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ON A SET OF DIOPHANTINE EQUATIONS

by

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We are interested in the following set of equations to be solved in integers:

\[ N + 1 = x^2, \]
\[ 3N + 1 = y^2, \]
\[ 8N + 1 = z^2. \]

The problem originates from the following one. The set \( A: \{0,1,3,8,120\} \) has the property that if \( x \) and \( y \) \((x \neq y)\) are taken from \( A \) then \( xy + 1 \) is a square. This implies that we know the solutions \( N = 0, N = 120 \) of (1.1). The question is whether more members can be added to \( A \) without losing the property mentioned above. Clearly any member that can be so added to \( A \) is a solution of (1.1).

This problem is mentioned in Dickson [1], vol 2, p 517. Apparently Diophantus first studied the problem of finding four numbers such that the product of any two increased by unity is a square. Fermat studied the equations (1.1) and found the solution \( N = 120 \). Euler was able to add a fifth positive number, which is not an integer, to the set \( A \). In the past few years the problem has appeared in many places. After it was stated as a problem in [2] we proved that (1.1) has no solutions \( N \) with \( 120 < N < 10^{200} \) ([4]). This result and our method were reported at the 1968 Oberwolfach meeting on Number Theory. There are several ways of reducing (1.1) in such a way that Thue's theorem can be applied. Hence it was known that (1.1) had only a finite number of solutions. The problem of finding lower bounds for \( N \) became more interesting when at the same Oberwolfach meeting A. Baker reported on his results concerning effective bounds in Thue's theorem. Application of his theorem yields a very large number \( C \) such that (1.1) has no solutions \( N \) with \( N > C \). It was remarked that such a number \( C \) can also be found by applying recent results of N.I. Fel'dman. Another approach is possible by using a method due to W. Ljunggren ([5]).

In a discussion with A. Baker we agreed that it might be possible to show that (1.1) has no other solutions than \( N = 0, N = 120 \) by increasing the lower bound \( 10^{200} \) and decreasing the constant \( C \). In this note we show that (1.1) has no solutions in the interval \( 120 < N < 10^{1700000} \). The computation of this upper bound on a EL-X8 computer took only 7 minutes. It can very easily be increased using methods described below.
2. A computer-search

We replace the first two equations of (1.1) by

$$3x - y = 2A,$$

$$y - x = 2B,$$

$$A^2 - 3B^2 = 1.$$  

The solutions $A_n, B_n$ of Pell's equation $A^2 - 3B^2 = 1$ are given by

$$A_n + B_n \sqrt{3} = (2 + \sqrt{3})^n.$$  

The quotients $A_n B_n^{-1}$ are convergents of the continued fraction for $\sqrt{3}$. We now find all solutions of the first two equations of (1.1) from (2.1) by taking $x_n = A_n + B_n$.

We find the sequence

$$x_0 = 1, x_1 = 3, x_2 = 11, \ldots,$$

satisfying

$$x_{n+1} = 4x_n - x_{n-1}.$$  

Explicitly, we have

$$x_n = \left(\frac{3 + \sqrt{3}}{6}\right) (2 + \sqrt{3})^n + \left(\frac{3 - \sqrt{3}}{6}\right) (2 - \sqrt{3})^n.$$  

In the same way, we treat the equations $x^2 = N + 1$ and $z^2 = 8N + 1$. Here we have $z^2 - x^2 = 7N$, hence $z \equiv \pm x \pmod{7}$.

If $z \equiv -x \pmod{7}$ we write

$$8x + z = 7C,$$

$$x + z = 7D,$$

$$C^2 - 8D^2 = 1.$$  

We find $C_n + D_n \sqrt{8} = (3 + \sqrt{8})^n$ and in the same way as above the sequence of solutions $\overline{x}_n$:

$$\overline{x}_0 = 1, \overline{x}_1 = 2, \overline{x}_2 = 11, \ldots,$$

satisfying

$$\overline{x}_{n+1} = 6\overline{x}_n - \overline{x}_{n-1}.$$
Explicitly:

\[(2.7) \quad \overline{x}_n = \left(\frac{4 + \sqrt{2}}{8}\right) (3 + \sqrt{8})^m + \left(\frac{4 - \sqrt{2}}{8}\right) (3 - \sqrt{8})^m.\]

If \( z \equiv x \mod (7) \) we transform to Pell's equation by

\[
8x - z = 7c, \\
z - x = 7d, \\
c^2 - 8d^2 = 1.
\]

