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Infinite divisible and stable distributions modulo 1

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1. Introduction.

Infinite divisibility and stability in the customary sense is extensively discussed in the literature (see e.g. Lukacs (1970), Feller (1971), Petrov (1975)). Schatte (1983) studies infinite divisibility modulo $2\pi$ (mod $2\pi$). He gives a representation theorem for infinite divisible (infdiv) (mod $2\pi$) Fourier-Stieltjes Sequences (FSS’s) and a limit theorem for sequences of infdiv (mod $2\pi$) FSS’s. Furthermore, under an infinite smallness (mod $2\pi$) condition he considers convergence of sums to infdiv (mod $2\pi$) distributions.

In this paper we consider distributions modulo $1$ (mod 1). In Section 2 we give some notations, definitions, and properties of FSS’s, and in Section 3 we reformulate Schatte’s results for infdiv (mod 1) distributions. From Schatte’s representation we deduce in Section 4 two other representations: one similar to the Levy-Khinchine canonical form, and the other to the Kolmogorov canonical form. In addition, we give a new characterization of infdiv (mod 1) distributions. In Section 5 we define stable (mod 1) distributions and characterize these distributions in two theorems. Finally, in Section 6, we generalize a limit theorem proved by Schatte.

2. Notation; definitions; properties of Fourier-Stieltjes Sequences.

We start by giving some notations and definitions. Throughout this paper, for $x \in \mathbb{R}$, $\xi \mathbb{Z}$ denotes the set ($\xi$ for $j \in \mathbb{Z}$). Let $X$ be a random variable (rv), and let $(X) \in [0,1)$ denote $X$ modulo 1. The left continuous distribution function (df) of a rv $X$ on $\mathbb{R}$ is denoted by $F_X$. Furthermore, $\mathcal{X}[0,1)$ will denote the set of rv’s $X$ with $P(0 \leq X < 1) = 1$. We recall the definition of the FSS of such rv’s.
Definition 2.1. Let $X \in \mathcal{Z}[0,1)$. The FSS $c_x : \mathbb{Z} \rightarrow \mathbb{C}$ of $X$ is defined by

$$c_x(k) = \int_{[0,1)} e^{2\pi ikx} dF_x(x) \quad (k \in \mathbb{Z}).$$

Clearly $c_x(0) = 1$, $|c_x(k)| \leq 1$, and $c_x(-k) = \overline{c_x(k)} \quad (k \in \mathbb{Z})$. Sometimes $c_x$ is written as $c_{F_x}$.

Since $e^{2\pi ikx} = e^{2\pi ikx} (x \in \mathbb{R})$, we have for any rv $X$ the trivial but useful identity

$$\varphi_x(2\pi k) = \mathbb{E}e^{2\pi ikX} = \mathbb{E} \varphi_x(2\pi k) (k \in \mathbb{Z}),$$

where $\varphi_x$ denotes the characteristic function (chf) of $X$.

Next, we state the uniqueness, continuity and convolution theorems for FSS’s. For the proofs we refer to Grenander (1963) or Schatte (1983); for other properties of FSS’s we refer to Wilms (1991).

Proposition 2.2. Let $X,Y \in \mathcal{Z}[0,1)$. Then

$$F_x = F_y \iff c_x = c_y.$$

Proposition 2.3. Let $(c_n)$ be a sequence of FSS’s and $(F_n)$ the corresponding sequence of df’s. The sequence $(F_n)$ converges weakly to a df $F$ iff $c_n(k) \to c(k) (k \in \mathbb{Z}, n \to \infty)$. The sequence $c$ is then the FSS of $F$.

Proposition 2.4. Let $X, X_1, X_2 \in \mathcal{Z}[0,1)$ with $X_1$ and $X_2$ independent. Further let $c, c_1$, and $c_2$ be the corresponding FSS’s. Then

$$X \overset{d}{=} (X_1 + X_2) \iff c(k) = c_1(k) \cdot c_2(k) (k \in \mathbb{Z}).$$

We say that a sequence $(G_n)$ of bounded nondecreasing functions with $G_n(-\infty) = 0$ converges weakly to $G$ (notation: $G_n \overset{w}{\to} G$) with $G(-\infty) = 0$, if for any two points $x,y$ of continuity of $G$

$$G_n(x) - G_n(y) \to G(x) - G(y) \quad (n \to \infty).$$

Furthermore, $d$ denotes equality in distribution.

Further we use the following notation: $U$ denotes the rv with (continuous) uniform distribution on $[0,1)$. For $r \in \mathbb{N}$, $U_r$ denotes the rv with discrete uniform distribution on $[0,1)$, i.e.
Lemma 2.5. (i) \( c_U(k) = 0 \) for all \( k \neq 0 \).

(ii) Let \( r \in \mathbb{N} \). Then 

\[
c_U(k) = \begin{cases} 
1 & \text{if } k \in \mathbb{Z} \\
0 & \text{otherwise.}
\end{cases}
\]

We now give a definition, due to Schatte (1983).

Definition 2.6. The replication number of a rv \( X \) (or \( (X) \) or \( c_X \)) is defined by

\[
\text{Rep}(X) = \sup\{r \in \mathbb{N} : \{X\} \overset{d}{=} \{X+1/r\} \}.
\]

We note that every \( X \) has a replication number at least 1, and that \( \text{Rep}(U) = 0 \). Next, we characterize distributions with finite replication number.

Lemma 2.7. Let \( r \in \mathbb{N} \), and let \( X \) be independent of \( U \). Then the following statements are equivalent:

(i) \( r \) is a divisor of \( \text{Rep}(X) \).

(ii) \( \{X^{1/r}\} \overset{d}{=} \{X\} \).

(iii) \( \{X+U\} \overset{d}{=} \{X\} \).

(iv) \( c_X(k) = 0 \) if \( k \in \mathbb{Z} \).

