Analysis of the outer product for the symmetric group

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Analysis of the outer product for the symmetric group

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Expressions are derived to write the basis vectors for an irreducible representation \( \mu \) of the symmetric group in terms of basis vectors for irreducible representations whose outer product yields \( \mu \).

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I. INTRODUCTION

It has been noticed\(^1\)\(^2\) that the symmetric group can be used to calculate recoupling coefficients for special unitary groups SU(\( N \)). The most obvious approach is to study the properties of the representations of the symmetric group in a tensor space. For this it is necessary to consider the outer product of the symmetric group in some detail. In particular, one must know how to express the basis vectors for an irreducible representation (irrep) \( \mu \) in basis vectors belonging to irreps whose outer product gives \( \mu \). The factors which give these relations are called outer coefficients. These outer coefficients are very important because the recoupling coefficients for SU(\( N \)) can be written\(^3\) as products of outer coefficients and Clebsch–Gordan coefficients for the symmetric group independent of \( N \).

The outer coefficients can be calculated in a number of ways. The first possibility is to use projection operators and the matrix form of the representations of the symmetric group. This is done in Sec. II. The second method generates the outer coefficients for \( S_p \) recursively from the outer coefficients for \( S_{p-1} \). Sections IV and V deal with this method. Section VI gives a graphical rule for a few special cases. Our notation for the representations of the symmetric group is given in Appendix A.

II. OUTER COEFFICIENTS

Suppose \( \mu \) is an irreducible representation (irrep) of \( S_p \). It is defined in a vector space \( V(\mu) \) with an orthonormal basis \( e_{\mu i}^{(\mu)} \). The matrix elements of \( \mu \) are written as

\[
D^{(\mu)}(s)|e_{\mu i}^{(\mu)}\rangle = \sum_M e_{\mu i}^{(M)} D^{(\mu)}(M)_M(s)
\]

for all elements \( s \) of \( S_p \). We choose the standard form for the vectors \( e_{\mu i}^{(\mu)} \). Standard means in this context that the basis vectors are labeled with Young's symbols and that the matrix elements of \( \mu \) are in the "Young's orthogonal form" (see Appendix A, Eq. (40)).

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III. OUTER COEFFICIENTS

The asterisk, denoting complex conjugation, is superfluous when the coefficients are real, as is the case for the symmetric group. From now on we will omit this asterisk everywhere.

The problem is: How to calculate these outer coefficients? Or to state it differently: How to construct the basis vectors \( e_{\mu i}^{(\kappa \times \lambda ; \mu\gamma)} \) for this we use the projection and shift operators defined in Appendix B.

\[
P^{(\kappa \times \lambda)}_{\mu}(|s\rangle) = \sum f(\kappa) f(\lambda) p_1|s_1\rangle + \ldots + \sum p_2|s_2\rangle D^{(\kappa \times \lambda)}_{\mu}(|s_1s_2\rangle)
\]

where \( s_1 \) and \( s_2 \) are elements of \( S_{p_1} \) and \( S_{p_2} \) respectively.

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We use only real matrix elements for the irreps of the symmetric group. From now on we will omit this asterisk everywhere.

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\]

where \( s_1 \) and \( s_2 \) are elements of \( S_{p_1} \) and \( S_{p_2} \) respectively.
The square of this norm is equal to
\[ P_{KL,11}^{(x,\lambda)} e_{\mu}^{(p_1)} e_{\nu}^{(p_2)} = \frac{\lambda}{p_2!} \sum_{s_2} D_{L,1}^{(\mu)}(s_2) D_{M,\lambda,1M'}(p_2)p_2'(s_2) \]
\[ \times \delta(K, \mu/M'(p_2)) \delta(1, \mu'(p_2)) e_{\mu}^{(p_2)} . \]

In this equation the labels \( K' \), \( L' = 1, 1 \) correspond to the first Yamanouchi symbols in the standard ordering for \( \kappa \) and \( \lambda \) (see Appendix A). We have used the fact that the product of the permutation \( s_2 \) and \( s_2' \) commute. Furthermore, the following general orthogonality relation has been used for the matrix elements of irreps of a group \( G \) of order \( f(G) \).

\[ \sum_{\sigma \in G} D_{\sigma}^{(\mu)}(g) D_{\sigma}^{(\nu)}(g^*) = \frac{f(G)}{f(\mu)} \delta(\mu, \mu') \delta(M, M') \delta(N, N') , \]

where \( f(\mu) \) is the dimension of the irrep \( \mu \).

