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Published: 01/01/2004

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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by

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A general compound multirate method for circuit simulation problems

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Summary. The "General Compound" multirate methods are attractive integration methods for the transient analysis of mixed analog-digital circuits. From a stability analysis, it follows that they have good stability properties.

1 Introduction

Electrical circuits consist of analog and digital sub-circuits. In analog circuits, the exact values of the voltages and currents are important, but in digital circuits only the logical state is important.

If the mixed analog-digital circuits have to be simulated in high accuracy, it is necessary to simulate the complete circuit on electrical level. In this case, the complete electrical circuit is modeled by the following differential-algebraic equation

\[ \frac{d}{dt} [q(t, x)] + j(t, x) = 0, \quad j(0, x(0)) = 0, \]  

(1)

where \( x \) consists of nodal voltages and some currents in the circuit.

Commonly, this IVP is solved by means of implicit integration methods, like BDF-methods. In each iteration all equations are discretized with the same step \( h_n \). Often, parts of electrical circuits have latency or multirate behaviour. Latency means that parts of the circuit are constant during a certain time interval. Multirate behaviour means that some variables are slowly-varying, compared to other variables. In both cases, it would be attractive to integrate the latent or slowly-varying sub-circuit with a larger step.

In section 2 we will show an attractive class of multirate methods for electrical circuits. Next we will study the stability for a two-dimensional linear test equation.
2 Multirate methods for circuits

2.1 Partitioning of variables and equations

For a multirate method it is necessary to partition the variables and equations into an active (A) and a latent (L) part. This can be done by the user or automatically. Then the DAE (1) is equivalent to the coupled system

\[
\frac{d}{dt}[q_A(t, x_A, x_L)] + j_A(t, x_A, x_L) = 0, \quad (2)
\]

\[
\frac{d}{dt}[q_L(t, x_A, x_L)] + j_L(t, x_A, x_L) = 0. \quad (3)
\]

It is necessary that the equations (2) and (3) are uniquely solvable. The partitioning is very important, because it affects the stability and the accuracy of the multirate method. Decomposing the DAE (1) into two nearly decoupled parts requires too much effort and hence approximation methods should be used.

2.2 Different multirate algorithms

There are many multirate methods for the system of equations (2),(3) [2, 6]. We will restrict our attention to multirate versions of the Euler Backward method. The time interval \([0, T]\) is discretized into the multirate time-grid \(\{t_n = nh = n\frac{H}{q} : n = 0, \ldots, N\}\) where the number \(q\) is called the multirate factor. The latent equations are integrated with one large step \(H\), but the active equations are integrated with a much smaller step \(h = \frac{H}{q}\) on a refinement of \([t_n, t_{n+q}]\).

The "Slowest First" (SF) method (algorithm 1) first integrates (3) with one large step \(H\), while \(x_A\) is approximated by means of extrapolation. Then equation (2) is integrated with the small step \(h\), while \(x_L\) is approximated by linear interpolation.

To improve the stability, the latent part can be integrated by an implicit compound step [4]. This "Compound Step" (CS) method first integrates (2) and (3) together with one large step \(H\), which results in \(x_A^{n+q}\) and \(x_L^{n+q}\). Then only equation (2) is integrated with the small step \(h\), while \(x_L^{n+q}\) is found by linear interpolation. Note that \(x_A^{n+q}\) is twice computed by the "Compound Step" method, which could be used to estimate the error. Another possibility is the "Mixed Compound Step" (MCS) method, which computes \(x_A^{n+q}\) and \(x_L^{n+q}\) simultaneously. This method corresponds to the multirate method for the Rosenbrock-Wanner methods described in [1]. The "Compound Step" has the advantage that it is easier to implement, while the "Mixed Compound Step" method is better scaled.

A generalized version is the "General Compound" (GC) method (algorithm 2) with \(\alpha \in \mathbb{R}\). This GC method contains the CS method (\(\alpha = 1\)) and the MCS method (\(\alpha = \frac{1}{q}\)).
Algorithm 1 The Slowest First (SF) method

Solve for $x_{A}^{n+q}$:

$$q_{A}(z_{A}^{n+q}, x_{L}^{n+q}) - q_{A}(x_{A}^{n}, x_{L}^{n}) + H_{j_{A}}(z_{A}^{n+q}, x_{L}^{n+q}) = 0$$  \hspace{1cm} (4)

$$x_{A}^{n+q} - x_{A}^{n} = 0$$  \hspace{1cm} (5)

