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Plenty of Franklin Magic Squares, but none of order 12

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Abstract

We show that a genuine Franklin Magic Square of order 12 does not exist. This is done by choosing a representation of Franklin Magic Squares that allows for an exhaustive search of all order 12 candidate squares. We further use this new representation (in terms of polynomials) to generate large classes of true Franklin Magic Squares of orders 8 and multiples of 16. Next we show how Franklin Magic Squares of orders \( n = 20 + 8k \) can be constructed. Finally we indicate how almost-Franklin Magic Squares of order 20 can be constructed in a general way.

1 Franklin Magic Squares

According to various descriptions a natural Franklin Magic Square of even size \( n \) is a square matrix \( M \) with \( n \) rows and columns with the properties

1. the entries of \( M \) are \( 1, 2, \ldots, n^2 \);

2. each row and each column has a fixed entry sum \( n(1 + n^2)/2 \);

3. each two by two sub-square \[ \begin{bmatrix} M_{i,j} & M_{i,j+1} \\ M_{i+1,j} & M_{i+1,j+1} \end{bmatrix} \] has sum \( 2(1 + n^2) \);

4. each half row starting in column 1 or \( n/2 + 1 \) has sum of entries equal to \( n(1 + n^2)/4 \), and similar for half columns starting in row 1 or \( n/2 + 1 \);

5. each half of the main diagonal (starting in column 1 or \( n/2 + 1 \)) together with each half of the back diagonal has total sum (such as \( \sum_{i=1}^{n/2} (M_{i,i} + M_{i,n+1-i}) \) equal to \( n(1 + n^2)/2 \). This construction is called a bent diagonal. The sum requirements also hold for so-called bent rows, which are translates of the two half-diagonals, possibly wrapping over the matrix sides.

These squares are called after the former US president and scientist Benjamin Franklin who constructed a few of such matrices, two of order eight, one of order 16. Note that the fourth property implies that \( n \) is a multiple of four. It turns out that the condition on 2x2 subsquares is the most prominent one and generates a lot of shapes with a constant-sum-property. In particular we have that

2x2 squares wrapping along one side of the matrix also have fixed sum \( 2(1 + n^2) \), and further that
for arbitrary $i, j, k$ the entries in

$$\begin{bmatrix}
M_{i,j} & M_{i,j+1+2k} \\
M_{i+1,j} & M_{i+1,j+1+2k}
\end{bmatrix}$$

have sum $2(1 + n^2)$. Combining this on two consecutive rows $i, i + 1$ we find that for arbitrary $i, j, k$,

$$M_{i,j} + M_{i,j+1+2k} \text{ and } M_{i+2,j} + M_{i+2,j+1+2k}$$

have equal values.

In the end this leads to the observation that for arbitrary $i, j, k, m$ we have that
each four-tuple $M_{i,j}, M_{i+2m+1,j}, M_{i,j+2}+1, M_{i+2m+1,j+2}+1$ has sum $2(1 + n^2)$.

This property of Franklin Magic Squares is often referred to as the mirroring property because its consequence is that on any shape that is symmetric horizontally and vertically along a line separating rows or columns, respectively, the entries of square add up to a number that is independent of the choice of the intersection of the axes of symmetry. Here we allow moving over the border of the square by embedding it on a torus. Note that this property is merely based on the 2x2 sub-square property.

Applying the above insights on the top halves of the main and back diagonal we find that the sum-of-half-diagonals property is equivalent (given the 2x2 square with fixed sum) to the statement that $M_{1,1}, M_{2,2}, M_{1,3}, M_{2,4}, \ldots, M_{1,n/2−1}, M_{2,n/2}$ and $M_{2,n/2+1}, M_{1,n/2+2}, \ldots, M_{2,n−1}, M_{1,n}$ together sum up to $n(1 + n^2)/2$.

Subtracting $n/4$ ‘subsquares’ $[M_{1,2k}, M_{2,2k}, M_{1,n+1−2k}, M_{2,n+1−2k}]$ of constant sum we find that $M_{1,1}−M_{1,2}+\ldots+M_{1,n/2−1}−M_{1,n/2}$ plus its mirror image $M_{1,n+1−2}−M_{1,n+1−2}+\ldots+M_{1,n−1−n/2−1}−M_{1,n−1−n/2}$ equals zero.

Adding a full row sum leads to a pattern of $n/2$ entries, with

$$(M_{1,1}+M_{1,3}+\ldots+M_{1,n/2−1})+(M_{1,n/2+2}+M_{1,n/2+4}+\ldots+M_{1,n}) = n(1+n^2)/4$$

which holds for ordinary magic squares with the 2x2 square property and the bent-diagonals-property.

Subtracting a fixed half row sum starting in column $n/2 + 1$ we finally obtain the property that

$$M_{1,1}+M_{1,3}+\ldots+M_{1,n/2−1} \text{ equals } M_{1,n/2+1}+M_{1,n/2+3}+\ldots+M_{1,n−1}$$

This alternate sum property is hence equivalent with the bent-diagonal property (in presence of the other franklin conditions), but much more easily checked. Obviously a similar reasoning is possible for vertical bent-diagonals, leading to columns having the alternate sum property.

All this leads to the following more compact definition of a Franklin Magic Square of arbitrary order $4k$, which is a matrix with properties:

1. entries are $1, \ldots, n^2$;
2. each 2x2 sub-square has entries summing up to $2(1 + n^2)$;
3. the first half of the first row, the second half of the first row, the first half of the first column, and the second half of the first column, each have entries that sum up to $n(1 + n^2)/4$;
4. entries on odd positions in the first half of the first row add up to the same value as entries on odd positions in the second half of the first row; similarly, entries on odd positions in the first half of the first column add up to the same value as entries on odd positions in the second half of the first column.

2 More compact representation

2.1 Isomorphisms

It turns out that any Franklin Magic Square maintains its magic properties under a number of matrix transformations, namely:

1. reflection along the horizontal, or vertical axis of symmetry;
2. permutation of row (column) indices within the sets $S_1 = \{2k + 1 | 0 \leq k < n/4\}$, $S_2 = \{2k | 1 \leq k \leq n/4\}$, $S_3 = \{2k + 1 | n/4 \leq k < n/2\}$ and $S_4 = \{2k | n/4 < k \leq n/2\}$;
3. exchanging the $n/4$ rows (columns) indexed by $S_1$ with those indexed by $S_3$; similarly, exchanging the $n/4$ rows (columns) indexed by $S_2$ with those indexed by $S_4$;
4. reflection along the diagonal;
5. replacing each entry $M_{ij}$ by $n^2 + 1 - M_{ij}$.

The first three properties suffice to prove that we can assume without loss of generality that the first entry $M_{11} = 1$. It is evident that the transformations above leave the compact definition of Franklin Magic Squares intact.

2.2 Bookkeeping

Based on the 2x2 sub-square property the square can be fixed by determining the entries on the first row and first column. For computational reasons it is more convenient to index rows and columns by $0, \ldots, n-1$, and to subtract 1 from each entry in the Franklin square, so that the entries become $0, \ldots, n^2 - 1$. Note that now the average entry value is $\nu = (n^2 - 1)/2$, instead of $(n^2 + 1)/2$. We now assume that the upper leftmost element is zero. We call this Franklin square basic instead of natural.

Next consider the following transformation $C(F)$ on any Franklin Magic Square $F$:

$$V_{ij} = C(F)_{ij} := \begin{cases} F_{ij} & \text{if } i + j \equiv 0 \text{ modulo } 2 \\ n^2 - 1 - F_{ij} & \text{if } i + j \equiv 1 \text{ modulo } 2 \end{cases}$$

which can be viewed as complementing entries on black positions (of the underlying chess board). Note that $F = C(V)$.

The 2x2 sub-square property of $F$ translates into a favorable property for $V$, namely: $V_{ij} + V_{i+1,j+1} - V_{i,j+1} - V_{i+1,j} = 0$, for all $i, j$. Based on this property, having the zero in $F_{00}$ gives that $V$ has the nice property that $V_{ij} = V_{0i} + V_{0j}$. Hence to generate candidate Franklin Magic Squares $F$ we enumerate all vectors $x = (x_0, \ldots, x_{n-1})$ and $y = (y_0, \ldots, y_{n-1})$, with properties:
1. $x_0 = y_0 = 0$;

2. $x_0 < x_2 < \ldots < x_{n/2-2}$,
   \[ x_1 < x_3 < \ldots < x_{n/2-1}, \]
   \[ x_{n/2} < x_{n/2+2} < \ldots < x_{n-2}, \]
   \[ x_{n/2+1} < x_{n/2+3} < \ldots < x_{n-1}; \]

3. $y_0 < y_2 < \ldots < y_{n/2-2}$,
   \[ y_1 < y_3 < \ldots < y_{n/2-1}, \]
   \[ y_{n/2} < y_{n/2+2} < \ldots < y_{n-2}, \]
   \[ y_{n/2+1} < y_{n/2+3} < \ldots < y_{n-1}; \]

4. $x_0 + x_2 + \ldots + x_{n/2-2} =
   \[ x_1 + x_3 + \ldots + x_{n/2-1} = \]
   \[ x_{n/2} + x_{n/2+2} + \ldots + x_{n-2} = \]
   \[ x_{n/2+1} + x_{n/2+3} + \ldots + x_{n-1}; \]

5. $y_0 + y_2 + \ldots + y_{n/2-2} =
   \[ y_1 + y_3 + \ldots + y_{n/2-1} = \]
   \[ y_{n/2} + y_{n/2+2} + \ldots + y_{n-2} = \]
   \[ y_{n/2+1} + y_{n/2+3} + \ldots + y_{n-1}; \]

6. $x_1 < x_{n/2+1};$

7. $y_1 < y_{n/2+1};$

8. $\max_i y_{2i} > \max_j x_{2j};$

9. $0 \leq y_i + x_j \leq n^2 - 1$, for all $i, j$;

10. the set \{ $y_i + x_j | i + j \equiv 0 \} \cup \{ n^2 - 1 - y_i - x_j | i + j \equiv 1 \}$ equals \{0, \ldots, $n^2 - 1$\}.

3 Not Finding the 12x12 Franklin Magic Square

Evidently the enumeration should be kept to a minimum by pruning the search for candidate Franklin Magic Squares as early as possible. For the 12 by 12 Franklin Square the following strategy turns out to lead to a manageable enumeration scheme.