According to the prescription given in Appendix B all we have to do is:

— Apply \( P_{11,11}^{(x,\lambda)} \) to all vectors \( e_{\mu}^{(p)} \).

— Orthonormalize the result. This means that the orthonormal vectors \( e_{\mu}^{(x,\lambda)} \) are the result of the action of the projection operator upon a certain linear combination of vectors \( e_{\mu}^{(p)} \). They can be expressed as

\[ e_{\mu}^{(x,\lambda)} = P_{11,11}^{(x,\lambda)} \sum_{\gamma} \alpha(\gamma, M') e_{\mu}^{(\gamma)} = \sum_{\gamma} S_{\mu \lambda}^{(\mu, \gamma)} e_{\mu}^{(\gamma)} . \]

— Let the other shift operators act upon the same linear combination of vectors \( e_{\mu}^{(p)} \). Again the resulting vectors \( e_{\mu}^{(x,\lambda)} \) are expressed as a linear combination of vectors \( e_{\mu}^{(p)} \).

\[ e_{\mu}^{(x,\lambda)} = P_{11,11}^{(x,\lambda)} \sum_{\gamma} \alpha(\gamma, M') e_{\mu}^{(\gamma)} = \sum_{\gamma} S_{\mu \lambda}^{(x,\lambda)} e_{\mu}^{(\gamma)} . \]

It is possible to simplify the outer coefficients. In Appendix A we show that for elements \( s_2 \) of \( S_{p_2} \) the matrix elements of the representation \( p_2 \) only depend upon the part of the Young diagram associated with \( M(p_2) \). So if we define \( \kappa \) as being the diagram belonging to the tableau \( K \), the following relation holds for the matrix elements appearing in (5):

\[ D_{M,\lambda,1M'}(p_2) = \delta(K, M(p_2)) D_{M,M'}(p_2)p_2'(s_2) . \]

The number \( \delta(K, M(p_2)) \) can always be factored out of \( S_{\mu \lambda}^{(x,\lambda)} \), in a trivial fashion. This means that the outer coefficients do not really depend upon \( K \). Therefore, we will replace the outer coefficient by the notation \( \gamma_{\kappa \lambda}^{\mu \nu}(p_2) \).

\[ S_{\kappa \lambda}^{\mu \nu} = \delta(K, M(p_2)) \gamma_{\kappa \lambda}^{\mu \nu}(p_2) . \]

We will now study the case in which there is no degeneracy \( \gamma \) present and derive an expression for the outer coefficients. When the product is not degenerate, it is sufficient to choose one vector \( e_{\mu}^{(p_1)} \) for which the result of the projection operator \( P_{11,11}^{(x,\lambda)} \) is unequal to zero. The normalization is then carried out by dividing by the norm \( N \) of the result.

The square of this norm is equal to

\[ N^2 = \frac{\lambda}{p_2!} \sum_{s_2} D_{L,1}^{(\mu)}(s_2) D_{M,\lambda,1M'}(p_2)p_2'(s_2) \]
\[ = \frac{\lambda}{p_2!} \sum_{s_2} D_{L,1}^{(\mu)}(s_2) D_{M,\lambda,1M'}(p_2)p_2'(s_2) . \]

The result of the other shift operators must be divided by the same norm. So the outer coefficient will be

\[ S_{\kappa \lambda}^{\mu \nu} = \frac{1}{N} \frac{\lambda}{p_2!} \sum_{s_2} D_{L,1}^{(\mu)}(s_2) D_{M,\lambda,1M'}(p_2)p_2'(s_2) . \]

For the shorthand outer coefficients defined in (10) a similar formula holds:

\[ \gamma_{\kappa \lambda}^{\mu \nu}(L, M(p_2)) = \frac{1}{N} \frac{\lambda}{p_2!} \sum_{s_2} D_{L,1}^{(\mu)}(s_2) D_{M,M'}(p_2)p_2'(s_2) . \]

We have fixed an overall phase by choosing some particular \( M' \) and dividing out the norm (instead of the norm times some phase factor). It turns out that in this particular case the sign of the result is independent of the choice of \( M' \) (provided, of course, that the result is unequal to zero).

