FINITE SUBGROUPS OF $F_4(\mathbb{C})$ AND $E_6(\mathbb{C})$

ARJEH M. COHEN *and* DAVID B. WALES

[Received 1 February 1995—Revised 14 November 1995]

ABSTRACT

The isomorphism types of Lie primitive finite subgroups of the complex Lie groups $E_6(\mathbb{C})$ and $F_4(\mathbb{C})$ are determined. Some additional information is provided, such as the characters of these finite subgroups on some small-dimensional modules for the Lie groups.

1. Introduction and statement of results

Having determined the finite subgroups of $G_2(\mathbb{C})$ in [18], we face the subsequent question of classifying the finite subgroups of $F = F_4(\mathbb{C})$ and $E = E_6(\mathbb{C})$. We will not attempt to determine all finite subgroups, but only those not contained in a proper closed Lie subgroup of positive dimension. We call a finite subgroup of $G = F$ or E *Lie primitive* if it is not contained in such a proper closed subgroup of G . Induction can be used to investigate subgroups which are not Lie primitive.

In studying these groups, we mainly use two rational linear representations of the universal covering group $\tilde{E}$ of E , namely a 27-dimensional module (there are two inequivalent ones), denoted by $\mathbf{K}$, and the adjoint module, denoted by $\mathbf{g}$. In particular, we shall make extensive use of the characters of $\tilde{E}$ on these modules, which will be denoted by ψ and χ , respectively. The group F occurs in $\tilde{E}$ as the stabilizer subgroup of a vector in $\mathbf{K}$. Our main result reads as follows.

1.1. MAIN THEOREM. *Suppose that G is the simple complex Lie group of type F_4 or E_6 , and B is a Lie primitive finite subgroup of G . Then B is contained in a subgroup of G isomorphic to an extension $A : L(3, 3)$ of a subgroup A of G , where $A \cong 3^3$ if G has type F_4 and $A \cong 3^{3+3}$ if G has type E_6 , or B has a simple socle L as described in Table 1. Conversely, every line in Table 1 corresponds to a simple subgroup L of G with the property that the restrictions to L of the characters ψ and χ of G on $\mathbf{K}$ and $\mathbf{g}$, respectively, are as described in that line.*

REMARKS. The first column of Table 1 indicates our knowledge of Lie primitivity of B . A plus + indicates that there is a finite subgroup of the indicated isomorphism type with the indicated character which is Lie primitive.

For the first three lines, where there is no +, we know of the existence of a simple subgroup L of indicated type with indicated characters $\psi|_L$ on $\mathbf{K}$ and $\chi|_L$ on $\mathbf{g}$, which is not Lie primitive. We think that no (other) occurrence of L in G with the same character restrictions on $\mathbf{K}$ and $\mathbf{g}$ leads to a normalizer that is Lie primitive but we cannot prove this.

During the course of the proof we generate a great deal of information about possible finite subgroups of E and F . In particular, we develop a complete list of all possible simple or quasisimple subgroups of E or F (whether Lie primitive or not). This is given in § 7. Information about subgroups which are not Lie

primitive can be obtained from knowing the maximal connected closed semi-simple Lie subgroups of positive dimension. This is given in § 4. There is a discussion of possibilities when $L \neq B$ in the remark before Lemma 8.2.

Information about the number of conjugacy classes of some of the groups, L , in which a + occurs is known and is found in § 6 along with the discussion of the group.

NOTATION. The symbol $3 \cdot 5_b$ for a character of a simple group L denotes 3 times the second irreducible character of degree 5 in the order of appearance in the Atlas [22].

TABLE 1. *Simple socles L of subgroups B of E and F as in the Main Theorem*

Lie primitivity	G	L	$\psi _L$	$\chi _L$	B
+	E	Alt_6	$5_a + 5_b + 8_a + 9_a$	$2 \cdot 8_a + 3 \cdot 8_b + 2 \cdot 9_a + 2 \cdot 10_a$	L or $L \cdot \sigma_f$
	F	Alt_6	$1_a + 2 \cdot 9_a + 8_a$	$2 \cdot 8_a + 3 \cdot 8_b + 2 \cdot 9_a + 2 \cdot 10_a$	L or $L \cdot \sigma_f$
	E	$L(2, 7)$	$2 \cdot 6_a + 7_a + 8_a$	$2 \cdot 3_a + 2 \cdot 3_b + 2 \cdot 6_a + 2 \cdot 7_a + 5 \cdot 8_a$	L or $\text{Aut } L$
	E	$L(2, 8)$	$9_a + 9_b + 9_c$	$9_a + 9_b + 9_c + 2 \cdot 8_a$ $+ 2 \cdot (7_b + 7_c + 7_d) + 2 \cdot 7_a$	L or $\text{Aut } L$
+	F	$L(2, 8)$	$1_a + 8_a + 9_a + 9_c$	$2 \cdot 7_a + 3 \cdot 7_{**} + 2 \cdot 8_a + 2 \cdot 9_a + 9_c$	L
+	F	$L(2, 13)$	$1_a + 12_a + 14_a$	$2 \cdot 12_a + \{12_b \text{ or } 12_c\} + 14_a + 2 \cdot 14_b$	L or $\text{Aut } L$
+	F	$L(2, 17)$	$1_a + 9_a + 17_a$	$9_a + 17_a + 16_a + 2 \cdot 18_{**}$	L
+	E	$L(2, 19)$	$9_a + 18_c$	$18_a + 20_b + 20_c + 20_d$	L
+	F	$L(2, 25)$	$1_a + 26_a$	$26_a + 26_d + 26_e$	L or $L \cdot \sigma_{fg}$
+	F	$L(2, 27)$	$1_a + 26_f$	$26_b + 26_c + 26_f$	L
+	E	$L(3, 3)$	27_a	$26_a + 26_b + 26_c$	L or $\text{Aut } L$
+	F	${}^3D_4(2)$	$1_a + 26_a$	$26_a + 52_a$	L or $\text{Aut } L$
+	E	${}^2F_4(2)'$	27_a	78_a	L or $\text{Aut } L$

Here σ_{fg} appearing in the $L(2, 25)$ line is the product of a field and a diagonal automorphism. The σ_f appearing in the Alt_6 lines is the field automorphism (of $L(2, 9)$). The proof of this theorem depends on the classification of the finite simple groups [31] which makes possible a determination of all finite simple groups having a non-trivial complex projective representation of degree at most 27.

The work on this paper started in 1986. In the meantime, much progress has been made on the determination of the maximal subgroups of a group of Lie type E_6 . Notably, Aschbacher [2]–[5] has given a list of all possible maximal subgroups of the simple group $E_6(\mathbf{F})$ for $\mathbf{F}$ either finite or algebraically closed, and so has Magaard [35] for $F_4(\mathbf{F})$. Also, [16] provides for all finite local maximal subgroups of Lie groups of exceptional type, redoing earlier work of Alekseevskii [1], as well as the analogue for finite groups. We feel that the merit of this paper with regard to these developments is twofold: it makes use of different techniques (primarily characters) thereby providing easier proofs (although, admittedly, only for characteristic zero), and it provides for new existence results (for all isomorphism types of simple groups occurring in the conclusion of the Main Theorem, we provide an embedding). Furthermore, results for characteristic zero can be used in finite characteristics particularly when the characteristic does not divide the group order. Some such information is given in the beginning of § 6.

The paper is organized as follows. In § 2, we introduce the 27-dimensional module $\mathbf{K}$ for the universal covering group $\tilde{E}$ (with centre of order 3) of E , and derive some of its properties. In § 3, we derive some information about elements

of small order and supersolvable subgroups. We also discuss a result of [1] (referred to above) leading to the determination of all Lie primitive finite subgroups of E that have a non-trivial normal abelian subgroup.

The next two sections recall some standard knowledge of Lie groups: they deal with maximal semisimple closed subgroups of G , and the geometry of the 27-dimensional module $\mathbf{K}$, respectively. Section 6 contains the constructive part of the proof of the main theorem in which embeddings of the groups in Table 1 are given—some for the first time. The subsequent section deals with the determination of finite simple subgroups of E . Finally, in § 8 and § 9, the proof of the Main Theorem is given. It uses a reduction to the study of a minimal normal subgroup of a hypothetical finite Lie primitive subgroup of E .

Acknowledgements. We are grateful to Michael Aschbacher and Kay Magaard for several useful conversations and for providing us with early versions of their preprints.

2. The module $\mathbf{K}$

Let $\mathbf{K}$ be the 27-dimensional complex vector space whose vectors are the triples (x_1, x_2, x_3) of 3 by 3 complex matrices x_i ($i = 1, 2, 3$), where addition and scalar multiplication are defined coordinatewise, and equip $\mathbf{K}$ with the cubic form $\langle \cdot \rangle$ given by

$$\langle x \rangle = \det x_1 + \det x_2 + \det x_3 - \text{trace } x_1 x_2 x_3$$

for $x = (x_1, x_2, x_3)$ in $\mathbf{K}$. The symmetric trilinear form corresponding to $\langle \cdot \rangle$ will be denoted by $\langle \cdot, \cdot, \cdot \rangle$. Thus,

$$\langle x, y, z \rangle = \frac{1}{6}(\langle x + y + z \rangle - \langle x + y \rangle - \langle x + z \rangle - \langle y + z \rangle + \langle x \rangle + \langle y \rangle + \langle z \rangle)$$

for x, y, z in $\mathbf{K}$. It is well known that $\text{Aut } \mathbf{K}$ (that is, the group of linear transformations of $\mathbf{K}$ preserving $\langle \cdot \rangle$) is the non-split central extension $\tilde{E}$ of $E_6(\mathbb{C})$ by a group of order 3. (In fact, this fact is easily derived from the information given below.) This description of the Lie group of type E_6 was first given in [29], but other descriptions were known as early as 1901 [25].

There are two bases we will use for $\mathbf{K}$. The first is a natural basis consisting of all triples of matrices $e_{j,k}^i$ ($1 \leq i, j, k \leq 3$) having 0 at every entry but for the (j, k) -entry of the i th matrix, which entry equals 1. We shall also use the basis and trilinear form described in [2, 3.1]. In particular, the 27-dimensional space $\mathbf{K}$ has a basis $X_i, X_{i'}, X_{jk}$, where $1 \leq i \leq 6$, $1 \leq j < k \leq 6$. For convenience, we shall adopt the conventions $X_{ji} = -X_{ij}$ (if $j > i$) and $X_{-i} = -X_i$, $X_{-i'} = -X_{i'}$. The trilinear form is non-zero only for the triples $X_i, X_{j'}, X_{ij}$ if $i \neq j$, and X_{ij}, X_{kl}, X_{mn} where $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$. The values are all ± 1 as follows: $\langle X_i, X_{j'}, X_{ij} \rangle = 1$ if $i < j$. To determine the sign of $\langle X_{ij}, X_{kl}, X_{mn} \rangle$ identify the triple of (ordered) pairs $\{ij, kl, mn\}$ in the triple below,

Triples of pairs		
12 34 56	14 23 56	16 25 34
12 35 64	14 25 63	16 23 45
12 36 45	14 26 35	16 24 53
13 24 65	15 26 43	
13 26 54	15 24 36	
13 25 46	15 23 64	

and then $\langle X_{ij}, X_{kl}, X_{mn} \rangle = 1$ if the triple of pairs is the same as in the table (up to an even number of inversions of pairs) and $\langle X_{ij}, X_{kl}, X_{mn} \rangle = -1$ if one or three of the pairs are inverted.

These bases can be related schematically using the following diagram:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 3' \\ 23 & 31 & 12 \end{pmatrix}, \quad \begin{pmatrix} 41 & 51 & 61 \\ 42 & 52 & 62 \\ 43 & 53 & 63 \end{pmatrix}, \quad \begin{pmatrix} 4' & -4 & 56 \\ 5' & -5 & 64 \\ 6' & -6 & 45 \end{pmatrix}.$$

Now $e_{j,k}^i = X_y$ if y occurs in the j, k -entry of the i th matrix of the diagram.

The subspace P of $\mathbf{K}$ consisting of all elements of the form $(x_1, 0, 0)$ will be identified with the vector space of all 3 by 3 complex matrices. The restriction of $\langle \cdot \rangle$ to P is just the determinant. Let e be the element of P for which x_1 is the identity matrix. Then the stabilizer F in $\tilde{E}$ of e is the Lie group $F_4(\mathbb{C})$. It leaves invariant the bilinear form

$$\langle x, y \rangle = \text{trace}(x_1 y_1 + x_2 y_3 + x_3 y_2) \quad (x = (x_i)_{1 \leq i \leq 3}, y = (y_i)_{1 \leq i \leq 3} \in \mathbf{K})$$

and can even be defined as the subgroup of $\tilde{E}$ of all elements preserving $\langle \cdot, \cdot \rangle$.

Using the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ we can define a multiplication on $\mathbf{K}$ by

$$\langle x \times y, z \rangle = \langle x, y, z \rangle \quad (x, y, z \in \mathbf{K}).$$

We shall write $x^\#$ instead of $\frac{1}{2}(x \times x)$. Although $\tilde{E}$ does not preserve $\times$, the relation $x \times y = 0$ is $\tilde{E}$ -invariant. On the other hand, F preserves both $\#$ and $\times$. An explicit form of $\#$ is provided by

$$x^\# = (x_1^\# - x_2 x_3, x_3^\# - x_1 x_2, x_2^\# - x_3 x_1),$$

where, for a 3×3 matrix $c = (c_{ij})$, the matrix $c^\#$ is the transpose of the matrix of the signed 2×2 minors of c , that is, $c_{ij}^\# = c_{kl} c_{mn} - c_{kn} c_{ml}$ whenever $(jkm) = (iln) = (123)$. An explicit expression for $\times$ can be obtained from this by use of the identity $x \times y = (x + y)^\# - x^\# - y^\#$ ($x, y \in \mathbf{K}$).

We shall describe some automorphisms of $\mathbf{K}$. First, if g_1, g_2, g_3 are elements of $\text{SL}(3, \mathbb{C})$, then the linear transformation s_{g_1, g_2, g_3} given by

$$s_{g_1, g_2, g_3}(x_1, x_2, x_3) = (g_1 x_1 g_2^{-1}, g_2 x_2 g_3^{-1}, g_3 x_3 g_1^{-1}) \quad (x = (x_i)_{1 \leq i \leq 3} \in \mathbf{K})$$

clearly preserves $\langle \cdot \rangle$, and so belongs to $\tilde{E}$. The representation of $\text{SL}(3, \mathbb{C}) \times \text{SL}(3, \mathbb{C}) \times \text{SL}(3, \mathbb{C})$ has kernel $\langle s_{\omega \cdot \text{id}, \omega \cdot \text{id}, \omega \cdot \text{id}} \rangle$, where ω stands for a cube root of unity and id is the identity matrix in $\text{SL}(3, \mathbb{C})$.

We denote by H the subgroup of $\tilde{E}$ consisting of all elements which are diagonal with respect to the standard basis $e_{j,k}^i$ ($1 \leq i, j, k \leq 3$). An arbitrary element $h \in H$ has the shape

$$h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) = \begin{pmatrix} \alpha\gamma^{-1}\delta^{-1} & \alpha\beta\gamma & \alpha\delta \\ \alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1} & \alpha^{-1}\gamma & \alpha^{-1}\beta^{-1}\delta \\ \gamma^{-1}\delta^{-1} & \beta\gamma & \delta \end{pmatrix},$$

$$\begin{pmatrix} \gamma\delta\zeta^{-1}\varepsilon^{-1} & \beta\gamma\delta\varepsilon & \gamma\delta\zeta \\ \beta^{-1}\gamma^{-1}\zeta^{-1}\varepsilon^{-1} & \gamma^{-1}\varepsilon & \beta^{-1}\gamma^{-1}\zeta \\ \delta^{-1}\zeta^{-1}\varepsilon^{-1} & \beta\delta^{-1}\varepsilon & \delta^{-1}\zeta \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{-1}\zeta\varepsilon & \alpha\beta\zeta\varepsilon & \varepsilon\zeta \\ \alpha^{-1}\beta^{-1}\varepsilon^{-1} & \alpha\varepsilon^{-1} & \beta^{-1}\varepsilon^{-1} \\ \alpha^{-1}\zeta^{-1} & \alpha\beta\zeta^{-1} & \zeta^{-1} \end{pmatrix}.$$

Here, each entry represents the scalar by which the corresponding basis element of $\mathbf{K}$ is multiplied in the action of h . In particular, $h(1, 1, \omega, \omega, \bar{\omega}, \bar{\omega}) = s_{\text{id}, \bar{\omega}, \omega}$, with ω as above, represents a central element of $\tilde{E}$ of order 3.

It is easy to see that H is a maximal torus of $\tilde{E}$ whose eigenspaces are the 1-dimensional linear subspaces spanned by the standard basis $(e_{j,k}^i)$ ($1 \leq i, j, k \leq 3$). The normalizer $N = N_{\tilde{E}}(H)$ of H in $\tilde{E}$ contains the elements $n_1 = s_{\text{id}, -(13), \text{id}}$, $n_2 = s_{-(12), \text{id}, \text{id}}$, $n_3 = s_{\text{id}, -(12), \text{id}}$, $n_5 = s_{\text{id}, \text{id}, -(12)}$, $n_6 = s_{\text{id}, \text{id}, -(13)}$ (where the permutations id , (12) , (13) are interpreted as permutation matrices from $\text{SL}(3, \mathbb{C})$, and $-(12)$ stands for the product of (12) with the homothety -1), as well as the transformation n_4 given by

$$x \mapsto \left(\begin{pmatrix} x_i^{1,1} & -x_{i+2}^{2,3} & x_i^{1,3} \\ -x_{i+1}^{3,2} & x_i^{2,2} & x_{i+1}^{1,2} \\ x_i^{3,1} & x_{i+2}^{2,1} & x_i^{3,3} \end{pmatrix}_{(i=1,2,3)} \right).$$

The group generated by the elements n_i ($i = 1, \dots, 6$) is a non-split extension $\tilde{W}$ of the Weyl group $W = W(E_6)$ by $\tilde{W} \cap H$, the elementary abelian subgroup of H of order 2^6 , consisting of all elements in H of order at most 2 (see [2] for a proof that the extension is non-split). The images of n_i ($i = 1, \dots, 6$) in W satisfy the Coxeter relations indicated by the diagram of type E_6 whose nodes are indexed as in [8]:

We shall also use

$$n_7 = n_2 n_4 n_5 n_3 n_4 n_1 n_3 n_2 n_4 n_5 n_6 n_5 n_4 n_3 n_2 n_4 n_5 n_1 n_3 n_4 n_2.$$

This element induces the reflection with respect to the highest root of a fundamental system determined by $n_1, \dots, n_6$.

The action of n_i ($1 \leq i \leq 7$) on the torus H can be given explicitly:

$$h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)^{n_i} = \begin{cases} h(\alpha, \beta, (\gamma\delta)^{-1}, \delta, \varepsilon, \zeta) & \text{if } i = 1, \\ h((\alpha\beta)^{-1}, \beta, \gamma, \delta, \varepsilon, \zeta) & \text{if } i = 2, \\ h(\alpha, \beta, (\beta\gamma\delta)^{-1}, \delta, \varepsilon, \zeta) & \text{if } i = 3, \\ h(\alpha, (\alpha\beta\gamma\varepsilon)^{-1}, \gamma, \delta, \varepsilon, \zeta) & \text{if } i = 4, \\ h(\alpha, \beta, \gamma, \delta, (\beta\varepsilon\zeta)^{-1}, \zeta) & \text{if } i = 5, \\ h(\alpha, \beta, \gamma, \delta, \varepsilon, (\varepsilon\zeta)^{-1}) & \text{if } i = 6, \\ h(\alpha^{-1}, \alpha^3\beta, \alpha^{-2}\gamma, \alpha\delta, \alpha^{-2}\varepsilon, \alpha\zeta) & \text{if } i = 7. \end{cases}$$

The full normalizer of H in $\tilde{E}$ is $N = H\tilde{W}$, and $H \cap F = \{h(\alpha, \beta, \alpha, 1, \varepsilon, \zeta) \mid \alpha, \beta, \varepsilon, \zeta \in \mathbf{F}^*\}$ is a maximal torus of F , with $N_F(H \cap F) = (H \cap F)(\tilde{W} \cap F)$. The subgroup $\tilde{W} \cap F$ is an extension by 2^4 of the Weyl group $W(F_4) = N_F(H \cap F)/(H \cap F)$ of F and is generated by the four elements n_5, n_4, n_2n_3, n_1n_7 , which induce a fundamental set of reflections for the Weyl group $W(F_4)$. The central element of $W(F_4)$ is induced by the transformation $x \mapsto (x_1^\top, x_3^\top, x_2^\top)$ ($x \in \mathbf{K}$) in F .

Recall that ψ and χ stand for the characters of $\tilde{E}$ on $\mathbf{K}$ and $\mathbf{g}$, respectively. Thus, ψ is distinct from $\bar{\psi}$ (that is, ψ takes complex values), and χ is a constituent of $\psi \otimes \bar{\psi}$. The restriction $\psi|_F$, however, is real valued and has an irreducible constituent of degree 26 occurring in $\chi|_F$.

To finish, note that all ingredients for the definitions of G involve only integers, so that, upon tensoring the $\mathbb{Z}$ -module spanned by one of the above bases in $\mathbf{K}$ with an arbitrary ring R , we can define the group $G(R)$ of points of the algebraic group scheme $G(\cdot)$ over R . Thus, for any field $\mathbf{F}$ we have constructions for the Chevalley groups $\tilde{E}_6(\mathbf{F})$ and $F_4(\mathbf{F})$. This observation will be used in § 6, where we construct certain finite subgroups of $G(\mathbb{C})$ by first embedding them in the group G over a suitable finite field, and subsequently lifting the embedding to one in $G(\mathbf{F})$ for $\mathbf{F}$ a field of characteristic 0.