1. generate a 3x6 sub-matrix on the 6 columns with even index and on rows indexed 0, 2, 4;

2. extend this to a 6x6 sub-matrix on the even columns and the even rows;

3. extend to a 9x9 sub-matrix adding rows and columns indexed 1, 3 and 5;

4. finally extend to a full 12x12 matrix

At each stage, before adding (three) more rows or (three) more columns, we update a list of candidate $x_j$ or $y_i$ values, given the partially filled $F$. Note that for instance, after the first step, possible values for $y_0, y_8, y_{10}$ come from a limited common domain, consistent with the 3x6 sub-matrix already filled.

Proceeding in this way we generate:
831083 tuples $x_2, x_4, x_6, x_8, x_{10}$;
40467771 extensions $y_4$,
1473501105 extensions $y_2, y_4$, i.e. 3x6 sub-matrices
25663243622 extensions $y_2, y_4, y_{10}$,
24473864360 extensions $y_2, y_4, y_6, y_8, y_{10}$, i.e. 6x6 squares,
22532519520 of which cannot be ruled out immediately;
121404978 9x9 extensions,
93083 of which might be extended to a 12x12 square.

In the end, none of these would lead to the desired Franklin Magic Square. Computation of these cases was carried out by a network of 50 computers. For this we split the work into 70 cases, corresponding with the possible settings for $x_2 \in \{1, \ldots, 70\}$. (For a higher value for $x_2$ we would have that $x_4 + y_{\text{max}} \geq 72 + 73 = 145 > \max\{0, \ldots, 12^2-1\}$. The total computation time was approximately 160 hours.

4 A generic scheme for building Magic Squares

The above described formulation in terms of vectors $x$ and $y$ can also be used in a generic way to generate (Franklin) Magic Squares with the 2x2 sub-square property for arbitrary even order, with or without additional properties. To this purpose we formulate the magic square properties in terms of an equation in polynomials.

4.1 An encoding in polynomials

The polynomials we consider have coefficients 0 and 1. Let $\delta(P)$ denote the degree of a polynomial $P$. Then for a polynomial in $z$, $P(z)$, of degree $\delta$, and any number $\nu \geq \delta$, let

$$P_{\nu}(z) = z^{\nu - \delta}P(1/z).$$

We are more or less writing $P$ backwards, or better, we are reflecting its exponents with respect to the value $\nu/2$. If we do not mention $\nu$ we take by convention the degree of the polynomial.

Let us now associate with each (Franklin) Magic Square, given in terms of $x$ and $y$, the polynomials

$$A(z) := \sum_{j=0}^n z^{x_j}, \quad B(z) := \sum_{i=0}^n z^{y_i}.$$

Let us further split these summations over odd and even indices: $A(z) = A_0(z) + A_1(z)$, with $A_k(z) := \sum_{j=0, j \equiv k} z^{x_j}$, for $k = 0, 1$; and $B(z) = B_0(z) + B_1(z)$, with $B_k(z) := \sum_{i=0, i \equiv k} z^{y_i}$, for $k = 0, 1$. A square with numbers $\{0, \ldots, n^2 - 1\}$ with the 2x2 sub-square property then satisfies the condition:

$$(A_0 B_0 + A_1 B_1)(z) + A_0 B_1 + A_1 B_0 \delta_{n^2-1}(z) = \frac{z^{n^2} - 1}{z - 1}$$

which simply stipulates that all numbers are present once in the matrix. Here $A_0, A_1, B_0, B_1$ are polynomials with $n/2$ terms each. Solving the above system (to find the square) is possible if one restricts to certain types of solutions. One such restriction (**Type 1a**) could be to choose

$$B_0 = B_1 = B_0^{\delta(B_0)}$$

which leads to the following simplification of the equation above:

$$(AB_0)(z) + \overline{A} B_0^{\delta(B_0)}(z) = (A + \overline{A}^{\delta(B_0)-1})(z) B_0(z) = \frac{z^{n^2} - 1}{z - 1}$$

A variant of this approach (**Type 1b**) would be to choose
\[ B_0 = B_1 \neq B_0^{\delta(B_0)} \text{ and } A = A^{\delta(A)} \] (4)

which leads to the simplification:

\[ (AB_0)(z) + AB_0^{n^2-1}(z) = A(z)(B_0 + B_0^{n^2-1-\delta(A)})(z) = \frac{z^{n^2} - 1}{z - 1} \] (5)

A second approach (Type 2) could be to assume the existence of a number \( \nu \) such that

\[ A_0 = A_0^{\nu^2-1-\nu} \quad A_1 = A_1^{\nu^2-1-\nu} \quad B_0 = B_0^{\nu} \quad B_1 = B_1^{\nu} \] (6)

A third approach (Type 3) is to assume an integer \( \nu \) exists such that

\[ A_0 = A_0^{\nu^2-1-\nu} \quad A_1 = A_1^{\nu^2-1-\nu} \quad B_0 = B_0^{\nu} \quad B_1 = B_0^{\nu} \] (7)

Both (6) and (7) translate equation (1) into the simple

\[ A(z)B(z) = \frac{z^{n^2} - 1}{z - 1} \] (8)

Now, for \( n = 2^q k \), with odd \( k \), the right hand side in equation (1) can be rewritten as

\[ \frac{z^{n^2} - 1}{z - 1} = \frac{z^{2^q k^2 2^q} - 1}{z^{2^q k^2 2^q} - 1} \frac{z^{2^q k^2 2^q} - 1}{z^{2^q k^2 2^q} - 1} \prod_{j=0}^{2^q-1} (1 + z^{2^j}) \]

Note that the first two factors in this decomposition are polynomials in \( z \) with \( k \) terms each, whereas the other factors are two-term polynomials. In case \( k = 1 \) this decomposition is unique, but for other values there are many possible decompositions.

Using the first method to solve (1), we look for a candidate polynomial \( B_0 \), with \( n/2 \) terms, by selecting one of the two first factors, and \( q - 1 \) factors from the other ones. Their product is indeed a polynomial in \( n/2 \) terms, and is symmetric (meaning \( B_0 = B_0^{\nu} \), for some \( \nu \)). The factors not selected form a product \( \Sigma(z) \) that is in fact a symmetric polynomial in \( 2n \) terms, and that we have to set equal to the sum \( A + A^{n^2-1-\delta(B_0)} \) by an appropriate choice for \( A \).

For the variant we select one factor from the first two and next \( q \) factors from the second part so as to build \( A \). The co-factor (with \( n \) terms) must then match \( B_0 + B_0^{n^2-1-\delta(A)} \) for an appropriate choice of \( B_0 \).

When using the second or third method, we may define \( B \) say, by taking one of the two first factors, and adding \( q \) factors from the other ones. Their product is then a symmetric polynomial in \( n \) terms, which can further be split into \( B_0 \) and \( B_1 \). The remaining factors are used to build \( A_0 \) and \( A_1 \).

In the remainder of the paper we show how to construct various types of Franklin Magic Squares. We first formulate how additional requirements on the constructed squares translate into conditions on the vectors \( x \) and \( y \), and hence on the polynomials \( A_0, A_1, B_0, B_1 \). Define \( X_{ik} = \sum_{j \equiv i, |2j/n|=k} x_j \) for \( i, k \in \{0, 1\} \). Similarly, let \( Y_{ik} = \sum_{j \equiv i, |2j/n|=k} y_j \).

**magic row sum** \( X_{00} + X_{01} = X_{10} + X_{11} \) or, equivalently, the sum of exponents in \( A_0 \) equals the sum of exponents in \( A_1 \);
**Magic column sum** \( Y_{00} + Y_{01} = Y_{10} + Y_{11} \) or, equivalently, the sum of exponents in \( B_0 \) equals the sum of exponents in \( B_1 \);

**Magic sum on horizontal bent diagonals** \( X_{00} = X_{11} \) and \( X_{01} = X_{10} \), or, equivalently, \( A_0 = A_0 + A_{01}, A_1 = A_{10} + A_{11} \) is a split into four polynomials of \( n/4 \) terms each with exponents in \( A_{00} (A_{10}) \) adding up to the same as those in \( A_{11} (A_{01}, \text{respectively}) \);

**Magic sum on vertical bent diagonals** \( Y_{00} = Y_{11} \) and \( Y_{01} = Y_{10} \), or, equivalently, \( B_0 = B_{00} + B_{01}, B_1 = B_{10} + B_{11} \) is a split into four polynomials of \( n/4 \) terms each with exponents in \( B_{00} (B_{10}) \) adding up to the same as those in \( B_{11} (B_{01}, \text{respectively}) \);

**Half the magic sum in first and second half row** \( X_{00} = X_{10} \) and \( X_{01} = X_{11} \), or, equivalently, there is a split of \( A \), as above, with exponents in \( A_{00} \) summing to the same as those in \( A_{10} \), and exponents in \( A_{01} \) summing to the same as those in \( A_{11} \);

**Half the magic sum in first and second half column** \( Y_{00} = Y_{10} \) and \( Y_{01} = Y_{11} \), or, equivalently, there is a split of \( B \), as above, with exponents in \( B_{00} \) summing to the same as those in \( B_{10} \), and exponents in \( B_{01} \) summing to the same as those in \( B_{11} \);

**Pan-diagonal magic sum** \( X_{00} + X_{01} + X_{10} + X_{11} + Y_{00} + Y_{01} + Y_{10} + Y_{11} = n(n^2 - 1)/2 \), or equivalently, exponents in \( A \) and \( B \) add up to the magic sum;

**Most-perfect** This means: complementary entries lie on the same diagonal, \( n/2 \) positions apart. That is, \( M_{i,j} + M_{i+n/2,j+n/2} = (n^2 + 1) \), for all \( i, j \). We then have \( x_j + x_{j+n/2} + y_i + y_{i+n/2} = n^2 - 1 \), for all \( i, j \), implying that \( x_j + x_{j+n/2} = \delta(A) \), for all \( j < n/2 \), and \( y_i + y_{i+n/2} = \delta(B) \), for all \( i < n/2 \), and each of \( A_0, A_1, B_0, B_1 \) must be symmetric;

**Four-on-a-row** This means: blocks of 4 consecutive entries partitioning a row (or column) each have magic entry sum. In other words, \( M_{i,4k+1} + M_{i,4k+2} + M_{i,4k+3} + M_{i,4k+4} = 2(n^2 + 1) \), for all \( i, k \). Then \( x_{4j} + x_{4j+2} = x_{4j+1} + x_{4j+3} \), for all \( j \), implying that pairs of exponents in \( A_{00} \) match with pairs of exponents in \( A_{10} \) having the same sum, etcetera.