Consider now the degenerate case. We adopt the following phase convention: any nonzero outer coefficient which has the following properties is positive:

— it must contain the first \( L \) in the standard ordering of the different \( L \)’s belonging to \( \lambda \);

— it has the first possible \( M \) for \( \mu \) (that means the outer coefficient is nonzero).

### III. SOME PROPERTIES OF THE OUTER COEFFICIENTS

We will derive some useful equations for the outer coefficients. Apply \( D_{L}^{(\mu)}(s_1,s_2) \) to (2b), where \( s_1 \) is an element of \( S_{p_1} \) and \( s_2 \) of \( S_{p_2} \). For the left-hand side of the equation this results in

\[ \sum_{\mu} e_{\mu}^{(p)} D_{M,M'} p_2'S_{\kappa \lambda}^{\mu \nu}(p_2,s_2) \]

for the right-hand side we find

\[ \sum_{\kappa \lambda} S_{\kappa \lambda}^{\mu \nu}(p_2,s_2) \]

Putting (14) and (15) together, we find, after removing the vector from the equation, interchanging the left- and right-hand side and choosing \( s_1 = e \),

\[ \sum_{\kappa \lambda} S_{\kappa \lambda}^{\mu \nu}(p_2,s_2) \]

From now on we will use the shorthand notation given in (10) for the outer coefficients. We also introduce the abbreviation

\[ \nu = \mu/\kappa \]

for the skew-symmetric Young diagram found by subtracting \( \kappa \) from \( \mu \). The label \( N \) is used to denote the corresponding part of the Yamanouchi symbol \( M \) [we used to write this in the form \( (p_2) \)]. Equation (16) will then look like

\[ \sum_{\nu} \gamma_{\kappa \lambda}^{\mu \nu}(L, N') \gamma_{\nu \lambda}^{\mu \nu}(L, N') \]

In the following we will also omit the argument \( s_2 \) from the representation matrices \( D \). Shifting the outer coefficient to the right yields
\[ D_{L,L'}^{(\lambda)} \delta(\lambda, \lambda') \delta(\gamma, \gamma') = \sum_{N,N'} \left( \begin{array}{c} \lambda \\ L \\ N' \\ \gamma \end{array} \right) \left( \begin{array}{c} \lambda' \\ N \\ \gamma' \end{array} \right) D_{N,N'}^{(\lambda, \lambda')}(L, N). \]  
(19)

We can shift the outer coefficient in (18) to the left:
\[ \sum_{\lambda \ell \gamma} \left( \begin{array}{c} \lambda \\ L \\ N' \\ \gamma \end{array} \right) D_{L,L'}^{(\lambda)} \left( \begin{array}{c} \lambda' \\ L \\ N \\ \gamma' \end{array} \right) = D_{N,N'}^{(\lambda, \lambda')}. \]  
(20)

**IV. RECURSION COEFFICIENTS**

The method described in Sec. II to calculate the outer coefficients has the disadvantage that for larger values of \( p_2 \) the work becomes extremely time-consuming. To solve this problem, we show that the outer coefficients for a given \( p_2 \) can be calculated recursively from coefficients for \( p_2 - 1 \). To obtain the recursion coefficients which relate the outer coefficients for \( p_2 \) and \( p_2 - 1 \), one has to solve a simple set of coupled linear equations.