Solve for $x_{A}^{n+j+1}$ ($j = 0, \ldots, q - 1$):

$$q_{A}(x_{A}^{n+j+1}, z_{A}^{n+j+1}) - q_{A}(x_{A}^{n+j}, z_{A}^{n+j}) + h_{j_{A}}(x_{A}^{n+j+1}, z_{A}^{n+j+1}) = 0$$  \hspace{1cm} (6)

$$z_{A}^{n+j+1} - z_{A}^{n+j} - \frac{1}{q}(z_{A}^{n+q} - z_{A}^{n}) = 0$$  \hspace{1cm} (7)

Algorithm 2 The General Compound (GC) method

Solve for $x_{L}^{n+q}$ and $x_{A}^{n+aq}$:

$$q_{A}(x_{A}^{n+aq}, z_{L}^{n+q}) - q_{A}(x_{A}^{n}, x_{L}^{n}) + H_{j_{A}}(x_{A}^{n+aq}, z_{L}^{n+q}) = 0$$  \hspace{1cm} (8)

$$x_{A}^{n+aq} - x_{A}^{n} - \frac{1}{q}(x_{A}^{n+q} - x_{A}^{n}) = 0$$  \hspace{1cm} (9)

$$q_{L}(z_{A}^{n+q}, x_{L}^{n+q}) - q_{L}(x_{A}^{n}, x_{L}^{n}) + H_{j_{L}}(z_{A}^{n+q}, x_{L}^{n+q}) = 0$$  \hspace{1cm} (10)

$$z_{A}^{n+q} - x_{A}^{n} = 0$$  \hspace{1cm} (11)

Solve for $x_{A}^{n+j+1}$ ($j = 0, \ldots, q - 1$):

$$q_{A}(x_{A}^{n+j+1}, z_{A}^{n+j+1}) - q_{A}(x_{A}^{n+j}, z_{A}^{n+j}) + h_{j_{A}}(x_{A}^{n+j+1}, z_{A}^{n+j+1}) = 0$$  \hspace{1cm} (12)

$$z_{A}^{n+j+1} - z_{A}^{n+j} - \frac{1}{q}(z_{A}^{n+q} - z_{A}^{n}) = 0$$  \hspace{1cm} (13)

3 Stability analysis of the SF and GC methods

Multirate methods have less good stability properties than ordinary integration methods. Therefore this section contains a stability analysis of the SF and GC methods.

3.1 A test equation

For ordinary integration methods absolute stability can be studied by looking at the scalar test equation $\dot{x} = \lambda x$ with $\lambda \in \mathbb{C}$. For multirate methods with two time-steps $h$ and $H$, the following (real) linear test equation is studied [5, 6], where $x_{A}$ and $x_{L}$ are the active and latent variable respectively.

$$\begin{pmatrix} \dot{x}_{A} \\ \dot{x}_{L} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{A} \\ x_{L} \end{pmatrix}$$  \hspace{1cm} (14)
Let $x_A^n$ and $x_L^n$ be the numerical approximations at the time-point $t_n = nh = \frac{n}{q} H$. The multirate method is absolutely stable when $x_A^n$ and $x_L^n$ tend to zero for $n \to \infty$ if $A$ is a stable matrix.

For $q = 1$, the stability behaviour of the multirate methods is independent of the used coordinate system. However, for $q > 1$ the stability does not only depend on the eigenvalues but also on the eigenvectors of the matrix $A$.

### 3.2 Analysis of the compound step

In both the SF and the GC methods the latent variable is first integrated. Using constant extrapolation of $x_A^n$ for the SF method we obtain the system

\[
\frac{x_A^{n+q} - x_A^n}{H} = a_{21} x_A^n + a_{22} x_L^{n+q}.
\]

From the equations (15), it follows that

\[
x_L^{n+q} = \rho x_A^n + \sigma x_L^n,
\]

where

\[
\rho = \frac{s_{21} H}{1 - a_{22} H}, \quad \sigma = \frac{1}{1 - a_{22} H}.
\]

For the GC method, we get another complete system of equations for $x_A^{n+q}$ and $x_L^{n+q}$:

\[
\begin{aligned}
\frac{x_A^{n+q} - x_A^n}{H} &= a_{21} x_A^n + a_{12} (x_L^n + a (x_L^{n+q} - x_L^n)), \\
\frac{x_L^{n+q} - x_L^n}{H} &= \rho (x_A^n + \frac{1}{\alpha} (x_A^{n+q} - x_A^n)) + a_{22} x_L^{n+q}.
\end{aligned}
\]