3. Nilpotent subgroups

According to [38], the number of elements in F of order a divisor of i is the coefficient of t^i in the power-series expansion of

$$\frac{1}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)}.$$

This series begins as follows:

$$1 + t + 3t^2 + 4t^3 + 8t^4 + 10t^5 + 17t^6 + 21t^7 + 32t^8 + 39t^9 \\ + 55t^{10} + 66t^{11} + 89t^{12} + 105t^{13} + \dots$$

Consequently, the number of elements of order i (for $i \leq 13$) is the coefficient of t^i in the polynomial

$$1 + t + 2t^2 + 3t^3 + 5t^4 + 9t^5 + 11t^6 + 20t^7 + 24t^8 \\ + 35t^9 + 43t^{10} + 65t^{11} + 67t^{12} + 104t^{13}.$$

Similarly, the number of elements in E of order a divisor of i is the coefficient of t^i in the power-series expansion of

$$\frac{1}{(1-t)(1-t^2)(1-t^3)(1-x^2t)(1-xt^2)(1-x^2t^2)},$$

where x is a variable with $x^3 = 1$. Consequently, the number of elements of order i (for $1 \leq i \leq 13$) is the coefficient of t^i in the polynomial

$$t + 2t^2 + 7t^3 + 11t^4 + 25t^5 + 39t^6 + 76t^7 + 110t^8 \\ + 187t^9 + 259t^{10} + 417t^{11} + 543t^{12} + 832t^{13}.$$

3.1. THEOREM. *The elements in $\tilde{E}$ of orders 2, 3, 4, 5, 6, and 7 (up to conjugacy) are given explicitly in Table 2. Each of the corresponding conjugacy classes meets F in at most one F -conjugacy class. An element from Table 2 has a conjugate in F if and only if it is real (that is, conjugate to its inverse).*

TABLE 2. Elements $h(\zeta^a, \zeta^b, \zeta^c, \zeta^d, \zeta^e, \zeta^f)$ of order $n \leq 7$ in $E_6(\mathbb{C})$,
where $\zeta = \exp(2\pi i/n)$

Label [exponents]	(a, b, c, d, e, f)	$\mathbf{K}$ -multiplicity	$\mathbf{g}$ -multiplicity	Centralizer type
2A	(1,0,1,0,1,1)	15 12	38 40	$A_5 A_1$
2B	(1,0,1,0,0,0)	11 16	46 32	$D_5 T_1$
3A	(1,1,1,0,2,2)	15 6 6	36 21 21	$A_5 T_1$
3B[2]	(1,1,2,1,1,1)	6 15 6	36 21 21	$A_5 T_1$
3C	(1,1,1,0,0,1)	9 9 9	24 27 27	$A_2 A_2 A_2$
3D	(1,0,1,0,0,0)	9 9 9	30 24 24	$D_4 T_2$
3E[2]	(0,0,1,1,2,2)	0 27 0	78 0 0	E_6
4A	(1,3,1,0,1,3)	15 6 0 6	36 20 2 20	$A_5 T_1$
4B[3]	(1,1,1,1,3,1)	5 2 10 10	28 20 10 20	$A_4 A_1 T_1$
4C[3]	(1,1,1,1,2,1)	5 6 6 10	26 16 20 16	$A_4 T_2$
4D	(1,1,1,0,2,1)	5 8 6 8	22 16 24 16	$A_3 A_1 A_1 T_1$
4E	(1,1,1,0,1,1)	7 6 8 6	20 20 18 20	$A_3 A_1 T_2$
4F	(1,1,1,0,1,3)	9 6 6 6	20 20 18 20	$A_2 A_2 A_1 T_1$
4G[3]	(1,2,2,1,0,2)	1 16 10 0	46 16 0 16	$D_5 T_1$
4H	(1,1,1,1,2,3)	9 8 2 8	30 16 16 16	$D_4 T_2$
5A[2]	(1,3,1,0,1,0)	15 6 0 0 6	36 20 1 1 20	$A_5 T_1$
5B[2,3,4]	(0,2,2,1,1,3)	2 5 10 0 10	28 5 20 20 5	$A_4 A_1 T_1$
5C[2,3,4]	(0,2,2,1,3,1)	5 5 6 1 10	26 11 15 15 11	$A_4 T_2$
5D[2,3,4]	(0,2,2,1,2,0)	4 6 8 3 6	20 13 16 6 13	$A_3 A_1 T_2$
5E	(1,1,1,0,2,4)	7 5 5 5 5	18 15 15 15 15	$A_3 T_3$
5F[2]	(1,1,1,0,4,2)	9 3 6 6 3	20 11 18 18 11	$A_2 A_2 A_1 T_1$
5G[2]	(1,1,1,0,1,1)	5 5 6 6 5	16 16 15 15 16	$A_2 A_1 A_1 T_2$
5H[2,3,4]	(0,3,3,1,1,2)	0 16 0 10 1	46 0 16 16 0	$D_5 T_1$
5I[2]	(1,1,1,0,1,2)	9 1 8 8 1	30 8 16 16 8	$D_4 T_2$
6A	(1,1,2,1,1,5)	3 5 4 6 4 5	18 10 14 12 14 10	$A_3 T_3$
6B	(1,1,1,1,1,1)	5 4 5 4 5 4	14 12 12 16 12 12	$A_1 A_1 A_1 A_1 T_2$
6C	(1,0,1,0,4,0)	5 4 5 4 5 4	12 14 13 12 13 14	$A_1 A_1 A_1 T_3$
6D	(1,1,4,1,1,5)	5 6 3 4 3 6	16 12 15 8 15 12	$A_2 A_1 A_1 T_2$
6E	(2,1,2,1,2,4)	7 4 2 8 2 4	20 8 13 16 13 8	$A_3 A_1 T_2$
6F	(2,1,5,1,2,4)	7 2 4 8 4 2	20 12 9 16 9 12	$A_3 A_1 T_2$
6G	(1,1,4,1,4,1)	7 5 4 2 4 5	18 14 10 12 10 14	$A_3 T_3$
6H	(1,1,5,1,5,1)	9 6 3 0 3 6	20 18 9 4 9 18	$A_2 A_2 A_1 T_1$
6I	(1,3,1,1,1,1)	9 3 3 6 3 3	18 11 10 18 10 11	$A_2 A_2 T_2$
6J	(1,5,6,1,1,5)	9 8 1 0 1 8	30 16 8 0 8 16	$D_4 T_2$
6K	(4,3,4,1,4,1)	15 6 0 0 0 6	36 20 1 0 1 20	$A_5 T_1$
6L[5]	(1,3,4,1,6,3)	0 6 15 6 0 0	36 20 1 0 1 20	$A_5 T_1$
6M[5]	(3,3,1,1,2,2)	0 12 0 0 15 0	38 0 0 40 0 0	$A_5 A_1$
6N[5]	(3,3,1,1,2,5)	0 16 0 0 11 0	46 0 0 32 0 0	$D_5 T_1$
6O[5]	(3,5,6,1,4,4)	1 1 5 5 5 10	26 11 10 10 10 11	$A_4 T_2$
6P[5]	(1,3,3,1,6,3)	1 8 9 8 1 0	30 16 8 0 8 16	$D_4 T_2$
6Q[5]	(1,1,1,1,2,1)	2 4 7 4 2 8	20 8 13 16 13 8	$A_3 A_1 T_2$
6R[5]	(1,1,1,1,2,4)	3 3 9 3 3 6	18 11 10 18 10 11	$A_2 A_2$
6S[5]	(1,1,1,1,2,5)	3 6 5 6 3 4	16 12 15 8 15 12	$A_2 A_1 A_1 T_2$
6T[5]	(1,2,4,1,1,2)	3 6 9 6 3 0	20 18 9 4 9 18	$A_2 A_2 A_1 T_1$
6U[5]	(1,1,1,1,3,4)	4 5 7 5 4 2	18 14 10 12 10 14	$A_3 T_3$
6V[5]	(1,1,1,1,4,1)	4 5 3 5 4 6	18 10 14 12 14 10	$A_3 T_3$
6W[5]	(1,1,1,1,5,1)	4 2 7 2 4 8	20 12 9 16 9 12	$A_3 A_1 T_2$
6X[5]	(1,1,4,1,5,1)	5 5 5 1 1 10	26 11 10 10 10 11	$A_4 T_2$
6Y[5]	(1,3,4,1,1,1)	5 5 5 10 1 1	26 11 10 10 10 11	$A_4 T_2$

TABLE 2 (*continued*).

Label [exponents]	(a,b,c,d,e,f)	$\mathbf{K}$ -multiplicity	$\mathbf{g}$ -multiplicity	Centralizer type
7A[2,4]	(1,5,6,0,6,6)	15 6 0 0 0 0 6	36 20 1 0 0 1 20	$A_5 T_1$
7B[2,3,4,5,6]	(1,2,2,1,3,2)	0 10 2 0 5 10 0	28 5 0 20 20 0 5	$A_4 A_1 T_1$
7C[2,3,4,5,6]	(1,1,5,1,6,1)	5 5 5 0 1 1 10	26 11 10 5 5 10 11	$A_4 T_2$
7D[2,3,4,5,6]	(1,1,5,1,4,5)	1 5 5 0 10 0 6	26 1 15 10 10 15 1	$A_4 T_2$
7E[2,3,4,5,6]	(1,1,5,1,2,5)	2 4 6 1 6 0 8	20 4 16 9 9 16 4	$A_3 A_1$
7F[2,3,4,5,6]	(1,1,6,1,1,1)	4 8 4 0 3 6 2	20 12 5 12 12 5 12	$A_3 A_1 T_2$
7G[2,3,4,5,6]	(1,1,5,1,1,1)	4 6 4 1 5 2 5	18 9 11 10 10 11 9	$A_3 T_3$
7H[2,4]	(1,1,5,1,5,1)	7 5 4 1 1 4 5	18 14 9 7 7 9 14	$A_3 T_3$
7I[2,4]	(1,1,6,1,6,1)	9 6 3 0 0 3 6	20 18 9 2 2 9 18	$A_2 A_2 A_1 T_1$
7J	(1,3,5,1,5,1)	9 3 3 3 3 3 3	18 10 10 10 10 10 10	$A_2 A_2 T_2$
7K[2,3,4,5,6]	(1,2,2,1,5,2)	2 5 1 6 3 6 4	16 8 14 9 9 14 8	$A_2 A_1 A_1 T_2$
7L[2,4]	(1,1,5,1,1,6)	5 6 3 2 2 3 6	16 12 12 7 7 12 12	$A_2 A_1 A_1 T_2$
7M[2,3,4,5,6]	(1,2,2,1,5,1)	3 3 3 3 6 5 4	14 12 10 10 10 10 12	$A_2 A_1 T_3$
7N	(1,1,1,1,1,3)	3 4 4 4 4 4 4	12 11 11 11 11 11 11	$A_2 T_4$
7O[2,4]	(1,1,1,1,1,6)	5 4 4 3 3 4 4	12 12 11 10 10 11 12	$A_1 A_1 A_1 T_3$
7P[2,3,4,5,6]	(1,5,4,1,6,4)	0 1 16 10 0 0 0	46 16 0 0 0 0 16	$D_5 T_1$
7Q[2,4]	(3,1,6,1,4,5)	9 1 0 8 8 0 1	30 8 0 16 16 0 8	$D_4 T_2$
7R[6]	(1,1,2,1,3,5)	0 8 8 1 8 1 1	30 8 8 8 8 8 8	$D_4 T_2$

Proof. Given an element $h = h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, its eigenvalues on $\mathbf{K}$ are immediate from its presentation in § 2, and its eigenvalues on $\mathbf{g}$ and centralizer type are determined by the fundamental roots with respect to the torus H (cf. [41]). Also, by [41], the centralizers are connected reductive Lie subgroups. The above formula shows that we have found a representative for each class provided that no two given elements are conjugate in $\tilde{E}$. Now from the data displayed in the table it is readily seen that this holds, except possibly for the two elements 3C and 3D of order 3 (with trace 0) and those of order 6 with trace 0. The two elements 3C and 3D cannot be conjugate as the restriction of the cubic form $\langle \cdot \rangle$ to the subspace of $\mathbf{K}$ of all fixed vectors is distinct: for 3C, the form on the 9-dimensional space of fixed vectors is the determinant (on the space of 3×3 matrices), for the other, it is a ‘star’ (that is, it vanishes on an 8-dimensional subspace). As for the elements of order 6 and trace 0, that is, types 6B and 6C, the square maps them into distinct conjugacy classes, viz. $(6B)^2 = 3D$ and $(6C)^2 = 3C$. This settles the first assertion of Theorem 3.1.

Since each semisimple element of F is real, only those $\tilde{E}$ -classes representing real elements may meet F . To see that a real class actually does meet F , one checks that, under the Weyl group action on H given above, it can be transformed to an element of $H \cap F$ (several class representatives are already inside this subgroup). Doing this for every real class, we find as many different classes in F for the given orders as there are in F (according to the above series expansion). Hence the second and the third assertion.

Kac theory (cf. [38]) gives a more direct construction of classes of elements of finite order.

In Table 2, and in the remainder of the paper, T_i stands for an i -dimensional torus.

3.2. DEFINITION (feasible characters). Let G be $\tilde{E}$ or F . A character ϕ of a finite group L of degree 27 will be called *feasible* for G if, for each $h \in L$ of order at most 7, there is $g \in G$ of the same order such that $\phi(h) = \psi(g)$. Similarly, a character ϕ of L of degree 78 will be called *feasible* for G , if, for each $h \in L$ of order at most 7, there is $g \in G$ of the same order such that $\phi(h) = \chi(g)$.

We shall reduce the possibilities for embedding finite groups L in G by determining whether L has feasible characters, and, if so, by studying possible embeddings on the basis of feasible characters found. The bulk of the paper will be concerned with the case where L is almost simple (that is, there is a non-abelian normal subgroup N of L such that L embeds in $\text{Aut } N$). The remainder of this section though is concerned with the other extreme, supersolvable groups. This study will be of help to the quasisimple case as the embedding in G of a quasisimple group L implies the embedding in G of each supersolvable subgroup of L .

3.3. LEMMA. Suppose that L is a finite supersolvable subgroup of $\tilde{E}$. Let H be a maximal torus of $\tilde{E}$. Then the following hold.

- (i) The subgroup L is conjugate to a subgroup of $N = N_{\tilde{E}}(H)$.
- (ii) If L is elementary abelian of order p^m for some p and some natural number m , then $m \leq 6$. Moreover, if $p \neq 2, 3, 5$, then L is conjugate to a subgroup of H .
- (iii) If L is a Klein four group (that is, elementary abelian of order 2^2), then L is conjugate to a subgroup of H , and we have one of the four possibilities given in Table 3.
- (iv) If L is a Klein four group in F , then L is conjugate to one of the groups of Table 3 with $L^\#$ having involution types BBB, BAA, AAA, and centralizer types D_4 , $A_1A_1B_2$, and A_2T_2 , respectively.

TABLE 3. Klein four subgroups of $\tilde{E}$ (up to conjugacy)

$L^\#$	Centralizer type	Automizer	Multiplicities on $\mathbf{g}$	Multiplicities on $\mathbf{K}$
BBB	D_4T_2	Sym_3	30 16 16 16	3 8 8 8
BBA	A_4T_2	$\mathbb{Z}_2$	26 20 20 12	5 6 6 10
BAA	$A_3A_1T_1$	$\mathbb{Z}_2$	22 24 16 16	7 4 8 8
AAA	$A_2A_2T_2$	Sym_3	18 20 20 20	9 6 6 6

Proof. (i) This result is due to [6].
(ii) This is immediate from [32].
(iii) By [41], the subgroup L is conjugate to a subgroup of H . Each element of L is an element of the 2-torsion of H , which under the action of N/H has the structure of elliptic orthogonal 6-space over $\mathbb{F}_2$. An involution in H corresponding to an isotropic point in this orthogonal space has $\tilde{E}$ -type 2B, and an involution

corresponding to a non-singular (anisotropic) point has $\tilde{E}$ type 2A. Thus the N -conjugacy classes of Klein four subgroups of H correspond to the various kinds of 2-spaces in the orthogonal space, viz., singular (BBB), hyperbolic (BBA), degenerate (BAA), and elliptic (AAA). The remaining details given in Table 3 follow by use of Table 2.

(iv) The proof is similar to that of (iii).

By a result ascribed to Jacobson (cf. [10, Chapter V, § 2, no. 8]), if L is a finite subgroup of G , then $C_G(L)$ is a closed reductive subgroup of G . If N is a closed subgroup of G , then we denote by N^0 the connected component of the identity in N .

3.4. LEMMA. *Let L be an elementary abelian 2-subgroup of E of rank at least 3 all of whose involutions are from the same conjugacy class in E . Then L has rank 3 and its involutions are of type 2A. Moreover, L is unique up to conjugacy, and $C_E(L)^0 \cong \mathrm{SL}(3, \mathbb{C})$.*

Proof. Suppose that L has rank 3. If the involution type is 2B, then the fixed point space of L on $\mathbf{K}$ has dimension $\frac{1}{8}(27 + 7(-5)) < 0$, which is absurd. Hence, the involutions have type 2A. Now, the fixed point space in $\mathbf{K}$ has dimension $\frac{1}{8}(27 + 7 \cdot 3) = 6$, and the fixed point space in $\mathfrak{g}$ has dimension $\frac{1}{8}(78 + 7(-2)) = 8$. Let M be a Klein four group in L and put $C = C_E(M)$. By Lemma 3.3(iii) we have that C^0 has Lie type $A_2A_2T_2$; by the uniqueness of M up to conjugacy, we may take M to be the Klein four subgroup of a 2-dimensional torus S of $\{s_{\mathrm{id}, \mathrm{id}, x} \mid x \in \mathrm{SL}(3, \mathbb{C})\}$. Then the semisimple part of C^0 coincides with the subgroup $\{s_{x, y, \mathrm{id}} \mid x, y \in \mathrm{SL}(3, \mathbb{C})\}$ of type A_2A_2 . It is straightforward to verify that $w \in \mathrm{GL}(\mathbf{K})$, given by $w(x_1, x_2, x_3) = (x_1^T, x_3^T, x_2^T)$, actually belongs to $\tilde{E}$. Moreover, from $ws_{a, b, c}w^{-1} = s_{b^{-T}, a^{-T}, c^{-T}}$ for $a, b, c \in \mathrm{SL}(3, \mathbb{C})$, we see that $w \in C$. As the centralizer in $\tilde{E}$ of an involution in M is connected of type A_5A_1 , a computation in the latter group shows that C/C^0 has order 2, so that $C = \langle C^0, w \rangle$.

Since a maximal torus does not contain a pure 2A-group of order 8, L meets the component wC^0 of C . We claim that, up to conjugacy, $L = \langle M, w \rangle$. For suppose that $s_{a, b, c}w \in C^0w$ is an element of L . The fact that it is an involution implies that $b = a^T$. For $x = a^{-1}$ and $z \in S$ such that $z^2 = c^{-1}$, we have $s_{x, \mathrm{id}, z}(s_{a, b, c}w)s_{x, \mathrm{id}, z}^{-1} = s_{xa, bx^T, z^2c}w = w$. This shows that each involution in C^0w is conjugate to w , whence the claim follows.

If L had rank 4, then it would have fixed point space on $\mathbf{K}$ of dimension $\frac{1}{16}(27 + 15 \cdot 3) = \frac{72}{16} \notin \mathbb{Z}$, a contradiction. Thus the rank of L is precisely 3. Clearly, $C_{\tilde{E}}(L)$ coincides with $\{s_{x, x^{-T}, \mathrm{id}} \mid x \in \mathrm{SL}(3, \mathbb{C})\} \times L$ and so $C_E(L)^0$ is of type A_2 .

3.5. THEOREM [1]. *If A is an elementary abelian subgroup of $G = E$ or F such that $N_G(A)$ is Lie primitive, then $A \cong 3^3$, while $N_F(A) \cong 3^3 : \mathrm{SL}(3, 3)$ and $N_E(A) \cong 3^{3+3} : \mathrm{SL}(3, 3)$. Moreover, there is a unique conjugacy class of such subgroups A . Each non-trivial element in such a subgroup A has type 3C.*

Proof. Though [1] provides a sketch of a beautiful proof, it is not hard to give an *ad hoc* proof by use of [42] and the results of § 2. Alternative proofs can be found in [2], [16], and [32].

As a consequence of this result, the Main Theorem holds for finite groups L containing an abelian normal subgroup L .

3.6. LEMMA. *If L is a finite p -subgroup of $\tilde{E}$, then either $p > 5$ and L is abelian of rank at most 6, or there is a power q of p such that L can be embedded in a group of shape*

(i) $\mathbb{Z}_q^6 \cdot 2^4 \cdot 2^2 \cdot 2$ for $p = 2$;

(ii) $\mathbb{Z}_q^6 : (3^3 : 3)$ for $p = 3$;

(iii) $\mathbb{Z}_q^6 : 5$ for $p = 5$.

Proof. Assertions (i), (ii), (iii) are immediate from the structure of $W(E_6)$. The first assertion about the rank is proved in [17].

For further information about elementary abelian subgroups of G , we refer to [32], [34], and [16].

4. Closed semi-simple Lie subgroups of $\tilde{E}$ of positive dimension

In [26], the connected semisimple (positive-dimensional) closed Lie subgroups of simple Lie groups have been determined. These are listed in [40] in Theorem 1 and the remark following it (see also [37]). In this section, the result for the groups G of type E_6 and F_4 will be recalled. We adopt the following notation: $\pi_i^{(j)}$ denotes the i th fundamental weight of the j th component of L , and the square brackets around a weight π stand for the character of L on the irreducible L -module with highest weight π .

4.1. THEOREM (Dynkin). *Suppose that L is a maximal connected closed semi-simple Lie subgroup of G of positive dimension.*

(i) *If $G = \tilde{E}$, then the triple $L, \psi|_L, \chi|_L$ is, up to contragredients, one of those shown in Table 4. In particular, if L is irreducible on $\mathbf{K}$, then L is one of $A_2(\mathbb{C})$, $G_2(\mathbb{C})$, $C_4(\mathbb{C})$.*

TABLE 4

Lie type L	$\psi _L$	$\chi _L$
A_2	$[2\pi_1 + 2\pi_2]$	$[\pi_1 + 4\pi_2] + [4\pi_1 + \pi_2] + [\pi_1 + \pi_2]$
G_2	$[2\pi_2]$	$[\pi_1 + \pi_2] + [\pi_1]$
C_4	$[\pi_2]$	$[\pi_4] + [2\pi_1]$
D_5	$[\pi_4] + [\pi_1] + [0]$	$[\pi_2] + [\pi_4] + [\pi_5] + [0]$
F_4	$[\pi_4] + [0]$	$[\pi_1] + [\pi_4]$
A_2G_2	$[\pi_2^{(1)} + \pi_2^{(2)}] + [2\pi_1^{(1)}]$	$[\pi_1^{(1)} + \pi_2^{(1)} + \pi_2^{(2)}] + [\pi_1^{(2)}] + [\pi_1^{(1)} + \pi_2^{(1)}]$
A_1A_5	$[\pi_2^{(2)}] + [\pi_5^{(2)} + \pi_1^{(1)}]$	$[\pi_1^{(1)} + \pi_3^{(2)}] + [\pi_1^{(2)} + \pi_5^{(2)}] + [2\pi_1^{(1)}]$
$A_2A_2A_2$	$[\pi_1^{(1)} + \pi_2^{(2)}] + [\pi_2^{(1)} + \pi_1^{(3)}] + [\pi_1^{(2)} + \pi_2^{(3)}]$	$[\pi_1^{(1)} + \pi_1^{(2)} + \pi_1^{(3)}] + [\pi_2^{(1)} + \pi_2^{(2)} + \pi_2^{(3)}] + [\pi_1^{(1)} + \pi_2^{(1)}] + [\pi_1^{(2)} + \pi_2^{(2)}] + [\pi_1^{(3)} + \pi_2^{(3)}]$

(ii) If $G = F$, then the triple $L, \psi|_L - 1, \chi|_L - (\psi|_L - 1)$ is one of those shown in Table 5.