Consider the elements \( s_2 \) of \( S_{p_2} \) which leave \( p \) invariant. They form a subgroup \( S_{p_2 - 1} \) of \( S_{p_2} \). For these elements we may write
\[ D_{L,L'}^{(\lambda)}(s_2) = D(L, L') D_{N,N'}^{(\lambda, \lambda')}(s_2) \]  
(21)

We have introduced here the subscript asterisk to denote that the last number of a Yamanouchi symbol \( M \) has been omitted: \( M' = M_{p_1} \cdots M_{p_2 - 1} \). Inserting the restriction (21) into (19), one finds
\[ D_{L,L'}^{(\lambda)}(p) \delta(\gamma, \gamma') = \sum_{N,N'} \left( \begin{array}{c} \lambda \\ L \\ N' \\ \gamma \end{array} \right) D_{N,N'}^{(\lambda, \lambda')}(L, N). \]  
(22)

We apply now Eq. (20) to representations \( \lambda / L_p \) or \( \rho / R_p \) of \( S_{p_2 - 1} \) and \( \nu / N_p \) of \( S_{p_2 - 1} \) (limited to the last \( p_2 - 1 \) objects). The corresponding Yamanouchi symbols are \( L \) or \( R \) and \( N \). We find
\[ D_{N,N'}^{(\nu / N_p)} = \sum_{R \ll p \rho / R_p} \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right) D_{R,R_p}^{(\rho / R_p)}(L, N). \]  
(23)

One now inserts (23) in (22) and shifts two outer coefficients to the left-hand side to find
\[ \sum_{L \ll N} \left( \begin{array}{c} \lambda \\ L \ll N' \gamma \end{array} \right) \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right) D_{L,L'}^{(\lambda)} = \sum_{R \ll N} \left( \begin{array}{c} \lambda \\ L \ll N' \gamma \end{array} \right) \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right) D_{R,R_p}^{(\rho / R_p)}. \]  
(24)

where \( L' \ll = L_p \) and \( N' \ll = N_p \) is assumed. The above equation can also be written in the following form:
\[ \sum_{L \ll N} D_{L,L'}^{(\lambda, \lambda')}(\lambda, L_p, \nu / N_p, \gamma / R_p, \beta) \]  
(25)

where we have defined
\[ Z(\lambda, L_p, \nu / N_p, \gamma / R_p, \beta) = \sum_{R \ll N} \left( \begin{array}{c} \lambda \\ L \ll N' \gamma \end{array} \right) \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right). \]  
(26)

Now we suppress all indices which are the same in the left- and right-hand side of (25). The simplified notation is
\[ \sum_{L \ll N} D_{L,L'}^{(\lambda, \lambda')}(L, \rho / R_p) \]  
(27)

The above equation is in fact nothing else than a matrix equation for \( D \) and \( Z \). We will apply Schur's lemma to (27). This lemma says that from (27) follows that either \( Z \) is zero when \( \rho / R_p \neq \lambda / L_p \) or else \( Z \) is a multiple of the unit matrix when \( \rho / R_p = \lambda / L_p \). Therefore,
\[ Z(\lambda, L_p, \nu / N_p, \gamma / R_p, \beta) = \delta(\lambda / L_p, \rho / R_p) \delta(L_p, R_p) \delta(N_p, N_p). \]  
(28)

Now we fill in the definition (26) of \( Z \):
\[ \sum_{L \ll N} \left( \begin{array}{c} \lambda \\ L \ll N \gamma \end{array} \right) \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right) = \delta(\lambda / L_p, \rho / R_p) \delta(L_p, R_p) \delta(N_p, N_p). \]  
(29)

Shifting one outer coefficient to the right-hand side of the equation yields the recursion relation we were looking for:
\[ \delta(L \ll N \lambda, \rho / R_p) = \sum_{R \ll N} R_{L, R_p}^{\nu / N_p} \left( \begin{array}{c} \lambda \ll N \gamma \end{array} \right) \left( \begin{array}{c} \rho / R_p \\ R \ll N_p \beta \end{array} \right) \]  
(30a)
or
\[ S_{K,M}^{\nu / N_p} = \sum_{R \ll N} R_{L, R_p}^{\nu / N_p} S_{K,L,M}^{\lambda, \lambda' / L_p}. \]  
(30b)

