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The LPG location-allocation problem

by

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Introduction.

This note gives a few reflexions on a type of location-allocation problems. It stems from a firm concerned with the distribution of LPG, but may be met in many other situations. Its model structure resembles largely that of the location-allocation model, described in [1]. It has a special property which allows its analysis by pure linear programming methods.

The LPG location-allocation model.

The problem is to prescribe some products, produced or imported at some locations, via a number of depots to a large number of customers. Each customer must receive its total demand from only one depot. How should the customers be assigned to the depots so as to minimize the total delivery costs taken proportional to the distance between production sites and depots and between depots and customers.

The model description is based on a proper coding of data, activities and restrictions and their storage in properly coded lists and tables.

Lists:
List C  -  Customers.
List T  -  Terminals.
List R  -  Refineries.

These lists are called by their codename C, T and R. In the model description these letters are used also as running indices (running over all members actually being in the list) and in the visualisation of the model in figure 1 as the total number of members in the lists.
Tables:

Table D[C]  Demand by Customers.
Table CAP[T]  Capacity of Terminals.
Table DCT[C,T]  Distance Customers to Terminals.
Table DRT[R,T]  Distance Refineries to Terminals.

The letters between brackets refer to the corresponding lists. They define the entries to the tables.

Activities:

TRT(R,T)  Amount of LPG transported from refinery R to terminal T.
ASCT(C,T)  Assignment of customer C to terminal T, a 0/1 decision activity.

Restrictions:

BALRT(T)  Balance at terminal T between total assigned customer demand and supply from the refineries.
CAPRT(T)  Capacity restriction for terminal T; if required a distinction may be made between the physical upper capacity UPCAPRT(T) and a required lower capacity LOCAPRT(T).
BALRR(R)  Balance at refinery R between the production of LPG (on the amount available for distribution) and the total demand at the terminals.
CAPRT(T)  Capacity restriction for terminal T; if required a distinction may be made between the physical upper capacity restriction UPCAPRT(T) and a required lower capacity restriction LOCAPRT(T).
BALRR(R) Balance at refinery R between the production of LPG and the total supply to the terminals.

ASRC(C) Assignment restriction: each customer assigned to exactly one terminal.

COST Name of the cost function.

The somewhat clumsy looking coding of activities and restrictions has found to be very useful in practice. The letter combination outside the bracket is actually a family name for a group of activities or restrictions. The members are identified by the running letters (running over the corresponding lists) between the brackets.

The formal model description is now:

\[
\begin{align*}
\text{BALRT(T)} & : \sum_{R,T} \text{TRT}(R,T) - \sum_{C} D(C)\text{ASCT}(C,T) = 0 \\
\text{BALRR(R)} & : \sum_{T} \text{TRT}(R,T) - \text{PR}(R) \leq 0 \\
\text{CAPRT(T)} & : \sum_{C} D(C)\text{ASCT}(C,T) \leq \text{CAPT}(T) \\
\text{ASRC(C)} & : \sum_{T} \text{ASCT}(C,T) = 1
\end{align*}
\]

plus the refinery restrictions by which the productions PR(R) are determined.

\[
\begin{align*}
\text{COST} & : \sum_{R,T} \text{DRT}(R,T)\text{TRT}(RT) + \\
& \sum_{C,T} \text{DCT}(C,T)\text{ASCT}(C,T) + \\
\end{align*}
\]

costs of refinery theory.

The structure of the model is clearly shown by the visualisation in figure 1.
Since the assignment activities are of the $0/1$ type, the analysis of this model leads to mixed integer linear programming problems. One may try to solve these problems by using a mixed integer programming routine and exploiting the 'special ordered sets' feature (see the restrictions ASRC(RC)). In the study [1] this proved possible at the cost, however, of considerable calculating time.

Solving the linear programming relaxation delivers in general a solution in which some customers are assigned to more than one terminal. We will show however that the number of non-uniquely assigned customers is always very small. This proof uses the well-known property that the number of non-zero activities in a basic feasible (hence in a basic optimal) solution does not exceed the number of restrictions involved in the problem.

One easily verifies that the number of restrictions involved is:

$$
\begin{align*}
T & \text{ of type BALRPT(T)} \\
+ R & \text{ of type BALRR(R)} \\
+ T & \text{ of type CAPRT(T)} \\
+ C & \text{ of type ASRC(C)} \\
+ r_1 & \text{ for refinery 1} \\
+ r_2 & \text{ for refinery 2} \\
+ r_3 & \text{ for refinery 3}
\end{align*}
$$

On the other hand, with respect to the number of non-zero activity levels, it is known that:

- at least $C$ assignment activities are positiv; if $C_1$ customers are not uniquely assigned to one terminal and $\lambda$ is the average number of terminal to which they are assigned, then $\lambda \geq 2$ and the total number of positiv assignment activities is at least equal to $C + (\lambda-1) C_1$.

- if $T_1$ is the number of terminals having some overcapacity, then $T_1$ slacks corresponding to the capacity restrictions CAPRT(T) are positiv.

- it is reasonable that each terminal is actually used; hence each terminal is supplied from at least one refinery, which means that at least $T$ of the activities TRT(R,T) have a positiv level.
the number of positive refinery activities is \( \bar{r}_1 \), \( \bar{r}_2 \), and \( \bar{r}_3 \) respectively where \( \bar{r}_R \leq \bar{r}_R \).

It follows that:

\[
2T + R + C + \sum R \bar{r}_R \geq C + (\lambda - 1) C_1 + T_1 + T + \sum R \bar{r}_R
\]

hence, since \( \lambda \geq 2 \)

\[
C_1 \leq (T-T_1) + \sum R (r_R - \bar{r}_R)
\]

where \( T-T_1 \) is the number of terminals used to their full capacity, and \( \sum R (r_R - \bar{r}_R) \) is some measure for the degree of degeneracy of the refinery models.

Since the number of terminals \( T \), and certainly the number of fully used terminals is very small compared to the total number \( C \) of customers, and also the refinery degeneracy, although in general not equal to zero, is low compared with \( C \), the number of non-uniquely assigned customers to terminals is small. In any case, as in the application described in [1], it may be expected that also for the LPG location-allocation problem, the linear programming solution can easily be adapted to a (near) optimal unique assignment of customers to terminals.

**Conclusion.**

I am not pessimistic about a linear programming approach to the LPG location-allocation problem.

From my experience with the project description in [1], where up to 50,000 zero-one assignment activities were involved, the large number of such variables in the LPG study need not necessarily present a serious bottleneck.

I am less sure about the influence of the three refinery parts in the model on the calculation time.
However, in a first study of the feasibility of this linear programming approach, it may be acceptable to delete the refineries from the model, using only lower and (or upperbounds) for the production activities PR(R).
This has also the advantage that the problem generation can be done by a very simple matrix generator.