Repetitive schemes for the single-machine multi-product lot-size scheduling problem

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Department of Mathematics

PROBABILITY THEORY, STATISTICS AND OPERATIONS RESEARCH GROUP

Memorandum COSOR 76-08
Repetitive schemes for the single-machine
multi-product lot-size scheduling problem

by

Th.H.B. Hendriks and J. Wessels

Eindhoven, April 1976

The Netherlands
Repetitive schemes for the single-machine multi-product lot-size scheduling problem*

by

Th.H.B. Hendriks and J. Wessels

Summary. Solution of the single-machine multi-product lot-size scheduling problem requires the interactive optimization of the cycle times for individual products and the scheduling of the cycles. Usually one presents procedures of the following form. Start by finding the best individual cycle times which satisfy some restrictions. Secondly, try to find an appropriate schedule (which does not exist necessarily) for the production cycles. In this paper we will present solutions which do not require extensive scheduling. Actually, our repetitive schemes may be seen as generalizations of the purely rotational scheme.

For the case of two homogeneous groups of products a systematic comparison of our solution with lower bounds for the costs is given. For some problems in the literature our solutions are compared with other solutions.

1. Introduction. In this paper we deal with the lot-size determination and production scheduling for one machine in the multi-product case. We make the following assumptions:

a) only one of \( N \) products can be produced at any time on the machine.

b) the demand rate \( d_i \) and the production rate \( p_i \) for product \( i \) are deterministic and constant in time.

Assume

\[
\rho_i := d_i p_i^{-1} < 1, \quad \rho := \sum_{i=1}^{N} \rho_i \leq 1.
\]

c) the set-up costs \( F_i \) and the set-up times \( t_i \) necessary for any production cycle of product \( i \) only depend on \( i \). (if \( \rho = 1 \) all \( t_i \) should equal 0).

d) the inventory costs per unit of time for a unit of product \( i \) are \( h_i \).

e) the production schedule should guarantee a service level for product \( i \) of at least \( \beta_i \) \((0 < \beta_i \leq 1)\). So \( \beta_i \) is the minimally required fraction of time with a positive inventory of product \( i \).

*The authors are grateful to H. Grünwald and B. Matzinger of Philips-ISA-R at Eindhoven for stimulating discussions.
If for each product there would be one machine available, then product \( i \) would be produced periodically in such a way that it gets a period of length 
\[
T_i := \left[ 2F_i \alpha_i \right]^{\frac{1}{2}}
\]
where \( \alpha_i := \beta_i^2 (1 - \rho_i) \delta_i \gamma_i \). For this optimal solution of the \( N \)-product \( N \)-machine problem the total costs per unit of time is equal to 
\[
K := \sum_{i=1}^{N} \left[ 2F_i \alpha_i \right]^{\frac{1}{2}}
\]
(see e.g. Hillier, Lieberman [3]).

\( K \) is a natural lower bound for the total costs per unit of time in the \( N \)-product \( 1 \)-machine problem.

For most values of the cycle lengths \( T_i \) and production quantities \( \alpha_i T_i \) (\( i = 1, \ldots, N \)) it will not be possible to realize the \( N \) cycles simultaneously on one machine. Realization is possible if we deviate from the optimal \( T_i \) and give all products the same cycle length. If we choose the cycle length optimal for this so-called purely rotational schedule we obtain a cycle length 
\[
T(I) := \left[ 2 \sum_{i=1}^{N} F_i \right]^{\frac{1}{2}} \left[ \sum_{i=1}^{N} \alpha_i \right]^{-\frac{1}{2}}
\]
and a cost per unit of time
\[
K(I) := \left[ 2 \sum_{i=1}^{N} F_i \right]^{\frac{1}{2}} \left[ \sum_{i=1}^{N} \alpha_i \right]^{-\frac{1}{2}}.
\]

If \( T(I) \) happens to be smaller than \( (1 - \rho)^{-1} \sum_{i=1}^{n} t_i \) a slight modification is necessary.

The advantage of the restriction to purely rotational schedules, is that no scheduling effort is necessary. Even if the set-up costs depend on the production transition, it is relatively easy to find the optimal rotational production schedule. Namely, the optimal order of the production in one rotational cycle may be determined first, whereas the optimal cycle length may be determined afterwards for a fixed order of production. The only disadvantage of the restriction to rotational schedules is that it might cost more than necessary.

In this paper we will try to find production schemes, which maintain the advantages of the rotational schemes, but have a better performance.