With the help of (30a) or (30b) we are able to calculate all outer coefficients for \( p_2 \) when the outer coefficients for \( p_2 - 1 \) and the recursion coefficients \( R_{L, R_p}^{\nu / N_p} \) are known. For \( p_2 = 1 \) we have
\[ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 1. \]  
(31)

It is straightforward to prove that the recursion coefficients satisfy the following orthogonality relations:
\[ \sum_{L \ll N} R_{L, R_p}^{\nu / N_p} R_{L, R_p}^{\nu / N_p} \delta(L / L_p, \lambda / L_p) \]  
(32a)
and
\[ \sum_{L \ll N} R_{L, R_p}^{\nu / N_p} R_{L, R_p}^{\nu / N_p} \delta(N_p, N_p) \delta(\beta, \beta) \]  
(32b)
In (32b) the factor \( \delta(L / L_p, \lambda / L_p) \) means that one has to
sum all \( \lambda \) and \( L_p \) for which \( \lambda / L_p \) is equal to some given \( \lambda' / L_p' \).

V. RELATIONS FOR THE RECURSION COEFFICIENTS

In this section we derive a set of equations that can be used to calculate the recursion coefficients. Consider the transposition \((p - 1, p)\). According to Eq. (40) the matrix element for this element of the symmetric group is given by

\[
D_{\lambda L_p}^{\lambda' L_p'}(p - 1, p) = (\pi_{\lambda L_p}^{\lambda L_p'}(p - 1, p)
+ \tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1})
+ \tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1}) \delta(L_{p - 1}, L_p)
+ \tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1}) \delta(L_{p - 1}, L_p)
+ \tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1}) \delta(L_{p - 1}, L_p)
\]

and the same with \( \nu, N \) instead of \( \lambda, L \). We have used the notation \( M_{\cdots} = M_1 \cdots M_{p - 2} \) for a Yamanouchi symbol \( M \) with the last two numbers omitted. Furthermore, \( \sigma \) is the inverse of the axial distance \( \rho \) defined in (A2) and \( \tau = (1 - \sigma^2)^{1/2} \).

Inserting (33) in (18) yields

\[
\tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1}) \delta(L_{p - 1}, L_p)
= \tau(\lambda, L_p, L_p - 1) \delta(\lambda', L_{p - 1}) \delta(L_p, L_{p - 1}) \delta(L_{p - 1}, L_p).
\]

We have used \( \tau(\lambda, L_p, L_p - 1) = \tau(\lambda, L_p - 1, L_p) \). For the recursion coefficients we find

\[
\{\lambda(\lambda, L_p, L_p - 1) - \sigma(\nu, N_p, N_p - 1)\} \sum_{\beta} R_{L_p, L_p'}^\nu N_p^\nu N_p^p \begin{pmatrix} \lambda / L_p & \nu / N_p \\ L_p \end{pmatrix}
+ \tau(\lambda, L_p, L_p - 1) \sum_{\beta} R_{L_p, L_p'}^\nu N_p^\nu N_p^p \begin{pmatrix} \lambda / L_p - 1 & \nu / N_p \\ L_p - 1 \end{pmatrix}
= \tau(\nu, N_p, N_p - 1) \sum_{\beta} R_{L_p, L_p'}^\nu N_p^\nu N_p^p \begin{pmatrix} \lambda / L_p - 1 & \nu / N_p - 1 \\ L_p \end{pmatrix}
\]

VI. GRAPHICAL RULES

A graphical rule to calculate the recursion coefficients for the case that \( \lambda = [p] \) or \( \lambda = [l^p] \) can be given.

Consider first the case \( \lambda = [p] \). Suppose one wants to calculate the recursion coefficient \( R_{L_p, L_p'}^\nu N_p^p \). For \( \lambda = [p] \) and also for \( \lambda = [l^p] \) there are no degeneracy labels \( \gamma \) and \( \beta \) present.