(v) \( \{X\} \overset{d}{=} U + \frac{1}{r}(Z) \) for some \( Z \) independent of \( U \).

Proof: The equivalence of (iii), (iv) and (v) is proved by Wilms and Thiemann (1993). The equivalence of part (ii) and (iii) follows immediately from

\[
c_X(k) = c_X(k)e^{2\pi i k/r} \quad (k \in \mathbb{Z})
\]

Proof of (i) \( \Rightarrow \) (ii): \( \text{Rep}(X) = mr \) for some \( m \in \mathbb{N} \); hence we have \( c_X(k) = 0 \) if \( k \in m\mathbb{Z} \); so \( c_X(k) = 0 \) if \( k \in \mathbb{Z} \). Then \( c_{X+1/r}(k) = c_X(k)e^{2\pi i k/r} = c_X(k) \). Hence \( \{X\} \overset{d}{=} \{X+1/r\} \).

Proof of (ii) \( \Rightarrow \) (i): It is sufficient to show the following assertion. Let \( r_1, r_2 \in \mathbb{N} \). If \( \{X^{1/r_1}\} \overset{d}{=} \{X\} \) and \( \{X^{1/r_2}\} \overset{d}{=} \{X\} \), then \( \{X^{1/q}\} \overset{d}{=} \{X\} \) with
q=lcm(r_1, r_2), where lcm(x,y) denotes the least common multiple of x and y (x,y∈N).

On account of part (iv) it suffices to prove c_{(x)}(k)=0 if k∈Z. Let k∈Z. Then k∈Z or k∈Z. Hence, by part (iv), c_{(x)}(k)=0.

Remarks 2.8. (i) In fact, in the proof of (ii)→(i) we show that

\[ \text{Rep}(X)=\text{lcm}(r∈\mathbb{N}: (X) \overset{d}{=} (X+1/r)). \]

(ii) In Wilms and Thiemann (1993) distributions X satisfying part (iii) are called U-shift-invariant.

(iii) If Rep(X)=∞, then for arbitrary large r we have c_{(x)}(k)=0 if k∉Z;

therefore c_{(x)}(k)=0 if k∈Z, i.e. (X) \overset{d}{=} U. In conclusion, we find

\[ (X) \overset{d}{=} U \iff \text{Rep}(X)=∞. \]

3. Review of Schatte's results.

3.1. Infinitely divisible (mod 1) distributions.

Here we review the results on infinite divisibility in modulo 1 sense proved by Schatte (1983); we give a representation theorem and a limit theorem for infdiv (mod 1) FSS's. We first define infinite divisibility (mod 1).

Definition 3.1. Let X∈[0,1). Then c:=c_{x, X} or F_{X, X} are said to be infinitely divisible (mod 1) if for each n∈N there exists a n-tuple (X_{n,m})_{m=1}^{n} (X_{n,m}∈[0,1)) of independent and identically distributed (iid) rv's such that

\[ X \overset{d}{=} (X_{n,1}+\ldots+X_{n,n}), \]

or equivalently, if for each n∈N there exist a FSS c_{n} with

\[ c(k)=(c_{n}(k))^{n} \quad (k∈\mathbb{Z}). \]

Clearly, U is infdiv (mod 1) with Rep(U)=∞.

Schatte proves the following results.
Proposition 3.2. (Schatte thm 4.1) Let $c_1$ and $c_2$ be infdiv (mod 1) FSS's. Then $c_1(k) \cdot c_2(k)$ is infdiv (mod 1) FSS. \qed

Proposition 3.3. (Schatte thm 4.2) Let $(c_n)$ be a sequence of infdiv (mod 1) FSS's. Let $c \to c \ (n \to \infty)$. Then $c$ is an FSS and is infdiv (mod 1). \qed

We now give the representation theorem.

Proposition 3.4. (Schatte thm 4.3) Let $X$ be a rv with $\text{Rep}(X)=r$. Then $X$ is infdiv (mod 1) iff its FSS $c$ can be written in the form

\[(3.1) \quad c(kr) = \exp\left(ik\alpha + \int_{[0,1)} \frac{e^{2\pi ikx} - 1 - iksin2\pi x}{1 - \cos 2\pi x} d\theta(x)\right) \quad (k \in \mathbb{Z}).\]

Here $\alpha$ is a real constant with $\alpha \in [0,2\pi)$ and $\theta$ is a nondecreasing left continuous bounded function on $[0,1)$ with $\theta(0)=0$. This representation is unique. \qed

We now formulate a limit theorem for infdiv (mod 1) df's.

Proposition 3.5. (Schatte thm 4.4) Let $F$ be a df, and let $(F_n)_{n=1}^{\infty}$ be a sequence of infdiv (mod 1) df's such that $F \to F \ (n \to \infty)$. Let $(c_n)_{n=1}^{\infty}$ be the corresponding sequence of FSS's all with replication number 1 and represented by

\[(3.1) \quad c_n(k) = \exp\left(ik\alpha_n + \int_{[0,1)} \frac{e^{2\pi ikx} - 1 - iksin2\pi x}{1 - \cos 2\pi x} d\theta_n(x)\right) \quad (k \in \mathbb{Z}).\]

Then the following assertions are true.

(i) $F(x) = x \ (x \in [0,1))$ iff

\[(3.2) \quad \int_{[0,1)} \frac{(1 - \cos 2\pi x)}{1 - \cos 2\pi x} d\theta_n(x) \to \infty \quad (n \to \infty, \ k \neq 0). \]

(ii) $c_F$ is represented by (3.1) with $\text{Rep}(c_F)=r$ iff (3.2) holds for $k=1, \ldots, r-1$, and

\[K_n \to \theta, \quad \beta_n \to \alpha \quad (n \to \infty),\]

where
\[ K_n(x) = \sum_{s=0}^{r-1} \int_0^x \frac{1 - \cos 2\pi t}{1 - \cos(2\pi(t+s)/r)} \, d\theta_n \left( \frac{t+s}{r} \right) \quad (x \in [0,1]), \]

\[ \beta_n = r\alpha + \int_0^1 \frac{\sin 2\pi x}{1 - \cos 2\pi x} \, d(K_n(x) - r\theta_n(x)). \]

We remark that if in part (ii) \( r = 1 \), then the conditions take the simple form \( \theta_n(x) \xrightarrow{w} \theta(x), \alpha \xrightarrow{n} (n \rightarrow \infty) \).