In the literature (see e.g. Doll and Whybark [2]) the main emphasis is on procedures which first find optimal cycle times for the individual products.
under some restriction and which secondly try to find a schedule (if there exists one) realizing the optimal individual cycle times.

In order to avoid the difficult scheduling operation we will restrict attention to generalizations of the rotational schemes. In doing so we take the risk that individual cycle times differ somewhat more than strictly necessary from the values $T_i$, however the costs are not very sensitive for such deviations. Note that for product $i$ the use of cycle time $\gamma T_i$ instead of $T_i$ gives a cost increase per time unit by a factor $\frac{1}{2}(\gamma^{-1} + \gamma)$, which is only $1.0167$ for $\gamma = 1, 2$.

In section 2 the performance of the optimal rotational scheme will be compared with the lower bound $K$ for the costs. This will be done for the case that the product mix consists of some homogeneous groups.

In section 3 the repetitive schemes will be introduced for the case of homogeneous groups. The comparison with $K$ will be given for the case of 2 groups.

In section 4 the repetitive schemes will be applied to some problems in which no natural grouping is available. The results will be compared with $K$ and results of other authors.

2. Rotational schemes for grouped products. In this section we will consider $K(1)K^{-1}$ i.e. the fractional deviation of the costs for the optimal rotational scheme from the lower bound $K$. This will be done especially for the situation that the products can be divided into groups, such that all products in the same group possess equal $\alpha_i$, $\rho_i$- and $F_i$-values. The case of two groups with $n_a$, $n_b$ products respectively is the most interesting one, since in the cases of more than two groups the adaptation of the cycle times will be less extreme.

Suppose $N = n_a + n_b$ and suppose that $\alpha_i = \alpha_a$, $F_i = F_a$ for $i \leq n_a$,

$\alpha_j = \alpha_b$, $F_j = F_b$ for $j > n_a$.

Define $T_i =: T_a$, $T_j =: T_b$ for $i \leq n_a$, $j > n_a$,

and $F =: n_a F_a^{-1} F_b^{-1}$, $\alpha := \alpha_a^{-1}$, $\tau := T_a T_b^{-1}$.

Then

$K(1)K^{-1} = [1 + F(\tau - 1)(F + \tau)^{-2}]^{1/2} =: g_1(\tau, F)$. 

Fig. 2.1, $g_1(\tau, F)$ for $\tau \geq 1$ and some values of $F$. 

<table>
<thead>
<tr>
<th>$g_1(\tau, F)$</th>
<th>$F = \frac{1}{3}$</th>
<th>$F = \frac{1}{2}$</th>
<th>$F = 1$</th>
<th>$F = \sqrt{2}$</th>
<th>$F = \frac{3}{2}$</th>
<th>$F = 2$</th>
<th>$F = 3$</th>
<th>$F = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau = \sqrt{2}$</td>
<td>1.009</td>
<td>1.012</td>
<td>1.015</td>
<td>1.015</td>
<td>1.015</td>
<td>1.015</td>
<td>1.013</td>
<td>1.012</td>
</tr>
<tr>
<td>$\tau = \frac{3}{2}$</td>
<td>1.012</td>
<td>1.016</td>
<td>1.020</td>
<td>1.021</td>
<td>1.021</td>
<td>1.020</td>
<td>1.018</td>
<td>1.016</td>
</tr>
<tr>
<td>$\tau = 2$</td>
<td>1.030</td>
<td>1.039</td>
<td>1.054</td>
<td>1.059</td>
<td>1.059</td>
<td>1.061</td>
<td>1.058</td>
<td>1.054</td>
</tr>
<tr>
<td>$\tau = \frac{5}{2}$</td>
<td>1.046</td>
<td>1.061</td>
<td>1.088</td>
<td>1.099</td>
<td>1.100</td>
<td>1.106</td>
<td>1.106</td>
<td>1.101</td>
</tr>
<tr>
<td>$\tau = 3$</td>
<td>1.058</td>
<td>1.078</td>
<td>1.118</td>
<td>1.136</td>
<td>1.139</td>
<td>1.149</td>
<td>1.155</td>
<td>1.152</td>
</tr>
<tr>
<td>$\tau = 4$</td>
<td>1.077</td>
<td>1.106</td>
<td>1.166</td>
<td>1.198</td>
<td>1.203</td>
<td>1.225</td>
<td>1.245</td>
<td>1.250</td>
</tr>
<tr>
<td>$+ \infty$</td>
<td>1.155</td>
<td>1.225</td>
<td>1.414</td>
<td>1.554</td>
<td>1.581</td>
<td>1.732</td>
<td>2.000</td>
<td>2.236</td>
</tr>
</tbody>
</table>

Table 2.1.
For fixed $\tau$ the function $g_1(\tau,F)$ attains its maximal value for $F = \tau$.