Choose the first possible \( N_p \) in the standard ordering to form a Yamanouchi symbol \( N \) with \( N_p \). Then the above recursion coefficient is equal to

\[
R_{L_p, L_p'}^\nu N_p^p = (p)^{-1/2} \prod_{q \neq p} \left[ 1 + \sigma(\nu, N_p, N_q) \right]^{1/2}. \tag{36}
\]

For \( \lambda = [l^p] \) the formula is

\[
R_{L_p, L_p'}^\nu N_p^p = (p)^{-1/2} \prod_{q \neq p} \left[ 1 - \sigma(\nu, N_p, N_q) \right]^{1/2}, \tag{37}
\]

where \( e(s) \) is the sign of the permutation \( s \) which transforms the first Yamanouchi symbol in the standard ordering into the Yamanouchi symbol \( N \). The label \( q \) runs from \( p + 1 \) to \( p - 1 \).

As an example we consider \( p = 5, p_2 = 3 \),

\[
\nu = \{221\}/\{11\} \quad \text{and} \quad N_p = 2. \text{ Then } N_p = 13. \text{ So the permutation } s \text{ which transforms the first Yamanouchi symbol for } \nu \text{ (which is equal to 123) into } N = 132 \text{ is equal to } s = (23). \text{ Therefore, } s \text{ is odd. The inverse axial distances in-}

\[
\text{are solved are} \quad \sigma(2,1) = -1 \text{ and } \sigma(2,3) = \frac{1}{2}.
\]

Therefore, for \( \lambda = [l^3] \) and \( L_p = 3 \),

\[
R_{L_p, L_p'}^\nu N_p^p = -(3)^{-1/2}(1 + 1)^{1/2}(1 - \frac{1}{2})^{1/2} = - (\frac{1}{2})^{1/2}.
\]

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APPENDIX A: THE SYMMETRIC GROUP

1. General remarks

A Young diagram \( \mu = [\mu_1, \cdots, \mu_p] \) is a figure containing \( p \) boxes ordered in \( p \) rows of length \( \mu_i \), with the properties:

\[
\mu_1 \geq \cdots \geq \mu_p > 0 \text{ and } \mu_1 + \cdots + \mu_p = p. \tag{A1}
\]

A standard Young tableau is a diagram which contains the numbers 1 to \( p \) in such a way that the numbers in each row increase from left to right and in each column increase from top to bottom.

Each standard tableau can be written in a compact way by a Yamanouchi symbol \( M = M_1, \cdots M_p \). This is an array of \( p \) numbers, the \( M_i \) being the rows in the standard tableau in which the number \( i \) appears. For example, the standard Young tableaux and Yamanouchi symbols for the diagram [31] are given by

\[
\begin{array}{c|c|c|c}
1 & 1 & 2 & 3 \\
\hline
4 & 5 & 8 & 9
\end{array} = 1121, \quad \begin{array}{c|c|c|c|c}
1 & 2 & 4 \\
3
\end{array} = 1121
\]

and
\[
\begin{bmatrix}
1 & 3 & 4 \\
2 &
\end{bmatrix} = 1211.
\]

Note that our notation differs from the one used by Hamermesh. The Yamanouchi symbols belonging to some diagram (and therefore the corresponding Young tableaux) can be ordered. The symbol \( M \) comes before the symbol \( N \) if \( M < N \) when the symbols are regarded as composite numbers (lexicographic ordering).

Consider two boxes \( x \) and \( y \) in a Young diagram \( \mu \). Box \( x \) is at the position \((a,b)\), where \( a \) denotes row and \( b \) column, and \( y \) at \((c,d)\). The axial distance \( \rho(\mu; x, y) \) between these boxes is equal to
\[
\rho(\mu; x, y) = |a-c| + |b-d|.
\]
It is the number of steps (horizontal or vertical) from \( x \) to \( y \). The steps are counted positive going down or to the left and negative when going up to or the right.