### 4.2 Simple Magic Squares

#### 4.2.1 Method 1a

If we pose no further restrictions, then for each symmetric polynomial \( \Sigma(z) \) of \( 2n \) terms we can easily find \( 2^n \) different solutions \( A(z) \) as follows: for the \( n \) smallest powers \( z^j \) in \( \Sigma \) we have that \( z^{N-j} \) is in \( \Sigma \) as well, where \( N = \delta(\Sigma) \). For each \( j \), select one of \( \{j, N-j\} \) to be in the set of powers of \( A \). For instance

\[
1 + z^2 + z^3 + z^6 + z^9 + z^{12} + z^{13} + z^{15} = (1 + z^3 + z^6 + z^{13}) + (1 + z^3 + z^6 + z^{13})^{15}
\]

Now \( A \) and \( B \) will lead to a square matrix of numbers \( \{0, \ldots, n^2 - 1\} \) satisfying the \( 2 \times 2 \) sub-square property. Each column will have fixed sum \( n(n^2 - 1)/2 \), for the simple reason that \( B_0 \) and \( B_1 \) are equal.

In order to have fixed row sums as well, we should be able to split \( A(z) = A_0(z) + A_1(z) \), with exponents in \( A_0 \) adding to the same sum as the exponents in \( A_1 \). This can be done in
general as follows. Let $1 + z^{2j}$ be a factor of $\Sigma$, that is $\Sigma(z) = (1 + z^{2j})\Omega(z)$, where $\Omega$ is symmetric as well, and with (even) $n$ terms. Pair the terms in $\Omega(z)$ with matching exponents $(z^t + z^{{\delta}(\Omega) - t})$. Taking $A(z) = \Omega(z)$ gives $\overline{A}^{\delta(\Omega)+2j}(z) = z^{2j}A(z)$. Now split $A$ into $A_0$ and $A_1$ by taking for $A_0$ half of the $n/2$ pairs in $\Omega(z)$, and for $A_1$ the remaining $n/4$ pairs. As each pair contributes $\delta(\Omega)$ to the sum of exponents, the split will be balanced. So the exponents in $A_0$ will add up to the same sum as those in $A_1$, and hence the resulting matrix will have constant row sums.

4.2.2 Method 1b

In order to find Type 1b squares with fixed row and column sum, we have to be able to split the $n$-term polynomial $A$ into two parts of equal exponent sum. This is easily achieved by the method described above: form pairs of matching terms $z^j, z^{{\delta}(A) - j}$, and divide these pairs over two groups of equal size. If $n$ is a multiple of four, such a split is possible in $\binom{n/2}{n/4}$ ways, for any given $A$.

In order to find a matching $B_0$ it suffices to pair matching terms $z^j, z^{2n-j-\delta(A) - j}$ and select one of each pair as an element of $B_0$. There are $2^{n/2}$ possible $B_0$'s, for a given $A$. By definition $B_0$ and $B_1$ have the same sum of exponents.

4.2.3 Method 2

After decomposing $\frac{z^{n^2+1}}{z-1} = A(z)B(z)$ where both $A$ and $B$ have $n$ terms, we have lots of ways to split $A$ into parts $A_0, A_1$ with $A_i = \overline{A_i}^{\delta(A)}$ by pairing the $j$th highest term in $A_i$ and then assign half of these pairs to $A_0$ and the other half to $A_1$. Similar for $B$. There are $\binom{n/2}{n/4}$ ways to generate $A_0$ and there are $\binom{n/2}{n/4}$ ways to generate $B_0$ (for fixed $A$ and $B$). We only need $n$ to be a multiple of 4.

Note that by keeping matching exponents close together method 2 as described above generates 4x4 blocks that have the property that each row, each column and each (broken) diagonal has magic sum $2(n^2 + 1)$. So if we apply this method to generate squares of order $4k$, only caring about 4 fields on-a-row having fixed sum $2(n^2 + 1)$, we get squares that have the property that each 8x8 sub-square has all the Franklin Magic Square properties as far as row, column and bent-diagonal sums are concerned, with average entry value equal to $(n^2 + 1)/2$. For instance take $n = 12$, $A(z) = (1 + z^{16} + z^{96})(1 + z^9)(1 + z^4)$, and $B(z) = (1 + z^{16} + z^{92})(1 + z^9)(1 + z)$. With $x = (0, 4, 108, 104, 8, 12, 100, 96, 48, 52, 60, 56)$ and $y = (0, 1, 35, 34, 2, 3, 33, 32, 16, 17, 19, 18)$ we obtain the square $M_{12,2}$ given in Figure 1. It contains four 8x8 subsquares, aligned with the 4x4 block structure, with all the Franklin Magic Square properties (except for containing 64 consecutive numbers). Notice that each 4x4 block in the structure is most-perfect in the sense that its 2x2 subsquares are one another’s complement. Further observe that in this example every 4x4 block with upper left entry in an odd row and an odd column has magic row and column sum! This can be enforced in general by building the $x$-vector in strips of four with values $j, j + \alpha, N - j, N - j - \alpha$, for some fixed $\alpha$, and similarly build the $y$-vector in strips of four of value $j, j + \beta, N' - j, N' - j - \beta$, for some fixed $\beta$. Here $N = \delta(A)$ and $N' = \delta(B)$. The features are highlighted in bold font.
4.2.4 Method 3

After decomposing \( \frac{z^n - 1}{z - 1} = A(z)B(z) \) where both \( A \) and \( B \) have \( n \) terms, we have lots of ways to split \( A \) into parts \( A_0, A_1 \) with \( A_0 = \overline{A_1} \). However we need to enforce equal sums of exponents. By extracting a factor \((1 + z^\alpha)(1 + z^\beta)\) from \( A(z) \):

\[
A(z) = (1 + z^\alpha)(1 + z^\beta)\Omega(z),
\]

we can take care for this. Match terms \( z^j \) and \( z^{N-j} \) in \( \Omega(z) \), where \( N = \delta(\Omega) \), and write \((1 + z^\alpha)(1 + z^\beta)(z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})\). Both four-tuples have exponent sum \( 2(N + \alpha + \beta) \). Assign one 4-tuple to \( A_0 \), and the other to \( A_1 \). There are \( 2^{n/8} \) such assignments, for fixed \( \alpha, \beta \), and there are many ways to choose \( \alpha \) and \( \beta \), for fixed \( A \).

Similarly, for \( B \) we find several ways to come up with a proper partition into \( B_0 \) and \( B_1 \).

4.3 Magic Squares with bent-diagonals with magic sum

As indicated before, the properties of 2x2 subsquares having fixed sum, and rows and columns having fixed magic sum, lead to the equivalence of the bent-diagonal property with the condition that odd positions in the first half and even positions in the second half of the first row add up to half the magic sum. This in terms of \( x \) means \( x_0 + x_2 + \ldots + x_{n/2-2} = x_{n/2+1} + \ldots + x_{n-1} \), and in terms of our polynomials this means that \( A_0 \) and \( A_1 \) must have a subset of \( n/4 \) terms each with the same exponent sum.

If \( n \) is a multiple of 8, the above construction of \( A \) by method 1a or 1b already provides such a decomposition of \( A \). And for \( B = B_0 + B_1 \) it is easy to distribute the terms of \( B_0 \) and \( B_1 \) in a symmetric way. Simply take \( y_i = y_{n-1-i} \), for all \( i \) (we had \( B_0 = B_1 \)).

If \( n \) is a multiple of 4, method 2 applied in the previous section yields pairs of terms each with the same exponent sum. Keeping these pairs adjacent (i.e. on positions \( i \) and \( i+2 \)) and
in the same half (i.e. \( i + 2 < n/2 \) or \( i \geq n/2 \)) yields \( x \) and \( y \) vectors with the right properties. This requires \( n \) to be a multiple of 8.

As method 3 generates 4-tuples of equal exponent sum we can nicely distribute such 4-tuples provided \( n \) is a multiple of 16. Simply keep 4-tuples adjacent (on positions \( i, i + 2, i + 4, i + 6 \)) and on the same half.

### 4.4 Magic Squares with half rows having half the magic sum

If we insist on the property of having half rows with half the magic sum, and not necessarily having the bent-row property, we can do the same as in the previous subsection. Indeed, in order to have half the magic sum in the first half of the first row it suffices to have a subset of \( n/4 \) terms in \( A_0 \) and a subset of \( n/4 \) terms in \( A_1 \) having the same sum of exponents. But this was exactly the same condition we needed for having bent-diagonals with magic sum.

By interchanging the columns \( 1, 3, \ldots, n/2 - 1 \) with the set of columns \( n/2 + 1, n/2 + 3, \ldots, n - 1 \) a magic square with magic sum on horizontal bent diagonals transforms into one with half the magic sum on half rows, and vice versa. Similarly for vertical bent diagonals and half columns with half the magic sum. Hence the construction for magic squares with bent diagonals having magic sum, can be used to generate magic squares with half the magic sum on half rows and half columns.

### 5 Full Franklin Magic Squares of order \( 8k \)

We can have both half rows and columns with half the magic sum, and bent-diagonals with magic sum, if both \( A \) and \( B \) can be split into four parts each with \( n/4 \) terms, such that the subsets of \( A \) have the same sum of exponents, and the subsets of \( B \) have the same exponent sum. As an exponent cannot appear four times a requirement is that \( n \) is at least 8.

In this section we discuss general construction methods for Franklin Magic Squares given that the order is a multiple of 8, with and without special features such as pan-diagonality and perfectness.

#### 5.1 Regular constructions of Franklin Magic Squares

##### 5.1.1 Method 1a

By construction along method 1a \( A \) already admits the partition into four parts of equal size and equal exponent sum. For \( B \) we merely have to pair each \( z^j \) in \( B_0(z) \) with \( z^{A(B_0) - j} \). Again if \( n \) is a multiple of 8, it is then possible to split \( B_0 \) into two sets of terms with \( n/8 \) pairs each.