Because of $g_1(\sqrt{2}, \sqrt{2}) = 1.015$ and $g_1(\frac{3}{2}, \frac{3}{2}) = 1.021$, it follows that

$1 \leq g_1(\tau,F) \leq 1.021$ for $1 \leq \tau < \frac{3}{2}$ and all $F > 0$,

$1 \leq g_1(\tau,F) \leq 1.015$ for $1 \leq \tau \leq \sqrt{2}$ and all $F > 0$.

So for $\tau = T_a T_b^{-1}$ in the neighbourhood of 1 (we need not consider $\tau < 1$) the restriction to rotational schemes will be quite reasonable.

For larger $\tau$-values ($\tau > \frac{3}{2}$ say), it seems reasonable to produce the products of the b-group two or more times in one cycle of the a-group. This will be investigated in the next section.

3. Repetitive schemes for grouped products. Again considering two homogeneous groups of products (with the same notations as in section 2), we will introduce the k-repetitive production scheme as a generalization of the rotational scheme. Suppose for the moment that $n_a = kr$ with $k$ and $r$ natural numbers. Then a cyclic production scheme of length $T$ may be applied, such that all the products are produced cyclically, with a cycle length $Tk^{-1}$ for the b-group and $T$ for the a-group. This can be achieved by producing $r$ products of the a-group together with each production round for the b-group (see fig. 3.1).

![Fig. 3.1, repetitive scheme for $k = 2$, $n_a = 4$, $n_b = 3$.](image)

Optimization of the costs per unit of time with respect to $T$ gives an optimal k-repetitive cycle of length $T(k)$ with costs per unit of time $K(k)$:
A 1-repetitive scheme is a rotational scheme.

If $T(k)$ happens to be smaller than $(1 - \rho)^{-1} [n_a t_a + n_b t_b]$ a slight modification is necessary.

$$K(k) = [2 (n_a F_a + n_b F_b)]^{1/2} \left[ n_a a_a + n_b k^{-1} a_b \right]^{-1/2}$$

A 1-repetitive scheme is a rotational scheme.

If $T(k)$ happens to be smaller than $(1 - \rho)^{-1} [n_a t_a + n_b t_b]$ a slight modification is necessary.

$$K(k)K^{-1}$$ gives the quality of the optimal $k$-repetitive scheme relative to the lower bound $K$.

$$K(k)K^{-1} = [1 + kF(\tau k^{-1} - 1)2(F + \tau)^{-2}]^{1/2} = g_k(\tau, F).$$

So for given $\tau, F$ the most favourable repetitive production scheme would be the optimal $k$-repetitive scheme with $\sqrt{(k-1)k} \leq \tau \leq \sqrt{k(k+1)}$. The restriction that $k$ divides $n_a$ is not essential. If the $a$-group can not be split up into $k$ equal subgroups it can be split up into $k$ subgroups which are as equal as possible. This gives no problems if $1 - \rho$ is sufficiently large; if $1 - \rho$ is too small it can easily be met by small variations in the length of the subcycles. The same holds for the assumption of equal $\rho_i$ within one group. If this assumption is not fulfilled subgroups may be made containing sums of $\rho_i$ which are nearly equal. We will not enter into these details. An example will be given at the end of this section.

Hence the optimal repetitive scheme for the two groups problem possesses a relative performance

$$g(\tau, F) := \min_k g_k(\tau, F) = \min_k K(k)K^{-1}$$

$$= g_k(\tau, F) \text{ if } (k_0 - 1)k_0 \leq \tau^2 \leq k_0(k_0 + 1).$$
For any $F$ the local maxima of $g(T,F)$ in $\tau = \sqrt{k(k+1)}$ are decreasing with $k$. So the maximum of $g(T,F)$ for $\tau \geq 1$, $F > 0$ is $g(\sqrt{2}, \sqrt{3}) = 1,015$. Applying the optimal repetitive scheme for two homogeneous products gives a performance of at most 1.5% above $K$.

From table 2.1 one learns that for $F = 4$ the relative performance lies at most 1.2% above $K$ for any $\tau \geq 1$; for $F = \frac{1}{3}$ this maximal deviation is 0.9%.