### 3.2. Convergence to infinitely divisible (mod 1) distributions.

Schatte considers a sequence \( (Y_{n,m})_{n,m=1}^{k_n} \) \((Y_n \in \mathbb{X}[0,1])\) of rv's in a triangular array, and looks for all limit distributions of

\[ Y_n = \{Y_{n,1} + \ldots + Y_{n,k_n}\} \]

as \( n \rightarrow \infty \), and assuming \( k_n \rightarrow \infty \) as \( n \rightarrow \infty \). He shows that under certain conditions, such as the condition of infinite smallness (mod 1) (i.e. for every \( c > 0 \))

\[ \max_{1 \leq m \leq k_n} P(c \leq Y_{n,m} \leq 1 - c) \rightarrow 0 \quad (n \rightarrow \infty), \]

the limit of a sequence of rv's in \( \mathbb{X}[0,1] \) is infdiv (mod 1), even if the elements of this sequence are not infdiv (mod 1) (cf. Section 5 Schatte (1983)).

### 4. Characterizations of infinite divisible (mod 1) distributions.

It is obvious that if \( X \) is infdiv in the customary sense, i.e. for each \( n \in \mathbb{N} \) there exists a n-tuple \((X_{n,m})_{m=1}^n\) of iid rv's such that

\[ X \overset{d}{=} X_{n,1} + \ldots + X_{n,n}, \]

then \((X)\) is infdiv (mod 1), since
Furthermore, we prove

**Lemma 4.1.** Let \( X \) be infdiv in the customary sense. Then \( \{X\} \) is infdiv \((\mod 1)\) with \( \text{Rep}(X)=l \).

**Proof:** From (4.1) we know that \( \{X\} \) is inf div \((\mod 1)\). Now suppose that \( \{X\} \) has \( \text{Rep}(X)=r>l \). Then \( \{X\} \overset{d}{=} \{X+1/r\} \) and hence \( \varphi_x(2\pi)=c_x(1)=0 \). This contradicts the fact that an inf div chf has no real zeros (see Lukacs (1970)). So \( \text{Rep}(X)=l \).

To characterize infdiv \((\mod 1)\) distributions, we need some auxiliary results. We first prove another representation theorem for infdiv \((\mod 1)\) FSS's.

**Lemma 4.2.** Let \( X \) be a rv with \( \text{Rep}(X)=r \). Then \( X \) is infdiv \((\mod 1)\) iff its FSS \( c \) can be written in the form

\[
(4.2) \quad c(kr)=\exp\left(ik\alpha + \int_{\left[\frac{-1}{2}, \frac{1}{2}\right]} \frac{e^{2\pi ikx} - 1 - iksin2\pi x}{1 - \cos2\pi x} \, dT(x)\right) \quad (k \in \mathbb{Z}).
\]

Here \( \alpha \) is a real constant with \( \alpha \in [0, 2\pi) \) and \( T \) is a nondecreasing left continuous bounded function on \( \left[\frac{-1}{2}, \frac{1}{2}\right] \) with \( T(-\frac{1}{2})=0 \). This representation is unique.

**Proof:** Let

\[
f(x):=\frac{e^{2\pi ikx} - 1 - iksin2\pi x}{1 - \cos2\pi x} \quad (x \in \mathbb{R}).
\]

\((\ast)\) By Proposition 3.4 \( c \) can be represented by (3.1). Using that \( f(x) \) is periodic on \( \left[\frac{1}{2}, 1\right] \) we obtain for \( k \in \mathbb{Z} \)

\[
\log c(kr)=ik\alpha + \int_{[0,1]} f(x) \, d\theta(x) \quad = ik\alpha - k^2(\theta(0+) - \theta(0)) + \int_{[-\frac{1}{2}, 0]} f(x) \, d\theta(x+1) + \int_{(0, \frac{1}{2})} f(x) \, d\theta(x)
\]
\[
= i k \alpha + \int_{[-\frac{1}{2}, 1]} f(x) dT(x),
\]
where
\[
T(x) = \begin{cases} 
\Theta(\frac{1}{2}, 1) - \Theta(x+1, 1) & \text{if } \frac{1}{2} \leq x < 0 \\
\Theta(\frac{1}{2}, 1) & \text{if } x = 0 \\
\Theta(\frac{1}{2}, 1) + \Theta(0, x) & \text{if } 0 < x < \frac{1}{2}
\end{cases}
\]
with
\[
\Theta(u, v) = \int_{[u, v)} d\Theta(y) \quad (u, v \in [0, 1), \ u < v).
\]
Obviously, the function \( T \) is bounded and nondecreasing, and \( T(-\frac{1}{2}) = 0 \).

(\(*\) Since \( f(x) \) is periodic on \([\frac{1}{2}, 0)\) we find
\[
\log c(kr) = ik \alpha - k^2 (T(0^+) - T(0^0)) + \int_{[0, \frac{1}{2}]} f(x) dT(x) + \int_{[\frac{1}{2}, 0)} f(x) dT(x-1)
\]
\[
= ik \alpha + \int_{[0, 1]} f(x) d\Theta(x),
\]
where
\[
\Theta(x) = \begin{cases} 
0 & \text{if } x = 0 \\
T(0, x) & \text{if } 0 < x < \frac{1}{2} \\
T(-\frac{1}{2}, \frac{1}{2}) - T(x-1, 0) & \text{if } \frac{1}{2} \leq x < 1
\end{cases}
\]
with
\[
T(u, v) = \int_{[u, v)} d(T(y)) \quad (u, v \in [-\frac{1}{2}, \frac{1}{2}), \ u < v).
\]
Obviously, the function \( \Theta \) is bounded and nondecreasing, and \( \Theta(0) = 0 \).

