U}^0 = \emptyset \quad \text{and} \quad \sigma^*(A,B,C,D) \cap \mathcal{U}^0 = \emptyset.$$

**Proof.** Let $\mathcal{V}^\varepsilon$ and $\mathcal{V}^\delta$ denote the output-nulling subspaces of $(A,B,C(\varepsilon),D(\varepsilon))$ and $(A,B,C,D)$ respectively. First note that, since $(A,B,C_0,D_0)$ is left-invertible, $(A,B,C,D)$ is left-invertible and there is $\delta > 0$ such that for all $\varepsilon \in [0,\delta]$ $(A,B,C(\varepsilon),D(\varepsilon))$ is left-invertible.
Consequently, \( \sigma^*(A, B, C(\varepsilon), D(\varepsilon)) = \sigma(A, B, C, D) \) for any \( F \) such that \((A + BF)\hat{V}^* \subseteq \hat{V}^* \) and \( \sigma^*(A, B, \hat{C}, \hat{D}) = \sigma(A, B, \hat{C}, \hat{D}) \) for any \( F \) such that \((A + BF)\hat{V}^* \subseteq \hat{V}^* \). Now, we contend that for all \( \varepsilon > 0 \) we have \( V_\varepsilon^* = \hat{V}^* \).

Let \( \varepsilon > 0 \). Then \( x_0 \in V_\varepsilon^* \) if and only if there is a regular \( u \) such that \( C(\varepsilon)x_0 + D(\varepsilon)u = 0 \). However, the latter equality is equivalent to \( \hat{C}x_0 + \hat{D}u = 0 \). Thus \( x_0 \in V_\varepsilon^* \) if and only if \( x_0 \in \hat{V}^* \). From this it follows that, for \( \varepsilon > 0 \), \( \sigma^*(A, B, C(\varepsilon), D(\varepsilon)) = \sigma^*(A, B, \hat{C}, \hat{D}) \). This proves our lemma.

Finally, for \( \varepsilon \in [0, 1] \) and \( x_0 \in \mathbb{R}^n \), let \( J^*(x_0) \) denote the optimal cost associated with the cost-functional (5.2) and denote by \( u^*_\varepsilon, x^*_\varepsilon, x_0 \) and \( y^*_\varepsilon, x_0 \) the associated optimal control, state-trajectory and output.

**COROLLARY 5.4.**

(i) \( J^*(x_0) \rightarrow J^0(x_0) \) \((\varepsilon \rightarrow 0)\) for all \( x_0 \in \mathbb{R}^n \).

(ii) There exists \( \delta > 0 \) such that for all \( \varepsilon \in [0, \delta] \) and \( x_0 \in \mathbb{R}^n \) \( u^*_\varepsilon, x_0 \) exists and is unique if and only if \((A, B, C_0, D_0)\) is left-invertible, \( \sigma^*(A, B, C_0, D_0) \cap t^0 = \emptyset \) and \( \sigma^*(A, B, \hat{C}, \hat{D}) \cap t^0 = \emptyset \).

(iii) Assuming this to be the case, for all \( x_0 \in \mathbb{R}^n \) we have \( u^*_\varepsilon, x_0 \rightarrow u^0, x_0 \) and \( x^*_\varepsilon, x_0 \rightarrow x^0, x_0 \) \((\varepsilon \rightarrow 0)\) in distributional sense and \( y^*_\varepsilon, x_0 \rightarrow y^0, x_0 \) \((\varepsilon \rightarrow 0)\) strongly.

6. **CONCLUSIONS**

In this paper we have investigated the parameter dependence of the optimal cost, optimal inputs, optimal state-trajectories and optimal outputs associated with a fixed linear system and a quadratic cost-functional depending on a real parameter. The questions we have considered were mainly inspired by similar ones that have been studied before in the context of the problem of 'cheap control' or the 'nearly singular optimal control problem'. 
It was shown that under two assumptions, namely continuity in $\varepsilon = 0$ of the output mappings $C(\varepsilon)$ and $D(\varepsilon)$ appearing in the cost-functional, and monotonicity in a neighbourhood of $\varepsilon = 0$ of the quadratic forms defined by these output mappings, the optimal cost is indeed continuous in $\varepsilon = 0$. This result generalizes and extends corresponding results obtained before in the 'cheap control' context. We stressed that our result is valid independent of any assumptions whatsoever concerning regularity or singularity of the underlying linear-quadratic problems.

Under the same assumptions as above it was shown that as $\varepsilon$ tends to zero then the optimal control inputs, optimal state-trajectories and optimal outputs for the linear-quadratic problems with $\varepsilon$ strictly positive converge to those associated with the linear-quadratic problem for $\varepsilon = 0$, provided that the optimal controls, state-trajectories and outputs exist and are unique for $\varepsilon$ sufficiently small (including $\varepsilon = 0$). Again, no assumptions were made on the regularity or singularity of the linear-quadratic problems. An important tool in our development was the dissipation inequality as introduced in [1] and studied before in [3] and [4].

Finally, we applied our very general results to derive the corresponding convergence results in the context of the 'cheap control' problem and in the context of an optimal control problem of 'priority control', where the cost-criterion was polynomially in powers of $\varepsilon^2$.

Many questions concerning the subject of this paper still need to be resolved. A very interesting open problem is the following. Again consider the systems $(A,B,C(c),D(c))$. Assume that for $\varepsilon > 0$ sufficiently small the optimal controls $u^*_C(\varepsilon)$ exist and are unique but that for $\varepsilon = 0$ the optimal controls exist but are not unique. This situation occurs if and only if there exists $\delta > 0$ such that $\sigma^*(A,B,C(\varepsilon),D(\varepsilon)) \cap \mathcal{C} = \emptyset$ for $\varepsilon \in [0,\delta]$, $(A,B,C(c),D(c))$ is left-invertible for $\varepsilon \in (0,\delta]$ and $(A,B,C(0),D(0))$ is not left-invertible. We may then pose the following questions: do the optimal controls $u^*_C(\varepsilon + 0)$ converge as $\varepsilon$ tends to zero? If they do, is their limit an optimal control for the linear-quadratic problem associated with $\varepsilon = 0$? Is it possible to give a characterization of the limit in terms of the system $(A,B,C(0),D(0))$? The above questions are closely connected with the existence of a nonzero controllable output-nulling subspace of the system $(A,B,C(0),D(0))$. In [16] some preliminary results concerning these questions were obtained, again in the context of 'cheap
control'. The general problem however seems to be quite difficult and its solution will undoubtedly involve extra assumptions on the $\varepsilon$-dependence of the mappings $C(\varepsilon)$ and $D(\varepsilon)$.

REFERENCES