The relative performance $g(T,F)$ can only be attained for two homogeneous groups of products, if the $a$-group can be split up into the relevant number ($k_0$) of subgroups in such a way that the scheme fits. If the set-up times $t_i$ cause troubles this can be met by a (usually) slight increase of the cycle length. If the division into subgroups causes the troubles, this can be met by other modifications as will be shown in an example. In the example $n_a = 5$; for larger values of $n_a$ the influence of subdivision problems on the relative performance is smaller.

**Example.** $n_a = n_b = 5$, $F_b = F_a$, $d_b = 2d_a = 2d$, $\beta_a = \beta_b$, $h_b = 2h_a$, $p_a = p_b = 1$, $t_a = t_b = 0$. Then

$$\tau = 2 \left( \frac{1 - 2d}{1 - d} \right)^{\frac{1}{2}}, F = 1.$$ 

The requirement $\rho \leq 1$ gives $d \leq \frac{1}{15}$ and $1.93 \leq \tau \leq 2$, hence a 2-repetitive scheme would be favourable. By dividing the $a$-group into two subgroups of sizes 3 and 2 respectively, we need an idle period in the second subcycle, which can only be realized if $d \leq \frac{1}{16}$. So problems arise only if $\frac{1}{16} < d \leq \frac{1}{15}$. 

---

**fig. 3.2, $g(\tau,F)$ for a fixed $F$.**

For any $F$ the local maxima of $g(\tau,F)$ in $\tau = \sqrt{k(k+1)}$ are decreasing with $k$. So the maximum of $g(\tau,F)$ for $\tau \geq 1$, $F > 0$ is $g(\sqrt{2}, \sqrt{3}) = 1,015$. Applying the optimal repetitive scheme for two homogeneous products gives a performance of at most 1.5% above $K$.
For $d = \frac{1}{15}$ the costs $K(2)$ is 0.03% above $K$.
For $d = \frac{1}{15}$ we try some slight modifications:

a) Bring one product of the a-group to the b-group, this makes the a-group dividable. Or
b) Adapt the lot-size of the b-products in their second run, by decreasing them such that the idle time can be avoided.

With these modifications one obtains the following relative performances for the 2-repetitive schemes in case $d = \frac{1}{15}$.

- Modification a: 1.45% above $K$.
- Modification b: 1.3% above $K$.

For more than two homogeneous groups of products a similar approach can be given. E.g. for three groups $(k,\ell)$-repetitive schemes can be constructed with $\ell$ subgroups for the b-group and $k\ell$ subgroups for the a-group. Then again no scheduling effort is needed and for each $(k,\ell)$ the optimal cycle length $T(k,\ell)$ may be determined easily. Using this the optimal $k$ and $\ell$ can be determined.

Summarizing one may say, that by choosing a fixed repetitive structure for the production scheme the optimal scheme with this structure can be determined easily. For the case of homogeneous groups of products such repetitive schemes have a relatively good performance. The problem now arises whether it is desirable or not to apply repetitive schemes in situations where no natural grouping is available. A partial answer to this question will be given in section 4.

4. Repetitive schemes for non-grouped problems. If the $N$ products do not naturally separate in some homogeneous groups, one might partition the set of products artificially according to the individual $T_i$-values. Instead of presenting a formal procedure for such a partitioning, we will illustrate the method with some examples from the literature.
a. Rogers' problem (see Rogers [5]):

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{product} & F_i & h_i & d_i & p_i & \rho_i & T_i \\
\hline
1 & 50 & 0.05 & 50 & 500 & 0.1 & 6.6 \\
2 & 75 & 0.02 & 100 & 1000 & 0.1 & 9.1 \\
3 & 50 & 0.10 & 200 & 800 & 0.25 & 2.6 \\
4 & 100 & 0.20 & 25 & 2500 & 0.01 & 6.4 \\
5 & 150 & 0.05 & 150 & 3750 & 0.04 & 6.4 \\
\hline
\end{array}$$

Table 4.1: Rogers' problem.

In this problem (with a day as unit of time) only one product may be produced per day. We first discard this restriction. When we try to partition the 5 products in 2 groups one naturally obtains for the a-group 1, 2, 4, 5 and for the b-group 3. For $k = 3$ the following subgroups are reasonable ($\rho$ is not critical) \{1, 2\}, \{4, 5\}. This results in the following cyclic production scheme 3132345. We now obtain the following results:

$$K(1) = 162.85 \ ; \ K(3) = 149.20 \ ; \ K = 148.09 \ ;$$

with \[ T(1) = 5.22 \ ; \ T(3) = 7.04 \ . \ So K(3)K^{-1} = 1.0075. \]

Taking the one-product-a-day restriction into account, we see that $T'(k)$ and $k^{-1}T'(k)$ should be integers. For $k = 3$ we see that $k^{-1}T'(k)$ should be at least 3, which leaves as only reasonable alternative $T'(3) = 9$.