**Remark 4.3.** Schatte (1983, p. 253) claims that the function
\[
f(2\pi t) = \exp \left( it \alpha + \int_{[0, 1]} e^{\frac{2\pi itx - 1 - itsin2\pi x}{1 - cos2\pi x}} d\Theta(x) \right) \quad (t \in \mathbb{R})
\]
is an infdiv chf. This is not true, because the integrand
\[
e^{\frac{2\pi itx - 1 - itsin2\pi x}{1 - cos2\pi x}}
\]
tends to infinity if \( x \uparrow 1 \) and \( t \in k\mathbb{Z} \).

As shown in Lemma 4.2, by transforming the function \( \Theta \) to the interval
We have that the integrand is bounded on \([-\frac{1}{2}, \frac{1}{2}]\) for all \(t \in \mathbb{R}\). This means that the function
\[
g(2\pi t) = \exp \left( \frac{\int_{\frac{1}{2}}^{1/2} \frac{e^{2\pi i tx} - 1 - i\sin(2\pi x)}{1 - \cos(2\pi x)} d\theta(x)}{[-\frac{1}{2}, \frac{1}{2}]} \right) \quad (t \in \mathbb{R})
\]
is an infdiv chf; this shall be used in the proof of Theorem 4.6.

Next, we prove two representations similar to the Levy-Khinchine canonical form and the Kolmogorov canonical form, respectively (see Lukacs (1970)).

**Lemma 4.4.** Let \(X\) be a rv with \(\text{Rep}(X) = r\). Then \(X\) is infdiv (mod 1) iff its FSS \(c\) can be written in the form
\[
c(kr) = \exp \left( \int_{\frac{1}{2}}^{1/2} (e^{2\pi ikx} - \frac{1}{1+x} - \frac{1}{2}) \frac{dH(x)}{x} \right) \quad (k \in \mathbb{Z}).
\]
Here \(\beta\) is a real constant with \(\beta \in [0, 2\pi)\) and \(H\) is a nondecreasing left continuous bounded function on \([-\frac{1}{2}, \frac{1}{2}]\) with \(H(-\frac{1}{2}) = 0\). The representation is unique.

**Proof:** \((\Rightarrow)\) By Lemma 4.2 \(c\) can be represented by (4.2); hence for \(k \in \mathbb{Z}\)
\[
\log c(kr) = i\beta + \int_{\frac{1}{2}}^{1/2} (e^{2\pi ikx} - \frac{1}{1+x} - \frac{1}{2}) \frac{dH(x)}{x},
\]
where
\[
\beta = \beta' \pmod{2\pi}, \quad \beta' = \alpha + \int_{\frac{1}{2}}^{1/2} \frac{2\pi x}{1+x} \frac{\sin(2\pi x)}{1 - \cos(2\pi x)} dT(x)
\]
with
\[
T(-\frac{1}{2}, 0) = \frac{1}{2} \quad T(x, 0) = \frac{1}{2}
\]
and
\[
H(x) = \begin{cases} 
T(-\frac{1}{2}, 0) - T(x, 0) & \text{if } \frac{1}{2} \leq x < 0 \\
T(-\frac{1}{2}, 0) & \text{if } x = 0 \\
T(-\frac{1}{2}, 0) + T(0, x) & \text{if } 0 < x < \frac{1}{2}
\end{cases}
\]
\[
T(u,v) = \int_{[u,v]} \frac{y^2}{2} \frac{1}{1-\cos2\pi y} \, dT(y) \quad (u,v \in [-\frac{1}{2}, \frac{1}{2}), \ u<v).
\]

Obviously, the function \(H\) is bounded and nondecreasing, and \(H(-\frac{1}{2}) = 0\).

(\ast) From (4.3) we find for \(k \in \mathbb{Z}\)
\[
\log c(kr) = ik\beta + \int_{[-\frac{1}{2}, \frac{1}{2})} \left( e^{2\pi ikx} - 1 - \frac{2\pi ikx}{1+x^2} \right) \frac{1+x^2}{x^2} \, dH(x)
\]
\[
= ik\alpha + \int_{[-\frac{1}{2}, \frac{1}{2})} \frac{e^{2\pi ikx} - 1 - ik\sin2\pi x}{1-\cos2\pi x} \, dT(x)
\]

where
\[
\alpha = \alpha' \pmod{2\pi}, \quad \alpha' = \beta + \int_{[-\frac{1}{2}, \frac{1}{2})} \left( \frac{2\pi kx}{1+x^2} - \sin2\pi x \right) \frac{1+x^2}{x^2} \, dH(x),
\]

\[
T(x) = \begin{cases} 
H(-\frac{1}{2},0) + H(x,0) & \text{if } -\frac{1}{2} < x < 0 \\
H(-\frac{1}{2},0) & \text{if } x = 0 \\
H(-\frac{1}{2},0) + H(0,x) & \text{if } 0 < x < \frac{1}{2}
\end{cases}
\]

with
\[
H(u,v) = \int_{[u,v]} (1-\cos2\pi y) \frac{1+y^2}{y^2} \, dH(y) \quad (u,v \in [-\frac{1}{2}, \frac{1}{2}), \ u<v).
\]

Obviously, the function \(T\) is bounded and nondecreasing, and \(T(-\frac{1}{2}) = 0\).