TABLE 5

Lie type L	$\psi _L - 1$	$\chi _L - (\psi _L - 1)$
A_1	$[16\pi_1] + [8\pi_1]$	$[22\pi_1] + [14\pi_1] + [10\pi_1] + [2\pi_1]$
B_4	$[\pi_1] + [\pi_4] + [0]$	$[\pi_2] + [\pi_4]$
A_1G_2	$[2\pi_1^{(1)} + \pi_2^{(2)}] + [4\pi_1^{(1)}]$	$[4\pi_1^{(1)} + \pi_2^{(2)}] + [\pi_1^{(2)}] + [2\pi_1^{(1)}]$
A_1C_3	$[\pi_2^{(2)}] + [\pi_1^{(1)} + \pi_1^{(2)}]$	$[\pi_1^{(1)} + \pi_3^{(2)}] + [2\pi_1^{(2)}] + [2\pi_1^{(1)}]$
A_2A_2	$[\pi_1^{(1)} + \pi_2^{(2)}] + [\pi_2^{(1)} + \pi_1^{(2)}] + [\pi_1^{(1)} + \pi_2^{(1)}]$	$[2\pi_2^{(1)} + \pi_2^{(2)}] + [2\pi_1^{(1)} + \pi_1^{(2)}] + [\pi_1^{(1)} + \pi_2^{(1)}] + [\pi_1^{(2)} + \pi_2^{(2)}]$

We shall describe the irreducible representations in a more mundane manner:

for A_2 , $(3_a^{2+} \otimes 3_b^{2+})$ modulo $(3_a \otimes 3_b)$, where 3_a denotes the natural representation (highest weight π_1) and 3_b its contragredient (highest weight π_2);

for G_2 , 7_a^{3-} modulo $(7_a \oplus 1_a)$, where 1_a and 7_a stand for the trivial (highest weight 0) and the standard representation (highest weight π_1), respectively;

for C_4 , 8_a^{2-} modulo (1_a) , where 1_a and 8_a stand for the trivial and the standard representation (highest weight π_1), respectively.

Let K be one of these Lie subgroups. The information readily unveils the eigenvalue multiplicities on $\mathbf{K}$ of semisimple elements of K . Take a surjective morphism $\lambda \mapsto t(\lambda)$ from $(\mathbb{C}^*)^l$, where l is the Lie rank of K , to a maximal torus T of K . Then, presenting the elements of T as diagonal matrices in the standard representation of K , we have:

for A_2 , $t(\lambda) = \text{diag}(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$ has eigenvalues 3×1 and each of the following twelve eigenvalues as well as their inverses: $\lambda_1^2\lambda_2^{-2}$, $\lambda_1^4\lambda_2^2$, λ_1^3 , $\lambda_1\lambda_2^{-1}$, $\lambda_1^2\lambda_2^4$, λ_2^3 , $\lambda_1^{-2}\lambda_2^{-1}$, $\lambda_1^{-1}\lambda_2^{-2}$, $\lambda_1^{-3}\lambda_2^{-3}$, $\lambda_1\lambda_2^{-1}$, $\lambda_1^{-1}\lambda_2^{-2}$, $\lambda_1^{-2}\lambda_2^{-1}$ (consideration of the values of these characters on the centre of A_2 shows that the embedded group is the adjoint group of type A_2);

for G_2 , $t(\lambda) = \text{diag}(1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_1^{-1}, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2^{-1})$ has eigenvalues 3×1 and each of the following twelve eigenvalues and their inverses: $\lambda_1^2\lambda_2^2$, λ_1^2 , λ_2^2 , $2 \times \lambda_1$, $2 \times \lambda_2$, $2 \times \lambda_1\lambda_2$, $\lambda_1\lambda_2^{-1}$, $\lambda_1^2\lambda_2$, $\lambda_1\lambda_2^2$;

for C_4 , $t(\lambda) = \text{diag}(\lambda_1, \dots, \lambda_4, \lambda_1^{-1}, \dots, \lambda_4^{-1})$ has eigenvalues 3×1 , $2 \times \lambda_i\lambda_j$ ($1 \leq i < j \leq 4$), and $\lambda_i\lambda_j^{-1}$ ($1 \leq i, j \leq 4$, $i \neq j$); in particular, the embedded group is the adjoint group $\text{PSp}(8, \mathbb{C})$.

There are automorphisms induced from G for some of the groups, L , appearing in the first columns in the tables above. The connected group A of type $A_2A_2A_2$ at the bottom of Table 4 is the centralizer of an element, h say, of order 3 (type 3C) outside the centre of $\tilde{E}$. However, the subgroup $\langle h \rangle$ is not unique with these properties, $\langle zh \rangle$ and $\langle z^2h \rangle$, where $z = s_{\text{id}, \bar{\omega}, \omega}$ (cf. § 2), being the two other central subgroups of A of order 3 outside $\langle z \rangle = Z(\tilde{E})$. These three groups are permuted by an element of order 3, which also permutes the three factors A_2 cyclically. To

determine further the structure of $N_{\tilde{E}}(A)/A$, we may assume that $\langle h \rangle$ is normalized. Thus we remain with a subgroup of $N_{\tilde{E}}(A)/A$ of order 2, corresponding to inversion of h . Since such an inversion is readily seen to exist and to interchange two factors A_2 of A , we conclude that $N_{\tilde{E}}(A)/A$ is the full permutation group on three letters, permuting the three factors A_2 of A , as well as their centres.

As for the connected group of type A_1A_5 , it is the centralizer and hence the normalizer of an involution and so has trivial component group.

The maximal connected semisimple closed subgroup D of $\tilde{E}$ of type D_5 (cf. Table 4) is the commutator subgroup of the connected group of type T_1D_5 , which is the centralizer of the central involution, z say, of D . Thus, $N_{\tilde{E}}(D) \leq C_{\tilde{E}}(z) \leq N_{\tilde{E}}(C_{\tilde{E}}(z)^\infty) = N_{\tilde{E}}(D)$, showing that $N_{\tilde{E}}(D)$ coincides with $C_{\tilde{E}}(z)$ and so has trivial component group.

We will also use this information in § 8 and § 9 when considering the possibilities for finite simple subgroups of E or F which are contained in a proper closed subgroup of positive dimension by using the next lemma.

4.2. LEMMA. *Suppose that H is a finite simple group which is contained in a proper closed subgroup of positive dimension of E or F . Then either $H \cong \text{Alt}_5$, Alt_6 and H normalizes a torus, or H is contained in a maximal connected closed semi-simple proper subgroup of positive dimension which is listed in Theorem 4.1.*

Proof. Let M be a proper closed subgroup of G of positive dimension containing H . Let M^0 be the connected component of the identity in M . If M^0 is not reductive, then H is contained in a maximal parabolic. As H is simple and the characteristic is 0, it is then in a Levi factor. This is a closed connected semi-simple proper subgroup, which is in turn contained in a maximal connected closed semi-simple proper subgroup of G and we have finished. Hence we may, and shall, assume that M^0 is reductive.

Denote by S the connected centre of M^0 and by X the product $X_1 \dots X_t$ of the simple factors X_i of M^0 . Then $H \leq N_G(X)$. In view of Theorem 4.1, there cannot be more than four factors. Hence H induces the trivial permutation representation on the simple factors of M^0 . If $S = 1$, then, as the outer automorphism groups of the X_i are all solvable, we have $H \leq (M^0)^\infty = X$, which is a closed connected semi-simple subgroup, and we have finished.

Next assume $S \neq 1$. Then $R = N_G(S)$ contains H . If H centralizes S , then H is in $C_G(S)^\infty$ which is closed, connected and semi-simple and so in a maximal such subgroup. If H does not centralize S , then it intersects $C_G(S)$ trivially. As M^0 is reductive, S must be in a maximal torus T which is in R . As all tori are conjugate in $C_G(S)$, we have $R = N_R(T)C_G(S)$ by the Frattini argument. Now H is embedded in $N_R(T)/(C_G(S) \cap N_R(T))T$, which is a quotient of a subgroup of $W(E_6)$. The order of H must divide the order of $W(E_6)$ which is $5 \cdot 3^4 \cdot 2^7$. By a result of Brauer (cf. [9]) or the classification, we obtain that $H \cong \text{Alt}_5$, Alt_6 , or $\text{PSU}(4, 2)$. But in the latter case we would have no proper subtorus of T left invariant by H , whence $S = T$ and $R = N_G(T)$. This implies an embedding in $\tilde{E}$ of $\text{PSU}(4, 2)$ normalizing a maximal torus. But then H permutes the twenty-seven eigenspaces of T , and because H has no proper subgroup of index smaller than 27, it must act transitively. Its character on $\mathbf{K}$ must be induced from a linear

character of a subgroup L isomorphic to $2^4 : \text{Alt}_5$. Since L is perfect, the linear character in question is the trivial one, and the representation on $\mathbf{K}$ is the permutation representation of H on L . From [22, p. 26], the corresponding character is $1 + 6 + 20$; but this character is not feasible as the character on the involution 2B in Atlas notation is 7 (see also Theorem 8.1(i) below). Consequently, $H \cong \text{Alt}_5$, Alt_6 , and H normalizes the torus S .

We will often use a standard method to guarantee that a finite subgroup H is in a proper closed subgroup of positive dimension. Suppose that H is a finite subgroup of E or F which fixes a non-zero vector in $\mathfrak{g}$. Its fixed space is a proper reductive subalgebra (cf. [10, Chapter V, § 2, no. 8]). Its group then centralizes H and, as it is reductive, contains a non-trivial torus. This shows that H is in the centralizer of a non-trivial torus. But this is a proper closed subgroup of positive dimension, which, by [41], is reductive.

5. The projective orbits on the standard modules

The structure of orbits of $\tilde{E}$ on $\mathbf{K}$ and F on $e^\perp$ has been studied in [36], [2]–[5], and [12]. We summarize the results of relevance to us.

The projective points $\mathbb{C}x$ with $x^\# = 0$ are called *singular points*, or *parabolic points* as their stabilizers are parabolic subgroups of $\tilde{E}$. The set P of all these points carries the structure of a partial linear space whose lines are projective lines $\mathbb{C}x + \mathbb{C}y$ with $x, y \in P$ such that $x \times y = 0$. This is the so-called ‘shadow geometry of type $E_{6,1}$ ’ (cf. [12]). A linear subspace of $\mathbf{K}$ on which $\#$ vanishes identically is called a *singular subspace*. Thus, singular subspaces of dimensions 1 and 2 are precisely the points and lines of the shadow geometry; they are the objects of the building of type E_6 corresponding to the nodes 1 and 3 in the diagram of § 2. The nodes labelled 2, 4 and 5 correspond to the collections of 6-dimensional singular subspaces, of 3-dimensional singular subspaces, and of 5-dimensional singular subspaces not contained in a 6-dimensional singular subspace, respectively. (The subspace $\mathbb{C}e_{1,1}^1 + \mathbb{C}e_{1,2}^1 + \mathbb{C}e_{1,3}^1 + \mathbb{C}e_{1,2}^3 + \mathbb{C}e_{2,2}^3 + \mathbb{C}e_{3,2}^3$ is an example of a 6-dimensional singular subspace, the subspace $\mathbb{C}e_{1,1}^1 + \mathbb{C}e_{1,2}^1 + \mathbb{C}e_{1,3}^1 + \mathbb{C}e_{1,2}^3 + \mathbb{C}e_{1,3}^3$ of a 5-dimensional singular subspace not contained in a 6-dimensional singular subspace.) There are no 7-dimensional singular subspaces. Finally, the node labelled 6 corresponds to the 10-dimensional subspaces of the form $y \times \mathbf{K}$ for $\mathbb{C}y$ a singular 1-space (a point of the geometry). Each of these six collections constitutes a single $\tilde{E}$ -orbit; their stabilizers are maximal parabolics whose types are determined by the corresponding nodes in the diagram for E_6 .

Another remarkable collection consists of all 3-dimensional subspaces of $\mathbf{K}$ on which $\langle \cdot \rangle$ does not vanish that contain precisely three singular points. A 3-space of this kind will be called a *special plane*. An example is $\mathbb{C}e_{1,1}^1 + \mathbb{C}e_{2,2}^1 + \mathbb{C}e_{3,3}^1$; its stabilizer in $\tilde{E}$ is a group of type $\tilde{D}_4$.

The G -orbits of projective points are described as follows.

5.1. THEOREM. (i) *There are three $\tilde{E}$ -orbits of projective points on $\mathbf{K}$ with the characteristics shown in Table 6.*

(ii) *There are five types of F -orbits of projective points on $e^\perp$ with the characteristics listed in Table 7.*

TABLE 6

Representative x	Orbit invariant	Stabilizer of x	$\mathbf{K}$ -decomposition of a reductive complement
$e = e_{11}^1 + e_{22}^1 + e_{33}^1$	$\langle x \rangle \neq 0$	F	$1_a + 26_a$
$e_{11}^1 + e_{22}^1$	$\langle x \rangle \neq 0; x^\# \neq 0$	$\mathbb{C}^{16} \cdot \text{Spin}(9, \mathbb{C})$	$2 \cdot 1_a + 9_a + 16_a$
e_{11}^1	$x^\# = 0$	$\mathbb{C}^{16} \cdot \text{Spin}(10, \mathbb{C}) \cdot \mathbb{C}^*$	$1_a + 10_a + 16_a$

TABLE 7

Representative x	Orbit invariant	Stabilizer $F_{\mathbb{C}x}$	$\mathbf{K}$ -decomposition of a reductive complement
$e_{11}^1 + \alpha e_{22}^1 + \beta e_{33}^1$ ($\alpha\beta \neq 0$)	$x, x^\#, e$ span a special plane	D_4 -type with possible outer automorphisms	$1_a^2 + 8_a + 8_b + 8_c$
$3e_{11}^1 - e$	$x^\# = x - e$	$\text{Spin}(9, \mathbb{C}) \cdot 2$	$1_a + 9_a + 16_a$
e_{12}^1	$x^\# = 0$	$\mathbb{C}^{15} \cdot \text{Spin}(7, \mathbb{C}) \cdot \mathbb{C}^*$	parabolic
$3e_{11}^1 - 3e_{23}^1 - e$	$x^\#, x, e$ span a plane with two singular points	$\mathbb{C}^7 \cdot \text{Spin}(7, \mathbb{C})$	$3 \cdot 1_a + 7_a + 2 \cdot 8_a$
$e_{21}^1 + e_{13}^1$	$x^\#, x, e$ span a plane with one singular point	$\mathbb{C}^{14} \cdot G_2(\mathbb{C}) \cdot \mathbb{C}^*$	$5 \cdot 1_a + 3 \cdot 7_a$

The last column in Tables 6 and 7 lists the decomposition of a reductive complement of the unipotent radical.

The above theorem will be useful in determining the structure of stabilizers of groups embedded in $\tilde{E}$ and F . Suppose, for instance, that L is a finite simple subgroup of $\tilde{E}$ fixing a non-zero vector v with $v^\# \neq 0$ and $\langle v \rangle = 0$. Then it follows from the first part of the above theorem (and the regular action of the unipotent radical of the stabilizer of v on the invariant 16-dimensional quotient of $\{x \in \mathbf{K} \mid x \times v^\# = 0\}$ by $\mathbb{C}v$) that L , being reductive, fixes a 2-space containing v and an $\tilde{E}_{\langle v \rangle}$ -conjugate of $v^\#$. Thus, L is a subgroup of a Levi-complement of type B_4 , and fixes a singular point.

6. Constructions

In this section, we establish the existence of the groups L appearing in the Main Theorem. We also indicate when there is an extension to a subgroup B of $\text{Aut } L$. It will be a case-by-case treatment of the groups L listed in Table 1, starting from the bottom up. In the proof of § 6.3, we also include a construction of the subgroup of F isomorphic to $3^3 : \text{SL}(3, 3)$.

Some of the constructions for finite subgroups L in G are done in $E_6(\mathbb{Z}/p)$ or $F_4(\mathbb{Z}/p)$ for a suitable prime p . We infer from this, provided p does not divide the order of L , that there are corresponding subgroups in E or F over the complex numbers by various lifting arguments. Accounts of these may be found in [32],

[15], and [20]. We refer to these references for details and state here what will be needed below.

LIFTING THEOREM. *Suppose that L is a finite group and p is a prime not dividing the order of L . Suppose further that L is a subgroup of $E_6(\mathbb{Z}/p)$ or of $F_4(\mathbb{Z}/p)$ with a specific action on $\mathbf{K}$ as a $\mathbb{Z}/p$ module. Then there is a subgroup of $E_6(\mathbb{C})$ or of $F_4(\mathbb{C})$ whose action on $\mathbf{K}$ reduces modulo p to L .*

6.1. *The Ree group $B = {}^2F_4(2)$, of order $2^{12} \cdot 3^3 \cdot 5^2 \cdot 13 = 35,942,400$, embeds in E with $\psi|_L = 27_a$.*

Proof. Denote by ξ the unique 78-dimensional irreducible character of the Tits group B' . We proceed in three steps.

Step 1. *The character ξ satisfies $(\xi^{2+}, 1) = (\xi^{3-}, 1) = 1$, and $(\xi^{4-}, 1) = 0$.*

To verify this, we use the following formulae for $g \in {}^2F_4(2)'$:

$$\begin{aligned}\xi^{2+}(g) &= \tfrac{1}{2}(\xi(g)^2 + \xi(g^2)), \\ \xi^{3-}(g) &= \tfrac{1}{6}(\xi(g)^3 - 3\xi(g^2)\xi(g) + 2\xi(g^3)), \\ \xi^{4-}(g) &= \tfrac{1}{24}(\xi(g)^4 - 6\xi(g)^2\xi(g^2) + 8\xi(g^3)\xi(g) + 3\xi(g^2)^2 - 6\xi(g^4)).\end{aligned}$$

Now direct computation (using the power maps of the character table of B' in [22]) yields

$$\begin{aligned}(\xi^{2+}, 1) &= \tfrac{1}{2}(1 + 1) = 1, \\ (\xi^{3-}, 1) &= \tfrac{1}{6}(1 - 3(-1) + 2 \cdot 1) = 1, \\ (\xi^{4-}, 1) &= \tfrac{1}{24}(10 - 6 \cdot 2 + 8 \cdot (-2) + 3 \cdot 10 - 6 \cdot 2) = 0.\end{aligned}$$

As a consequence, we obtain, using an argument of S. Norton (similar arguments were found by R. L. Griess Jr in 1977), the next step.

Step 2. *Let M be the complex 78-dimensional B' -module with character ξ . Then M is real (that is, has a real invariant subspace of dimension 78), and leaves invariant a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and a Lie algebra structure $[\cdot, \cdot]$.*

Since $(\xi^{2+}, 1) = 1$, the Frobenius–Schur index is 1. Hence the first statement and the existence of a non-trivial symmetric bilinear form $(\cdot, \cdot)$ follow. Since B' is irreducible, the form must be non-degenerate. In view of $(\xi^{3-}, 1) = 1$, there is, up to scalar multiples, a unique alternating trilinear form $\langle \cdot, \cdot, \cdot \rangle$ invariant under B' . Define a bilinear multiplication $[\cdot, \cdot]$ by means of $([x, y], z) = \langle x, y, z \rangle$ for all $x, y, z \in M$. Since $(\cdot, \cdot, \cdot)$ is alternating and $(\cdot, \cdot)$ is symmetric, we get $[x, y] = -[y, x]$ (anticommutativity) and $([x, y], z) = (x, [y, z])$ (associativity) for all $x, y, z \in M$. Now the form $(\cdot, \cdot, \cdot, \cdot)$ given by $(x, y, z, u) = ([x, y], [z, u]) + ([y, z], [x, u]) + ([z, x], [y, u])$ is easily seen to be a 4-linear alternating B' -invariant form. Since $(\xi^{4-}, 1) = 0$, the form $(\cdot, \cdot, \cdot, \cdot)$ vanishes identically. Rewriting the identity by means of associativity, we get

$$([x, y], z) + ([y, z], x) + ([z, x], y), u) = 0,$$

and hence (by non-degeneracy of $(\cdot, \cdot)$) the Jacobi identity. Consequently $[\cdot, \cdot]$ defines a Lie algebra structure.

Step 3. The Lie algebra M of Step 2 is the complex simple Lie algebra of type E_6 , and $B' = {}^2F_4(2)'$ is embeddable in E .

Since M is an irreducible B' -module, its (Lie algebraic) radical is trivial. Thus M is semisimple, whence there exist ideals $M_1, \dots, M_t$ of M , which are simple Lie algebras, such that $M = M_1 \oplus \dots \oplus M_t$ and every simple ideal of M coincides with one of the M_i . In particular, B' permutes $\{M_i \mid i = 1, \dots, t\}$. But $t \leq 78$ and B' has no subgroup of index $t \leq 78$, as can be seen by considering the non-trivial characters of degree less than 78 and noting that all of them have negative values on the element $2A$. This means there is no permutation character corresponding to any proper subgroup of index less than or equal to 78. Consequently, B' stabilizes each M_i ($i = 1, \dots, t$). Again by irreducibility of B' , we must have $M = M_1$, so that M is a simple complex Lie algebra of dimension 78. The simple complex Lie algebras of dimension greater than 1 are well known to be those of type A_l ($l \geq 1$), B_l ($l \geq 2$), C_l ($l \geq 2$), D_l ($l \geq 4$), E_6 , E_7 , E_8 , F_4 and G_2 of dimension $l(l+2)$, $2l^2+l$, $2l^2+l$, $2l^2-l$, 78, 133, 248, 52 and 14, respectively (cf. [33]). Thus M is isomorphic to the Lie algebra of type B_6 , C_6 or E_6 , so that $(\text{Aut } M)^0 = B_6(\mathbb{C})$, $C_6(\mathbb{C})$, $E_6(\mathbb{C})$ in the respective cases. Now, B' , being simple and clearly not contained in $\text{Aut } M/(\text{Aut } M)^0$, is a subgroup of $(\text{Aut } M)^0$. It has trivial Schur multiplier (see [22]). Since $B_6(\mathbb{C})$ and $C_6(\mathbb{C})$ have faithful projective representations of degree 13 and 12, respectively, and B' has no such representation, it follows that we must have $(\text{Aut } M)^0 = E_6(\mathbb{C})$. Thus B' is a subgroup of $E_6(\mathbb{C})$ and M is the Lie algebra of type E_6 .

Step 4. Conclusion. Let ρ be the irreducible character of $B = {}^2F_4(2)$ of degree 27 as given in [22] under χ_4 with $\rho(4D) = -1 - 4i$. Then $(\rho^{2+}, \bar{\rho}) = (\rho^{2+}|_L, \rho|_L) = 1$, where $L = B'$, so B preserves the unique cubic form with stabilizer $\tilde{E}_6(\mathbb{C})$ that its commutator subgroup L preserves (by the previous step). Consequently, $B = {}^2F_4(2)$ embeds in $\tilde{E}$.