The solution satisfies again equation (16) with different values for $\rho$ and $\sigma$:

\[
\rho = \frac{a_{21} H + a_{12} (a_{11} + 1) a H^2}{1 - (a_{11} + a_{22}) H + (a_{11} a_{22} - a_{12} a_{21}) a H^2}, \quad \sigma = \frac{1 - a_{11} H + a_{12} (1 - a) H^2}{1 - (a_{11} + a_{22}) H + (a_{11} a_{22} - a_{12} a_{21}) a H^2}.
\]

### 3.3 Stability conditions

For both methods $x_A^{n+j}$ is estimated for $j \in \{1, \ldots, q - 1\}$ employing $x_A^n$ and $x_L^{n+q}$.

\[
x_A^{n+j} = x_A^n + \frac{j}{q} (x_A^{n+q} - x_A^n) = x_A^{n+j} + \frac{j}{q} x_L^{n+q}.
\]

Finally, the active part is integrated along the time window $[t_n, t_n + H]$ with $q$ steps $h$.

\[
\frac{x_A^{n+j+1} - x_A^{n+j}}{h} = a_{11} x_A^{n+j+1} + a_{12} x_L^{n+j+1}.
\]

Equation (21) is equivalent to
\[
x^{n+j+1}_A = \frac{1}{k-a_{11}} x^{n+j}_A + \frac{a_{12}}{k-a_{11}} x^{n+j+1}_L = \gamma x^{n+j}_A + \delta x^{n+j+1}_L, \quad (22)
\]

where
\[
\gamma = \frac{1}{k-a_{11}}, \quad \delta = \frac{a_{12}}{k-a_{11}}. \quad (23)
\]

For \( j \in \{0, \ldots, q-1\} \) we have
\[
x^{n+j+1}_A = \gamma x^{n+j}_A + \delta (1 - \frac{j+1}{q}) x^{n+q}_L + \delta \frac{j+1}{q} x^{n+q}_L
\]
\[
= \gamma^{j+1} x^{n}_A + \sum_{k=0}^{q-1} \gamma^{j-k} \left( (1 - \frac{q-1}{q} \delta \frac{k+1}{q}) x^{n+k}_L + \delta \frac{k+1}{q} x^{n+q}_L \right). \quad (24)
\]

Inserting (16) into (24) for \( j = q-1 \) results in
\[
x^{n+q}_A = \nu x^{n}_A + \tau x^{n}_L,
\]
where
\[
\nu = \gamma^q + \sum_{i=0}^{q-1} \gamma^i \rho \delta (1 - \frac{i}{q}), \quad \tau = \sum_{i=0}^{q-1} \gamma^i \delta (\frac{1}{q} (1 - \sigma) + \sigma). \quad (26)
\]

From equations (16) and (25) it follows that
\[
\begin{pmatrix} x^{n+q}_A \\ x^{n+q}_L \end{pmatrix} = \begin{pmatrix} \sigma & \rho \\ \tau & \nu \end{pmatrix} \begin{pmatrix} x^{n}_A \\ x^{n}_L \end{pmatrix}. \quad (27)
\]

The methods are A-stable if \( \rho(M) < 1 \) for all \( H, q > 0 \) and stable matrices \( A \) [6]. Let \( \lambda = \det(M - \lambda I) = \lambda^2 - \text{tr}(M) \lambda + \det(M), \) where \( M \in \mathbb{R}^{2 \times 2}. \) One can easily show that [3]
\[
\rho(M) < 1 \Leftrightarrow \begin{cases} \\
\phi(-1) = 1 + \text{tr}(M) + \det(M) > 0, \\
\phi(0) = \det(M) < 1, \\
\phi(1) = 1 - \text{tr}(M) + \det(M) > 0.
\end{cases} \quad (28)
\]

Because
\[
\text{tr}(M) = \sigma + \gamma^q + \sum_{i=0}^{q-1} \gamma^i \rho \delta (1 - \frac{i}{q}), \\
\det(M) = \sigma \gamma^q + \sigma \sum_{i=0}^{q-1} \gamma^i \rho \delta (1 - \frac{i}{q}) - \rho \gamma^q \sum_{i=0}^{q-1} \gamma^i \delta (\frac{1}{q} (1 - \sigma) + \sigma) \quad (29)
\]