Similarly, we can prove

**Lemma 4.5.** Let \(X\) be a rv with \(\text{Rep}(X) = r\).

Then \(X\) is infdiv \(\pmod{1}\) iff its FSS \(c\) can be written in the form
\[
(4.4) \quad c(kr) = \exp \left( ik\gamma + \int_{[-\frac{1}{2}, \frac{1}{2})} e^{2\pi ikx} - 1 - 2\pi ikx \frac{1+x^2}{x^2} \, dK(x) \right) \quad (k \in \mathbb{Z}).
\]

Here \(\gamma\) is a real constant with \(\gamma \in [0,2\pi)\) and \(K\) is a nondecreasing left continuous bounded function on \([\frac{1}{2}, \frac{1}{2})\) with \(K(-\frac{1}{2}) = 0\). The representation is unique.
We now give the main theorem of this paper, which characterizes infdiv (mod 1) distributions.

**Theorem 4.6.** Let $X$ be a rv with $\text{Rep}(X) = r$. Then $X$ is infdiv (mod 1) iff

$$X \overset{d}{=} \frac{U_r}{r} + \frac{1}{r}Y$$

for some infdiv $Y$ independent of $U_r$.

**Proof:** $(\Rightarrow)$ By Lemma 4.4 we have that $c_X$ can be represented by (4.3); so

$$c_X(kr) = \exp\left(ik\beta + \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]} (e^{2\pi ikx} - 1 - \frac{2\pi ix}{2} \frac{1+x^2}{x^2} \text{d}H(x)) \right) \quad (k \in \mathbb{Z}).$$

We define a rv $Y$, independent of $U_r$, by its chf (see Remark 4.3)

$$\varphi_Y(2\pi it) = \exp\left(it\beta + \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]} (e^{2\pi itx} - 1 - \frac{2\pi itx}{2} \frac{1+x^2}{x^2} \text{d}H(x)) \right) \quad (t \in \mathbb{R});$$

the Levy-Khinchine canonical representation yields then that $Y$ is infdiv in the customary sense (see Lukacs (1970)). Furthermore, we find

$$c_X(kr) = \varphi_Y(2\pi kr) = c_Y(k) \quad (k \in \mathbb{Z}).$$

Since $c(k) = 0$ if $k \in \mathbb{Z}$ we have from Lemma 2.7

$$X \overset{d}{=} \frac{U_r}{r} + \frac{1}{r}Z$$

for some $Z$ independent of $U_r$, and thus

$$c_X(kr) = c_Y(k) = c_Y(k) = c_Z(k) = c_Z(kr) = c_Z(kr) = c_Z(kr) = c_Z(k) \quad (k \in \mathbb{Z}).$$

Combining (4.6), (4.7) and Proposition 2.2 we find $Y \overset{d}{=} Z$. Hence (4.5) holds.

$(\Leftarrow)$ From Lemma 2.7 we find $c_X(k) = 0$ if $k \in \mathbb{Z}$, and using Lemma 2.5(ii) it follows $c_X(kr) = c_Y(k) \quad (k \in \mathbb{Z})$. Lemma 4.1 implies then that $c_Y$ can be represented by (3.1) with $\text{Rep}(Y) = 1$. So $X$ is infdiv (mod 1) with $\text{Rep}(X) = r$.

$\Box$

**Remarks 4.7.** (1) Analogous to the proof of Theorem 4.6, and now applying Lemma 4.5 and using the Kolmogorov canonical representation, we can prove that $X$ is infdiv (mod 1) with $\text{Rep}(X) = r$ iff

$$X \overset{d}{=} \frac{U_r}{r} + \frac{1}{r}Y$$

for some infdiv $Y$ independent of $U_r$ with $\mathbb{E}Y^2 < \infty$.

(11) We claim that $U \overset{d}{=} Y$ for any infdiv rv $Y$. Suppose that there is a
rv Y such that $U \overset{d}{=} (Y)$. Then $\varphi_Y(2\pi k) = 0 \ (k \in \mathbb{Z})$, and this contradicts the fact that an infdiv chf has no real zeros (see Lukacs (1970)).

(iii) From (4.5) it follows that $(rX) \overset{d}{=} (Y)$.

(iv) We give here another proof of (i): Since $Y$ is infdiv, there exists a n-tuple $(Y, n_i)_{i=1}^n$ of iid rv's such that $Y \overset{d}{=} Y_{n,1} + \ldots + Y_{n,n}$. Let $(U(i))_{i=1}^n$ be a n-tuple of iid rv's distributed as $U$, and independent of $(Y, n_i)$. Take $X \overset{d}{=} U_{r,1} + \frac{1}{n} (Y, n_i)$; then $X \overset{d}{=} X_{n,1} + \ldots + X_{n,n}$.

The following corollary is a consequence of Theorem 4.6 and Remark 4.7(i).

**Corollary 4.8.** Let $Y$ be infdiv with $EY^2 = \omega$. Then there is an infdiv rv $X$ such that $EX^2 < \omega$ and $(X) \overset{d}{=} (Y)$.

Conversely, let $X$ be infdiv with $EX^2 < \omega$. Then there is an infdiv rv $Y$ such that $EY^2 = \omega$ and $(Y) \overset{d}{=} (X)$.

**Proof:** By Lemma 4.1 we have $(Y)$ is infdiv (mod 1) with $\text{Rep}(Y) = 1$. Then from (4.8) we find $(Y) \overset{d}{=} (X)$ for some infdiv $X$ with $EX^2 < \omega$.

If $X$ is infdiv (mod 1), then from Lemma 4.1 we know $(X)$ is infdiv (mod 1) with $\text{Rep}(X) = 1$. Theorem 4.6 yields that $(X) \overset{d}{=} (Y)$ for some infdiv $Y$ with $EY^2 = \omega$.