REMARK. There is one conjugacy class of these groups as for each there is a unique form fixed by the group which must be the E_6 form. (Any two are conjugate by an element in $\text{GL}(27, \mathbb{C})$ which fixes the form and so lies in $E_6(\mathbb{C})$.)

6.2. The group $B = \text{Aut}^3 D_4(2)$ embeds in F with $\psi|_L = 1_a + 26_a$.

Proof. See [14] or [39].

6.3. The group $L = L(3, 3)$ embeds in F with $\psi|_L = 1_a + 26_a$. In [21], its image is shown to normalize an elementary abelian subgroup of F of order 27. In particular, the Main Theorem holds for such a subgroup L of F .

Proof. The existence of $3^3 : \text{SL}(3, 3)$ is observed in [1]; a construction was given in [32]. An explicit construction runs as follows: take $\bar{r}_1 = s_{\text{id}, \text{id}, \bar{\omega}}$, $\bar{r}_2 = s_{\sigma, \sigma, \sigma}$, where

σ is the permutation matrix $(3, 2, 1)$, and $\bar{r}_3 = s_{\tau, \tau, \tau}$, where τ is the diagonal matrix with entries $(1, \omega, \bar{\omega})$. Thus, for $x = (x_1, x_2, x_3) \in \mathbf{K}$,

$$\bar{r}_1(x) = (x_1, \omega x_2, \bar{\omega} x_3), \quad \bar{r}_2(x) = (x_1^{(123)}, x_2^{(123)}, x_3^{(123)}), \quad \bar{r}_3(x) = (x_1^\omega, x_2^\omega, x_3^\omega)$$

with the understanding that, for a 3×3 -matrix x ,

$$\omega x = \begin{pmatrix} \omega x_{11} & \omega x_{21} & \omega x_{31} \\ \omega x_{12} & \omega x_{22} & \omega x_{32} \\ \omega x_{13} & \omega x_{23} & \omega x_{33} \end{pmatrix}, \quad x^{(123)} = \begin{pmatrix} x_{22} & x_{23} & x_{21} \\ x_{32} & x_{33} & x_{31} \\ x_{12} & x_{13} & x_{11} \end{pmatrix},$$

and

$$x^\omega = \begin{pmatrix} x_{11} & \bar{\omega} x_{12} & \omega x_{13} \\ \omega x_{21} & x_{22} & \bar{\omega} x_{23} \\ \bar{\omega} x_{31} & \omega x_{32} & x_{33} \end{pmatrix}.$$

Now, $\bar{U} = \langle \bar{r}_1, \bar{r}_2, \bar{r}_3 \rangle$ is elementary abelian of order 3^3 contained in F , and all of its non-trivial elements are of type 3C. It is readily seen that its normalizer in F is finite, and that it coincides with its centralizer in F . We shall explicitly describe its normalizer in F . We shall find that it is of type $3^3 : L(3, 3)$ thereby establishing that it is the subgroup found by [1], and appearing in the conclusion of Theorem 3.5.

We first give a basis of $\bar{U}$ -eigenvectors e_a ($a \in \mathbf{F}_3^3$) of $\mathbf{K}$:

$$e_a = e_{(i,j,k)} = \omega^{(i+1)jk} (e_{1,1-k}^{i+1} + \omega^j e_{2,2-k}^{i+1} + \omega^{-j} e_{3,3-k}^{i+1}) \quad (a = (i, j, k) \in \mathbf{F}_3^3);$$

here indices are taken modulo 3 with values in $\{1, 2, 3\}$.

The following two elements of F permute this basis up to multiplications by ω :

$$\begin{aligned} \bar{b} &= (e_{(0,0,1)}, e_{(0,0,2)})(e_{(0,1,0)}, e_{(2,1,0)}, e_{(1,0,0)}, e_{(1,1,0)}, e_{(0,2,0)}, e_{(1,2,0)}, e_{(2,0,0)}, e_{(2,2,0)}) \\ &\quad (e_{(1,0,1)}, e_{(1,1,2)}, e_{(0,2,1)}, e_{(1,2,2)}, e_{(2,0,1)}, e_{(2,2,2)}, e_{(0,1,1)}, e_{(2,1,2)}) \\ &\quad (e_{(2,0,2)}, e_{(2,2,1)}, e_{(0,1,2)}, e_{(2,1,1)}, e_{(1,0,2)}, e_{(1,1,1)}, e_{(0,2,2)}, e_{(1,2,1)}), \\ \bar{l} &= (e_{(0,0,2)}, e_{(1,0,0)}, e_{(0,1,0)})(e_{(0,0,1)}, e_{(2,0,0)}, e_{(0,2,0)})(e_{(1,0,1)}, e_{(1,1,0)}, e_{(0,1,1)}) \\ &\quad (e_{(2,0,2)}, e_{(2,2,0)}, e_{(0,2,2)})(e_{(1,0,2)}, e_{(2,1,0)}, e_{(0,2,1)})(e_{(2,0,1)}, e_{(1,2,0)}, e_{(0,1,2)}) \\ &\quad (e_{(1,1,2)}, \omega e_{(2,1,1)}, \bar{\omega} e_{(1,2,1)})(e_{(2,2,1)}, \bar{\omega} e_{(1,2,2)}, \omega e_{(2,1,2)}) \\ &\quad [e_{(1,1,1)} \mapsto \bar{\omega} e_{(1,1,1)}][e_{(2,2,2)} \mapsto \omega e_{(2,2,2)}]. \end{aligned}$$

These elements induce an action on $\bar{U}$ given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{respectively,}$$

with respect to the basis $\{\bar{r}_i\}$ if $\bar{U}$ is viewed as a vector space over $\mathbf{F}_3$.

It is straightforward (and tedious) to check that the following relations are satisfied:

$$\bar{b}^8 = \bar{l}^3 = (\bar{b}\bar{l})^{13} = (\bar{b}^4\bar{l})^3, \quad \bar{b}\bar{l}\bar{b}^2\bar{l}\bar{b}^3\bar{l}\bar{b}^4\bar{l}^2 = 1.$$

By enumerating cosets over the subgroup $\langle \bar{b}^4, \bar{l} \rangle$ of order 12 (using a computer) we find that these relations provide a presentation for $L(3, 3)$. Consequently, the group $\langle \bar{b}, \bar{l} \rangle$ is isomorphic to $L(3, 3)$.

6.4. The group $L = L(3, 3)$ embeds in E with $\psi|_L = 27_a$.

Proof. Let B be a subgroup of E isomorphic to ${}^2F_4(2)$; cf. § 6.1. By [44], [22], or [43], there is a subgroup isomorphic to $\text{Aut } L(3, 3)$ in B ; in [32], the existence of $L(3, 3)$ as a subgroup of ${}^2F_4(2)'$ is established directly (without the use of computers), as part of a general method. From consideration of the restriction of ψ to a subgroup of B isomorphic to $L(3, 3)$, it follows that any subgroup of B isomorphic to $L(3, 3)$ is irreducible in each of the two 27-dimensional projective representations of $\tilde{E}$.

6.5. The group $L = L(2, 27)$ embeds in F with $\psi|_L = 1_a + 26_f$.

Proof. Set $L = L(2, 27)$. For the time being, let $\mathbf{F}$ be an arbitrary field containing a third root of unity ω , and let $\mathbf{K}$ be the 27-dimensional vector space with cubic form $\langle \cdot \rangle$ as given in § 2, but now over $\mathbf{F}$ instead of $\mathbb{C}$. We shall construct an embedding of L in $F_4(\mathbf{F})$ for the case $\mathbf{F} = \mathbf{F}_{547}$, from which, by the facts reported in the beginning of this section, and the fact that 547 does not divide $|L|$, it follows that L can be embedded in $F \cong F_4(\mathbb{C})$.

Denote by u_a and h_b , where $a, b \in \mathbf{F}_{27}$ with $b \neq 0$, the elements of L determined by

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \quad \text{respectively.}$$

The group L has a Borel subgroup $B = UH$, where $U = \{u_a \mid a \in \mathbf{F}_{27}\}$ and $H = \{h_b \mid b \in \mathbf{F}_{27}^*\}$. The normalizer $N_L(H)$ in L of H is the subgroup $H\langle w \rangle$, where $w \in L$ is given by

$$w = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We select specific elements generating L . To this end, let α be a generator of $\mathbf{F}_{27}$ satisfying $\alpha^3 = \alpha^2 - \alpha - 1$. Then the distinguished generating set consists of

$$r_1 = u_1, \quad r_2 = u_\alpha, \quad r_3 = u_{\alpha^3}, \quad t = h_\alpha^8, \quad \text{and} \quad w.$$

The following relations on these five elements are actually defining relations:

$$[r_i, r_j] = r_i^3 = 1 \quad (1 \leq i \leq j \leq 3), \quad (1)$$

$$t^{13} = 1, \quad tr_1t^{-1} = r_3^2, \quad tr_2t^{-1} = r_1r_2^2, \quad tr_3t^{-1} = r_1r_2, \quad (2)$$

$$w^2 = 1, \quad wtw = t^{-1}, \quad (3)$$

$$wr_1w = r_1^2wr_1^2, \quad wr_2w = t^5r_2^2wr_1^2r_3, \quad wr_3w = t^2r_3^2wr_2^2r_3^2. \quad (4)$$

To see that these are actual defining relations, note that, with $U = \langle r_1, r_2, r_3 \rangle$, each element outside the subgroup $B = U\langle t \rangle$ can be written in the form uwb for some $u \in U$ and $b \in B$.

We shall construct a 27-dimensional representation of L which decomposes into the trivial and an irreducible 26-dimensional representation with character ξ such

that

$$\begin{aligned}\xi(w) &= 2, & \xi(r_i) &= -1, & \xi(t) &= 0, & \xi(r_1w) &= -1, \\ \xi(r_2w) &= -y, & \xi(r_3w) &= -y^3 + 3y,\end{aligned}\tag{5}$$

where y is a solution of $y^3 + y^2 - 2y + 1 = 0$. (Thus $y = \psi + \psi^{-1}$ where ψ is a primitive 7th root of 1.) We observe that a representation with this character lifts to a characteristic zero representation with character $1_a + 26_d$ in the notation of [22].

First consider $U = \langle r_1, r_2, r_3 \rangle$. Define $\bar{r}_i$ ($1 \leq i \leq 3$) as in § 6.3, taking $\omega \in \mathbf{F}$. Then $\bar{U} = \langle \bar{r}_i \mid 1 \leq i \leq 3 \rangle$ is an elementary abelian group of order 27 all of whose non-trivial elements are conjugate in F , so the $\bar{r}_i$ satisfy (1) (with $\bar{r}_i$ replacing r_i). Moreover, $N_F(\bar{U})/\bar{U} \cong L(3, 3)$ as we have seen above (there is no obstruction from taking the particular field $\mathbf{F}$ since the transformations $\bar{b}$ and $\bar{l}$ of § 6.3 are defined over $\mathbf{F}$ by specifying ω as indicated). Clearly, the map $u \mapsto \bar{u}$ on the generators r_i of U extends to an isomorphism $U \rightarrow \bar{U}$. We proceed to extend this map to B .

Since the normalizer in F of U permutes the twenty-seven 1-dimensional eigenspaces of U , it is not hard to find an element $\bar{t} \in N_F(U)$ of order 13 satisfying the relations (2) with $\bar{r}_i$ instead of r_i and $\bar{t}$ instead of t . In fact the element $\bar{t} = \bar{b}\bar{l}$ from § 6.3 suffices. This element induces an action on $\bar{U}$ given by the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

and so $\bar{t}, \bar{r}_i$ satisfy (2). It now follows directly that the map $u \mapsto \bar{u}$ extends to an isomorphism $B \rightarrow \bar{B} = \bar{U} \cdot \langle \bar{t} \rangle$.

It remains to find the appropriate image $\bar{w}$ of w . To this end, we start by determining the eigenspaces of $\bar{t}$ on $\mathbf{K}$. Each eigenspace of $\bar{t}$ on the subspace of $\mathbf{K}$ orthogonal to e has dimension 2, and contains precisely two singular 1-spaces. It is straightforward to compute the eigenspaces of $\bar{t}$ (using the Reynolds operator $x \mapsto \sum_{i=1}^{i=|t|} \zeta^{-i} \bar{t}^i x$). In each of these eigenspaces Y , the two singular points are found as solutions of the quadratic equations $x^\# = 0$ for $x \in Y$. The twenty-six 1-spaces so obtained, together with the 1-space $\mathbf{F}e$, are the twenty-seven eigenspaces of the maximal torus $C_F(\bar{t})$. Hence, by identifying the mutual geometric positions of the twenty-seven 1-spaces just found (that is, the graph on these twenty-seven points in which two points are adjacent whenever the points span a singular 2-space) with those of the 1-spaces from the basis $e_{j,k}^i$, we can find an element $v \in F$ mapping each $e_{j,k}^i$ to a vector in one of these 1-spaces for each $i, j, k \in \{1, 2, 3\}$.

At this point we specify the field $\mathbf{F}$ as $\mathbb{Z}/547$ and let ω be 40. This field was chosen because it contained the appropriate cube, seventh, and thirteenth roots of unity. The computations involved were carried out on a desktop computer using the programming language ABC; cf. [30]. The outcome of these computations is the matrix opposite for v with respect to the basis f_a ($1 \leq a \leq 27$) given by $e_{jk}^i = f_a$ if $a = i + 3(j-1) + 9(k-1)$.

The computations regarding v were later repeated in Maple, and can easily be repeated in similar packages, such as Mathematica. What is needed beyond a linear algebra package is an implementation of the Jordan algebra multiplication.

465	206	513	185	381	77	308	406	190	170	191	325	96	5	419	359	338	408	337	266	327	160	222	322	470	144	146
523	459	317	430	169	421	29	123	432	457	176	390	381	324	393	463	355	231	540	353	222	372	495	138	138	170	79
503	442	467	0	409	337	79	76	354	340	298	207	62	541	362	381	225	427	211	479	544	433	459	425	490	109	530
41	516	229	165	365	380	221	146	117	196	457	343	377	329	173	497	111	285	237	341	455	302	263	506	303	156	526
511	504	192	180	486	57	347	474	356	3	136	178	323	490	0	284	322	355	124	109	63	126	400	544	480	150	317
219	128	4	273	68	345	405	64	452	476	508	104	239	364	137	218	191	490	346	513	395	70	529	51	410	536	194
337	465	140	245	3	521	29	310	106	383	122	317	258	179	508	160	452	206	467	226	52	343	328	281	41	269	518
396	254	422	296	458	276	216	451	173	184	98	189	412	373	524	456	384	329	444	312	447	324	307	452	56	447	526
361	290	263	507	461	291	332	494	388	284	486	381	424	35	541	532	474	63	358	190	503	399	269	381	450	533	462
209	200	57	81	252	327	71	444	396	280	0	471	53	366	324	80	301	45	130	532	485	406	116	176	45	537	259
396	104	137	173	545	12	262	262	111	293	79	31	412	233	254	327	421	136	304	431	504	393	108	476	56	351	500
150	518	160	137	40	502	146	357	462	72	182	214	446	507	227	99	483	51	283	194	471	144	393	484	30	257	116
168	261	3	303	283	33	545	137	84	428	382	99	526	348	192	135	386	87	18	340	254	498	75	168	271	369	122
538	433	480	316	87	426	174	365	307	290	427	380	50	164	418	444	87	122	303	206	434	124	64	42	96	213	286
9	380	517	177	112	207	462	144	12	260	503	249	497	467	245	260	403	376	447	426	528	145	180	444	451	399	137
142	301	10	293	546	313	114	238	259	369	382	522	185	543	354	189	20	518	526	315	347	270	32	15	40	42	24
186	118	314	271	106	19	333	507	438	426	303	286	123	307	163	346	174	412	98	456	49	105	209	477	97	78	355
8	210	292	4	414	105	476	370	5	163	386	262	261	318	511	403	510	415	469	355	195	35	272	463	537	264	487
142	484	108	110	83	488	531	534	354	410	306	296	185	538	217	201	323	391	259	60	373	180	359	451	40	22	401
186	89	1	343	445	287	82	87	183	266	175	125	123	406	23	21	95	83	199	71	149	2	253	322	97	322	123
219	371	279	380	117	261	419	75	41	272	195	275	239	251	33	67	330	363	287	220	421	248	543	292	410	357	130
337	214	495	96	374	398	116	521	120	458	56	50	258	7	269	370	376	461	326	44	450	200	35	278	41	174	512
174	460	402	48	503	225	293	365	417	429	524	75	65	529	422	317	99	383	467	70	261	166	92	173	439	185	457
361	151	66	391	369	426	541	93	418	187	479	441	424	111	325	512	192	368	256	237	40	165	57	306	450	327	480
100	80	31	59	430	437	241	4	273	496	521	123	422	194	483	53	370	52	192	488	513	436	250	57	352	34	279
369	438	399	49	457	359	434	290	443	415	420	158	138	430	87	10	102	167	497	311	366	492	282	544	440	195	206
360	518	422	499	544	355	458	187	527	93	245	190	188	231	441	175	52	26	482	476	327	100	105	372	536	459	300

The matrix for v as in § 6.5

By checking that $(vf_a) \times (vf_b) = v(f_a \times f_b)$ for $1 \leq a \leq b \leq 27$, it is verified that v belongs to F .

Now $\bar{t} = v\bar{h}v^{-1}$, where $\bar{h} = h(46, 1, 46, 1, 353, 261)$. We need to find an involution $\bar{w} \in \bar{E}$ inverting $\bar{t}$ and satisfying (4). First, we note that the element $\bar{w}_0 \in F$ given by

$$\bar{w}_0(x) = (x_1^\top, x_3^\top, x_2^\top) \quad (x = (x_1, x_2, x_3) \in \mathbf{K})$$

is an involution of F inverting the conjugate $\bar{h}$ of $\bar{t}$. Thus, the element $\bar{w}$ we are looking for must be in $v\bar{w}_0v^{-1} \cdot C$, where $C = C_F(\bar{t})$. As this is a maximal torus in F , it coincides with vTv^{-1} , where $T = H \cap F$ is the maximal torus of F described in § 2. Hence $\bar{w} = v\bar{w}_0h(\alpha, \beta, \alpha, 1, \varepsilon, \zeta)v^{-1}$ for certain $\alpha, \beta, \varepsilon, \zeta \in \mathbf{F}^*$. To determine $\alpha, \beta, \varepsilon, \zeta$, note that they are a solution of the equations

$$\text{trace}(v^{-1}\bar{u}_a\bar{w}_0vh(\alpha, \beta, \alpha, 1, \varepsilon, \zeta)) = 1 + \xi(u_a w) \quad (a \in \mathbf{F}_{27}). \quad (6)$$

Here, the right-hand side is known from (5). Now, taking $y = 54$ (and using computers), we found that $\alpha = 348, \beta = 206, \varepsilon = 371, \zeta = 232$ is a solution to (5). Finally, straightforward computations show that

$$\bar{w} = \bar{w}_0vh(348, 206, 348, 1, 371, 232)v^{-1}$$

satisfies (the ‘barred’ version of) equations (4). Thus, $w \mapsto \bar{w}$ determines an extension of the map $z \mapsto \bar{z}$ on B to an isomorphism $L \rightarrow \bar{L} = \langle \bar{B}, \bar{w} \rangle$, and hence to an embedding of L in F .

6.6. *The group $B = L(2, 25) \cdot \sigma_{fg}$ embeds in F with $\psi|_L = 1_a + 26_a$.*

Proof. By [44], [24], [43], or [22], there is a subgroup L of ${}^2F_4(2)'$ isomorphic to $L(2, 25)$. The restrictions to L of the characters χ and ψ of ${}^2F_4(2)'$ (which are in turn restrictions of the 78- and a 27-dimensional non-trivial representation of $\bar{E}$), are readily seen to be $\psi|_L = 1 + 26_a$ and $\chi|_L = 26_a + 26_a + 26_e$. Consequently, L stabilizes a non-zero vector in the 27-dimensional representation of ${}^2F_4(2)'$ corresponding to ψ . Since F is the only vector stabilizer in $\bar{E}$ with an irreducible complement of dimension 26 (cf. Theorem 5.1), it follows from the decomposition of $\psi|_L$ that $L \cong L(2, 25)$ is contained in $F_4(\mathbb{C})$. The existence of $L(2, 25) \cdot \sigma_{fg}$ as a subgroup of $F_4(2)$ (see [22]) shows that $L(2, 25) \cdot \sigma_{fg}$ is also in $F_4(\mathbb{C})$.

6.7. *The group $L = L(2, 19)$ embeds in E with $\psi|_L = 9_a + 18_e$.*

Proof. Let u, t, w be the elements of $L(2, 19)$ corresponding to the matrices over $\mathbf{F}_{19}$:

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 2 & 0 \\ 0 & 9 \end{pmatrix}.$$

We shall construct an embedding $L(2, 19) \rightarrow \bar{E}$ with image L . Denote by $\bar{u}, \bar{w}, \bar{t}$ the images of u, w , and t , respectively. By Lemma 3.3(i), up to $\bar{E}$ -conjugacy, we may take $\bar{u} \in H$ and $\bar{t} \in N_{\bar{E}}(H)$. Now $\bar{t}$ is of order 9 and as the Weyl group has a unique conjugacy class of groups of order 9, there is a unique choice. It acts without fixed points on the twenty-seven lines and so its centralizer in the torus is finite. By Lang’s theorem, $\bar{t}H$ is a single H -conjugacy class. Thus, again up to $\bar{E}$ -conjugacy, we may take $\bar{t} = n_1n_5n_3n_4n_3n_2n_4n_6$. As $\bar{u} = h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ for certain $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{C}^*$, we see that

$$\bar{t}\bar{u} = h(\beta\gamma\varepsilon, \beta^{-1}\delta\varepsilon^{-2}, \alpha^{-1}\delta^{-1}\varepsilon, \alpha\varepsilon^{-1}, \beta\delta^{-1}\varepsilon^2\zeta, \varepsilon^{-1}\zeta^{-1}).$$

Now the relation $tut^{-1} = u^4$ forces $\alpha = \gamma^{14}, \beta = \gamma^{15}, \gamma = \gamma, \delta = \gamma^3, \varepsilon = \gamma^2$, and $\zeta = \gamma^{11}$. Now let $\phi = \gamma$ and $\bar{u} = h(\phi^{14}, \phi^{15}, \phi, \phi^3, \phi^2, \phi^{11})$.

The element $\bar{u}$ may be replaced by an even (non-trivial) power due to the $\bar{t}$ -action. Replacing it by an odd power corresponds to replacement by its inverse, which corresponds to replacing $\tilde{E}$ by its complex conjugate. Thus, we have fully determined the subgroup $\langle \bar{t}, \bar{u} \rangle$ of $\tilde{E}$ which is isomorphic to the Borel subgroup of $L(2, 19)$. (It is unique up to conjugacy in $\text{Aut } \tilde{E}$.)