For example, for \( n = 8 \) one can take

\[
\frac{z^{64} - 1}{z - 1} = (z^{32} + 1)(z^{16} + 1)(z^8 + 1)(z^4 + 1)(z^2 + 1)(z + 1)
\]

which yields \( A(z) = (1 + z^{56}) + (z^8 + z^{48}) + (z^{16} + z^{40}) + (z^{24} + z^{32}) \), and \( B_0(z) = (1 + z^3) + (z^1 + z^2) \). Via vectors \( x = (0, 16, 56, 40, 8, 24, 48, 32) \), and \( y = (0, 2, 3, 1, 1, 3, 2, 0) \) we obtain the 8x8 squares.
and finally we obtain a square $M_{1a}$ given in Figure 2.

\begin{figure}[h]
\centering
\begin{tabular}{cccccccc}
1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\
62 & 19 & 6 & 43 & 54 & 27 & 14 & 35 \\
4 & 45 & 60 & 21 & 12 & 37 & 50 & 31 \\
63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\
2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\
61 & 20 & 5 & 44 & 53 & 28 & 13 & 36 \\
3 & 46 & 59 & 22 & 11 & 38 & 51 & 30 \\
64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\
\end{tabular}
\hspace{1cm}
\begin{tabular}{cccccccc}
1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\
60 & 21 & 4 & 45 & 52 & 29 & 12 & 37 \\
6 & 43 & 62 & 19 & 14 & 35 & 54 & 27 \\
63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\
2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\
59 & 22 & 3 & 46 & 51 & 30 & 11 & 38 \\
5 & 44 & 61 & 20 & 13 & 36 & 53 & 28 \\
64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\
\end{tabular}
\caption{Franklin Magic Squares obtained by methods 1a and 1b}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{cccccccc}
0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\
2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\
3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\
1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\
1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\
3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\
2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\
0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\
\end{tabular}
\hspace{1cm}
\begin{tabular}{cccccccc}
0 & 47 & 56 & 23 & 8 & 39 & 48 & 31 \\
61 & 18 & 5 & 42 & 53 & 26 & 13 & 34 \\
3 & 44 & 59 & 20 & 11 & 36 & 51 & 28 \\
62 & 17 & 6 & 41 & 54 & 25 & 14 & 33 \\
1 & 46 & 57 & 22 & 9 & 38 & 49 & 30 \\
60 & 19 & 4 & 43 & 52 & 27 & 12 & 35 \\
2 & 45 & 58 & 21 & 10 & 37 & 50 & 29 \\
63 & 16 & 7 & 40 & 55 & 24 & 15 & 32 \\
\end{tabular}
\caption{Figures 2: Franklin Magic Squares obtained by methods 1a and 1b}
\end{figure}

### 5.1.2 Method 1b

When we apply method 1b we again have to able to split $A$ into four parts with equal exponent sum, and $B_0$ into two parts with equal exponent sum. The first part is easy because $A$ is symmetric and if $n$ is a multiple of 8 we can easily create $n/2$ pairs and partition them over 4 groups. As $B_0$ is not symmetric in all cases we have to enforce this by defining $(B_0 + B_0^n - 1 - \delta(A))(z) = (1 + z^\alpha)\Omega(z)$, and take $B_0 = \Omega$. Match complementary terms and split the set of pairs in two.

For an example of method 1b let us consider $A(z) = (z^{32} + 1)(z^{16} + 1)(z^8 + 1)$ as above, and $(B_0 + \overline{B_0^n})(z) = (z^4 + 1)(z^2 + 1)(z + 1) = (z^2 + 1)B_0(z)$, with $B_0(z) = B_1(z) = (1 + z^5) + (z^1 + z^4)$. With vectors $x = (0, 16, 56, 40, 8, 24, 48, 32)$ and $y = (0, 4, 5, 1, 1, 5, 4, 0)$ this leads to matrix $M_{1b}$ given in Figure 2.

Note that both method 1a and 1b lead to symmetry along the horizontal axis: each entry $f$ mirrors its complement $n^2 + 1 - f$.

### 5.1.3 Method 2

Application of method 2 immediately generates $A$ and $B$ consisting of pairs of terms with sums of exponents equal to $\delta(A)$ and $\delta(B)$, respectively. As a side-result, all matrices obtained
in this way will be magic squares that are pan-diagonal. Further, keeping the pairs adjacent (i.e. on positions $i$ and $i+2$) and on the same half (either $i+2 < n/2$ or $i > n/2$) yields $x$ and $y$ vectors with the right properties, in particular they yield matrices with the bent-diagonal property. For the latter to be true, $n$ must be a multiple of 8.

For an example of method 2 let us consider $A(z) = (z^{32} + 1)(z^{16} + 1)(z^8 + 1)$ as above, and $B(z) = (z^2 + 1)(z^2 + 1)(z + 1)$, with $B_0(z) = (1 + z^7) + (z^1 + z^6)$, and $B_1(z) = (z^2 + z^5) + (z^3 + z^4)$. With vectors $x = (0, 16, 56, 40, 8, 24, 48, 32)$ and $y = (0, 2, 7, 5, 1, 3, 6, 4)$ this leads to matrix $M_2$ given in Figure 3. Again, notice the most-perfectness of the 4x4 blocks.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>48</th>
<th>57</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>62</td>
<td>19</td>
<td>6</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>41</td>
<td>64</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>22</td>
<td>3</td>
<td>46</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3: Franklin Magic Square obtained by method 2

5.1.4 Method 3

Application of method 3 yields 4-tuples of the same exponent sums equal to $2\delta(A)$ and $2\delta(B)$, hence for multiples of 16 it works. As an example let us take $A(z) = (z^{128} + 1)(z^{32} + 1)(z^{16} + 1)(z^8 + 1)$, and $B = (z^{64} + 1)(z^4 + 1)(z^2 + 1)(z + 1)$. For splitting $A$ we take $\alpha = 128$, $\beta = 32$, $N = 24$ and we get $A_{00}(z) = z^0 + z^{152} + z^{56} + z^{160}$, $A_{10}(z) = z^{24} + z^{128} + z^{32} + z^{184}$, $A_{01}(z) = z^8 + z^{144} + z^{48} + z^{164}$, $A_{11}(z) = z^{16} + z^{136} + z^{40} + z^{176}$. For splitting $B$ we have $\alpha = 64$, $\beta = 4$, $N = 3$ yielding $B_{00}(z) = z^0 + z^{67} + z^7 + z^{68}$, $B_{10}(z) = z^3 + z^{64} + z^4 + z^{71}$, $B_{01}(z) = z^1 + z^{56} + z^5 + z^{69}$, $B_{11}(z) = z^2 + z^{65} + z^5 + z^{70}$. With vectors $x = (0, 24, 152, 128, 56, 32, 160, 184, 8, 2, 16, 144, 136, 48, 40, 168, 176)$, and $y = (0, 3, 67, 64, 7, 4, 68, 71, 1, 2, 66, 65, 6, 5, 69, 70)$ this yields $M_3$ as given in Figure 4.

Notice that each 8x8 quadrant is rotationally anti-symmetric: rotating the quadrant by 180 degrees maps each entry on its complement.

5.2 Pan-diagonal Franklin Magic Squares

We may also want to enforce squares with diagonals having the magic sum. Then in addition to the previous conditions we have to restrict ourselves to polynomials $A$ and $B$ each splittable in four subsets of equal exponent sum, such that the sum of exponents of $A_0$ and $B_0$ add up to the desired value $n(n^2 - 1)/4$. Application of methods 2 and 3 directly leads to pan-diagonal Franklin Magic Squares, as by construction the average values of the $x_j$ and $y_i$ add up to $(n^2 - 1)/2$. 

\[
M_2 = \begin{pmatrix}
1 & 48 & 57 & 24 \\
62 & 19 & 6 & 43 \\
8 & 41 & 64 & 17 \\
59 & 22 & 3 & 46 \\
2 & 47 & 58 & 23 \\
61 & 20 & 5 & 44 \\
7 & 42 & 63 & 18 \\
60 & 21 & 4 & 45 \\
\end{pmatrix}
\]
\[
M_3 = \begin{array}{cccccccc}
1 & 232 & 153 & 128 & 57 & 224 & 161 & 72 \\
253 & 28 & 101 & 132 & 197 & 36 & 93 & 188 \\
68 & 165 & 220 & 61 & 124 & 157 & 228 & 5 \\
192 & 89 & 40 & 193 & 136 & 97 & 32 & 249 \\
8 & 225 & 160 & 121 & 64 & 217 & 168 & 65 \\
252 & 29 & 100 & 133 & 196 & 37 & 92 & 189 \\
69 & 164 & 221 & 60 & 125 & 156 & 229 & 4 \\
185 & 96 & 33 & 200 & 129 & 104 & 25 & 256 \\
\end{array}
\begin{array}{cccccccc}
9 & 240 & 145 & 120 & 49 & 216 & 169 & 80 \\
245 & 20 & 109 & 140 & 205 & 44 & 85 & 180 \\
76 & 173 & 212 & 53 & 116 & 149 & 236 & 13 \\
184 & 81 & 48 & 201 & 144 & 105 & 24 & 241 \\
16 & 233 & 152 & 113 & 56 & 209 & 176 & 73 \\
244 & 21 & 108 & 141 & 204 & 45 & 84 & 181 \\
77 & 172 & 213 & 52 & 117 & 148 & 237 & 12 \\
177 & 88 & 41 & 208 & 137 & 112 & 17 & 248 \\
\end{array}
\begin{array}{cccccccc}
2 & 231 & 154 & 127 & 58 & 223 & 162 & 71 \\
254 & 27 & 102 & 131 & 198 & 35 & 94 & 187 \\
67 & 166 & 219 & 62 & 123 & 158 & 227 & 6 \\
191 & 90 & 39 & 194 & 135 & 98 & 31 & 250 \\
7 & 226 & 159 & 122 & 63 & 218 & 167 & 66 \\
251 & 30 & 99 & 134 & 195 & 38 & 91 & 190 \\
70 & 163 & 222 & 59 & 126 & 155 & 230 & 3 \\
186 & 95 & 34 & 199 & 130 & 103 & 26 & 255 \\
\end{array}
\end{array}
\]

Figure 4: Franklin Magic Square by method 3
For methods 1a and 1b we can enforce this feature in various ways.