we obtain the following three constraints which ensure absolutely stability
\[
1 + \text{tr}(M) + \det(M) = 1 + (1 + \sigma) \gamma^q + \sigma - \rho \delta \sum_{i=0}^{q-1} \gamma^i (\frac{2H}{q} - 1) > 0, \\
\det(M) = \sigma \gamma^q - \frac{\rho \delta}{q} \sum_{i=0}^{q-1} \gamma^i \delta (\frac{1}{q} (1 - \sigma) + \sigma) < 1, \quad (30)
\]
\[
1 - \text{tr}(M) + \det(M) = 1 + (\sigma - 1) \gamma^q - \sigma - \rho \delta \sum_{i=0}^{q-1} \gamma^i > 0.
\]
Thus we get the following stability conditions for the investigated multirate methods

\begin{align}
(1 + \sigma)\left[1 + \gamma^q\right] - \rho \delta \sum_{l=0}^{q-1} \gamma^l \left(\frac{2l}{q} - 1\right) &> 0, \\
\frac{\rho \delta}{q} \sum_{l=0}^{q-1} \gamma^l l - \sigma \gamma^q + 1 &> 0, \\
(1 - \sigma)(1 - \gamma^q) - \rho \delta \sum_{l=0}^{q-1} \gamma^l &> 0.
\end{align}

(31)

### 3.4 Asymptotic stability conditions

Because the stability conditions (31) are rather complex, we will derive more compact stability conditions by means of asymptotical analysis.

#### Stability for $H \to 0$ (fixed $q$)

The multirate methods are conditionally stable if the stability conditions are valid for $H \to 0$. Therefore we will derive asymptotic approximations of these conditions. It easily follows that $\rho \delta = a_{12} a_{21} H^2 + O(H^3)$, $\gamma = 1 + a_1 H + O(H^2)$, $\sigma = 1 + a_{22} H + O(H^2)$ and $\gamma^q = 1 + a_{11} H + O(H^2)$. Using these approximations, we obtain

\begin{align}
(1 + \sigma)\left[1 + \gamma^q\right] - \rho \delta \sum_{l=0}^{q-1} \gamma^l \left(\frac{2l}{q} - 1\right) &= 4 + O(H), \\
\frac{\rho \delta}{q} \sum_{l=0}^{q-1} \gamma^l l - \sigma \gamma^q + 1 &= -(a_{11} + a_{22}) H + O(H^2), \\
(1 - \sigma)(1 - \gamma^q) - \rho \delta \sum_{l=0}^{q-1} \gamma^l &= (a_{11} a_{22} - a_{12} a_{21}) H^2 + O(H^3).
\end{align}

(32)

After inserting these asymptotic expressions into (31), we obtain the following asymptotic stability conditions for $A$

\begin{align}
\text{tr}(A) = a_{11} + a_{22} &< 0, \\
\text{det}(A) = a_{11} a_{22} - a_{12} a_{21} &> 0.
\end{align}

(33)

Thus the SF method and the GC methods are stable for $H \to 0$ if $A$ is a stable matrix.

#### Stability for $q \to \infty$ (fixed $H$)

If the multirate factor $q \to \infty$, it is necessary that $|\gamma^q| < 1$ such that $\gamma^q \to 0$. This means that the Euler Backward method is stable for the active part, which is the case if $a_{11} < 0$. Taking the limit $q \to \infty$, we obtain

\begin{align}
(1 + \sigma)\left[1 + \gamma^q\right] - \rho \delta \sum_{l=0}^{q-1} \gamma^l \left(\frac{2l}{q} - 1\right) &\to 1 + \sigma + \rho \delta \frac{1}{1-\gamma}, \\
\frac{\rho \delta}{q} \sum_{l=0}^{q-1} \gamma^l l - \sigma \gamma^q + 1 &\to 1, \\
(1 - \sigma)(1 - \gamma^q) - \rho \delta \sum_{l=0}^{q-1} \gamma^l &\to 1 - \sigma - \rho \delta \frac{1}{1-\gamma}.
\end{align}

(34)

This means that for $q \to \infty$ we have the following stability conditions
Because \( \frac{1}{1-\gamma} = -\frac{a_{11}}{a_{11}} \), we get

\[
| - \frac{a_{12}}{a_{11}} \rho + \sigma | < 1. \tag{36}
\]

Using (17) for the SF method, condition (36) is equivalent to

\[
| \mathcal{P}_{SF}(H) | = \left| - \frac{a_{12}}{a_{11}} \rho + \sigma \right| = \left| \frac{1 - \frac{a_{12} a_{21}}{a_{11}} H}{1 - a_{22} H} \right| < 1.
\]

If this rational function \( \mathcal{P}_{SF}(H) \) has a negative pole and \( \lim_{H \to \infty} |\mathcal{P}_{SF}(H)| = \frac{a_{12} a_{21}}{|a_{11}|} < 1 \), the method is unconditionally stable. Thus, if \( a_{11} < 0 \), \( a_{22} < 0 \) and \( |a_{12} a_{21}| < |a_{11} a_{22}| \), the SF method is unconditionally stable for \( q \to \infty \).