In the following theorems we give some properties of inf div (mod 1) distributions. We first prove an auxiliary result. Here the greatest common divisor of $x$ and $y$ is denoted by $\text{gcd}(x,y)$, and the least common multiple of $x$ and $y$ by $\text{lcm}(x,y) \ (x, y \in \mathbb{N})$.

**Lemma 4.9.** Let $p,q,r,s \in \mathbb{N}$, $a = \text{lcm}(p,q)$ and $\text{gcd}(p,s) = 1$. Then

(i) $(U + U) \overset{d}{=} U$,

(ii) $(p\overline{U}) \overset{d}{=} p\overline{U}$.

(iii) $(s\overline{U}) \overset{d}{=} s\overline{U}$.

**Proof:** (i) If $k \in \mathbb{Z}$, then $k = am + mb_1 = mb_2 + p$ for some $m, b_1, b_2 \in \mathbb{N}$; so $c_U(k) = c_U(k) = 1$. If $k \in \mathbb{Z}$, then $c_U(k) = 0$ or $c_U(k) = 0$ since $pq = aw$. Hence $c_U(k) = c_U(k) = c_U(k) \ (k \in \mathbb{Z})$. 

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(ii) It suffices to show that $c_{qr}(pkr) = c_{qr}(pk) \ (ke\mathbb{Z})$.

If $pk \in \mathbb{Z}$, then $c_{qr}(pkr) = 1$. If $pk \not\in \mathbb{Z}$, then $kr \not\equiv qr \ (i.e. \ c_{qr}(pkr) = 0)$. Hence $c_{qr}(pkr) = c_{qr}(pk) \ (ke\mathbb{Z})$.

(iii) Clearly, we have $c_{sk}(k) = c_{pk}(k) \ (ke\mathbb{Z})$.

If $ke \in \mathbb{Z}$, then $c_{p}(sk) = 1$. If $ke \not\in \mathbb{Z}$, then $sk \not\in \mathbb{Z}$ since $gcd(p, s) = 1$, so $c_{sk}(sk) = 0$. Hence $c_{sk}(sk) = c_{sk}(k) \ (ke\mathbb{Z})$.

It is well known that if $Y_1$ and $Y_2$ are independent and infdiv in the customary sense, then this is also true for $Y := aY_1 + bY_2$ for any $a, b \in \mathbb{R}$. Furthermore, let $r \in \mathbb{N}$; if $Y_1$ and $Y_2$ are independent of $U$ with $EY_m < \infty \ (m=1,2)$, then this is also true for $Y$. These properties will be used in the proof of the following two theorems.

**Theorem 4.10.** Let $X_1$ be infdiv (mod 1) with $Rep(X_1) = r_1$.

(i) Let $X_2$ be infdiv (mod 1) with $Rep(X_2) = r_2$ and independent of $X_1$. Then $(X_1 + X_2)$ is infdiv (mod 1) with $Rep(X_1 + X_2) = lcm(r_1, r_2)$.

(ii) Let $s \in \mathbb{R}$. Then $(X_1 + s)$ is infdiv (mod 1) with $Rep(X_1 + s) = r_1$.

**Proof:** (i) Let $q := lcm(r_1, r_2)$ and $p := gcd(r_1, r_2)$. Let further $c_m$ be the FSS of $X_m \ (m=1,2)$. From (4.8) we have

$$c_{1}(k)c_{2}(k) = c_{r_1}(k)c_{r_2}(k) E_{m}(r_1/r_1, r_2/r_2)$$

for some infdiv $Y_m$ independent of $U$ and $U$ with $EY_m < \infty \ (m=1,2)$. From Lemma 4.9(i) we find $c_{r_1}(k)c_{r_2}(k) = c_{r_1}(k)$. Hence

$$(X_1 + X_2) = U + \frac{1}{q}((r_1 Y_1 + r_2 Y_2)/p).$$

(ii) This is a consequence of part (i). Take $P(X_2 = (s)) = 1$. Then $X_2$ is infdiv (mod 1) with $Rep(X_2) = 1$.

We remark that if $max(r_1, r_2) = \infty$, then $(X_1 + X_2)$ is infdiv (mod 1) with $Rep(X_1 + X_2) = \infty$ since $(X + U) \overset{d}{=} U$ for any rv $X$ independent of $U$.
Theorem 4.11. Let \( q \in \mathbb{N} \), and let \( X \) be infdiv \((\text{mod } 1)\) with \( \text{Rep}(X) = r \). If \( \gcd(q,r) = p \), then \( \{qX\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qX) = r/p \).

Proof: From (4.8) we obtain
\[
X \overset{d}{=} U_r + \frac{1}{r} \{Y\}
\]
for some infdiv \( Y \) independent of \( U_r \) with \( EY^2 < \infty \). Hence we obtain
\[
(4.9) \quad c_{\{qX\}}(k) = c_{U_r}(qk) E e^{2\pi i k Y / r}.
\]

We distinguish three cases: 1. \( p=r \); 2. \( p=1 \); 3. \( 1<p<r \).

1. If \( \gcd(q,r) = r \), then \( q = ar \) for some \( a \in \mathbb{N} \). Hence from (4.9) we find
\[
c_{\{qX\}}(k) = c_{U_r}(ark) E e^{2\pi i k Y / r} = c_{\{aY\}}(k).
\]
So \( \{qX\} \overset{d}{=} \{aY\} \); from (4.8) we have that \( \{qX\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qX) = 1 \).

2. If \( \gcd(q,r) = 1 \), then from Lemma 4.9(iii) we know that \( c_{U_r}(qk) = c_{U_r}(k) \).
So, from (4.9) it follows \( \{qX\} \overset{d}{=} U_r + \frac{1}{r} \{qY\} \). Expression (4.8) implies that \( \{qX\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qX) = r \).