Next consider $\bar{w}$. Set $\bar{w}_0 = n_1 n_2 n_5$. This is an element of $\tilde{E}$ inverting $\bar{t}$. Now $\bar{w} \in \bar{w}_0 C$, where C is the centralizer in $\tilde{E}$ of $\bar{t}$. Denote by v an element conjugating $\bar{t}$ to an element of H :

$$v^{-1}tv = h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta).$$

Then, it is clear that $C = vH\{s_{1,1,g} \mid g \in A\}v^{-1}$ where A is the subgroup of $\text{SL}(3, \mathbb{C})$ fixing the decomposition $\langle \varepsilon_1 \rangle \oplus \langle \varepsilon_2, \varepsilon_3 \rangle$ if ε_i denotes the standard basis. Denote by u_1 and w_1 the element $s_{1,1,g}$ with g acting trivially on ε_1 and as follows on the basis $\varepsilon_2, \varepsilon_3$:

$$u_1: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then every element of $v^{-1}Cv$ can be written in the form $h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)u_1^\xi w_1^\sigma u_1^\eta$ for certain non-zero complex numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, \eta \in \mathbb{C}$, and $\sigma \in \{0, 1\}$. Setting $\bar{w} = \bar{w}_0 v h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) u_1^\xi w_1^\sigma u_1^\eta v^{-1}$, we must have

$$\text{trace}(\bar{u}^a \bar{w}) = \text{trace}(v^{-1} \bar{u}^a \bar{w}_0 v h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) u_1^\xi w_1^\sigma u_1^\eta) = \chi(u^a w) \quad (7)$$

for each a ($0 \leq a \leq 18$). The right-hand side is known from the character table of $L(2, 19)$ and the fact that $\chi|_L = 9_a + 18_c$ (cf. Theorem 8.1 below).

Set $p = 6841$. As for $L(2, 27)$, we proceed modulo p . (Again, an embedding in $E_6(\mathbb{F}_p)$ implies the existence of an embedding in $\tilde{E}$ since p does not divide the order of $L(2, 19)$.) We take $\gamma = 5832$, and 19th root of 1 in $\mathbb{F}_p$. Then $\bar{u} = h(2361, 5260, 5832, 831, 5613, 2596)$. Now the matrix v , shown on p. 128, with the same convention as for § 6.5, puts $\bar{t}$ in diagonal form. The computations involved (including the verification that v belongs to F) were carried out as described in § 6.5. The same holds for the computation of solutions to equations (7) over $\mathbb{F}_p$.

Using that $\bar{w}^2 = 1$ (which implies in particular that $\gamma = \alpha$ and $\delta = 1$), we found, for $\sigma = 1$, a unique solution, namely $\alpha = \gamma = 1492$, $\beta = 631$, $\delta = 1$, $\varepsilon = 2146$, $\zeta = 818$, $\xi = 4372$, and $\eta = 1744$. Finally, using straightforward matrix computations (performed by computer), we verified the relations $\bar{w}^2 = 1$, $(\bar{u}\bar{w})^3 = 1$, and $(\bar{u}^3\bar{w})^9 = 1$. As can be shown by a Todd–Coxeter enumeration, together with $\bar{t}^9 = \bar{u}^{19} = (\bar{w}\bar{t})^2 = 1$ and $\bar{t}\bar{u}\bar{t}^{-1} = \bar{u}^4$, they provide a full set of defining relations for $L(2, 19)$. Since all these relations hold, the conclusion is that $\langle \bar{u}, \bar{w}, \bar{t} \rangle \cong L(2, 19)$.

6.8. THEOREM. *The group $L = L(2, 17)$ embeds in F with $\psi|_L = 1_a + 9_a + 17_a$.*

Proof. Let u, t, w be the elements of $L(2, 17)$ corresponding to the respective matrices:

$$u \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t \leftrightarrow \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}, \quad w \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1402	5652	2967	6485	5528	5455	1999	2363	2938	4461	0	5094	6722	1906	3159	3006	153	3889	6760	2779	6181	285	4042	4866	5746	4998	984
1001	4816	4215	3343	2438	4551	3046	6289	74	6754	2785	6724	5487	2576	1363	2140	0	748	4686	5863	892	3376	6618	4034	6818	4506	6503
1001	4474	3632	2206	2242	1983	1918	2984	4871	1980	451	6073	5487	2037	2664	1979	0	2521	5951	6211	2421	5023	4973	850	6818	6544	2364
6719	3925	2357	2026	5025	2727	1734	1153	5562	3983	823	4444	4233	0	2829	5479	1604	1703	3611	1801	635	2347	5958	2703	4891	2856	4549
6719	5348	1002	678	5923	5797	5121	1897	3357	6070	740	579	4233	0	2026	1739	1206	3188	1904	1370	6210	2493	3187	3561	4891	2335	2227
6719	1329	5005	4137	1937	4023	6827	3950	687	3629	1017	6556	4233	0	5230	6464	5378	1105	1326	5858	1822	2001	6124	3798	4891	6215	2064
1001	3354	790	3343	6340	220	3046	4502	2562	6754	3039	6673	5487	1794	3185	2140	0	197	4686	3961	930	3376	3876	5617	6818	5337	1795
5439	2367	3209	862	4599	6487	5571	3857	4778	5078	0	768	119	2037	1233	946	4258	2356	1695	2880	4420	168	1068	4336	1095	4555	919
5439	304	1965	356	6446	6096	4842	466	2596	2380	0	4515	119	4455	854	3835	1359	5965	81	2149	6812	6556	6124	276	1095	3348	292
6719	3487	6051	4137	501	725	6827	2339	6220	3629	3039	168	4233	0	3398	6464	3337	1869	1326	3500	5911	2001	4759	3378	4891	329	3392
5439	3925	2357	862	5025	2441	5571	1153	640	5078	0	4444	119	820	4890	946	5237	4084	1695	978	635	168	2586	3997	1095	4611	4427
6719	304	1965	678	6446	3237	5121	466	6399	6070	1004	4515	4233	0	4137	1739	5482	3876	1904	3153	6812	2493	2586	4587	4891	1504	742
6719	1189	3874	678	1313	4648	5121	4478	3926	6070	5097	1747	4233	0	678	1739	153	6618	1904	2318	660	2493	1068	5534	4891	3002	3872
6719	2025	2626	4137	4403	2093	6827	552	6775	3629	2785	117	4233	0	5054	6464	4967	3867	1326	4324	5949	2001	2799	6506	4891	297	1385
1001	2916	4484	2206	1816	5168	1918	5688	6202	1980	823	2397	5487	820	878	1979	0	5787	5951	2779	6206	5023	1703	6700	6818	626	2135
122	6292	5566	4815	2783	5064	5107	5010	5655	2858	1274	5212	2608	0	4260	1362	4187	1868	3230	5132	5055	4494	3022	6058	1950	4845	5994
1001	1493	5839	1292	918	5156	1877	4944	3699	4948	740	6262	5487	4292	4331	2722	0	1112	3045	2000	631	5283	1105	1310	6818	3985	2919
5439	1329	5005	5623	1937	4888	3269	3950	3518	6224	0	6556	119	2471	3904	2060	1463	4791	5065	4841	1822	117	3905	233	1095	4465	3320
6719	2367	3209	2026	4599	2337	1734	3857	93	3983	451	768	4233	0	1431	5479	2583	165	3611	3331	4420	2347	3905	3355	4891	1989	1445
1402	1493	5839	6485	918	641	1999	4944	6499	4461	0	6262	6722	2549	4536	3006	1206	2076	6760	6211	631	285	2082	2251	5746	5191	6149
5840	1329	5005	3498	1937	4771	3795	3950	2636	87	5824	6556	1354	4370	4548	4701	0	945	2155	2983	1822	3465	3653	2810	23	3002	1457
1402	3354	790	1218	6340	505	3572	4502	3658	617	0	6673	6722	5047	213	4781	3337	1672	1776	6380	930	6724	883	4602	5746	1833	1597
1402	6292	5566	5979	2783	2087	1270	5010	5418	1763	0	5212	6722	2857	6123	5895	2654	6440	5146	3858	5055	6673	3654	1492	5746	2325	5346
5840	304	1965	5549	6446	4349	4964	466	4687	1893	5837	4515	1354	2386	1850	4119	0	3841	3796	1539	6812	1558	4972	1978	23	1989	5807
1001	5652	2967	1292	5528	6034	1877	2363	988	4948	5097	5094	5487	4935	4360	2722	0	2729	3045	6380	6181	5283	3867	668	6818	4845	2888
1402	4816	4215	1218	2438	4383	3572	6289	6701	617	0	6724	6722	4265	3691	4781	4967	3119	1776	5302	892	6724	3022	2472	5746	2632	1723
1001	6292	5566	2206	2783	6531	1918	5010	2609	1980	5567	5212	5487	3984	3299	1979	0	5374	5951	4692	5055	5023	165	6132	6818	6512	2342

The matrix for v as in § 6.7

Observe that the relations $t^8 = u^{17} = w^2 = 1$ and $t^{-1}ut = u^9$ hold. We construct an $L(2, 17)$ by first finding a copy $\langle \bar{u}, \bar{t} \rangle$ of the Borel subgroup $\langle u, t \rangle$ in F and then adjoining an involution $\bar{w}$ corresponding to w . The following element $\bar{t}$ of order 8, given on a basis of § 2, is in $\bar{E}$ as a check of the trilinear form will show:

$$\begin{aligned} X_4 &\rightarrow X_4, & X_{3'} &\leftrightarrow -X_{34}, \\ X_{1'} &\rightarrow X_{46} \rightarrow X_{2'} \rightarrow X_{45} \rightarrow X_{14} \rightarrow -X_{6'} \rightarrow X_{24} \rightarrow -X_{5'}, \\ X_{13} &\rightarrow -X_{25} \rightarrow -X_5 \rightarrow -X_{4'} \rightarrow X_1 \rightarrow X_{16} \rightarrow X_{35} \rightarrow X_3, \\ X_2 &\rightarrow X_{15} \rightarrow X_6 \rightarrow X_{12} \rightarrow X_{23} \rightarrow X_{26} \rightarrow -X_{36} \rightarrow -X_{56}. \end{aligned}$$

There is a unique class in $W(E_6)$ of elements of order 8 and so as an element of $N_{\bar{E}}(H)$, this is a representative. As $\langle u, t \rangle$ is a Frobenius group of order $17 \cdot 8$, by Lemma 3.3 any embedding in F is F -conjugate to one in $N_F(H \cap F)$, and so we may assume that it is $\bar{t}$ up to power and ignoring the scale factors on the individual vectors.

As for the image $\bar{u}$ in $N_{\bar{E}}(H)$ of u , we must have $\bar{u} \in H$ and $\bar{t}^{-1}\bar{u}\bar{t} = \bar{u}^9$. Let ε be a primitive 17th root of 1. Suppose that $X_{1'}$ has eigenvalue ε and X_{13} has eigenvalue ε^α . Then as $\bar{u}\bar{t} = \bar{t}\bar{u}^9$, the eigenvalues for $\bar{u}$ acting on $\mathbf{K}$ of $X_i, X_{i'}, X_{ij}$ are as follows where i below an X denotes the eigenvalue ε^i :

X_4	$X_{3'}$	X_{34}						
0	0	0						
$X_{1'}$	X_{46}	$X_{2'}$	X_{45}	X_{14}	$-X_{6'}$	X_{24}	$-X_{5'}$	
1	2	4	8	16	15	13	9	
X_{13}	$-X_{25}$	$-X_5$	$-X_{4'}$	X_1	X_{16}	X_{35}	X_3	
α	2α	4α	8α	16α	15α	13α	9α	

As $\{X_{1'}, X_{13}, X_3\}$ is a special triple, that is, a triple of singular points of a special plane, $1 + \alpha + 9\alpha \equiv 0 \pmod{17}$ and so $\alpha = 5$. Now as $\langle X_{1'}, X_5, X_{15} \rangle \neq 0$, X_{15} has eigenvalue 13 and $X_{1'}$, X_5 , and X_{15} must have eigenvalues ε , ε^3 , and ε^{13} , respectively. The element $\bar{u}$ is now completely determined. Note that if $\bar{u}$ is an element of order 17 normalized by $\bar{t}$ with a fixed space of dimension 3 as in $L(2, 17)$, then we may take powers if necessary so that $X_{1'}$ is an eigenvector with eigenvalue ε . Thus, without loss of generality, we may assume that

$$\bar{u} = h(\varepsilon^{10}, \varepsilon^8, \varepsilon^{14}, \varepsilon, \varepsilon^7, \varepsilon^9).$$

Clearly $\mathbb{C}X_4 + \mathbb{C}(X_{3'} - X_{34})$ is a 2-dimensional space fixed by $\langle \bar{u}, \bar{t} \rangle$ and the vector $X_{3'} + X_{34}$ is fixed by $\langle \bar{u}, \bar{t}^2 \rangle$ and negated by $\bar{t}$. The $L(2, 17)$ acts on $\mathbf{K}$ with invariant subspaces of dimensions 1, 9, 17. Let $\langle v_* \rangle$, W_9 , V_{17} be these spaces. Because v_* is fixed by $\langle \bar{u}, \bar{t} \rangle$, we know that $v_* = \alpha X_4 + A(X_{3'} - X_{34})$. As v_* cannot be parabolic or singular since the $L(2, 17)$ is in F_4 , we know that α and A are non-zero. We may assume $\alpha = 1$ and $v_* = X_4 + A(X_{3'} - X_{34})$. The space V_{17} is spanned by eigenvectors for $\bar{u}$ with all possible eigenvalues. Let $v_0 = KX_4 + B(X_{3'} - X_{34})$, $v_1 = CX_{1'} + DX_{12}$, and $v_3 = -FX_5$ for certain $K, B, C, D, F \in \mathbb{C}$. Then the v_i are eigenvectors for $\bar{u}$ with eigenvalues ε^i ($0 \leq i \leq 2$). Acting by $\bar{t}$ gives the remaining basis vectors for V_{17} . Let $w_0 = X_{3'} + X_{34}$ and $w_1 = GX_{1'} + HX_{12}$, where $G, H \in \mathbb{C}$, be eigenvectors for $\bar{u} \in W_9$ with eigenvalues 0 and 1. Then W_9 is spanned by w_0 , w_1 and vectors $\bar{t}^i w_1$, for $2 \leq i \leq 7$.

Let h_α be the element $h(1, \alpha^3, \alpha^{-2}, \alpha, \alpha^{-1}, \alpha^2)$ of H . Writing out h_α on the X basis, we see that it centralizes $\langle \bar{t}, \bar{u} \rangle$. Since $\mathbb{C}$ is algebraically closed, we may assume that A is any element we wish except 0. We take $A = -12$.

We know $\langle v_*, v_*, v \rangle = 0$ for $v \in V_{17}$ as the action is irreducible and non-trivial on V_{17} . Expanding this, we obtain $0 = \langle X_4 + A(X_{3'} - X_{34}), X_4 + A(X_{3'} - X_{34}), KX_4 + B(X_{3'} - X_{34}) \rangle = 2A(KA + 2B)$. Consequently $B = 6K$. Since $v_0 \neq 0$, we have $K, B \neq 0$, so we can assume that $K = 1, B = 6$.

The F_4 we will be looking at is the stabilizer of v_* . The centralizer of $\bar{t}$ in the torus $H \cap \tilde{E}_{v_*}$ is h_{-1} as is easy to check since $X_4, X_{3'}, X_{34}$ must be fixed. As this is finite, Lang's theorem may be used to see that after conjugation by an element of the torus, the signs of $\bar{t}$ are as above.

We may now act on v_1, v_3, w_1 by $\bar{t}$ to obtain bases for V_{17} and W_9 . For reasons which will become apparent below we define $w_{i+1} = (-1)^i \bar{t}^i w_1$ for $0 \leq i \leq 7$. Here w_{i+1} is an eigenvector with eigenvalue ε^{2^i} .

The v_i are chosen as $\bar{t}^j v_1$ or $\bar{t}^j v_3$ for appropriate j so that v_i is an eigenvector with eigenvalue ε^i . In fact, $v_{2^i} = \bar{t}^{(i-1)} v_1$ and $v_{3 \cdot 2^i} = \bar{t}^{(i-1)} v_3$, where the 2^i is taken modulo 17. Thus, we obtain the following basis of $\mathbf{K}$, where the integer to the right of each vector is the power of ε which is the eigenvalue for that vector under $\bar{u}$:

$$\begin{aligned}
 v_* &= X_4 - 12(X_{3'} - X_{34}) & 0, \\
 v_0 &= X_4 + 6(X_{3'} - X_{34}) & 0, & w_0 &= X_{3'} + X_{34} & 0, \\
 v_1 &= CX_{1'} + DX_{12} & 1, & w_1 &= GX_{1'} + HX_{12} & 1, \\
 v_2 &= CX_{46} + DX_{23} & 2, & -w_2 &= GX_{46} + HX_{23} & 2, \\
 v_3 &= -FX_5 & 3, & w_3 &= GX_{2'} + HX_{26} & 4, \\
 v_4 &= CX_{2'} + DX_{26} & 4, & -w_4 &= GX_{45} - HX_{36} & 8, \\
 v_5 &= FX_{13} & 5, & w_5 &= GX_{14} - HX_{56} & 16, \\
 v_6 &= -FX_{4'} & 6, & -w_6 &= -GX_{6'} + HX_2 & 15, \\
 v_7 &= FX_{16} & 7, & w_7 &= GX_{24} + HX_{15} & 13, \\
 v_8 &= CX_{45} - DX_{36} & 8, & -w_8 &= -GX_{5'} + HX_6 & 9, \\
 v_9 &= -CX_{5'} + DX_6 & 9, \\
 v_{10} &= -FX_{25} & 10, \\
 v_{11} &= FX_3 & 11, \\
 v_{12} &= FX_1 & 12, \\
 v_{13} &= CX_{24} + DX_{15} & 13, \\
 v_{14} &= FX_{35} & 14, \\
 v_{15} &= -CX_{6'} + DX_2 & 15, \\
 v_{16} &= CX_{14} - DX_{56} & 16.
 \end{aligned}$$

It is our aim to determine C, D, H, G, F such that the linear transformation $\bar{w}$ given by its action on the above basis via the known representations of $L(2, 17)$ on $\mathbb{C}v_*$, W_9 , and V_{17} belongs to $\tilde{E}$. We shall find these parameters by identifying

the values of the trilinear form on triples from the above basis with the values on their images under $\bar{w}$.

We first describe the action of $\bar{w}$ on W_9 and V_{17} . The action of $L(2, 17)$ on V_{17} is that of the permutation representation on the cosets of $B = \langle t, u \rangle$ with the trivial action factored out. From this, identification of the v_j within that module yields

$$\bar{u}v_i = \varepsilon^i v_i, \quad \bar{t}v_i = v_{2i},$$

$$\bar{w}v_0 = \frac{1}{17} \left(-v_0 + 18 \sum_{i=1}^{16} v_i \right), \quad \text{and, for } i \neq 0, \quad \bar{w}v_i = \frac{1}{17} \left(v_0 + \sum_{l=1}^{16} a_{li} v_l \right),$$

where $a_{li} = \sum_{j=1}^{16} \varepsilon^{-ij} \varepsilon^{(-j)^{-1}l}$.

The action on W_9 is equivalent to the action on an invariant subspace of a certain induced module. Let (S) denote the non-zero squares in $\text{GF}(17)$ and (NS) the non-squares. Write

$$r = \sum_{i \in (\text{S})} \varepsilon^i - \sum_{j \in (\text{NS})} \varepsilon^j.$$

Then $r^2 = 17$. The action of $L(2, 17)$ is now given on the basis $(w_j)_{0 \leq j \leq 8}$ as follows:

$$\bar{u}w_0 = w_0, \quad \bar{u}w_i = \varepsilon^{2(i-1)} w_i \quad (1 \leq i \leq 8),$$

$$\bar{t}w_0 = -w_0, \quad \bar{t}w_i = -w_{i+1} \quad (1 \leq i \leq 7), \quad \bar{t}w_8 = -w_1,$$

$$\bar{w}w_0 = \frac{1}{17} \left(rw_0 + \sum_{i=1}^8 34w_i \right), \quad \bar{w}w_i = \frac{1}{17} \left(w_0 + \sum_{l=1}^8 b_{li} w_l \right),$$

where $b_{li} = \sum_{j=1}^{16} \varepsilon^{-2(i-1)j} \varepsilon^{(-j)^{-1}2^{l-1}} \varphi(j)$ and $\varphi(j) = 1$ if $j \in (\text{S})$ and $\varphi(j) = -1$ if $j \in (\text{NS})$.

We shall use the following relations between the coefficients:

$$\sum_{j=1}^8 b_{j1} a_{-2(j-1), 16} = r, \quad \sum_{j=1}^8 (-1)^{j-1} b_{j1} b_{1+((j+3) \bmod 8), 1} = -34r,$$

$$\sum_{j=1}^8 b_{1j} = -r, \quad \sum_{j=1}^8 (-1)^{j-1} b_{j1} a_{-2(j-1), 14} = -102.$$

The index $k = 1 + ((j+3) \bmod 8)$ occurring in the second row is the index for which the eigenvalue of $\bar{u}$ acting on w_k is the inverse of $\bar{u}$ acting on w_j .

For any $C, D, H, G, F \in \mathbb{C}^*$, we have an element $\bar{w} \in \text{GL}(\mathbf{K})$, and the group $\langle \bar{u}, \bar{t}, \bar{w} \rangle$ is isomorphic to $L(2, 17)$. The strategy is to take vectors y_1, y_2, y_3 in $\mathbf{K}$ and derive consequences from

$$\langle \bar{w}y_1, \bar{w}y_2, \bar{w}y_3 \rangle = \langle y_1, y_2, y_3 \rangle \quad (8)$$

for $\bar{w}$ to belong to $\tilde{E}$. We already know that $\langle gy_1, gy_2, gy_3 \rangle = \langle y_1, y_2, y_3 \rangle$ for $g \in \langle \bar{t}, \bar{w} \rangle$ as $\bar{t}$ and $\bar{w}$ are in $\tilde{E}$. Since $\bar{w}^2 = I$, a variant is to show that $\langle y_1, \bar{w}y_2, \bar{w}y_3 \rangle = \langle \bar{w}y_1, y_2, y_3 \rangle$. Equations using this are sometimes more tractable. Of course $\langle v_*, w, v \rangle = 0$ for any $w \in W_9$ and $v \in V_{17}$ as the representations are distinct.

By choosing appropriate y_i and using the relations above, we shall now derive equations for C, D, H, G, F .