5.2.1 Method 1a

Consider in the decomposition of \( \frac{z^{n^2-1}}{z-1} \) a factor product of the form

\[
W_{\alpha,\beta}(z) = (1 + z^\alpha)(1 + z^\beta) = 1 + z^\alpha + z^\beta + z^{\alpha+\beta}
\]

We choose \( W_{\alpha,\beta}(z) \) to be a factor of \( A + A^{n^2-1-\delta(B_0)} \). In \( (A + A^{n^2-1-\delta(B_0)})(z) \) let us pair up terms \( z^j \) and \( z^{N-j} \), where \( N \) is the degree of the co-factor.

Notice that \( W_{\alpha,\beta}(z)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j}) \). The first four terms have exponent sum \( 2N + 2\alpha + 2\beta \), and the same holds for the last four terms. Hence the average exponent value is \( (N + \alpha + \beta)/2 \), which is half the degree of \( A + A^{n^2-1-\delta(B_0)} \). Note that the two parts are each others complement (with respect to power \( \nu = N + \alpha + \beta \)).

One further observation is that in both 4-tuples, the first two terms have exponents adding up to \( N + \alpha \), whereas the second pair has exponent sum \( N + \alpha + 2\beta \). Assign one of the two parts to \( A \). This split is actually already possible for \( n \) being a multiple of four. If \( n \) is a multiple of 16, the aforementioned method allows us to generate polynomials \( A \) that can be split up in four groups with \( n/4 \) terms each, such that within each group the average exponent equals \( (n^2 - 1 - \delta(B_0))/2 \). Now, together with averaged exponents in \( B_0 \) this leads to Franklin Magic Squares that have the additional property that all diagonals have the magic sum, and all half diagonals (i.e. diagonals within each quadrant) have half the magic sum.

Working out the above approach for \( n = 16 \) yields 40320 different pan-diagonal Franklin Magic Squares the first of which is generated by:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>96</th>
<th>225</th>
<th>129</th>
<th>226</th>
<th>130</th>
<th>3</th>
<th>99</th>
<th>160</th>
<th>32</th>
<th>65</th>
<th>193</th>
<th>66</th>
<th>194</th>
<th>163</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>4</td>
<td>28</td>
<td>24</td>
<td>8</td>
<td>12</td>
<td>20</td>
<td>16</td>
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<td>20</td>
<td>12</td>
<td>8</td>
<td>24</td>
<td>28</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>0</td>
<td>225</td>
<td>226</td>
<td>3</td>
<td>160</td>
<td>65</td>
<td>66</td>
<td>163</td>
<td>96</td>
<td>129</td>
<td>130</td>
<td>99</td>
<td>32</td>
<td>193</td>
<td>194</td>
<td>35</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>96</td>
<td>129</td>
<td>130</td>
<td>99</td>
<td>32</td>
<td>193</td>
<td>194</td>
<td>35</td>
<td>0</td>
<td>28</td>
<td>8</td>
<td>20</td>
<td>16</td>
<td>12</td>
<td>24</td>
<td>4</td>
</tr>
<tr>
<td>( B_0 )</td>
<td>0</td>
<td>28</td>
<td>8</td>
<td>20</td>
<td>16</td>
<td>12</td>
<td>24</td>
<td>4</td>
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<td>12</td>
<td>28</td>
<td>0</td>
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</tr>
</tbody>
</table>

which yields a square \( M_{pd1a} \) given in Figure 5.

The square contains all numbers from 1 to 256, with rows, columns and diagonals each summing to 2056; with half rows, half columns and half main and back diagonal summing to 1028; with bent-diagonals summing to 2056, and with each 2x2 square having sum 514. Each four-on-a-row has sum 514. The four sub-matrices are magic themselves, with constant row, column and diagonal sums, including parallels of the diagonals and back diagonals. The matrix is anti-symmetric along the horizontal line of symmetry, opposite entries add up to 257.

5.2.2 Method 1b

In this case we have to be able to split \( B_0 \) into two parts with equal exponent sum and we like to retain the horizontal axis of symmetry. We borrow from the trick we applied for method 1a, and identify a factorization of \( (B_0 + B_0^{n^2-1-\delta(A)})(z) = W_{\alpha,\beta}(z)\Omega(z) \). We pair up terms
Figure 5: Pan-diagonal Franklin Magic Square by method 1a

\[
M_{pd1a} = \begin{array}{cccccccccccc}
1 & 160 & 226 & 127 & 227 & 126 & 4 & 157 \\
252 & 101 & 27 & 134 & 26 & 135 & 249 & 104 \\
29 & 132 & 254 & 99 & 255 & 98 & 32 & 129 \\
232 & 121 & 7 & 154 & 6 & 155 & 229 & 124 \\
9 & 152 & 234 & 119 & 235 & 118 & 12 & 149 \\
244 & 109 & 19 & 142 & 18 & 143 & 241 & 112 \\
21 & 140 & 246 & 107 & 247 & 106 & 24 & 137 \\
240 & 113 & 15 & 146 & 14 & 147 & 237 & 116 \\
\end{array}
\begin{array}{cccccccccccc}
161 & 224 & 66 & 63 & 67 & 62 & 164 & 221 \\
92 & 37 & 187 & 198 & 186 & 199 & 89 & 40 \\
189 & 196 & 94 & 35 & 95 & 34 & 192 & 193 \\
72 & 57 & 167 & 218 & 166 & 219 & 69 & 60 \\
169 & 216 & 74 & 55 & 75 & 54 & 172 & 213 \\
84 & 45 & 179 & 206 & 178 & 207 & 81 & 48 \\
181 & 204 & 86 & 43 & 87 & 42 & 184 & 201 \\
80 & 49 & 175 & 210 & 174 & 211 & 77 & 52 \\
\end{array}
\begin{array}{cccccccccccc}
17 & 144 & 242 & 111 & 243 & 110 & 20 & 141 \\
236 & 117 & 11 & 150 & 10 & 151 & 233 & 120 \\
13 & 148 & 238 & 115 & 239 & 114 & 16 & 145 \\
248 & 105 & 23 & 138 & 22 & 139 & 245 & 108 \\
25 & 136 & 250 & 103 & 251 & 102 & 28 & 133 \\
228 & 125 & 3 & 158 & 2 & 159 & 225 & 128 \\
5 & 156 & 230 & 123 & 231 & 122 & 8 & 153 \\
256 & 97 & 31 & 130 & 30 & 131 & 253 & 100 \\
\end{array}
\begin{array}{cccccccccccc}
177 & 208 & 82 & 47 & 83 & 46 & 180 & 205 \\
76 & 53 & 171 & 214 & 170 & 215 & 73 & 56 \\
173 & 212 & 78 & 51 & 79 & 50 & 176 & 209 \\
88 & 41 & 183 & 202 & 182 & 203 & 85 & 44 \\
185 & 200 & 90 & 39 & 91 & 38 & 188 & 197 \\
68 & 61 & 163 & 222 & 162 & 223 & 65 & 64 \\
165 & 220 & 70 & 59 & 71 & 58 & 168 & 217 \\
96 & 33 & 191 & 194 & 190 & 195 & 93 & 36 \\
\end{array}
As before, rewrite $W_{α, β}(z)(z^j + z^{N-j}) = (z^j + z^{α+N-j} + z^{β+N-j} + z^{α+β+j}) + (z^{N-j} + z^{α+j} + z^{β+j} + z^{α+β+N-j})$. The first four terms have exponent sum $2N + 2α + 2β$, and the same holds for the last four terms. Hence the average exponent value is $(N + α + β)/2$, which is half the degree of $B_0 + B_0^{n^2-1-δ(A)}$. Note that the two parts are each others complement (with respect to power $ν = N + α + β$). Select one of the four-tuples to be a part of $B_0$.

If $n$ is a multiple of 16, $Ω(z)$ contains an even number of matched pairs $z^j, z^{N-j}$. The four-tuples destined for $B_{00}$ remain as they are, the four-tuples for $B_{01}$ should be reversed in order, that is, rewritten as $(z^{α+β+j} + z^{β+N-j} + z^{α+N-j} + z^{j})$. By doing so the pairs adjacent terms in $B_0$ will nicely match with the pairs of adjacent terms in $B_1$. Again we define $y_{n-1-i} = y_i$, for each even $i$.

Application of method 1b is illustrated by the following example. Let us take $A(z) = (z^{128} + 1)(z^{32} + 1)(z^{16} + 1)(z^8 + 1)$, and $B_0 + B_0^{71} = (z^{64} + 1)(z^4 + 1)(z^2 + 1)(z + 1)$. For splitting $A$ into four parts we simply take matching pairs $z^{1}$, $z^{184-j}$ and distribute these pairs evenly. We may obtain $A_{00}(z) = z^0 + z^{184} + z^8 + z^{176}$, $A_{10}(z) = z^{16} + z^{168} + z^{24} + z^{160}$, $A_{01}(z) = z^{32} + z^{152} + z^{40} + z^{144}$, $A_{11}(z) = z^{48} + z^{136} + z^{56} + z^{128}$. To obtain $B_0$ let us take $α = 64$, $β = 4$, $N = 3$. We may get $B_{00}(z) = z^0 + z^{67} + z^1 + z^{68}$, $B_{10}(z) = z^3 + z^{66} + z^6 + z^{69}$, $B_{01}(z) = z^{69} + z^6 + z^{66} + z^1$, $B_{11}(z) = z^{68} + z^7 + z^{67} + z^0$. Now $A$ has average exponent 92 and $B$ has average exponent 71/2 which sums up to 255/2 = ($n^2 - 1$)/2. With vectors $x = (0, 16, 184, 168, 8, 24, 176, 160, 32, 48, 152, 136, 40, 56, 144, 128)$ and $y = (0, 1, 67, 66, 7, 6, 68, 69, 69, 68, 6, 7, 66, 67, 1, 0)$ we obtain matrix $M_{pd1b}$ in Figure 6.