The different irreducible representations (irreps) of \( S_p \) can be represented by Young diagrams. Let the irrep \( \mu \) of \( S_p \) be defined in a vector space \( V(\mu) \). The orthonormal basis vectors \( e^{(\mu)}_v \) in this space can be labeled by the Yamanouchi symbols \( \mu \). The matrices of the transpositions \((i, i + 1)\) in the "Young's orthogonal form"\(^5,6\) are given by
\[
D^{(\mu)}_{(i, i + 1)} = \frac{1}{\rho(\mu; M_i, M_{i+1})} e^{(\mu)}_{M_i} - e^{(\mu)}_{M_{i+1}},
\]
\[
D^{(\mu)}_{(i, i + 1)} = \frac{1}{\rho(\mu; M_i, M_{i+1})} e^{(\mu)}_{M_i} - e^{(\mu)}_{M_{i+1}},
\]
where \( \rho(\mu; M_i, M_{i+1}) \) is the inverse axial distance between the boxes corresponding to \( M_i \) and \( M_{i+1} \).

2. Subgroup representations

Consider subgroups \( S_{p_1} \) and \( S_{p_2} \) of \( S_p \), where \( p_1 + p_2 = p \). Here \( S_{p_1} \) and \( S_{p_2} \) are the permutation groups of the first \( p_1 \) objects and the last \( p_2 \) objects. The Yamanouchi symbol \( M \) for the symmetric group \( S_p \) is adapted for the subgroups \( S_{p_1} \) and \( S_{p_2} \).

\[
M = M_{p_1} \cdots M_{p_1} \cdots M_{p_2} \cdots M_{p_2} \equiv (M_{p_1} | M_{p_2}).
\]

The part \( (M_{p_1}) \) of the Yamanouchi symbol \( M \) forms again a valid Yamanouchi symbol for the group \( S_{p_1} \). It belongs to a Young tableau for \( S_{p_1} \), which can be obtained from the tableau of \( M \) by removing the boxes with the numbers \( p_1 + 1, \ldots, p \). This new tableau for \( S_{p_1} \) belongs to a Young diagram that is denoted as \( M_{p_1} | M_{p_2} \) or \( M_{p_1} M_{p_1} \cdots M_{p_2} \). From (A3) it follows immediately that the matrix elements of an irreducible representation of any transposition of the subgroup \( S_{p_1} \) depend only upon that part of the Young diagram where the numbers \( 1, \ldots, p_1 \) have been put. Hence the same holds for general permutations in the subgroup \( S_{p_1} \). So the elements \( s_i \) of \( S_{p_1} \) leave the last \( p_2 \) numbers in the Yamanouchi symbol invariant. We have
\[
D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_v = \sum_{M_{p_1}} e^{(\mu)}_{M_{p_1}} D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_{M_{p_1}} e^{(\mu)}_v,
\]
\[
= \sum_{M_{p_1}} e^{(\mu)}_{M_{p_1}} D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_v.
\]

For transpositions (and also for general elements) of the subgroup \( S_{p_1} \) holds analogously that the matrix elements of the irreducible representations only depend upon the form of that part of the diagram where the numbers \( p_1 + 1, \ldots, p \) have been placed. These elements \( s_i \) of \( S_{p_1} \) leave the first \( p_1 \) numbers in the Yamanouchi symbol invariant:
\[
D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_v = \sum_{M_{p_1}} e^{(\mu)}_{M_{p_1}} D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_{M_{p_1}} e^{(\mu)}_v,
\]
\[
= \sum_{M_{p_1}} e^{(\mu)}_{M_{p_1}} D^{(\mu)}_{(i, i + 1)} e^{(\mu)}_v.
\]

The skew-symmetric diagram obtained from the diagram \( \mu \) by omitting the boxes \( M_{p_1}, \ldots, M_{p_1} \) will be written down as \( \mu \) or \( \mu / M_{p_1} \) or \( \mu / M_{p_1} \) or \( \mu / / \). The Young diagram which corresponds to the boxes which contain the numbers \( 1, \ldots, p \).