3. If \( \gcd(q,r) = p \), then \( r = mp \) and \( q = np \) for some \( m, n \in \mathbb{N} \). Further we have \( \gcd(m,n) = 1 \). Applying part (ii) and (iii) of Lemma 4.9 it follows
\[
\{qU_r\} \overset{d}{=} \{npU_m\} \overset{d}{=} \{nU_m\} \overset{d}{=} U_r \overset{d}{=} U_r / p.
\]
Consequently, from (4.9) we find
\[
\{qX\} \overset{d}{=} U_r / p + \frac{p}{r} \{qY/p\}.
\]
Then (4.8) implies that \( \{qX\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qX) = r/p \).

We note that the assertion of Theorem 4.11 is also true if \( q \in \mathbb{Z} \setminus \mathbb{N} \); if \( q=0 \), then \( \{qY\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qY) = 1 \). Furthermore, if \( X \overset{d}{=} U \), then \( \{qX\} \overset{d}{=} U \), i.e. \( \{qX\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(qX) = \infty \) (\( q \in \mathbb{Z} \setminus \{0\} \)).

In general \( \{qX\} \) is not infdiv \((\text{mod } 1)\) if \( q \in \mathbb{R} \setminus \mathbb{Z} \) and \( X \) is infdiv \((\text{mod } 1)\). For example, if \( X \overset{d}{=} U_2 \) and \( q = 3/2 \), then \( c_{\{3X/2\}}(1) \neq 0 \) and \( c_{\{3X/2\}}(2) = 0 \); hence \( \text{Rep}(3X/2) = 1 \). Since \( c_{\{3X/2\}}(2) = 0 \), \( (3X/2) \) is not infdiv \((\text{mod } 1)\).

This is also true for rv's \( X \) with \( \text{Rep}(X) = 1 \): Let \( Y \) be exponentially distributed with \( EY = 1/\lambda \). Then \( \{Y\} \) is infdiv \((\text{mod } 1)\) with \( \text{Rep}(Y) = 1 \); it is easy to verify that \( \{Y/2\} \) is not infdiv \((\text{mod } 1)\).
4.12. Examples

(i) Let $X$ have a normal distribution with $E X = \mu$, $\text{Var} X = \sigma^2$. Then $c_{(X)}(k) = \exp\left(2\pi i k \mu - 2(k \sigma)^2\right)$, and $(X)$ is infdiv (mod 1) with $\text{Rep}(X) = 1$.

(ii) Let $r, k \in \mathbb{N}$ with $k \geq r + 1 \geq 2$, and $X \overset{d}{=} U + \frac{1}{r}$. Then $X \overset{d}{=} U + \frac{1}{r} \frac{1}{k}$, and hence $X$ is infdiv (mod 1) with $\text{Rep}(X) = r$.

(iii) Let $X$ have FSS $c$. The FSS $\tilde{c}$ of the rv $Y$ with compound Poisson distribution generated by $c$ is defined by $\tilde{c}(k) = \exp(\alpha(c(k) - 1))$ $(k \in \mathbb{Z}, \alpha > 0)$.

Since the chf of $Y$ is infdiv (cf. Feller (1971)), we have that $\tilde{c}$ is infdiv (mod 1) with $\text{Rep}(Y) = 1$.

Let $m \in \mathbb{N}$, and take $X \overset{d}{=} U_m$. Then

$$c(k) = e^{-\lambda} + \left(1 - e^{-\lambda}\right) \sum_{j=0}^{m-1} P(Y=j) = e^{-\lambda} + \frac{1}{m} \left(1 - e^{-\lambda}\right)$$

for $j=0, \ldots, m-1$.

5. Stable (mod 1) distributions.

Stable characteristic functions (or distributions) in the customary sense are defined as follows: a chf $\phi$ is said to be stable if for every $b_1, b_2 > 0$ there exist constants $b > 0, \alpha \in \mathbb{R}$ such that

$$\phi(b_1 t) \phi(b_2 t) = \phi(bt) \exp(i \alpha t) \quad (t \in \mathbb{R}).$$

In modulo 1 sense FSS's are very useful, but unfortunately it is not possible to give a definition similar to (5.1) since a FSS is only defined for integers $k \in \mathbb{Z}$ (cf. definition 2.1). When trying to define self-decomposability modulo 1 the same problem arises. Therefore, we consider stability modulo 1 according to the following definition.

Definition 5.1. Let $X \in \mathbb{I}[0,1)$. Then $c := c_X$, $X$ or $F_X$ are said to be stable (mod 1) if there exists a real number $\alpha \in (0,1)$ such that
Clearly, from relation (5.2) it follows that for each \( n \in \mathbb{N} \) there exists a real number \( \alpha_n^* \in [0,1) \) such that
\[
c^n(k) = c(k) \exp(2\pi ik\alpha_n^*) \quad (k \in \mathbb{Z}),
\]
or equivalently, for each \( n \in \mathbb{N} \) there exists a sequence \( (X_j) \) of iid rv's in \( X(0,1) \) and a real number \( \alpha_n \in [0,1) \) such that the equation
\[
\{X_1 + \ldots + X_n + \alpha_n\} \overset{d}{=} X_1
\]
holds. It is clear that \( U \) is stable (mod 1) (take \( \alpha_n = 0 \)). Furthermore, it is easy to verify that if \( X \) is stable (mod 1), then \( X \) is infdiv (mod 1).

In the following theorems we characterize stable (mod 1) distributions. We note that in Wilms and Thiemann (1993) solutions of equation (5.3) are characterized.

**Theorem 5.2.** Let \( r \in \mathbb{N} \), and \( X \in X(0,1) \). Then the following statements are equivalent:
(a) \( X \) is stable (mod 1).
(b) \( X \overset{d}{=} U \) or \( X \overset{d}{=} U_r + \beta \) for some \( \beta \in [0,1/r) \).