Using (19) for the GC methods, condition (36) is equivalent to

\[
| \mathcal{P}_{GC}(H) | = \left| - \frac{a_{12}}{a_{11}} \rho + \sigma \right| = \left| \frac{1 - \frac{a_{12} a_{21}}{a_{11}} + \alpha a_{11} H}{1 - (\alpha a_{11} + a_{22}) H + \alpha (a_{11} a_{22} - a_{12} a_{21}) H^2} \right| < 1.
\]

It can be shown that this is the case if \( \frac{a_{12} a_{21}}{a_{11}} + \alpha a_{11} < |a_{11} a_{22}|, \alpha a_{11} + a_{22} < 0 \) and \( \alpha (a_{11} a_{22} - a_{12} a_{21}) > 0 \). Because \( \alpha a_{11} + a_{22} < 0 \), we find

\[
a_{11} + a_{22} < \frac{a_{12} a_{21}}{a_{11}}, \quad a_{11} < -a_{11} - a_{22}. \tag{37}
\]

From the left inequality in (37) we can derive \( a_{22} - \frac{a_{12} a_{21}}{a_{11}} < 0 \) or

\[
\frac{1}{a_{11}} (a_{11} a_{22} - a_{12} a_{21}) < 0.
\]

The other inequality in (37) gives \( \frac{a_{12} a_{21}}{a_{11}} < -a_{22} - 2\alpha a_{11} \) or

\[
a_{12} a_{21} > -a_{11} a_{22} - 2\alpha a_{11}^2.
\]

Because \( \alpha > 0 \), the GC method is always stable if

\[
|a_{12} a_{21}| < |a_{11} a_{22}|. \tag{38}
\]

### 4 Numerical example

Consider for \( 0 \leq t \leq 10 \)

\[
\begin{pmatrix}
\dot{x}_A \\
\dot{x}_L
\end{pmatrix} =
\begin{pmatrix}
-1 & \mu \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
x_A \\
x_L
\end{pmatrix}, \quad \begin{pmatrix}
x_A(0) \\
x_L(0)
\end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{39}
\]

For \( \mu < 0 \) it is a stable system with eigenvalues \(-1 \pm i\sqrt{-\mu}\). The system is solved by the SF method and the GC method for \( \alpha = 1 \) and \( \alpha = \frac{1}{2} \). A sufficient stability condition is \( |\mu| < 1 \), but for the GC methods \( \mu > -1 - 2\alpha \) suffices.
Table 1. Sufficient stability conditions for the SF method and the GC method.

<table>
<thead>
<tr>
<th>SF</th>
<th>GC</th>
<th>GC (α = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{11} &lt; 0$</td>
<td>$a_{11} &lt; 0$</td>
<td>$a_{11} &lt; 0$</td>
</tr>
<tr>
<td>$a_{22} &lt; 0$</td>
<td>$a_{11} + a_{22} &lt; 0$</td>
<td>$a_{11} + a_{22} &lt; 0$</td>
</tr>
<tr>
<td>$a_{12} a_{21} &lt; a_{11} a_{22}$</td>
<td>$-a_{11} a_{22} - 2a_{11}^2 a_{21} &lt; a_{12} a_{21}$</td>
<td>$a_{12} a_{21} &lt; a_{11} a_{22}$</td>
</tr>
</tbody>
</table>

Table 2. Stability of multirate methods ($H = 0.1, q = 10$).

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>SF</th>
<th>GC (α = $\frac{1}{2}$)</th>
<th>GC (α = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-10$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$-100$</td>
<td>-</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$-1000$</td>
<td>-</td>
<td>-</td>
<td>✓</td>
</tr>
</tbody>
</table>

5 Conclusions

We have derived stability conditions (Table 1) for the SF and GC methods if $H \to 0$ or $q \to \infty$. These results are presently be generalized to the general multi-dimensional case. The GC methods have the advantage that they do not require that $a_{22} < 0$, but only $a_{11} + a_{22} < 0$. If $A$ is stable, this condition is always satisfied for $\alpha = 1$. Because the GC methods it is sufficient if $a_{12} a_{21} > -a_{11} a_{22} - 2a_{11}^2$, large values for $\alpha$ are preferable.

References