For $k = 2$ the subgroups \{1,4\} and \{4,5\} would be possible with again $k^{-1}T'(k) \geq 3$. So $T'(2) = 6$ would be reasonable.

In this way we obtain:

$$K'(1) = 162.97 \ ; \ K'(2) = 150.27 \ ; \ K'(3) = 153.73 .$$

For this situation Rogers [5] gives a solution with costs 152.28; the solution of Doll and Whybark [2] costs 152.68; the solution of Madigan [4](see [2]) costs 150.19.

$$K'(2)K^{-1} = 1.015 .$$

Note that without the one-product-a-day restriction we obtain for the 2-repetitive cycle: $K(2) = 150.05$ and $T(2) = 6.33$. 

b. Bomberger's problem (see Bomberger [1]):

<table>
<thead>
<tr>
<th>product</th>
<th>$F_i$</th>
<th>$2400 h_i$</th>
<th>$p_i$</th>
<th>$d_i$</th>
<th>$8 t_i$</th>
<th>$\rho_i$</th>
<th>$T_i$</th>
<th>$K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>0.0065</td>
<td>30000</td>
<td>400</td>
<td>1</td>
<td>0.0133</td>
<td>167.53</td>
<td>0.179</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0.1775</td>
<td>8000</td>
<td>400</td>
<td>1</td>
<td>0.05</td>
<td>37.73</td>
<td>1.060</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>0.1275</td>
<td>9500</td>
<td>800</td>
<td>2</td>
<td>0.0842</td>
<td>39.26</td>
<td>1.528</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>0.1000</td>
<td>7500</td>
<td>1600</td>
<td>1</td>
<td>0.213</td>
<td>19.53</td>
<td>1.024</td>
</tr>
<tr>
<td>5</td>
<td>110</td>
<td>2.7850</td>
<td>2000</td>
<td>80</td>
<td>4</td>
<td>0.04</td>
<td>49.68</td>
<td>4.428</td>
</tr>
<tr>
<td>6</td>
<td>50</td>
<td>0.2675</td>
<td>6000</td>
<td>80</td>
<td>2</td>
<td>0.0133</td>
<td>106.61</td>
<td>0.938</td>
</tr>
<tr>
<td>7</td>
<td>310</td>
<td>1.500</td>
<td>2400</td>
<td>24</td>
<td>8'</td>
<td>0.01</td>
<td>204.33</td>
<td>3.034</td>
</tr>
<tr>
<td>8</td>
<td>130</td>
<td>5.900</td>
<td>1300</td>
<td>340</td>
<td>4</td>
<td>0.262</td>
<td>20.52</td>
<td>12.671</td>
</tr>
<tr>
<td>9</td>
<td>200</td>
<td>0.9000</td>
<td>2000</td>
<td>340</td>
<td>6</td>
<td>0.17</td>
<td>61.48</td>
<td>6.506</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0.0400</td>
<td>15000</td>
<td>400</td>
<td>1</td>
<td>0.0267</td>
<td>39.26</td>
<td>0.255</td>
</tr>
</tbody>
</table>

Table 4.2: Bomberger's problem.

In this problem $\rho = 0.8825$ which is relatively critical because of the positive set-up times.

With two groups the following partition is reasonable: a-group = \{1,3,5,6,7,9\}

b-group = \{2,4,8,10\}. For $k = 2$ the a-group may be partitioned into \{9\} and \{1,3,5,6,7\}.

This gives

$$K(1) = 41.17; \quad K(2) = 35.72; \quad K = 31.62 .$$

With three groups: $a = \{1,6,7\}, b = \{2,3,5,9,10\}, c = \{4,8\}$ a subpartitioning into 4 and 2 subgroups of the a- and b-group seems reasonable ($k = 2, \ell = 2$).

b is split into \{9,10\} and \{2,3,5\}. This gives $K(2,2) = 32.96$.

With four groups: $A = \{1,7\}, b = \{6\}, c = \{2,3,5,9,10\}, d = \{4,8\}$ and $k = \ell = m = 2$ one obtains $K(2,2,2) = 32.07$.

For this problem Bomberger [1] gives a solution with costs 36.65;

the solution of Stankard and Gupta [6] costs 36.24;

Madigan's [4] method gives 33.94;

Doll and Whybark [2] find $K(2,2,2)K^{-1} = 1.014$. 
References.