### 5.3 Most-perfect Squares

Sometimes we like to have yet another even stronger requirement for symmetry: diagonals should be composed of pairs of complementary integers, at distance $n/2$. Complementary integers are pairs of entries with sum $(n^2 + 1)$. Being $n/2$ apart (which is even) they must match with exponents $x_j$, $x_{j+n/2}$ both in $A_0$ or both in $A_1$. Hence, methods 1a, 1b and 3 cannot yield such solutions. Method 2 does create solutions that have the right property. It is a matter of ordering the coefficients in $x$ and $y$ respectively in the right way so as to have $x_j + x_{j+n/2} = δ(A)$, for all $j < n/2$ and $y_i + y_{i+n/2} = δ(B)$, for all $i < n/2$. For a given $A$, as before, write $A(z) = (1 + z^α)(1 + z^β)Ω(z)$, and consider matching terms $z^j$ and $z^{N-j}$.

Now we rewrite $(1 + z^α)(1 + z^β)(z^j + z^{N-j}) = (z^j + z^{α+N-j} + z^{β+N-j} + z^{α+β+j}) + (z^{α+β+N-j} + z^{β+j} + z^{α+j} + z^{N-j})$. Now the order in the second 4-term has been rearranged such that complementary terms can be offset in the $x$-vector by $n/2$ positions. The first 4-tuples are used for building the polynomials $A_{00}$ and $A_{10}$, the second 4-tuples are used for $A_{01}$ and $A_{11}$.

For $n = 16$ this approach leads to 1260 different most-perfect Franklin Magic Squares, with the additional property of four-on-a-row. The first in the series was generated by $A, B, x$ and $y$ given by

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<tr>
<td>$B_1$</td>
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<td>7</td>
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</tbody>
</table>
Figure 6: Pan-diagonal Franklin Magic Square obtained with method 1b

\[
M_{pd1b} =
\begin{array}{cccccccc}
1 & 240 & 185 & 88 & 9 & 232 & 177 & 96 \\
255 & 18 & 71 & 170 & 247 & 26 & 79 & 162 \\
68 & 173 & 252 & 21 & 76 & 165 & 244 & 29 \\
190 & 83 & 6 & 235 & 182 & 91 & 14 & 227 \\
8 & 233 & 192 & 81 & 16 & 225 & 184 & 89 \\
250 & 23 & 66 & 175 & 242 & 31 & 74 & 167 \\
69 & 172 & 253 & 20 & 77 & 164 & 245 & 28 \\
187 & 86 & 3 & 238 & 179 & 94 & 11 & 230 \\
\end{array}
\begin{array}{cccccccc}
33 & 208 & 153 & 120 & 41 & 200 & 145 & 128 \\
223 & 50 & 103 & 138 & 215 & 58 & 111 & 130 \\
100 & 141 & 220 & 53 & 108 & 133 & 212 & 61 \\
158 & 115 & 38 & 203 & 150 & 123 & 46 & 195 \\
40 & 201 & 160 & 113 & 48 & 193 & 152 & 121 \\
218 & 55 & 98 & 143 & 210 & 63 & 106 & 135 \\
101 & 140 & 221 & 52 & 109 & 132 & 213 & 60 \\
155 & 118 & 35 & 206 & 147 & 126 & 43 & 198 \\
\end{array}
\begin{array}{cccccccc}
70 & 171 & 254 & 19 & 78 & 163 & 246 & 27 \\
188 & 85 & 4 & 237 & 180 & 93 & 12 & 229 \\
7 & 234 & 191 & 82 & 15 & 226 & 183 & 90 \\
249 & 24 & 65 & 176 & 241 & 32 & 73 & 168 \\
67 & 174 & 251 & 22 & 75 & 166 & 243 & 30 \\
189 & 84 & 5 & 236 & 181 & 92 & 13 & 228 \\
2 & 239 & 186 & 87 & 10 & 231 & 178 & 95 \\
256 & 17 & 72 & 169 & 248 & 25 & 80 & 161 \\
\end{array}
\begin{array}{cccccccc}
102 & 139 & 222 & 51 & 110 & 131 & 214 & 59 \\
156 & 117 & 36 & 205 & 148 & 125 & 44 & 197 \\
39 & 202 & 159 & 114 & 47 & 194 & 151 & 122 \\
217 & 56 & 97 & 144 & 209 & 64 & 105 & 136 \\
99 & 142 & 219 & 54 & 107 & 134 & 211 & 62 \\
157 & 116 & 37 & 204 & 149 & 124 & 45 & 196 \\
34 & 207 & 154 & 119 & 42 & 199 & 146 & 127 \\
224 & 49 & 104 & 137 & 216 & 57 & 112 & 129 \\
\end{array}
\]
Figure 7: Most-perfect Franklin Magic Square, by method 2

and the resulting square $M_{pf2}$ is given in Figure 7.

6 Franklin Magic Squares of order 20 and higher

In section 3 it was shown that no 12 by 12 Franklin Magic Square exists. It turns out that this is a unique exception. Below we show how to construct a Franklin Magic Square of order $20 + 8k$, for $k \geq 0$. We first construct two squares of order 20.

6.1 Franklin Magic Squares of order 20

Using method 1a we aim for a polynomial $A$ of 20 terms, and a polynomial $B_0$ of 10 terms, such that $(A + \overline{A}^{399 - \delta(B_0)})(z)B_0(z) = \frac{z^{400}}{z^{400}} - 1$. We need that $A$ can be split into four parts of five terms with equal exponent sum, and $B_0$ must be split into two parts of 5 terms each, again with equal exponent sum.

A candidate solution for $B_0$ is of the form $(1 + z\gamma + z^2\gamma + z^3\gamma + z^4\gamma)(1 + z^{10\gamma})$ which can be split into $(z^0 + z\gamma + z^{10\gamma} + z^{11\gamma} + z^{13\gamma}) + (z^{2\gamma} + z^3\gamma + z^4\gamma + z^{12\gamma} + z^{14\gamma})$. Each part has exponent sum $35\gamma$.

A candidate solution for $A$ is derived from the general form $(A + \overline{A}^\nu)(z) = (1 + z^\alpha)(1 + z^\beta + z^{2\beta} + \ldots + z^{19\beta})$. One possible solution is $A_{\alpha, \beta}(z) := (z^0 + z^\beta + z^{7\beta} + z^{16\beta} + z^{17\beta}) + (z^\alpha + z^{4\beta} + z^{8\beta} + z^{11\beta} + z^{18\beta}) + (z^{4\beta} + z^{5\beta} + z^{9\beta} + z^{10\beta} + z^{13\beta}) + (z^{2\beta} + z^{5\beta} + z^{6\beta} + z^{12\beta} + z^{16\beta})$. Here each part has sum $\alpha + 41\beta$. Take $\nu = \alpha + 19\beta$, then $A$ and $\overline{A}^\nu$ have no term in common.
These partial solutions can be combined for $(\alpha, \beta, \gamma) = (100, 1, 20)$ or $(\alpha, \beta, \gamma) = (5, 20, 1)$. We obtain solution $(x_1, y_1)$ with

$x_1 = (0, 100, 1, 4, 7, 8, 116, 11, 17, 18, 104, 2, 5, 105, 9, 6, 10, 12, 13, 16)$, and

$y_1 = (0, 280, 20, 240, 200, 80, 220, 260, 40, 40, 260, 60, 220, 80, 200, 240, 20, 280, 0)$.

This yields matrix $M_{20.1}$, depicted in Figure 8. The second solution $(x^2, y^2)$ is given by

$x^2 = (0, 5, 20, 80, 140, 160, 325, 220, 340, 360, 85, 40, 100, 105, 180, 120, 200, 240, 260, 320)$, and

$y^2 = (0, 14, 1, 12, 10, 4, 11, 3, 13, 2, 2, 13, 3, 11, 4, 10, 12, 1, 14, 0)$,

yielding matrix $M_{20.2}$. The last square is given in Figure 9.

### 6.2 Franklin Magic Squares of order $20 + 8k$

The construction of 20 by 20 squares given above can be extended to yield an $n$ by $n$ Franklin Magic Square for any $n = 20 + 8k$, with $k \geq 0$. Again we use method 1a.
Figure 9: 20x20 Franklin Magic Square $M_{20,2}$, constructed by method 1a
For $B_0$ we need a polynomial with $10 + 4k$ terms that can be split into two parts with
equal exponent sum. We choose $B_0$ to be of the form $(1 + z^{10+4k}) (1 + z^7 + \ldots + z^{(5+2k-1)\gamma})$,
where the latter factor has $5 + 2k$ terms. Now a possible split into two parts may be
$B_{00} = (1 + z^{10} + [z^{3\gamma} + z^{6\gamma} + \ldots + z^{(5+2k-3)\gamma}]) + z^{10+4k}(1 + z^{3\gamma} + \ldots + z^{(5+2k-2)\gamma})$, and $B_{01} = (z^{2\gamma} + [z^{3\gamma} + z^{6\gamma} + \ldots + z^{(5+2k-2)\gamma}] + z^{10+4k}(z^{2\gamma} + [z^{3\gamma} + z^{6\gamma} + \ldots + z^{(5+2k-1)\gamma}])$.
Each part has exponent sum $(5 + 2k)(10 + 4k)\gamma/2 + (5 + 2k)(5 + 2k - 1)\gamma/2$.

For $A$ to be derived from $A + A^{n^2-1-\delta(B_0)} = (1 + z^{(10+4k)\gamma})(1 + z^{a} + z^{2a} + \ldots + z^{(n-1)\alpha})$, we can choose either $\gamma = 1, \alpha = 15 + 6k$, or $\alpha = 1, \gamma = n = 20 + 8k$. Now write $(1 + z^{a} + z^{2a} + \ldots + z^{(n-1)\alpha}) = (1 + z^{a} + z^{2a} + \ldots + z^{(4k-1)\alpha}) + z^{4k\alpha}(1 + z^{a} + z^{2a} + \ldots + z^{(n-8k-1)\alpha}) + z^{(n-4k)\alpha}(1 + z^{a} + z^{2a} + \ldots + z^{(4k-1)\alpha})$.

Now define $A(z) = (1 + z^{a} + z^{2a} + \ldots + z^{(4k-1)\alpha}) + z^{(10+4k)\gamma}(1 + z^{a} + z^{2a} + \ldots + z^{(4k-1)\alpha}) + z^{4k\alpha}A_{\gamma,\alpha}(z)$. Here the last part is taken from the general solution for $n = 20$ in
the previous subsection.

It is not difficult to see that both solutions generate an $n$ by $n$ Franklin Magic Square
with the symmetry property along the horizontal middle line.