This leads to the sequence \( x_k^* \) given by:

\[(2.8) \quad x_0^* = 1, \quad x_1^* = 4, \quad x_2^* = 23, \ldots ,\]

\[(2.9) \quad x_{k+1}^* = 6x_k^* - x_{k-1}^* ,\]

\[(2.10) \quad x_k^* = \left(\frac{4 + \sqrt{2}}{8}\right) (3 + \sqrt{8})^k + \left(\frac{4 - \sqrt{2}}{8}\right) (3 - \sqrt{8})^k.\]

On a EL-X8 computer all terms of the sequences (2.2), (2.5) and (2.8) less than \( 10^{1200} \) were generated. The following pairs \((n,m)\) for which \(|\log x_m^* - \log x_n^*| < 10^{-3}\) and the pairs \((n,k)\) for which \(|\log x_n^* - \log x_k^*| < 10^{-3}\) were found:

\[(2.11) \quad n = 2, \quad m = 2, \quad \overline{x}_2 = \overline{x}_2 = 11,\]

\[(2.12) \quad n = 1125, \quad m = 841 , \quad \log \overline{x}_m^* - \log x_n^* = 0.000725 \quad (\text{rounded upwards}),\]

\[(2.13) \quad n = 1643, \quad m = 1228, \quad \log x_n^* - \log \overline{x}_m^* = 0.000307 \quad (""""""),\]

\[(2.14) \quad n = 289 , \quad k = 216 , \quad \log x_n^* - \log x_k^* = 0.000456 \quad (""""""),\]

\[(2.15) \quad n = 1412, \quad k = 1055, \quad \log x_k^* - \log x_n^* = 0.000706 \quad (""""""),\]

\[(2.16) \quad n = 1930, \quad k = 1442, \quad \log x_n^* - \log x_k^* = 0.000330 \quad ("""""").\]
3. A continued fraction method

Using continued fractions we shall extend the results of section 2. Let \((v, \mu)\) be a pair of integers such that \(x_v = (1 + \varepsilon)x_\mu\) where \(|\varepsilon|\) is small. Consider (2.4) and (2.7) for \(n, v (n > v)\) and \(m, \mu (m > \mu)\) respectively and take logarithms. If \(x_n = x_m\) then we have

\[(n - v) \log(2 + \sqrt{3}) + \log\{1 + (2 - \sqrt{3})^{2n+1}\} - \log\{1 + (2 - \sqrt{3})^{2v+1}\} =
\]

\[= (m - \mu) \log(3 + \sqrt{8}) + \log\{1 + \frac{9 + 4\sqrt{2}}{7}(3 - \sqrt{8})^{2m}\} - \log\{1 + \frac{9 + 4\sqrt{2}}{7}(3 - \sqrt{8})^{2v}\} - \log(1 + \varepsilon)
\]

i.e.

\[(3.1) \quad \left| \frac{n - v}{m - \mu} - \frac{\log(3 + \sqrt{8})}{\log(2 + \sqrt{3})} \right| = c(\varepsilon, v, \mu). \]

We shall use the following theorems (cf [3] chapter 10):

(3.2) If \(\xi\) is irrational and \(\left| \frac{p}{q} - \xi \right| < \frac{1}{2q^2}\) then \(\frac{p}{q}\) is a convergent of the continued fraction for \(\xi\).

(3.3) If \(\frac{p_n}{q_n}(p_n, q_n = 1), n = 1, 2, \ldots\) are the convergents of \(\xi\) then

\[\frac{1}{q_n q_{n+2}} < |\frac{p_n}{q_n} - \xi| < \frac{1}{q_n q_{n+1}}.\]

We apply these theorems to

\[\xi = \frac{\log(3 + \sqrt{8})}{\log(2 + \sqrt{3})} = [1, 2, 1, 20, 1, 5, 3, 8, 5, 1, 2, 1, 1, 1, 1, 4, 3, \ldots].\]
The first 16 convergents of $\xi$ are:

\[
\frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{83}{62}, \frac{87}{65}, \frac{518}{387}, \frac{1641}{1226}, \frac{13646}{10195}, \frac{69871}{52201}, \frac{83517}{62396}, \frac{236905}{176993}, \frac{320422}{239389},
\]

\[
\frac{557327}{416382}, \frac{877749}{655771}, \frac{1435076}{1072153}, \frac{6618053}{4944383} \quad (27 \text{ seconds computation on EL-X8}).
\]