APPENDIX B: PROJECTION AND SHIFT OPERATORS

Consider a vector space \( V \) where a representation \( D \) of a group \( G \) is defined. Suppose \( D \) contains the irrep \( \Gamma(\mu) \). The shift operator \(^7\) is defined by
\[
P^{(\mu)}_{\mu} = \frac{f(\mu)}{f(G)} \sum_{G} D^{(\mu)}_{\mu} (g) D^{(\mu)}_{\mu} (g),
\]

where \( f(\mu) \) is the number of \( \mu \) and \( f(G) \) is the number of elements \( g \) of \( G \). \( P \) is a projection operator if \( \mu = M' \). When the basis vectors of the irreducible subspaces are given by \( e^{(\mu)}_v \), the following relations hold:
\[
P^{(\mu)}_{\mu} e^{(\nu)}_v = \delta(\mu, \nu) e^{(\mu)}_v,
\]
\[
P^{(\mu)}_{\mu} e^{(\nu)}_v = \delta(\mu, \nu) e^{(\mu)}_v.
\]

The procedure for constructing a basis for the irreducible subspaces is as follows:
—apply \( P^{(\mu)}_{\mu} \) to all vectors of \( V \) (1 means the first basis vector);
—orthonormalize the result; this will lead to vectors \( e^{(\nu)}_v \), where \( \nu \) runs from 1 to \( \Gamma(\mu) \);
—compute the vectors \( e^{(\nu)}_v = P^{(\mu)}_{\mu} e^{(\nu)}_v \);
—the space \( V^{(\nu)} \) spanned by the vectors \( e^{(\nu)}_v \) will form invariant subspaces of \( V \), such that the restriction of \( D \) to \( V^{(\nu)} \) is equivalent to \( \mu \).

APPENDIX C: TABLE OF RECURRENCE COEFFICIENTS

We have tabulated the recursion coefficients \( R^{(\mu)}_{\mu} \) for \( p_1 = 2 \) and \( p_2 = 3 \). In these cases there is no degeneracy label \( \beta \) present. In Table I we denote the inverse of the axial distance \( \rho \) and the box in the upper right corner by \( x \). For example, \( v_1 \) (see Fig. 1) could stand for \( \mu \), where \( x \) is equal to \(-1\). One sees here that when boxes of a diagram \( v \) in the table touch only at the corners they can be shifted with respect to each other. For the diagram \( v_2 \) (see Fig. 1) the situation is somewhat more complicated. Number the boxes according to the first Yamanouchi symbol. This means that 1 is the upper
### Table I. Table of recursion coefficients.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$N_2$</th>
<th>$L_2$</th>
<th>$[2]$</th>
<th>$[11]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(1 + x)/2</td>
<td>(1 - x)/2</td>
<td>(1 - x)/2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$N_3$</th>
<th>$L_3$</th>
<th>$[3]$</th>
<th>$[21]$</th>
<th>$[21]$</th>
<th>$[111]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
<td>1</td>
<td>(1 + 2x)/3</td>
<td>2(1 - x)/3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td>(1 + 2x)/3</td>
<td>2(1 - x)/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
<td>(1 - x)/3</td>
<td>2(1 - x)/3</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$L_3$</th>
<th>$\gamma$</th>
<th>$N_3$</th>
<th>$R_{L_3, N_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>3</td>
<td>3</td>
<td>(1 + y)(1 + x)/3</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>3</td>
<td>(1 + y)(1 + x)/3</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1</td>
<td>3</td>
<td>(1 - y)(1 + x)/3</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td>1</td>
<td>(1 + y)(1 + x)/3</td>
</tr>
</tbody>
</table>

**FIG. 1.** Graphical representation of the diagrams $\nu_1, \mu_1$, and $\nu_2$.  

box, 2 is the middle box, and 3 is the lower box. Then $x$ is the inverse axial distance from 3 to 1, $y$ from 3 to 2, and $z$ from 2 to 1. The variables $x$, $y$, and $z$ are related via $1/x = 1/y + 1/z$.  

Throughout the table we have used the abbreviation $a = -zy^2 - 2yz + 2z - y^2 + 2y$.  

The recursion coefficients in the table yield outer coefficients.

$$a = -zy^2 - 2yz + 2z - y^2 + 2y.$$
with phases according to the convention of Sec. II. A √ must be added over each entry in the table. For example, 
− (1 + x)/2 means − [(1 + x)/2]^{1/2}.