From $\langle v_*, w_1, v_{16} \rangle = 0$ we obtain

$$12HD + GC = 0. \quad (9)$$

The equation $\langle \bar{w}w_0, w_1, v_{16} \rangle = \langle w_0, \bar{w}w_1, \bar{w}v_{16} \rangle$ gives $-rHD/17 = (2 + rHD)/17^2$, whence

$$16rHD - 2 = 0. \quad (10)$$

Next, $\langle \bar{w}w_0, w_4, w_4 \rangle = \langle w_0, \bar{w}w_4, \bar{w}w_4 \rangle$ gives

$$34(-2H^2G)/17 = (-H^2)(34r)/17^2,$$

whence

$$G = r/34. \quad (11)$$

Furthermore, $\langle \bar{w}w_0, w_1, v_0 \rangle = \langle w_0, \bar{w}w_1, \bar{w}v_0 \rangle$ gives

$$-34(G^2 + \frac{1}{2}AH^2) = (2 + HD r 18)/17,$$

which from (10) and (11), simplifies to

$$68AH^2 = -3. \quad (12)$$

From $\langle \bar{w}v_0, w_1, v_{14} \rangle = \langle v_0, \bar{w}w_1, \bar{w}v_{14} \rangle$, we derive

$$\frac{18}{17}(-CHF) = \frac{(-102)}{17^2}(GC - 6HD)(-1).$$

To see this, note that $\bar{w}v_0 = (-v_0 + \sum_{i=1}^{16} 18v_i)/17$ and so $17^2\langle \bar{w}v_0, w_1, v_{14} \rangle = 17 \cdot 18\langle v_2, w_1, v_{14} \rangle = 17 \cdot 18(-CHF)$, which explains the left-hand side. As for the right-hand side, use the fact that $\langle \bar{t}^i v_0, \bar{t}^i w_1, \bar{t}^i v_{16} \rangle = \langle v_0, w_1, v_{16} \rangle$ and so $\langle v_0, w_j, v_{-2^{j-1}} \rangle = (-1)^{j-1}\langle v_0, w_1, v_{16} \rangle$. Then,

$$\begin{aligned} \langle v_0, 17\bar{w}w_1, 17\bar{w}v_{14} \rangle &= \langle v_0, w_0, -v_0 \rangle + \left\langle u_0, \sum_{j=1}^8 b_{j1}w_j, \sum_{k=1}^{16} a_{k,14}v_k \right\rangle \\ &= \left(\sum_{j=1}^8 (-1)^{j-1} b_{j1} a_{-2^{j-1}, 14} \right) \langle v_0, w_1, v_{16} \rangle \\ &= \left(\sum_{j=1}^8 (-1)^{j-1} b_{j1} a_{-2^j, 14} \right) (-GC + 6HD) \\ &= -102(-GC + 6HD). \end{aligned}$$

Now, using (10) we find that the above identity simplifies to

$$2HFC = -GC. \quad (13)$$

As can be easily checked, there are exactly two solutions to (9)–(13), with $A = -12$ which are

$$C = -3, \quad D = \mp \frac{1}{2}, \quad H = \mp r/68, \quad G = r/34, \quad F = \pm 1.$$

These are conjugate under h_{-1} . This shows that there is at most one conjugacy class of $L(2, 17)$ s in $E_6(\mathbb{C})$ with this character. There is another class obtained by applying a field isomorphism.

We computed the trilinear form for all triples of basis vectors and showed that (8) holds for all of them and so $\bar{w}$ is in $F_4(\mathbb{C})$. Now the elements $\bar{u}, \bar{t}, \bar{w}$ generate a subgroup of $F_4(\mathbb{C})$ isomorphic to $L(2, 17)$.

REMARK. The identical argument works for algebraically closed fields, k , of characteristic prime to $|L(2, 17)|$. Since the elements of the embedded Borel subgroup of $L(2, 17)$ are semisimple, [41] can be applied to replace Lemma 3.3 for establishing that B embeds in the normalizer of a torus. Up to algebraic conjugacy, there is a unique subgroup isomorphic to $L(2, 17)$ in $F_4(k)$ containing

a given Borel subgroup. Arguing as in [19], we see there is a unique subgroup in $F_4(k)$ isomorphic to $L(2, 17)$ with character $1 + 9_a + 17_a$.

6.9. *The group $B = \text{PGL}(2, 13)$ embeds in F with $\psi|_L = 1_a + 12_a + 14_a$. Up to algebraic conjugacy, there is a unique class of subgroups of F isomorphic to $L(2, 13)$ with the given character on $\mathbf{K}$. Each embedding of $L(2, 13)$ with this character extends to an embedding of $\text{PGL}(2, 13)$.*

Proof. See [19]. (There, embeddability in fields of characteristic prime to $|L(2, 13)|$ is also discussed.)

6.10. **THEOREM.** *The group $L = L(2, 8)$ embeds in F with $\psi|_L = 1_a + 8_a + 9_a + 9_c$.*

Proof. Set $L = L(2, 8)$. Denote by u, t , and w , respectively, the generators of L with corresponding matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $x \in \mathbb{F}_8$ satisfies $x^3 + x + 1 = 0$. The generators satisfy the following relations:

$$\begin{aligned} w^2 = u^2 = t^7 = 1, \quad wtw = t^{-1}, \\ ut^{-2}ut^2 = t^{-3}ut^3, \quad (wu)^3 = 1. \end{aligned} \quad (14)$$

This is a set of defining relations for $L(2, 8)$, as can be verified by a Todd–Coxeter enumeration with respect to the subgroup $B = \langle t, u \rangle$, or by directly establishing that each element can be written in the form bwb' with $b, b' \in B$. The representation of L in $\mathbf{K}$ with character $1_a + 8_a + 9_a + 9_c$ will be given by matrices for the images $\bar{u}, \bar{t}, \bar{w}$ of u, t, w in $\text{GL}(\mathbf{K})$.

Set $\rho = s_{(123), (123), 1}$, so that ρ is a permutation matrix with respect to the basis $(f_j)_{1 \leq j \leq 27}$ (cf. § 6.5) and put

$$\begin{aligned} e_0 &= -e_{2,3}^1 - e_{3,2}^1 = -f_{22} - f_{16}, \\ e_1 &= -ie_{1,2}^2 + ie_{2,1}^3 = -if_{11} + if_6, \\ e_2 &= -ie_{1,3}^2 + ie_{3,1}^3 = -if_{20} + if_9, \\ e_3 &= -e_{1,3}^2 - e_{3,1}^3 = -f_{20} - f_9, \\ e_4 &= ie_{1,1}^2 - ie_{1,1}^3 = if_2 - if_3, \\ e_5 &= -e_{1,2}^2 - e_{2,1}^3 = -f_{11} - f_6, \\ e_6 &= e_{1,1}^2 + e_{1,1}^3 = f_2 + f_3, \\ e_7 &= ie_{2,3}^1 - ie_{3,2}^1 = if_{22} - if_{16}. \end{aligned}$$

Then $e_i, \rho e_i, \rho^2 e_i$ form a basis of $\{e_{i,i}^1\}_{i=1,2,3}^\perp$ with the property

$$\left. \begin{aligned} e_i \times \rho e_j &= \rho^2 e_k \\ e_j \times \rho e_i &= -\rho^2 e_k \end{aligned} \right\} \quad \text{for all } (ijk) \in \{(431), (542), (653), (764), (175), (216), (327)\}.$$

These equations put a multiplicative structure on the span U of all e_i for $i = 1, \dots, 7$ via $x \cdot y = \rho(x \times \rho(y))$, which correspond to the multiplication rules $e_i e_j = e_k$ for the same cyclically ordered triples as above, for the 7-dimensional

orthogonal complement of the identity in the standard Cayley division algebra over $\mathbb{C}$ (cf. [23]). The automorphism group of this Cayley algebra is well known to be a group isomorphic to $G_2(\mathbb{C})$. From these observations, the presence of a subgroup H of $\tilde{E}$ isomorphic to $G_2(\mathbb{C})$ is evident. It acts irreducibly on each of the three 7-spaces $U, \rho U, \rho^2 U$, and trivially on the complement spanned by the remaining standard basis elements (that is, the subspace of $\mathbf{K}$ of all $(X_1, 0, 0)$ with $X_1^7 = X_1$). In particular, the subgroup H is contained in F . Since the action of H commutes with ρ , an element of H is fully determined by its action on U . Thus, to check whether a linear transformation of U extends to a transformation of $\mathbf{K}$ lying in F , one only needs to verify whether it preserves the multiplication $\cdot$ defined on U . Below, we use this observation to find $\bar{t}$ and $\bar{u}$.

The centralizer of H in F contains all $s_{f,f,1}$ with f an orthogonal 3×3 matrix. From this and the description of the group of type $A_1 G_2$ in Theorem 4.1(ii), it is readily seen that the connected component of 1 of the centralizer in F of H is a subgroup of F of Lie type A_1 . Denote this subgroup by A . Let σ be the permutation matrix of order 7 inducing the permutation (7654321) on the indices i of $\rho^j e_i$ for each j . Thus $\sigma \in H$. Set $\bar{t} = s_{g,g,1} \sigma$ where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi/7 & \sin 2\pi/7 \\ 0 & -\sin 2\pi/7 & \cos 2\pi/7 \end{pmatrix}. \quad (15)$$

As $s_{g,g,1} \in A$, we see that the transformations $s_{g,g,1}$ and σ belong to F and commute. In particular, $\bar{t}$ has order 7. It can be easily checked that the diagonal transformation $\bar{u}$ fixing the indices $\{0, 3, 5, 6\}$ and negating the indices $\{1, 2, 4, 7\}$ for $\{e_i, \rho e_i, \rho^2 e_i\}$ and fixing e is an element of H . The $\bar{t}$ conjugates of $\bar{u}$ form an elementary abelian group of order 8 normalized by $\bar{t}$. Thus, $\bar{u}$ and $\bar{t}$ generate a Frobenius group, to be denoted by B , of order 56; it is in F by our construction.

We will now identify the invariant subspaces for the action of L on $\mathbf{K}$ in terms of certain vectors and parameters. Let

$$\begin{aligned} v_\infty^1 &= 2\sqrt{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ v_\infty^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} \quad \text{and} \quad v_{-\infty}^2 = \lambda_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix}, \\ v_\infty^3 &= \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \text{and} \quad v_{-\infty}^3 = \lambda_3 \begin{pmatrix} 0 & 1 & -i \\ 1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \end{aligned}$$

for certain λ_2 and λ_3 . Here and elsewhere we identify 3×3 matrices with the appropriate element in the span of e_{ij}^1 ($1 \leq i, j \leq 3$). These vectors are all fixed by $\bar{u}$ and are eigenvectors for $\bar{t}$ as follows:

$$\bar{t}v_\infty^1 = v_\infty^1, \quad \bar{t}v_{\varepsilon\infty}^2 = \zeta^{2\varepsilon}v_{\varepsilon\infty}^2, \quad \bar{t}v_{\varepsilon\infty}^3 = \zeta^\varepsilon v_{\varepsilon\infty}^3.$$

Here $\varepsilon = \pm 1$. The conjugates of $\bar{u}$ by $\bar{t}$ also fix them. These vectors together with e span the complete subspace of $\mathbf{K}$ fixed by the $\bar{t}$ conjugates of $\bar{u}$.

We will construct L to fix e . The remaining invariant subspaces are V_1 of dimension 8, and V_2, V_3 of dimension 9. The subspace V_1 contains the unique vector fixed by $\langle \bar{u}, \bar{t} \rangle$ and orthogonal to e which is v_∞^1 . By interchanging V_2 and V_3 if necessary we may assume that V_2 contains v_∞^2 and a multiple of $v_{-\infty}^2$ which we assume is $v_{-\infty}^2$ by fixing the parameter λ_2 later. Similarly V_3 contains v_∞^3 and $v_{-\infty}^3$.

Now let

$$m_7 = -i(ae_7 + bpe_7 + c\rho^2e_7)$$

and $\bar{t}^k m_k = m_7$ ($k = 1, \dots, 7$). Here the subscripts are taken modulo 7 with 0 being 7. The actions of $\bar{u}$ and $\bar{t}$ are as follows:

$$\begin{aligned} \bar{t}m_i &= m_{i-1}, \\ \bar{u}m_i &= \begin{cases} m_i & \text{if } i = 3, 5, 6, \\ -m_i & \text{if } i = 1, 2, 4, 7. \end{cases} \end{aligned}$$

For any $\{a, b, c\}$ the span of the m_i is a 7-dimensional space invariant under $\langle \bar{u}, \bar{t} \rangle$ with the given monomial action. The space V_1 must contain v_∞^1 and vectors m_i^1 for some choice of a, b, c which we take to be a_1, b_1, c_1 . Similarly V_j contains $\{v_{\varepsilon_\infty}^j\}$ and some vectors m_i^j for choices a_j, b_j, c_j for $j = 2, 3$. The basis we need is given by

$$e, v_\infty^1, m_1^1, \dots, m_6^1, m_7^1, v_\infty^2, v_{-\infty}^2, m_1^2, \dots, m_6^2, m_7^2, v_\infty^3, v_{-\infty}^3, m_1^3, \dots, m_6^3, m_7^3.$$

We shall refer to this basis as the m -basis. Note that

$$\begin{aligned} m_{7-k}^j &= s_{g,g,1}^k \sigma^k m_7^j \\ &= -i \left(a_j e_k + \left(b_j \cos \frac{2\pi k}{7} + c_j \sin \frac{2\pi k}{7} \right) \rho e_k + \left(-b_j \sin \frac{2\pi k}{7} + c_j \cos \frac{2\pi k}{7} \right) \rho^2 e_k \right), \end{aligned} \tag{16}$$

which enables us easily to compute the base transformation. Here $\cos(2\pi k/7) = \frac{1}{2}(\zeta^k + \zeta^{-k})$, and $\sin(2\pi k/7) = (\zeta^k - \zeta^{-k})/(2i)$. With respect to this basis, $\bar{u}$ is diagonal with entries

$$1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1,$$

and $\bar{t}$ is determined by the definition of the m -basis and (15), (16). By construction, both $\bar{u}$ and $\bar{t}$ belong to F .

Using representation theory for $L(2, 8)$, we can find an involutory transformation $\bar{w} \in \text{GL}(\mathbf{K})$ which fixes e , and stabilizes the three subspaces of $\mathbf{K}$ spanned by the elements $v_{\varepsilon_\infty}^i, m_j^i$ ($\varepsilon \in \{1, -1\}$, $j = 1, \dots, 7$) of the m -basis for each $i = 1, 2, 3$, and satisfies the relations (12) defining $L(2, 8)$ together with $\bar{t}$ and $\bar{u}$. This is particularly easy in this case as the two 9-dimensional subspaces are induced from linear characters of a Borel subgroup and the 8-dimensional subspace is the permutation representation minus the trivial representation.

At this stage, we have that, for each non-zero λ_2, λ_3 and parameters $a_1, \dots, c_3$ chosen so that the vectors are linearly independent, there is a morphism of groups $x \mapsto \bar{x}$ from L to $\text{GL}(\mathbf{K})$ determined by $t \mapsto \bar{t}$, $u \mapsto \bar{u}$, $w \mapsto \bar{w}$. This defines a representation of L in $\mathbf{K}$ with the property that the Borel subgroup generated by $\bar{t}$ and $\bar{u}$ belongs to F .

We claim that, conversely, every Borel subgroup of a subgroup L of F with $\psi|_L = 1_a + 8_a + 9_a + 9_c$ is conjugate in F to this one.

To prove this claim note that by Lemma 3.4, there is a unique class of

elementary subgroups of order 8 all of whose involutions have trace 3 on $\mathbf{K}$. In a standard $G_2(\mathbb{C})$, we can find such a subgroup. Up to conjugacy in F , we may take this standard $G_2(\mathbb{C})$ to be H above, so that the order 8 subgroup of the Borel subgroup of L is conjugate to $\bar{U} = \langle \bar{u}^{\bar{r}^j} \mid j = 1, \dots, 6 \rangle$. Character computations show that the connected centralizer of $\bar{U}$ in F coincides with the connected centralizer A in F of H . According to [18], the full normalizer in H of U is isomorphic to $K = 2^3 \cdot L(2, 7)$. By Sylow theory, up to conjugacy, the latter group contains a unique $2^3 : 7$. The full normalizer of $\bar{U}$ in F lies in AH . Hence the Borel subgroup of L lies in AK and after conjugation in K we may take the projection of the element $\bar{t} \in AK$ on the second factor to coincide with σ above. The projection of $\bar{t}$ on the factor A must have shape $s_{h,h,1}$ for an orthogonal 3×3 matrix h . In particular, it has eigenvalues $1, \alpha, \alpha^{-1}$ for a 7th root of 1. But then in view of the character of $\psi|_L$ given in Theorem 4.1(ii) for L of type $A_1 G_2$, the transformation $\bar{t}$ has trace

$$2 + \alpha + \alpha^{-1} + \alpha^2 + \alpha^{-2} + \text{tr}_{7_a}(\sigma)(1 + \alpha + \alpha^{-1}) = 2 + \alpha + \alpha^{-1} + \alpha^2 + \alpha^{-2} \quad \text{on } \mathbf{K}.$$

On the other hand, the character value of $\bar{t}$ on $\mathbf{K}$ should be $2 + \zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2}$, whence $\alpha = \zeta, \zeta^{-1}$; so, indeed $\bar{t}$ is A -conjugate to $s_{g,g,1}\sigma$ defined above. The claim is proved.

It remains to find values of the parameters $a_1, \dots, c_3, \lambda_2, \lambda_3$ for which $\bar{w}$ belongs to F . To facilitate computations, we compute modulo $p = 337$ (see the beginning of this section). More precisely, the generating matrices for $\bar{L}$ will be defined over $\mathbb{Z}[\sqrt{3}, \zeta, i]$, where ζ is a primitive 7th root of 1, and we shall compute modulo the ideal generated by 337, $i - 189$, $\sqrt{3} - 45$, and $\zeta - 64$. We use the convention $\cos 2\pi k/7 = \frac{1}{2}(\zeta^k + \zeta^{-k})$ and $\sin 2\pi k/7 = (\zeta^k - \zeta^{-k})/(2i)$. This implies (up to a sign) that we compute modulo $\sqrt{7} - 218$ as well. This means that modulo 337, ζ may be replaced by 64, i by 189, $\sqrt{3}$ by 45, and $\sqrt{7}$ by 21.

The matrices for $\bar{w}$ with respect to the three bases of the respective subspaces are

$$\begin{pmatrix} 42 & 41 & 41 & 41 & 41 & 41 & 41 & 41 \\ 42 & 42 & 42 & 210 & 211 & 211 & 42 & 211 \\ 42 & 42 & 210 & 211 & 211 & 42 & 211 & 42 \\ 42 & 210 & 211 & 211 & 42 & 211 & 42 & 42 \\ 42 & 211 & 211 & 42 & 211 & 42 & 42 & 210 \\ 42 & 211 & 42 & 211 & 42 & 42 & 210 & 211 \\ 42 & 42 & 211 & 42 & 42 & 210 & 211 & 211 \\ 42 & 211 & 42 & 42 & 210 & 211 & 211 & 42 \end{pmatrix}, \begin{pmatrix} 0 & 295 & 336 & 285 & 329 & 258 & 273 & 42 & 162 \\ 1 & 0 & 315 & 194 & 250 & 277 & 284 & 161 & 204 \\ 42 & 247 & 36 & 76 & 253 & 217 & 196 & 57 & 177 \\ 273 & 38 & 76 & 253 & 217 & 196 & 57 & 177 & 36 \\ 258 & 291 & 253 & 217 & 196 & 57 & 177 & 36 & 76 \\ 329 & 304 & 217 & 196 & 57 & 177 & 36 & 76 & 253 \\ 285 & 306 & 196 & 57 & 177 & 36 & 76 & 253 & 217 \\ 336 & 73 & 57 & 177 & 36 & 76 & 253 & 217 & 196 \\ 162 & 89 & 177 & 36 & 76 & 253 & 217 & 196 & 57 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 295 & 162 & 258 & 336 & 273 & 285 & 42 & 329 \\ 1 & 0 & 250 & 204 & 277 & 315 & 284 & 194 & 161 \\ 162 & 291 & 76 & 57 & 253 & 177 & 217 & 36 & 196 \\ 329 & 89 & 57 & 253 & 177 & 217 & 36 & 196 & 76 \\ 42 & 304 & 253 & 177 & 217 & 36 & 196 & 76 & 57 \\ 285 & 247 & 177 & 217 & 36 & 196 & 76 & 57 & 253 \\ 273 & 306 & 217 & 36 & 196 & 76 & 57 & 253 & 177 \\ 336 & 38 & 36 & 196 & 76 & 57 & 253 & 177 & 217 \\ 258 & 73 & 196 & 76 & 57 & 253 & 177 & 217 & 36 \end{pmatrix}.$$

We found suitable parameter choices for $a_1, \dots, c_3, \lambda_2, \lambda_3$ by solving

$$\langle x, y, z \rangle = \langle \bar{w}x, \bar{w}y, \bar{w}z \rangle \quad (17)$$

for several choices of vectors x, y, z . Note that the trilinear form evaluated on vectors from different invariant subspaces must be 0. It is straightforward to check that $\langle e, m_i^l, m_j^l \rangle = a_i a_j + b_i b_j + c_i c_j$. Consequently the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is subject to the orthogonality relations

$$a_i a_j + b_i b_j + c_i c_j = 0 \quad \text{for all distinct } i, j \in \{1, 2, 3\}. \quad (18)$$

Using the triples $\langle e, v_\infty^1, v_\infty^1 \rangle, \langle v_\infty^1, v_\infty^1, v_\infty^1 \rangle, \langle m_1^1, m_2^1, m_6^1 \rangle$ gives $a_1 = -24$ and a value for $b_1^2 + c_1^2$. Various other triples can be used to obtain the following solution and exactly three others with $\lambda_2 = 133$:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} -24 & 196 & 38 \\ 31 & 0 & -211 \\ -293 & 247 & 155 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 133, \lambda_3 = 306.$$

We checked that $\bar{w}x^\# = (\bar{w}x)^\#$ for $x = e_i + e_j$ ($i, j \in \{1, \dots, 27\}$) hold and that $\bar{w}e_i^\# = (\bar{w}e_i)^\# = 0$. This means that $\bar{w} \in F$ for these values.