We know from section 2 that $x_n = \overline{x}_m > 11$ implies $m > 1500$. Now assume $m < 3364$ and apply (3.1) with $v = 1643, \mu = 1228$. For $c(\varepsilon, v, \mu)$ we find 0.000234. Hence

\[
\left| \frac{n - 1643}{m - 1228} - \xi \right| = \frac{0.000234}{m - 1228} < \frac{1}{2(m - 1228)^2}.
\]

By (3.2) and (3.3) this is only possible if

\[
m - 1228 = 2 \times 387 \quad \text{and} \quad n - 1643 = 2 \times 518,
\]

or

\[
m - 1228 = 1226 \quad \text{and} \quad n - 1643 = 1641.
\]

Since $x_{1679} \equiv x_{3254} \equiv 1 \pmod{4}$ and $\overline{x}_{2002} \equiv \overline{x}_{2454} \equiv 3 \pmod{4}$ these two possible cases are excluded and hence we have

\[
(3.4) \quad x_n \neq \overline{x}_m \quad \text{if } 2 < m < 3364.
\]

In the same way we treat $x_n = \overline{x}_k$, starting from (2.16). The result then is

\[
(3.5) \quad x_n \neq \overline{x}_k \quad \text{if } k < 3442.
\]

From (3.4) and (3.5) we see that:

\[
(3.6) \quad \text{The system (1.1) has no solutions } N \text{ with } 120 < N < 10^{5000}.
\]
4. A sieve method

The best results we have found up to now were obtained by the following sieve method. Each of the sequences \( \{x_n\}, \{x_r\}, \{x^*_k\} \) is periodic mod \( p \) (for every integer \( p \)). The condition \( x_n = \overline{x_m} \) implies, by considering \( x_n \equiv \overline{x_m} \mod (4) \):

\[ [n \equiv 0 \text{ or } 2 \mod (4) \text{ and } m \equiv 0 \mod (4)] \text{ or } [n \equiv 1 \text{ or } 2 \mod (4) \text{ and } m \equiv 2 \mod (4)] \]

and also, by considering \( x_n \equiv \overline{x_m} \mod (3) \):

\[ [n \equiv 1 \text{ or } 5 \mod (6) \text{ and } m \equiv 1 \text{ or } 3 \mod (4)] \text{ or } [n \equiv 2 \text{ or } 3 \mod (4) \text{ and } m \equiv 1 \text{ or } 2 \mod (4)] \]

Combination of these yields

\[ [n \equiv 0 \text{ or } 11 \mod (12) \text{ and } m \equiv 0 \mod (4)] \text{ or } [n \equiv 2 \text{ or } 9 \mod (12) \text{ and } m \equiv 2 \mod (4)] \]

By considering \( x_n \mod p \) for successive primes new conditions for \( n \) and \( m \) are found. This was executed on the EL-X8 for the primes \( \leq 57 \) (5 minutes computing time). The same thing was done for \( x_n \) and \( x^*_k \) and the primes \( \leq 53 \) (2 minutes computing time).

The results were:

\[ x_n = \overline{x_m} \text{ implies:} \]

\[ n \text{ or } -n - 1 \mod (2550240) \text{ is one of the following numbers} \]

(a) 0, 191520, 695519, 887040;
(b) 80640, 110880, 584639, 776160;
(c) 2, 665277, 927357, 957602;
(d) 110882, 776157, 816477, 846722;
(e) 151197, 181442, 776162, 1108802;
(f) 665279, 856800, 997919, 1189440;

and

\[ m \mod (36340920) \text{ is one of the following numbers} \]

(a) 0, 5191560, 26860680, 32052240;
(b) 6320160, 11511720, 20540520, 25732080;
If $n$ is taken from the set (a) then $m$ must be in (a) etc.

(4.2) $x_n = x_k^2$ implies for $n$ what was stated above and $k \equiv r \pmod{36340920}$ is one of the following numbers:

(a) 0, 4288680, 9480240, 31149360;
(b) 10608840, 15800400, 24829200, 30020760;
(c) 3837238, 14220358, 25957798, 36340918;
(d) 10157398, 19637638, 20540518, 30020758;
(e) 13317478, 16477558, 23700598, 26860678;
(f) 1128600, 3160080, 6320160, 34309440.

Again set (a) for $n$ is combined with set (a) for $k$ etc.

These conditions can be combined with the results of section 3 but even without doing that we immediately see that

(4.3) the system (1.1) has no solutions $N$ with $120 < N < 10^{1700000}$.

With very little extra work this bound can be improved to $10^{10^{10}}$. 
References


\[ ax^2 - by^2 = c, \quad a_1 z^2 - b_1 y^2 = c_1, \]