6.3 Huub Reijnders’ method for a 20 by 20 Franklin Magic Square

The first known 20 by 20 Franklin square was constructed by Huub Reijnders, who did this
apparently from scratch. It appears that his solution falls in the scheme set above. The
exception is that he has a special way of solving $A + A^{n^2-1-\delta(B_0)} = (1 + z^{n/4})(1 + z^{n} + z^{2n} + \ldots + z^{(n-1)n})$. His solution for $n = 20$ is $A_{20}(z) = (1 + z^{5})(1 + z^{20} + \ldots + z^{140}) + (z^{160} + z^{180} + z^{200} + z^{220})$ which splits into $(1 + z^{40} + z^{120} + z^{140}) + z^{180}, z^{5}(1 + z^{40} + z^{120} + z^{140}) + z^{160},
(z^{20} + z^{60} + z^{80} + z^{100}) + z^{220}$, and $z^{5}(z^{20} + z^{60} + z^{80} + z^{100}) + z^{200}$, each with exponent sum
480.

The split for $B_0$ is the same as in the subsection above.

In terms of vectors Reijnders’s solution is given by

\[
x = (0, 2, 1, 3, 10, 4, 11, 12, 13, 14, 14, 13, 12, 11, 4, 10, 3, 1, 2, 0), \text{ and}
\]
\[
y = (0, 5, 60, 65, 80, 85, 220, 200, 120, 125, 140, 145, 180, 160, 100, 105, 40, 45, 20, 25)
\]
yielding matrix $M_{20,r}$ given in Figure 10.

This solution approach can be extended to $n = 20 + 8k$ by realizing that the above trick
works by matching four exponents $n/4$ against one exponent $n$. In the remainder of solution
$A$ one needs only exponents that are multiples of $n$. This is easily realized by considering the
solution $A_{n}(z) = (1 + z^{n/4})(1 + z^{n} + \ldots + z^{7n}) + (z^{(n-4)n/2} + z^{(n-2)n/2} + z^{(n+0)n/2} + z^{(n+2)n/2}) + Q_{n}(z)$ where $Q_{n}(z) = z^{8n} + z^{9n} + \ldots + z^{(n-6)n/2} + z^{(n+4)n/2} + \ldots + z^{(n-9)n}$. Note that $Q(z)$ contains $n - 20 = 8k$ terms with an average exponent of $(n - 1)n/2$. The terms in $Q$ can be
paired up in $4k$ pairs each with exponent sum $(n - 1)n$, and these pairs can be evenly divided
over four sets with equal exponent sum.

7 Almost-Franklin Magic Squares of order 12

It was proved in section 3 that no true Franklin Magic Squares of order 12 exist. Hence, one
may try to construct Magic Squares that are as ‘Franklin’ as possible. We will stick to the
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Figure 10: 20x20 Franklin Magic Square $M_{20,r}$, constructed by Reijnders
property of 2x2 squares having constant sum. Further we will stick to the typical Franklin feature of having bent diagonals with the magic sum. As order 12 Franklin Magic Squares do not exist we have to give up on having magic half rows and magic half columns. Actually we may stick to having Franklin half rows and Franklin bent-diagonals if we just give up Franklin half columns. Another opportunity is to have Franklin half rows and Franklin half columns and only horizontal Franklin bent-diagonals.

We may abandon the requirement of having magic half rows and magic half columns, and turn to having either the four-on-a-row property or having most-perfectness.

7.1 Horizontally correct Franklin Magic Squares

Application of method 1a yields a polynomial \( A \) of 24 terms and a polynomial \( B_0 \) of 6 terms. If \( A + \overline{A} \) is of the form \((1 + z^\alpha)(1 + z^\beta + \ldots + z^{11\beta})\), with \( \alpha < \beta \) or \( \alpha \geq 12\beta \), then a solution \( A \) exists that can be split into four parts of equal exponents sum. For instance \( A(z) = (z^{3\beta} + z^{9\beta} + z^{5\beta + \alpha}) + (z^{3\beta} + z^{5\beta} + z^{11\beta + \alpha}) + (z^{4\beta} + z^{10\beta} + z^{3\beta + \alpha}) + (z^{2\beta} + z^{11\beta} + z^{4\beta + \alpha}) \). Here each part has exponent sum \( \alpha + 17\beta \).

A matching \( B_0 \) of the form \((1 + z^\delta)(1 + z^\gamma + z^{2\gamma})\) leads to a vector \( y \), with \( y_{11-i} = y_i \), and thus yields a square with magic bent-diagonals (both horizontally and vertically). Furthermore this square has a horizontal line of symmetry reflecting complementary entries.

By properly ordering the exponents one even gets columns with the four-on-a-row property: take \( y = (0, \delta, \delta + \gamma, \gamma, 2\gamma, \delta + 2\gamma, \delta + 2\gamma, 2\gamma, \gamma, \delta + \gamma, \delta, 0) \).

An example, with \( \beta = 1, \alpha = 72, \gamma = 12, \delta = 36 \), yields

\[
\begin{align*}
    x &= (3, 1, 9, 5, 77, 83, 4, 2, 10, 11, 75, 76) \\
    y &= (0, 36, 48, 12, 24, 60, 60, 24, 12, 48, 36, 0).
\end{align*}
\]

The result is the square \( M_{12s} \) given in Figure 11.

By interchanging rows 2, 4, 6 with 8, 10, 12 the square changes into one which has magic half columns, instead of having vertical magic bent-diagonals.

7.2 Decomposition and basic arrangements

In table 1 we list the possible decompositions of \( z^{144}_{s-1} \) into two- and three term factors with coefficients 1. There are \( \binom{144}{6} = 15 \) of such decompositions.

They are labeled by a sequence of 2s and 3s that indicate the place of the factors with three terms.

Table 2 displays all possible permutations of numbers 0 up to 11 that have the properties

\[
\begin{align*}
    v_{4i} - v_{4i+1} + v_{4i+2} - v_{4i+3} &= 0, \quad \text{for } i = 0, 1, 2, \quad (9) \\
    v_0 + v_2 + v_4 &= v_7 + v_9 + v_{11}, \quad (10)
\end{align*}
\]

up to isomorphism. These permutations were found by enumeration.

All rows except the ones marked by an asterisk have the property that for each pair \( j, 11-j \) both entries are on an even position, or both are on an odd position.

Evidently, when properties (9) and (10) hold for a certain vector \( v \), then they also hold for \( w = Cv \), where \( C \) is an arbitrary scalar. The right-most entries \( s - t \) in the table denote that for some vectors properties (9) and (10) also hold for exponents in the polynomials \((1 + z^\alpha + \ldots + z^{(s-1)\alpha})(1 + z^\beta + \ldots + z^{(t-1)\beta})\) according to the conversion table 3.
Figure 11: As Franklin as possible, no magic half columns

Table 1: Possible decompositions of $\frac{z^{144} - 1}{z - 1}$
Table 2: Possible arrangements with 4-on-a-row and bent-diagonal properties

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_9$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
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<tbody>
<tr>
<td>0</td>
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<td>7</td>
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<td>6-2</td>
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<td>5</td>
<td>11</td>
<td>8</td>
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<td>1</td>
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<td></td>
</tr>
</tbody>
</table>

Table 3: Conversion table for order 12 sequences

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t$</th>
<th>conversion of $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>$\lfloor k/6 \rfloor \alpha + (k % 6) \beta$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$\lfloor k/4 \rfloor \alpha + (k % 4) \beta$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>$\lfloor k/3 \rfloor \alpha + (k % 3) \beta$</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>$\lfloor k/2 \rfloor \alpha + (k % 2) \beta$</td>
</tr>
</tbody>
</table>

25
7.3 Method 2 for bent-diagonal and 4-on-a-row properties

Application of method 2 on any vector $x$ taken from Table 2, together with a vector $y$ obtained by taking any row of this table and multiplying it by 12 directly leads to a pan-diagonal 12x12 Magic Square with the bent-diagonals property as well as the four-on-a-row property. One should not take any of the rows marked by an asterisk.

Now we show how method 2 can be applied on a less trivial factorization. Consider the decomposition $z_{144} - z^{-1} = A(z)B(z)$, with $A(z) = (1 + z + z^2)(1 + z^{36} + z^{72} + z^{108})$ and $B(z) = (1 + z^3 + z^6 + \ldots + z^{33})$. For $B$ any row from the table not marked by an asterisk, multiplied by 3 will do. Let us take the last one: $y = (18, 21, 30, 27, 3, 0, 9, 12, 15, 6, 24, 33)$. For $A$ pick a row marked 3-4 or 4-3, let us say the one but last row. We have $\alpha = 1, \beta = 36, s = 3, t = 4$. The row is converted to $x = (1 + 36, 1 + 108, 2 + 108, 2 + 36, 0 + 0, 0 + 36, 0 + 108, 0 + 72, 1 + 72, 1 + 0, 2 + 0, 2 + 72) = (37, 109, 110, 38, 0, 36, 108, 72, 73, 1, 2, 74)$. The resulting square $M_{12,V}$ is depicted in Figure 12.

Similarly, an even more complicated decomposition can be base of a pan-diagonal 12x12 square with the bent-diagonals property as well as the four-on-a-row property. One should not take any of the rows marked by an asterisk.

$$
M_{12,V} = \begin{bmatrix}
56 & 17 & 129 & 88 & 19 & 90 & 127 & 54 & 92 & 125 & 21 & 52 \\
86 & 131 & 13 & 60 & 123 & 58 & 15 & 94 & 50 & 23 & 121 & 96 \\
68 & 5 & 141 & 76 & 31 & 78 & 139 & 42 & 104 & 113 & 33 & 40 \\
80 & 137 & 7 & 66 & 117 & 64 & 9 & 100 & 44 & 29 & 115 & 102 \\
41 & 32 & 114 & 103 & 4 & 105 & 112 & 69 & 77 & 140 & 6 & 67 \\
107 & 110 & 34 & 39 & 144 & 37 & 36 & 73 & 71 & 2 & 142 & 75 \\
47 & 26 & 120 & 97 & 10 & 109 & 118 & 63 & 83 & 134 & 12 & 61 \\
95 & 122 & 22 & 51 & 132 & 49 & 24 & 85 & 59 & 14 & 130 & 87 \\
53 & 20 & 126 & 91 & 16 & 93 & 124 & 57 & 89 & 128 & 18 & 55 \\
101 & 116 & 28 & 45 & 138 & 43 & 30 & 79 & 65 & 8 & 136 & 81 \\
62 & 11 & 135 & 82 & 25 & 84 & 133 & 48 & 98 & 119 & 27 & 46 \\
74 & 143 & 1 & 72 & 111 & 70 & 3 & 106 & 38 & 35 & 109 & 108
\end{bmatrix}
$$

Figure 12: 12x12 Magic Square with bent-diagonals and 4-on-a-row, by method 2
For bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property, i.e. complementary entries will reflect in the horizontal axis of symmetry.