The centralizer of $\langle \bar{t}, \bar{u} \rangle$ in F_4 consists of all $s_{h,h,1}$ where h is orthogonal and centralizes g . The set of such matrices h is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & d & -c \\ 0 & c & d \end{pmatrix}$$

where $d^2 + c^2 = 1$. The action of $s_{h,h,1}$ on v_∞ and $v_{-\infty}$ is as follows:

$$\begin{aligned} s_{h,h,1} v_\infty^2 &= (d^2 - c^2 - 2cdi) v_\infty^2, \\ s_{h,h,1} v_{-\infty}^2 &= (d^2 - c^2 + 2cdi) v_{-\infty}^2, \\ s_{h,h,1} v_\infty^3 &= (d - ci) v_\infty^3, \\ s_{h,h,1} v_{-\infty}^3 &= (d + ci) v_{-\infty}^3. \end{aligned}$$

Consequently we can choose $\lambda_2 = 133$ if we wish. The elements of the centralizer of B which scale v_∞^2 and $v_{-\infty}^2$ by the same scalar are

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and its powers. Conjugating by S_1 fixes λ_2 and negates λ_3 . Conjugating by S_1^2 fixes λ_2 and λ_3 and changes some values for a_i, b_i, c_i . The solutions we found have a fixed λ_2 and correspond to exactly these four solutions. This shows that there is a unique such conjugacy class of $L(2, 8)$ subgroups with this character in $F_4(\mathbb{Z}/337)$ and the same elimination shows that there is a unique such conjugacy class in $F_4(\mathbb{C})$.

To enable independent checks, we present on p. 138 the matrix whose columns contain the basis vectors of the m -basis for the given values of the parameters.

[illegible]

The m -basis found in § 6.10

In particular, if W is the matrix affording $\bar{w}$ as above on V_1, V_2, V_3 , and M is the transpose of the matrix opposite, then $W' = M^{-1}WM$ has the property that $W'(x^\#) = (W'x)^\#$ for $x = f_i + f_j$ and f_i . As a check, we showed the same for $M^{-1}TM$ and $M^{-1}UM$ for the corresponding matrices T and U for $\bar{t}$ and $\bar{u}$. This establishes the unique embedding in F of $L(2, 8)$ with this character.

6.11. *The group $B = \text{Aut } L(2, 8)$ embeds in $\tilde{E}$ with $\psi|_L = 9_a + 9_b + 9_c$.*

Proof. The group $L(2, 8)$ exists via the subgroup $G_2(\mathbb{C})$ which is irreducible on $\mathbf{K}$ (cf. § 4). The existence of $\text{Aut } L(2, 8)$ is due to [4].

6.12. *The group $L = L(2, 7)$ embeds in $\tilde{E}$ with $\psi|_L = 2 \cdot 6_a + 7_a + 8_a$.*

Proof. This is via the irreducible $A_2(\mathbb{C})$ or the A_5 factor of the subgroup $A_1(\mathbb{C})A_5(\mathbb{C})$ each described in § 4. The irreducible A_2 gives the character listed in Theorem 1.1.

6.13. *The group $L = \text{Alt}_6$ embeds in F with $\psi|_L = 1_a + 2 \cdot 9_a + 8_a$.*

Proof. This is via the subgroup $A_2(\mathbb{C})A_2(\mathbb{C})$ of F described in § 4. This amounts to the same as an embedding via $3_a \otimes 3_a \otimes 3_b$ for $3 \cdot \text{Alt}_6$ in a subgroup of $\tilde{E}$ of type $A_2A_2A_2$.

6.14. *The group $L = \text{Alt}_6$ embeds in $\tilde{E}$ with $\psi|_L = 5_a + 5_b + 8_a + 9_a$.*

Proof. This is via the irreducible $A_2(\mathbb{C})$ in E described in § 4. Alternatively, the embedding is realized via the irreducible subgroup of type C_4 (irreducible on the standard module for $\text{Sp}(8, \mathbb{C})$).

7. Finite simple subgroups of E

The tables in this section list various embeddings of groups in proper closed subgroups of $E_6(\mathbb{C})$ and $F_4(\mathbb{C})$. No attempt has been made to give all of the embeddings. We use the list of finite non-abelian simple groups (see, for example, [31]) in order to prove the following theorem.

7.1. THEOREM. (i) *Let L be a finite simple non-abelian subgroup of $\tilde{E}$ and let ψ be the character of the representation of $\tilde{E}$ on $\mathbf{K}$. Then $L, \psi|_L$ is (up to Galois conjugacy) one of the pairs indicated in Table 8.*

(ii) *Let $\hat{L}$ be a subgroup of $\tilde{E}$ which is a perfect central extension of a finite non-abelian simple group L by a group of order 3 with $Z(\hat{L}) = Z(\tilde{E})$. Then, up to Galois conjugacy, $\hat{L}, \psi|_{\hat{L}}$ is one of the pairs indicated in Table 9.*

Before proving this result, for inductive purposes, we shall establish the following lemma.

7.2. LEMMA. *Let $\hat{L}$ be a subgroup of $\tilde{E}$ and a perfect central extension of a finite non-abelian simple group L by a subgroup $Z \not\subseteq Z(\tilde{E})$. Then $\hat{L}$ is one of the subgroups indicated in Table 10.*

TABLE 8. *Finite non-abelian simple subgroups of $\tilde{E}$*

L	$\psi _L$	Realization
Alt ₅	$2 \cdot 1_a + 3_a + 3_b + 4_a + 3 \cdot 5_a$	$1_a + 5_a$ in $SL(6, \mathbb{C})$
Alt ₅	$1_a + 3_a + 2 \cdot 4_a + 3 \cdot 5_a$	via $3_a \otimes (3_b + 4_a)$ in $A_2(\mathbb{C})G_2(\mathbb{C})$
Alt ₅	$1_a + 7 \cdot 3_a + 5_a$	$2 \cdot 3_a$ in $SL(6, \mathbb{C})$
Alt ₅	$3 \cdot 1_a + 3 \cdot 3_a + 3 \cdot 5_a$	standard $G_2(\mathbb{C})$
Alt ₅	$6 \cdot 1_a + 3 \cdot 3_a + 3 \cdot 4_a$	$3_a + 4_a$ in $G_2(\mathbb{C})$
Alt ₅	$5 \cdot 1_a + 3_a + 3_b + 4 \cdot 4_a$	$2 \cdot 1_a + 4_a$ in $SL(6, \mathbb{C})$
Alt ₅	$3 \cdot 3_a + 3 \cdot 3_b + 4_a + 5_a$	$3_a + 3_b$ in $SL(6, \mathbb{C})$
Alt ₅	$9 \cdot 1_a + 6 \cdot 3_a$	irr. in standard $SL(3, \mathbb{C})$
Alt ₆	$2 \cdot 1_a + 9_a + 8_a + 8_b$	$1_a + 9_a$ in $D_5(\mathbb{C})$
Alt ₆	$1_a + 5_a + 5_b + 8_a + 8_b$	${}^2F_4(2)'$
Alt ₆	$1_a + 2 \cdot 9_a + 8_a$	see §6.13
Alt ₆	$5_a + 5_b + 8_a + 9_a$	see §6.14
Alt ₆	$2 \cdot 1_a + 3 \cdot 5_a + 10_a$	irr. in Levi $D_5(\mathbb{C})$
Alt ₆	$3 \cdot 1_a + 3 \cdot 8_a$	$3_a \otimes 3_a \otimes 3_a$ for $3 \cdot \text{Alt}_6$ in $A_2(\mathbb{C})A_2(\mathbb{C})A_2(\mathbb{C})$
Alt ₇	$2 \cdot 6_a + 15_a$	6-dim. irr. in $SL(6, \mathbb{C})$
$L(2, 7)$	$6 \cdot 1_a + 3 \cdot 7_a$	7-dim. irr. in $G_2(\mathbb{C})$
$L(2, 7)$	$3 \cdot 1_a + 3 \cdot 8_a$	8-dim. irr. in $D_4(\mathbb{C})$
$L(2, 7)$	$2 \cdot 6_a + 7_a + 8_a$	see §6.12
$L(2, 7)$	$7 \cdot 3_a + 6_a$	$2 \cdot 3_a$ in $SL(6, \mathbb{C})$
$L(2, 7)$	$9 \cdot 1_a + 3 \cdot 3_a + 3 \cdot 3_b$	$1 + 3_a + 3_b$ in $G_2(\mathbb{C})$
$L(2, 7)$	$1_a + 3 \cdot 3_a + 3 \cdot 3_b + 8_a$	$3_a + 3_b$ in $SL(6, \mathbb{C})$
$L(2, 7)$	$1_a + 3_a + 3_b + 2 \cdot 6_a + 8_a$	$A_2(\mathbb{C})A_2(\mathbb{C})$ in F (§4)
$L(2, 8)$	$6 \cdot 1_a + 3 \cdot 7_b$	7-dim. irr. in stand. $G_2(\mathbb{C})$
$L(2, 8)$	$9_a + 9_b + 9_c$	see §6.11
$L(2, 8)$	$1_a + 8_a + 9_a + 9_c$	see §6.10
$L(2, 8)$	$2 \cdot 1_a + 7_a + 9_a + 9_c$	irr. in $B_4(\mathbb{C})$
$L(2, 8)$	$4 \cdot 1_a + 7_a + 2 \cdot 8_a$	irr. in $B_3(\mathbb{C})$
$L(2, 11)$	$1_a + 5_a + 10_b + 11_a$	10-dim. irr. in $D_5(\mathbb{C})$
$L(2, 11)$	$2 \cdot 1_a + 3 \cdot 5_a + 10_a$	$1_a + 5_a$ in $SL(6, \mathbb{C})$
$L(2, 13)$	$6 \cdot 1_a + 3 \cdot 7_a$	7-dim. irr. in $G_2(\mathbb{C})$
$L(2, 13)$	$13_a + 14_a$	7-dim. irr. in irr. $G_2(\mathbb{C})$
$L(2, 13)$	$1_a + 12_a + 14_a$	see §6.9
$L(2, 17)$	$2 \cdot 1_a + 9_a + 16_b$	9-dim. irr. in $\text{Spin}(9, \mathbb{C})$
$L(2, 17)$	$1_a + 9_a + 17_a$	see §6.8
$L(2, 19)$	$9_a + 18_c$	see §6.7
$L(2, 25)$	$1_a + 26_a$	${}^2F_4(2)'$ (see §6.6)
$L(2, 27)$	$1_a + 26_d$	see §6.5
$L(3, 3)$	27_a	${}^2F_4(2)'$ (see §6.4)
$L(3, 3)$	$1_a + 26_a$	see §6.3
PSU(3, 3)	27_a	7-dim. irr. in irr. $G_2(\mathbb{C})$
PSU(3, 3)	$1_a + 2 \cdot 6_a + 14_a$	8-dim. irr. in $C_4(\mathbb{C})$ or ${}^3D_4(2)$
PSU(3, 3)	$6 \cdot 1_a + 3 \cdot 7_a$	irr. in stand. $G_2(\mathbb{C})$
PSU(4, 2)	$2 \cdot 6_a + 15_a$	$SL(6, \mathbb{C})$
PSU(4, 2)	$2 \cdot 1_a + 3 \cdot 5_a + 10_a$	$\text{Spin}(10, \mathbb{C})$
${}^3D_4(2)$	$1_a + 26_a$	see §6.2
${}^2F_4(2)'$	27_a	see §6.1
M_{11}	$1_a + 10_a + 16_a$	10-dim. irr. in $\text{Spin}(10, \mathbb{C})$

TABLE 9. *Finite quasisimple subgroups $\hat{L}$ with $Z(\hat{L}) = Z(\hat{E})$*

$\hat{L}$	$\psi _{\hat{L}}$	Realization
$3 \cdot \text{Alt}_6$	$3 \cdot (3_a + 3_b) + 9_a$	$3_a + 3_b$ in $A_5(\mathbb{C})$
$3 \cdot \text{Alt}_6$	$6_a + 7 \cdot 3_a$	$3_a \otimes (3_a + 3_b + 1_a)$ in $A_2(\mathbb{C})G_2(\mathbb{C})$
$3 \cdot \text{Alt}_6$	$2 \cdot 6_a + 15_a$	6_a in $SL(6, \mathbb{C})$
$3 \cdot \text{Alt}_7$	$2 \cdot 6_a + 15_a$	$SL(6, \mathbb{C})$

TABLE 10. *Finite quasisimple subgroups $\hat{L}$ with $1 \neq Z(\hat{L}) \neq Z(\tilde{E})$*

$\hat{L}$	Type of element $Z^\#$	Group containing $\hat{L}$
$2 \cdot \text{Alt}_n$ ($n \leq 9$)	2B	$\text{Spin}(8, \mathbb{C})$
$2 \cdot \text{Alt}_5$	2A	$\text{SL}(2, \mathbb{C})$
$3 \cdot \text{Alt}_6$	3C	$\text{SL}(3, \mathbb{C})$
$6 \cdot \text{Alt}_6$	6M	$\text{SL}(6, \mathbb{C})$
$6 \cdot \text{Alt}_7$	6M	$\text{SL}(6, \mathbb{C})$
$\text{SL}(2, 7)$	2A	$\text{SL}(6, \mathbb{C})$
$\text{SL}(2, 7)$	2B	$\text{SL}(4, \mathbb{C})$
$\text{SL}(2, 11)$	2A	$\text{SL}(6, \mathbb{C})$
$\text{SL}(2, 13)$	2A	$\text{SL}(6, \mathbb{C})$
$6 \cdot \text{L}(3, 4)$	6M	$\text{SL}(6, \mathbb{C})$
$2 \cdot \text{PSU}(4, 2)$	2B	$\text{SL}(4, \mathbb{C})$
$6 \cdot \text{PSU}(4, 3)$	6M	$\text{SL}(6, \mathbb{C})$
$2 \cdot \text{PSp}(6, 2)$	2B	$\text{Spin}(8, \mathbb{C})$
$2^2 \cdot \text{PS}\Omega^+(8, 2)$	2B	$\text{Spin}(8, \mathbb{C})$
$2 \cdot J_2$	2A	$\text{SL}(6, \mathbb{C})$

Proof of Lemma 7.2. By the assumption $Z \not\subseteq Z(\tilde{E})$, the subgroup $\hat{L}$ can be embedded in the centralizer Y of a non-central semisimple element of $\tilde{E}$. As discussed in § 4 using [41], Y is a proper reductive subgroup of $\tilde{E}$, whose type corresponds to a closed root subsystem of the root system of type E_6 . Hence its simple components have types A_n or D_n ($n \leq 5$). By projecting $\hat{L}$ onto a suitable component, we can find an embedding of a quotient of $\hat{L}$ with non-trivial centre not contained in the centre of $\tilde{E}$. This means the group has a projective representation of dimension at most 10. These groups are listed in [27] and [28], and using the classification [31] we find that the list is complete. In particular, L is one of Alt_n ($n \leq 11$), $L(n, q)$ ($n = 2, q \leq 19$ or $n = 3, q \leq 4$ or $n = 4, q \leq 3$), $\text{PSU}(n, q)$ ($n = 3, q \leq 3$ or $n = 4, q \leq 3$), $\text{PS}\Omega^+(8, 2)$, $\text{PSp}(6, 2)$, J_2 . Now, by feasibility checks on characters given by [22] and use of Theorem 3.1, the result follows. Note that the subgroup $3 \cdot \text{Alt}_7$ does not occur as any embedding factors through $\text{SL}(6, \mathbb{C})$, which forces the centre to coincide with that of $\tilde{E}$.

Proof of Theorem 7.1. Let $\hat{L}$ be a finite subgroup of $\tilde{E}$ which is a non-split central extension of $L = \hat{L}/(\hat{L} \cap Z(\tilde{E}))$. (We allow for $\hat{L} \cong L$.) We check that $\hat{L}$ is isomorphic to one of the groups listed in Tables 8 and 9. For each of the groups listed, the feasible characters are easily determined (cf. Theorem 3.1), and we shall rarely comment on this further. The problem, then, is to show that groups which may have feasible characters cannot be subgroups of E or F as appropriate. For this we use frequently the results of § 4 including the remarks at the end. We do not give details here as the techniques are routine and occur extensively in [17]. We use for $L(2, 27)$ and $L(2, 25)$ the fact that, due to a remark near the end of § 2 for groups in F , the 26-dimensional character on $\mathbf{K}$ must be a constituent on $\mathbf{g}$.

7.3. COROLLARY FOR F_4 . *Every non-abelian finite simple subgroup of F is isomorphic to one of Alt_n ($5 \leq n \leq 6$), $L(2, q)$ ($q \in \{7, 8, 13, 17, 25, 27\}$), $L(3, 3)$, $\text{PSU}(3, 3)$, or ${}^3D_4(2)$, with character on $\mathbf{K}$ as listed in Table 8.*

Proof. Obviously, any finite simple subgroup L of F should be listed in Table 8 and have a fixed vector in $\mathbf{K}$. We shall prove the corollary by ruling out the

possibilities for L arising in that table with a trivial constituent in a character $\psi|_L$ but not appearing in the conclusion.

Case $L \cong L(2, 11)$ or M_{11} . The characters $\psi|_L$ in Table 8 all have trivial constituents but they also all have complex values and so are not feasible as restrictions of a character of F on $\mathbf{K}$.

Case $L \cong \text{PSU}(4, 2)$. If L is a subgroup of F , then it must stabilize a vector in $e^\perp \cap \mathbf{K}$ (as the character on $\mathbf{K}$ is $2 \cdot 1_a + 3 \cdot 5_a + 10_a$). By Theorem 5.1(ii), this implies that $\text{PSU}(4, 2) \leq G_2(\mathbb{C})$ or $\text{Spin}(9, \mathbb{C})$, a contradiction with the character decomposition given. Hence $\text{PSU}(4, 2)$ is not embeddable in F .

8. Feasibility of normalizers and maximality

8.1. PROPOSITION. *Suppose that B is a subgroup of G having a proper subgroup L appearing in Table 1 with $L \subset B \subseteq \text{Aut } L$. Then L and B appear on a line of Table 11.*

REMARKS. The Atlas [22] lists only one character for two characters which are irreducible and the same when restricted to a subgroup of index 2. We distinguish these characters by a superscript. For example, the notation 8_a^+ for $\text{PGL}(2, 7)$ is the first character of degree 8 listed in the Atlas. The character 8_a^- is this character negated outside the subgroup of index 2. Thus $8_a^+|_{L(2,7)} = 8_a^-|_{L(2,7)}$ but $8_a^+(\tau) = -8_a^-(\tau)$ for $\tau \in \text{PGL}(2, 7) - L(2, 7)$. The notation 1_a^ω and 26_a^ω for ${}^3D_4(2)$ are corresponding characters using a cube root of 1 for ${}^3D_4(2) \cdot 3$. There is no sign to a sum of algebraic conjugates such as $3_a + 3_b$ in $\text{PGL}(2, 7)$ as there is a unique character which vanishes outside the normal subgroup. The definitions of σ_{fg} and σ_f were given in Table 1. The bracketed characters in the last column for the Alt_6 lines means the $*$ must be chosen so that the character is feasible which means the value of χ on the field involution is -2 . This can be obtained by various combinations.

TABLE 11. Possible normalizers B of finite simple subgroups L of G appearing in Table 1

L	B	$\psi _B$	$\chi _B$
Alt_6	$\text{Alt}_6 \cdot \sigma_f$	$(5_a + 5_b) + 8_a^+ + 9_a^-$	$9_a^+ + 9_a^- + \{2 \cdot 10_a^* + 2 \cdot 8_a^* + 3 \cdot 8_b^*\}$
Alt_6	$\text{Alt}_6 \cdot \sigma_f$	$1_a^+ + 9_a^+ + 9_a^- + 8_a^+$	$9_a^+ + 9_a^- + 8_a^+ + \{2 \cdot 10_a^* + 8_a^* + 3 \cdot 8_b^*\}$
$L(2, 7)$	$\text{PGL}(2, 7)$	$6_a^\pm + 6_b^\pm + 7_a^+ + 8_a^+$	$2 \cdot (3_a + 3_b) + 6_a^\pm + 6_b^\pm + 7_a^+ + 7_b^- + 2 \cdot 8_a^+ + 3 \cdot 8_b^-$
$L(2, 8)$	$\text{PGL}(2, 8)$	$(9_a + 9_b + 9_c)$	$(9_a + 9_b + 9_c) + 2 \cdot (7_b + 7_c + 7_d) + 2 \cdot 7_a^+ + 2 \cdot 8_a^+$
$L(2, 13)$	$\text{Aut } L(2, 13)$	$1_a^+ + 12_a^+ + 14_a^+$	$12_a^+ + 12_a^- + \{12_b^- \text{ or } 12_c^-\} + 14_a^+ + 14_b^\pm + 14_b^\mp$
$L(2, 25)$	$L(2, 25) \cdot \sigma_{fg}$	$1_a^+ + 26_a^-$	$26_a^- + (26_d + 26_e)$
$L(3, 3)$	$\text{Aut } L(3, 3)$	27_a^+	$26_a^- + (26_b + 26_c)$
${}^3D_4(2)$	${}^3D_4(2) \cdot 3$	$1_a^+ + 26_a^+$	$26_a^+ + 52_a^+$
${}^2F_4(2)'$	${}^2F_4(2)$	27_a^-	78_a^+

Proof. Several groups can be eliminated because the character of L has no feasible extension to B . This holds for $L(2, 8)$ with character $1_a + 8_a + 9_a + 9_c$, $L(2, 17)$ and $L(2, 19)$. The embeddings of $L(2, 25)$ and $L(2, 27)$ do not extend to PGL as the involution $2B$ in Atlas notation would have trace ± 1 on the 27-space.

The group $\text{Aut } L(2, 25)$ has three subgroups of index 2; each has $L(2, 25)$ as a subgroup of index 2. One is $\text{PGL}(2, 25)$, a second is $L(2, 25)$ extended by a field automorphism and the third is $L(2, 25)$ extended by the product of a field automorphism and a diagonal automorphism. There is a feasible character for this last group, which appears last in the Atlas as indicated. For the field automorphism case, which appears second in the Atlas, there is no possibility for $26_a^\pm + (26_d + 26_e)$, in which the involutions 2C and 2D in Atlas notation have trace -2 or 14 .

For Alt_6 there are three possible extensions. The character extends only for $\text{Alt}_6 \cdot \sigma_f$ which appears second in the Atlas. There are feasible characters as indicated.

There is no possible extension to $L(2, 27) \cdot 3$ as the character 26_f does not extend.

The remaining groups have feasible characters as indicated. To see this for ${}^3D_4(2)$, note that the characters are $1_a^* + 26_a^*$ and $26_a^* + 52_a^*$. In order that 3C (Atlas notation) be feasible, the characters must be $1_a^+ + 26_a^+$ or $1_a^\omega + 26_a^\omega$ and $26_a^+ + 52_a^+$. In the first case, 3C, 3D, 6C, 6D, 6E in Atlas notation can be identified as 3A, 3C, 6E, 6I, 6C in E_6 notation. The power maps fit. The characters are as follows:

Atlas	3C	3D	6C	6D	6E
$1_a^+ + 26_a^+$	9	0	1	3	0
$26_a^+ + 52_a^+$	15	-3	-1	1	1
E_6	3A	3C	6E	6I	6C

There are two conjugates for which 3A, 6E, 6I are replaced by 3B, 6Q, 6R, and $1_a^+ + 26_a^+$ is replaced by $1_a^\omega + 26_a^\omega$. However these characters are not real and the corresponding group is not in $F_4(\mathbb{C})$.