**Proof:** Let \( c \) denote the FSS of \( X \). Let \( X \) be stable (mod 1). Then (5.2) holds, i.e. \( c(k)(c(k) - \exp(2\pi ik\alpha)) = 0 \) (\( k \in \mathbb{Z} \)). If \( c(k) = 0 \) for all \( k \neq 0 \), then \( X \overset{d}{=} U \). Alternatively, suppose there exists an integer \( k \in \mathbb{N} \) such that \( c(k) \neq 0 \). Take \( r = \min\{k \in \mathbb{Z} : c(k) \neq 0\} \). Hence \( c(r) \exp(-2\pi i r \alpha) = 1 \). Thus \( \{X - \alpha\} \overset{d}{=} U_r \). Then there exist an integer \( s \in \{0, \ldots, r-1\} \) and \( \beta \in [0,1/r) \) such that \( \alpha = \beta + s/r \). Therefore \( X \) has its distribution concentrated on the set \( \{\beta, \beta + 1/r, \ldots, \beta + (r-1)/r\} \). Hence \( X \overset{d}{=} U_r + \beta \). So (a) \(\Rightarrow\) (b).

Let \( X \overset{d}{=} U \). From Lemma 2.5(i) we have \( c(k) = 0 \) for \( k \neq 0 \). Thus if we take \( \alpha = 0 \), then relation (5.2) holds, i.e. \( U \) is stable (mod 1).

Let \( X \overset{d}{=} U_r + \beta \) for some \( \beta \in [0,1/r) \). Using Lemma 2.5(ii) we have
\[
c(k) = \begin{cases} 
\exp(2\pi ik\beta) & \text{if } k \in \mathbb{Z} \\
0 & \text{otherwise}.
\end{cases}
\]

So if we take \( \alpha = \beta \), then relation (5.2) holds, i.e. \( X \) is stable (mod 1). \( \square \)
Theorem 5.3. The set of stable (mod 1) distributions coincides with the set of distributions that are limits of \((X_1+\ldots+X_n+\alpha_n)\), where \((X_i)_{n=1}^\infty\) are iid rv's in \([0,1)\), \(\alpha_n \in (0,1)\), and equation (5.3) holds.

Proof: Let \(F\) be stable (mod 1). Then (5.3) holds. Hence the distribution of \(X_1\) is a limit of the distribution of \((X_1+\ldots+X_n+\alpha_n)\).

Suppose there exist a sequence \((X_i)\) of iid rv's in \([0,1)\) and \(Y \in \mathcal{L}(0,1)\) such that \((X_1+\ldots+X_n+\alpha_n) \overset{d}{\to} Y\) (\(n \to \infty\)), or equivalently

\[
c_n(k) \exp(2\pi ik\alpha_n) \to c_Y(k) \quad (k \in \mathbb{Z}, \ n \to \infty).
\]

If \(c_Y(k) = 0\) for all \(k \neq 0\), then \(Y \overset{d}{=} U\). Hence \(Y\) is stable (mod 1).

Alternatively, suppose there exists an integer \(k \in \mathbb{N}\) such that \(c_Y(k) \neq 0\). Take \(r = \min\{k \in \mathbb{N}: c_Y(k) \neq 0\}\). Then \(|c_Y(r)| = 1\), and thus \(|c_Y(r)| = 1\). So there exists a real number \(\beta \in (0,1)\) such that \(c_Y(r) = \exp(2\pi ir\beta)\). Thus \((Y-\beta) \overset{d}{=} U\). As in the proof of Theorem 5.2 we find \(Y \overset{d}{=} U + \beta\) for some \(\beta \in (0,1/r)\). Then Theorem 5.2 yields \(Y\) is stable (mod 1).

6. A generalization of a limit theorem by Schatte.

In this section we generalize the result as given by Schatte in Proposition 3.5.

Theorem 6.1. Let \(F\) be a df, and let \((F_n)_{n=1}^\infty\) be a sequence of infdiv (mod 1) df's such that \(F_n \overset{w}{\to} F\) (\(n \to \infty\)). Let \((c_n)_{n=1}^\infty\) be the corresponding sequence of FSS's all with finite replication number \(r\) and represented by

\[
c_n(kr) = \exp \left(ik\alpha_n + \int_{0,1} \frac{\exp \left(-i k \sin 2\pi x\right)}{1 - \cos 2\pi x} \, d\theta_n(x)\right) \quad (k \in \mathbb{Z}).
\]

Then \(c_F\) is represented by (3.1) with \(\text{Rep}(c_F) = pr = m\) for some \(p \in \mathbb{N}\) iff (3.2) holds for \(k = 1, \ldots, m-1\), and

\[
K_n \overset{w}{\to} \theta, \quad \beta_n \overset{a}{\to} \alpha \quad (n \to \infty),
\]

where
\[ K_n(x) = \sum_{s=0}^{m-1} \int_{0}^{1} \frac{1-\cos 2\pi t}{1-\cos 2\pi (t+s)/m} \, d\theta_n \left( \frac{t+s}{m} \right) (x \in [0,1]), \]

\[ \beta_n = \max_n \int_{0}^{1} \sin 2\pi x \, d(K_n(x)-m \theta_n(x)). \]

**Proof:** The proof of this theorem is analogous to the proof of Schatte. The only modification, which needs some attention, is the following expression:

\[ c_n(km) = \exp\left( ik \alpha_n + \int_{0}^{1} \frac{e^{2\pi i k x} - 1 - ik \sin 2\pi x}{1 - \cos 2\pi x} \, d\theta_n(x) \right) (k \in \mathbb{Z}), \]

which can be written as

\[ c_n(km) = \exp\left( ik \beta_n + \int_{0}^{1} \frac{e^{2\pi i k x} - 1 - ik \sin 2\pi x}{1 - \cos 2\pi x} \, dK_n(x) \right) (k \in \mathbb{Z}). \]

\[ \Box \]

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**7. References.**


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