In general this procedure will not yield a square which is pan-diagonal. If we want to enforce this property we have to be more restrictive in the choice for $A_i$. In particular we need that the average exponent in $B$ equals half the degree of $B_0 + B_0^{143-\delta(A)}$. Choose $B_0$ in such a way that the six terms have exponents two pairs of which have the same sum. This is often possible in many ways. Let $e_0, \ldots, e_5$ be the exponents in $B_0$ and assume $e_0 + e_1 = e_2 + e_3$. Define $y = (e_0, e_2, e_1, e_3, e_4, e_5, e_4, e_3, e_1, e_2, e_0)$. This arrangement will yield a square which has bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property.

From table 2 pick the first row: $0, 1, 7, 6, 11, 8, 2, 5, 4, 3, 9, 10$ to arrange the exponents of $B$ in such a way that

$$
(0 + z^0 + z^2 + z^3 + z^4 + z^6 + z^7 + z^8 + z^{10}) (1 + z^0 + z^2 + z^3 + z^4 + z^6 + z^7 + z^8 + z^{10})
$$

Choose $B_0$ in such a way that the six terms have exponents two pairs of which have the same sum. This is often possible in many ways. Both exponents in $B_0$ and the sum $e_0 + e_1 = e_2 + e_3$. Define $y = (e_0, e_2, e_1, e_3, e_4, e_5, e_4, e_3, e_1, e_2, e_0)$. This arrangement will yield a square which has bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property, i.e. complementary entries will reflect in the horizontal axis of symmetry.

Remark: the HSA-square, designed by a group of Dutch high school students and publicized in March 2007, fits in this scheme. As an example, let us select the second row in the decomposition table $B_0 + B_0^\nu = (1 + z^2)(1 + z^4)(1 + z^{48} + z^{96})$ and $A(z) = (1 + z)(1 + z^8 + z^{16})(1 + z^{24})$. Writing out the consecutive factors we obtain $A(z) = (1 + z)(1 + z^8 + z^{16} + z^{24} + z^{32} + z^{40})$ and $B_0 + B_0^\nu = (1 + z^2 + z^4 + z^6)(1 + z^{48} + z^{96})$, with $\nu = 143 - 41 = 102$. From table 2 pick the first row: $0, 1, 7, 6, 11, 8, 2, 5, 4, 3, 9, 10$ to arrange the exponents of $B$ for $A$.

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_9$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>8</td>
<td>6</td>
<td>7</td>
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<td>4-3</td>
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<td>8</td>
<td>10</td>
<td>2</td>
<td>0</td>
<td>3-4</td>
</tr>
</tbody>
</table>

Table 4: Arrangements of $B$ with 4-on-a-row, bent-diagonal and symmetry properties

### 7.4 Method 1b for bent-diagonal and 4-on-a-row properties

Using method 1b we start again from a decomposition into four factors as above. Given the decomposition select two factors with $3 + 4$ or $2 + 6$ terms, whose product will be $A$, and use the conversion table 3 to build an appropriate vector $x$.

The other two factors will have as product $B_0 + B_0^{143-\delta(A)}$. Choose $B_0$ in such a way that the six terms have exponents two pairs of which have the same sum. This is often possible in many ways. Let $e_0, \ldots, e_5$ be the exponents in $B_0$ and assume $e_0 + e_1 = e_2 + e_3$. Define $y = (e_0, e_2, e_1, e_3, e_4, e_5, e_4, e_3, e_1, e_2, e_0)$. This arrangement will yield a square which has bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property, i.e. complementary entries will reflect in the horizontal axis of symmetry.
A. It has a 6-2 generalization, with \( \alpha = 8 \) and \( \beta = 1 \). We obtain \( x = (0\alpha + 0\beta, 0\alpha + 1\beta, 3\alpha + 1\beta, 3\alpha + 0\beta, 5\alpha + 1\beta, 4\alpha + 0\beta, 1\alpha + 0\beta, 2\alpha + 1\beta, 2\alpha + 0\beta, 1\alpha + 1\beta, 4\alpha + 1\beta, 5\alpha + 0\beta) \)
\[
= (0, 1, 25, 41, 32, 8, 17, 16, 9, 33, 40).
\]
To build \( B_0 \) pick the first row from table 4 0, 2, 8, 6, 7, 10, 10, 7, 8, 2, 0. With the 4-3 factorization with \( \alpha = 2 \) and \( \beta = 48 \) this leads to \( y = (0\alpha + 0\beta, 0\alpha + 2\beta, 2\alpha + 0\beta, 2\alpha + 0\beta, 2\alpha + 0\beta, 2\alpha + 2\beta, 0\alpha + 2\beta, 0\alpha + 0\beta) = (0, 96, 100, 4, 52, 54, 54, 52, 4, 100, 96, 0) \).

Plugging in these vectors yields a square \( M_{pd12.4} \) depicted in Figure 13.

### 7.5 Method 1a for bent-diagonal and 4-on-a-row properties

Using method 1a we start from a decomposition into factors \( A + A \) and \( B_0 \), where the first has 24 terms and the second only 6. If we take for the first factor \( 1+z^\alpha \) times a factor representable (by conversion) with a row from table 2, we can take for \( A \) this second factor. If \( B_0(z) = (1+z\beta)(1+z\gamma + z^{2\gamma}) \), a proper reordering gives \( B_0(z) = (1+z\beta+\gamma) + (z^{2\gamma} + z^{2\gamma+\beta}) + (z^{\gamma} + z^{\beta}) \) and \( B_1(z) = (z^{\beta} + z^{\gamma}) + (z^{2\gamma+\beta} + z^{2\gamma}) + (z^{\beta+\gamma} + 1) \). The resulting square will have bent-diagonal properties, four-on-a-row properties and symmetry along the horizontal axis of symmetry. The result will in general not be pan-diagonal.

To enforce pan-diagonality, the choice for \( A + A^{143-\delta(B_0)} \) is restricted to be of the form \((1+z^\alpha)\) times a 12-term representable by a row from table 4.

### 7.6 Method 2 for constructing most-perfect order 12 Magic Squares

It is possible to impose on the 12 by 12 Magic Square that it has the most-perfectness property. For this to be true one has to have \( x_j + x_{j+6} \) equal to \( \delta(A) \). Such an arrangement for \( A(z) = 1 + z + \cdots + z^{11} \) can explicitly be found by complete enumeration.
This yields the table 5, in which the remark section, as before, indicates how to use the conversion table 3 to get even more polynomials with the property of providing a most-perfect arrangement.

As an example, take the first row and 12 times the last row of Table 5 to get

\[ x = (0, 2, 6, 7, 8, 10, 11, 9, 5, 4, 3, 1), \] 
\[ y = (0, 84, 24, 96, 72, 120, 132, 48, 108, 36, 60, 12). \]

The resulting square \( M_{12,p} \) has magic row and column sums, magic bent-diagonals, and has complementary entries in opposite quadrants, as seen from Figure 14.

## 8 Conclusions

The existence of Franklin Magic Squares of order \( n = 4k \), with \( n \neq 4 \) and \( n \neq 12 \) has been shown. Multiples of 8 pose no problems. Orders \( 20 + 8k \) are more difficult to realize, but not impossible. We have described four methods by which one can construct many Franklin Magic Squares. We are not aware of any Franklin Magic Square that does not fit into one of these four schemes.

The non-existence of a 12 by 12 Franklin Magic Square has been demonstrated by an exhaustive search that was only possible by maximal use of symmetry arguments as well as aggressive pruning.

I like to thank Andries Brouwer and Tonny Hurkens for fruitful discussions, and of course Arno van den Essen, and students Petra, Jesse and Willem, for the hype and interest they created.
Figure 14: Most-perfect 12 by 12 Magic Square with bent-diagonals

\[
M_{12, p} =
\begin{array}{cccccccc}
1 & 142 & 7 & 137 & 9 & 134 & 12 & 135 \\
60 & 87 & 54 & 92 & 52 & 95 & 49 & 94 \\
25 & 118 & 31 & 113 & 33 & 110 & 36 & 111 \\
48 & 99 & 42 & 104 & 40 & 107 & 37 & 106 \\
73 & 70 & 79 & 65 & 81 & 62 & 84 & 63 \\
24 & 123 & 18 & 128 & 16 & 131 & 13 & 130 \\
\end{array}
\begin{array}{cccccccc}
5 & 141 & 2 & 144 & 3 & 138 & 8 & 136 \\
65 & 81 & 62 & 84 & 63 & 78 & 68 & 76 \\
24 & 123 & 18 & 128 & 16 & 131 & 13 & 130 \\
\end{array}
\begin{array}{cccccccc}
132 & 15 & 126 & 20 & 124 & 23 & 121 & 22 \\
108 & 39 & 102 & 44 & 100 & 47 & 97 & 46 \\
61 & 82 & 67 & 77 & 69 & 74 & 72 & 75 \\
\end{array}
\begin{array}{cccccccc}
132 & 15 & 126 & 20 & 124 & 23 & 121 & 22 \\
108 & 39 & 102 & 44 & 100 & 47 & 97 & 46 \\
61 & 82 & 67 & 77 & 69 & 74 & 72 & 75 \\
\end{array}
\begin{array}{cccccccc}
132 & 15 & 126 & 20 & 124 & 23 & 121 & 22 \\
108 & 39 & 102 & 44 & 100 & 47 & 97 & 46 \\
61 & 82 & 67 & 77 & 69 & 74 & 72 & 75 \\
\end{array}
\begin{array}{cccccccc}
132 & 15 & 126 & 20 & 124 & 23 & 121 & 22 \\
108 & 39 & 102 & 44 & 100 & 47 & 97 & 46 \\
61 & 82 & 67 & 77 & 69 & 74 & 72 & 75 \\
\end{array