The corresponding tabulations for the remaining groups can be computed similarly and are as follows:

${}^2F_4(2)'$	Atlas	4D	4D*	4F	4G
	27_a^-	$-1 + 4i$	$-1 - 4i$	3	-1
	78_a^+	6	6	2	-2
	E_6	4C		4F	4D

$L(3, 3)$	Atlas	2B	4B	6B
	27_a^+	3	3	0
	$26_a^- + (26_b + 26_c)$	-2	2	1
	E_6	2A	4F	6C

$L(3, 3)$	Atlas	2B	4B	6B
	$1_a^+ + 26_a^+$	3	-1	0
	$26_a^- + (26_b + 26_c)$	-2	2	1
	E_6	2A	4E	6C

$L(2, 7)$	Atlas	2B	6A
	$6_a^\pm + 6_a^\pm + 7_a^+ + 8_a^+$	3	0
	$2 \cdot (3_a + 3_b) + 6_a^\pm + 6_a^\pm + 7_a^+ + 7_a^- + 2 \cdot 8_a^+ + 3 \cdot 8_a^-$	-2	1
	E_6	2A	6C

$L(2, 13)$	Atlas	2B	4A
	$1_a^+ + 12_a^+ + 14_a^+$	3	3
	$12_a^+ + 12_a^- + \{12_b^- \text{ or } 12_c^-\} + 14_a^+ + 14_b^\pm + 14_b^\pm$	-2	2
	E_6	2A	4F

$L(2, 25)$	Atlas	4C
	$1_a^+ + 26_a^-$	3
	$26_a^- + (26_d + 26_e)$	2
	E_6	4F

$L(2, 25)$	Atlas	4C
	$1_a^+ + 26_a^+$	-1
	$26_a^- + (26_d + 26_e)$	2
	E_6	4E

$L(2, 8)$	Atlas	3B	6A
	$9_a + 9_b + 9_c$	0	0
	$9_a + 9_b + 9_c + 2 \cdot (7_b + 7_c + 7_d) + 2 \cdot 7_a^+ + 2 \cdot 8_a^+$	6	-2
	E_6	3D	6B

Alt_6	Atlas	2D
	$(5_a + 5_b) + 8_a^+ + 9_a^-$	3
	$9_a^+ + 9_a^- + \{2 \cdot 10_a^* + 2 \cdot 8_a^* + 3 \cdot 8_b^*\}$	-2
	E_6	2A

Alt_6	Atlas	2D
	$1_a^+ + 9_a^+ + 9_a^- + 8_a^+$	3
	$9_a^+ + 9_a^- + 8_a^+ + \{2 \cdot 10_a^* + 8_a^* + 3 \cdot 8_b^*\}$	-2
	E_6	2A

The second $L(3, 3)$ and the second $L(2, 25)$ are ruled out because the 26-dimensional character on $\mathbf{K}$ must also occur as a constituent on $\mathfrak{g}$ but they do not. For these tabulations we have used that, for a group B to be in F , the trivial character must be a constituent of $\psi|_B$ rather than 1_a^- .

REMARK. Let L and B be as in the Main Theorem. If L is non-abelian simple with $\psi|_L$ and $\chi|_L$ as in a row of Table 1, then we know of the existence of a subgroup B of G strictly containing L with the indicated properties in all but the $L(2, 7)$ and the Alt_6 cases. For $L(2, 8)$ this follows by [4], for $L(2, 13)$ by [19], for $L(2, 25)$ by § 6.6, for $L(3, 3)$ by § 6.4, for ${}^3D_4(2)$ by § 6.2, and for ${}^2F_4(2)$ by § 6.1. In the remaining cases $L = L(2, 7)$ and $L = \text{Alt}_6$, the known embeddings, via the maximal connected semi-simple Lie subgroups of type A_1A_5 and $A_2A_2A_2$, do not extend within the Lie subgroups.

8.2. LEMMA (maximality in E). *Suppose that B is a finite subgroup of E which is Lie primitive and has non-abelian simple socle L . Then the isomorphism type of L and $\psi|_L$, $\chi|_L$ are as listed in Table 1. Moreover, in the cases listed with + in the first column, the converse holds in the sense that there is a subgroup L whose normalizer in E is Lie primitive in E .*

Proof. Observe that L is one of the groups of Table 8 or 9. Set $\hat{L}$ for the perfect group in $\tilde{E}$ mapping onto L . The characters $\psi|_{\hat{L}}$ and $\chi|_L$ must be free of the trivial character. For, otherwise, as B/L is abelian, B fixes a 1-space in $\mathbf{K}$ or $\mathbf{g}$, and so is a subgroup of a proper closed Lie subgroup of positive dimension as discussed in § 4 for χ and in Theorem 5.1 for ψ .

We deploy Theorem 7.1, and check the various possibilities for L .

$L \cong \text{Alt}_5$. The only character $\psi|_L$ in Table 8 without the trivial character is $3 \cdot 3_a + 3 \cdot 3_b + 4_a + 5_a$. Straightforward computation (using Theorem 3.1) shows that $\chi|_L$ has the trivial character with multiplicity at least 5. Hence, $L \not\cong \text{Alt}_5$.

$L \cong \text{Alt}_6$. The only character $\psi|_L$ in Table 8 without the trivial character is $5_a + 5_b + 8_a + 9_a$. The requirement $(\chi|_L, 1) = 0$ then forces $\chi|_L = 2 \cdot 8_a + 3 \cdot 8_b + 2 \cdot 9_a + 2 \cdot 10_a$, and so the elements of L with Atlas type 2A, 3A, 3B, 4A, 5A have E -type 2A, 3C, 3C, 4E, 5G, respectively. As for the case where $\hat{L}$ is a 3-cover of L , the trivial character always occurs in $\chi|_L$.

$L \cong \text{Alt}_7$. The character $\psi|_L$ from Table 8 forces $\chi|_L$ to have values $-2, 15, -3$ or $6, -2, 3, 7, 1$ on the elements of L of type 2A, 3A, 3B, 4A, 5A, 6A, 7A/7B (as for the element of order 4, notice that it has trace -1 on $\mathbf{K}$ and must square to an element of E -type 2A and so must itself be of E -type 4E). This implies that the trivial character occurs in $\chi|_L$ with multiplicity at least 2. Next, if $\hat{L} \cong 3 \cdot \text{Alt}_7$, then again the trivial character occurs in $\chi|_L$. Thus, $L \not\cong \text{Alt}_7$.

$L \cong L(2, 7)$. The only character $\psi|_L$ without trivial constituents corresponding to a character $\chi|_L$ without trivial constituents is the one listed in Table 1.

$L \cong L(2, 8)$. Here also there is only one character without trivial constituents. The extension to $L \cdot 3$ has been constructed by [4] (see § 6.11). The $L \cdot 3$ is Lie primitive for if it were not, as in § 4, then L would be in a maximal connected semisimple subgroup of positive dimension. To have consistent action on $\mathbf{K}$ it could only be irreducible and so be A_2 , G_2 , and C_4 . But, $L \cdot 3$ is not a subgroup of any of these Lie groups.

$L \cong L(2, 11)$. In each case, there is a character with a trivial constituent.

$L \cong L(2, 13)$. The only fixed point free representation is dealt with in [19] and must be in G_2 .

$L \cong L(2, 17)$. In this case, there is no character on $\mathbf{K}$ without trivial constituents.

$L \cong L(2, 19)$. This was constructed in § 6.7. Straightforward feasibility checks show that $\chi|_L$ is either $18_a + 3 \cdot 20_a$ or $18_a + 20_b + 20_c + 20_d$ (up to conjugacy). But the Borel subgroup of L embeds uniquely into $\tilde{E}$ (up to $\text{Aut } \tilde{E}$ -conjugacy, as we have seen in § 6.7). The element $\bar{t}^3$ of order 3 in § 6.7 has centralizer of type $A_2 A_2 A_2$, and so is of type 3C. It follows that χ has value -3 on elements of order 3, and hence that $\chi|_L = 18_a + 20_b + 20_c + 20_d$.

Next we establish maximality of L as a closed subgroup of $\tilde{E}$. As discussed in § 4, if L is in a closed subgroup of positive dimension, then it is in a maximal connected semisimple closed subgroup M of positive dimension. Because the restriction of $\psi|_L = 9_a + 18_c$, the action of $\psi|_M$ must be consistent with this. As the smallest dimension of a representation of L is 9, L is not contained in Lie groups of type A_2 , G_2 , or C_4 . But now Theorem 4.1(ii) gives that M must be reducible on $\mathbf{K}$, and so $\psi|_M$ must have two constituents of dimensions 9 and 18. But this is not consistent with any of the remaining decompositions in 4.1. In fact dimension 9 alone rules out all but D_5 which has a fixed space. This settles maximality of $L \cong L(2, 19)$ in E .

$L \cong L(2, 25)$ and $L \cong L(2, 27)$. These have only characters with trivial constituents.

$L \cong L(3, 3)$. The only fixed point free character was dealt with in § 6.4. This embedding was by inclusion in ${}^2F_4(2)$. If L were in a closed subgroup of E of positive dimension, then, as it acts irreducibly on $\mathbf{K}$, by Lemma 4.2, it must be in $A_2(\mathbb{C})$, $G_2(\mathbb{C})$, or $C_4(\mathbb{C})$; but the smallest degree of a non-trivial character is 12 and so it is in none of these. Hence L is Lie primitive.

$L \cong \text{PSU}(3, 3)$. The only character $\chi|_L$ without trivial constituents is 27_a with $\chi|_{\tilde{L}} = 14_a + 32_a + 32_b$. Consider the L -invariant 14-dimensional subspace $\mathbf{v}$ of $\mathbf{g}$. A direct check from the character table for $\text{PSU}(3, 3)$ shows that the inner product of 32_* and 14_a^2 is zero, so that $\mathbf{v}$ must be a Lie subalgebra. In fact, as 14_a has inner product 1 with its wedge square, this representation carries a unique Lie algebra structure (up to scalar multiples), which must be that of G_2 (cf. [18]). Therefore, L embeds in a closed Lie subgroup of type G_2 .

$L \cong \text{PSU}(4, 2)$. The only character to be considered here is $\psi|_L = 2 \cdot 6_a + 15_a$, again for L on $\mathbf{K}$ and on its dual. Let U be an L -invariant subspace of dimension 6. Then B also stabilizes U (since the character table for $\text{PSU}(4, 2)$ has no irreducible 12-dimensional characters whose restrictions to L have 6_a as a constituent). But $6_a^{2+} = 1_a + 20_a$ has no constituent in common with $\psi|_L$, so $U^\# = 0$ and B is contained in a maximal parabolic of type A_5 . Thus $L \not\cong \text{PSU}(4, 2)$.

$L \cong {}^3D_4(2)$. This has no fixed point free character.

$L \cong {}^2F_4(2)'$. The existence was discussed in § 6.1. The derived subgroup of the Tits group is Lie primitive in E since, by Theorem 4.1, the only closed Lie subgroups of E which are irreducible in a 27-dimensional projective representation of E have type A_2 , G_2 , and C_4 , and L is a subgroup of none of these.

$L \cong M_{11}$. This has no fixed point free character.

8.3. **LEMMA** (maximality in F). *Suppose that B is a finite subgroup of F which is Lie primitive and has non-abelian simple socle L . Then L is either isomorphic to $L(3, 3)$ and embeds in a subgroup of F of type $3^3 : L(3, 3)$ or L is one of the groups listed in Table 1. Moreover, in the cases listed in Table 1 with $+$ in the first column, the converse holds in the sense that there is a subgroup L whose normalizer in F is Lie primitive in F .*

Proof. Here the condition that B be Lie primitive forces B to fix a unique vector in $\mathbf{K}$ and fix no non-zero vectors in $\mathfrak{g}$ (cf. Theorem 5.1 and § 4). If the simple socle L fixed a non-trivial subspace $\mathfrak{s}$ of $\mathfrak{g}$, then B would normalize the closed connected subgroup S of G of positive dimension whose Lie algebra in $\mathfrak{g}$ coincides with $\mathfrak{s}$, and so be contained in $N_G(S)$.

Some groups have been ruled out by Corollary 7.3.

$L \cong \text{Alt}_5$. The only possibility for $\psi|_L$ with multiplicity 1 for the trivial character and corresponding $\chi|_L$ without fixed vectors on $\mathfrak{g}$ is $\psi|_L = 1_a + 3_a + 2 \cdot 4_a + 3 \cdot 5_a$, with $\chi|_L = 4 \cdot 3_a + 5 \cdot 3_b + 4 \cdot 4_a + 7 \cdot 5_a$. We refer to [35] to see that it must be in a closed subgroup of positive dimension in F . This action is not the restriction of an action of the symmetric group Sym_5 to Alt_5 for either $\mathbf{K}$ or $\mathfrak{g}$ and so there is no possible B properly containing L .

$L \cong \text{Alt}_6$. There is a single character $\psi|_L$ having the trivial character with multiplicity 1 and without non-zero fixed vectors in $\mathfrak{g}$. Thus, we cannot exclude that a corresponding embedding of L in F yields a non-Lie primitive subgroup, and so it is listed in Table 1.

$L \cong L(2, 7)$. Again there is a single character $\psi|_L$ having the trivial character with multiplicity 1 and with corresponding $\chi|_L$ having no trivial constituent, viz. $\psi|_L = 1_a + 3_a + 3_b + 2 \cdot 6_a + 8_a$. This can also be ruled out using the results and arguments of [35, § 10] regarding subgroup-invariant subalgebras of $\mathbf{K}$. To be more specific, let S be a 3-dimensional L -submodule of $\mathbf{K}$. Then, by inspection of the symmetric square of 3_a for L , the subspace S satisfies the hypothesis of Lemma 10.7 in [35] or L is contained in a closed Lie subgroup of $\tilde{E}$ of type $A_2 A_2 A_2$. In the former case, [35] gives that L is contained in a closed Lie subgroup of $\tilde{E}$ of type $A_2 G_2$. By the decomposition of $\mathbf{K}$ into irreducible L -modules, one finds that, in both cases, L centralizes a subgroup of F of order 3 fixing an L -invariant Jordan subalgebra, J say, of dimension 9 (isomorphic to $1_a + 8_a$ as an L -module). Now J , being the unique L -invariant subalgebra of dimension 9, is also left invariant by the normalizer in F of L . As the stabilizer in F of J is a closed Lie subgroup of F of type $A_2 A_2$, neither $L(2, 7)$ with this character nor $\text{Aut } L(2, 7)$ is possible here.

$L \cong L(2, 8)$. The only character is handled in § 6.11. We now prove maximality of the subgroup L isomorphic to $L(2, 8)$ of F embedded with character $\psi|_L = 1_a + 8_a + 9_a + 9_c$. As above, suppose that M is in a maximal connected proper semisimple Lie subgroup of F of positive dimension containing L . Its action on $\mathbf{K}$ must be consistent with one of the Lie subgroups given in Theorem 4.1(ii). As the smallest dimension of a non-trivial character of L is 7, the only possibilities are $B_4(\mathbb{C})$ and $G_2(\mathbb{C})$. But $B_4(\mathbb{C})$ fixes a vector and $A_1(\mathbb{C})G_2(\mathbb{C})$ has a submodule of dimension 5, namely $4\pi_1^{(1)}$. Hence L is Lie primitive in F .

$L \cong L(2, 13)$. This was constructed in [19]. As in the $L(2, 8)$ case, L must be in a connected semisimple subgroup consistent with Theorem 4.1(ii) with subspaces of dimensions 12 and 14. The only one consistent with this is A_1C_3 . But this is a faithful representation of $SL(2, 13)$ in which the involution negates the 14 space which is not the action constructed. Hence L is a Lie primitive subgroup of F .

$L \cong L(2, 17)$. This has been constructed in § 6.8. If L were in a subgroup of positive dimension, let M be a semisimple one containing it. Then M acts consistently with Theorem 4.1(ii). But as the smallest dimension of even a projective character of L is 8, it could only be B_4 which fixes a 2 space in $\mathbf{K}$.

$L \cong L(2, 27)$. This has been constructed in § 6.5. Maximality follows because no closed Lie subgroup of F of positive dimension acts irreducibly on the 26-dimensional module.

$L \cong L(2, 25)$. This has been discussed in § 6.6. The proof of maximality is similar to the one for $L(2, 27)$.

$L \cong L(3, 3)$. By [21], any such subgroup is contained in a $3^3 : L(3, 3)$. As seen in § 8.1, $\text{Aut}(L)$ has no feasible character.

$L \cong \text{PSU}(3, 3)$. Here L stabilizes a 6-dimensional subspace U on which it acts with character 6_a . Since $6_a^{2+} = 21_a$ is irreducible and not a constituent of $\psi|_L$, the subspace U satisfies $u^\# = 0$ for each $u \in U$. Hence it is normalized in $\tilde{E}$ by a parabolic of type A_5 . As B has no irreducible character of degree 12, it also stabilizes U , and so is contained in this parabolic and so is contained in a closed subgroup of positive dimension in F_4 .

$L \cong {}^3D_4(2)$. See [11]. Maximality follows as above.

9. Proof of the Main Theorem

First, let $G = E$ and suppose that B is a finite Lie primitive subgroup of E . Let L be its generalized Fitting subgroup. If L is simple or has a normal abelian subgroup, then the Main Theorem holds by Theorem 3.5 or Lemma 8.2. Otherwise it is a product of at least two groups which contradicts each of [4, Theorem 4], [7], and [14]. Alternatively we could argue here as we do below for F . We have finished with this case.

Next, let $G = F$ and suppose that B is a finite Lie primitive subgroup of F . As above using Theorem 3.5 or Lemma 8.3, we may assume that its generalized Fitting subgroup L is a product of at least two simple groups. Suppose that there are two simple factors; denote them by L_1 and L_2 . The list of Theorem 7.1 and Corollary 7.3 and the fact that $SL(2, \mathbb{C})$ does not contain non-abelian finite simple groups, implies that L_i is isomorphic to one of Alt_5 , $L(2, 7)$, $L(2, 8)$, $L(2, 13)$, or $\text{PSU}(3, 3)$. This follows because the irreducible constituents of L_1 acting on $\mathbf{K}$ have degree at most the multiplicity of some irreducible constituent of L_2 acting on $\mathbf{K}$. Each of these simple groups has an elementary abelian subgroup X_i of order 4 all of whose involutions are conjugate and so the same is true of L_i . Now, by Lemma 3.3(iv), $L_2 \leq C_F(L_1) \leq C_F(X_1)$ and $C_F(X_1)$ has type D_4 or A_2T_2 . On the other hand, all non-central involutions in D_4 (Spin!) have trace 3 on $\mathbf{K}$ so we may assume without loss of generality that X_1 has type AAA and centralizer type A_2T_2 . But the centralizer Y_2 of L_1 must contain the simple group L_2 . Thus the factor L_2 embeds in the component of type A_2 so has a 3-dimensional projective

representation, whence $L_2 \cong \text{Alt}_5$ or $L(2, 7)$. Since L_2 is perfect, we must have $L_2 \leq [C, C] \cong A_2(\mathbb{C})$, with representation on $\mathbf{K}$ equal to $3_a \otimes 3_b + 3 \cdot 3_a + 3 \cdot 3_b = 1_a + 8_a + 3 \cdot 3_a + 3 \cdot 3_b$, where 3_a denotes the standard representation of $A_2(\mathbb{C})$, 3_b its contragredient, and 8_a the adjoint representation (cf. § 4).

First, consider $L_2 \cong L(2, 7)$. Then $\psi|_{L_2} = 1_a + 8_a + 3 \cdot 3_a + 3 \cdot 3_b$, and so $C_F(L_2) = C_F(Y_2) \cong A_2(\mathbb{C})$ with representation $9 \cdot 1_a + 3 \cdot 3_a + 3 \cdot 3_b$, and the representation of L_1 is different from that of L_2 . In particular, B stabilizes the unique L_2 invariant 8-space and hence is contained in a closed Lie subgroup of strictly positive dimension, a contradiction.

It remains to consider $L_2 \cong \text{Alt}_5$. Then $\psi|_L = 1_a + 7 \cdot 3_a + 5_a$. Again we see that L_1 cannot have the same decomposition as L_2 , and the same argument applies. The proof of Theorem 1.1 is now complete.

References

1. A. V. ALEKSEEVSKII, 'Finite commutative Jordan subgroups of complex simple Lie groups', *Functional Anal. Appl.* 8 (1974) 277–279.
2. M. ASCHBACHER, 'The 27-dimensional module for E_6 , I', *Invent. Math.* 89 (1987) 159–195.
3. M. ASCHBACHER, 'The 27-dimensional module for E_6 , II', *J. London Math. Soc.* (2) 37 (1988) 275–293.
4. M. ASCHBACHER, 'The 27-dimensional module for E_6 , III', *Trans. Amer. Math. Soc.* 321 (1990) 45–84.
5. M. ASCHBACHER, 'The 27-dimensional module for E_6 , IV', *J. Algebra* 191 (1991) 23–39.
6. A. BOREL and J.-P. SERRE, 'Sur certains sous-groupes de groupes de Lie compacts', *Comment. Math. Helv.* 27 (1953) 128–139.
7. A. V. BOROVIK, 'Finite subgroups of simple algebraic groups', *Soviet Math. Dokl.* 40 (1990) 570–573.
8. N. BOURBAKI, *Groupes et algèbres de Lie, Chapitres IV, V et VI* (Hermann, Paris, 1968).
9. R. BRAUER, 'On simple groups of order $5 \cdot 3^a \cdot 2^b$ ', *Bull. Amer. Math. Soc.* 74 (1968) 900–903.
10. C. CHEVALLEY, *Théorèmes généraux sur les algèbres de Lie, Théorie des groupes de Lie III* (Hermann, Paris, 1955).
11. A. M. COHEN, 'Exceptional presentations of three generalized hexagons of order 2', *J. Combin. Theory Ser. A* 35 (1983) 79–88.
12. A. M. COHEN and B. N. COOPERSTEIN, 'The 2-spaces of the standard $E_6(q)$ -module', *Geom. Dedicata* 24 (1988) 467–480.
13. A. M. COHEN and R. L. GRIESS JR, 'On finite simple subgroups of the complex Lie group of type E_8 ', *The Arcata Conference on Representations of Finite Groups 1986* (ed. P. Fong), Proceedings of Symposia in Pure Mathematics 47, part 2 (American Mathematical Society, Providence, R. I., 1987), pp. 367–405.
14. A. M. COHEN and R. L. GRIESS JR, 'Non-local Lie primitive subgroups of Lie groups', *Canad. J. Math.* 45 (1993) 88–103.
15. A. M. COHEN, R. L. GRIESS JR, and B. LISSER, 'The group $L(2, 61)$ embeds in the Lie group of type E_8 ', *Comm. Algebra* 21 (1993) 1889–1907.
16. A. M. COHEN, M. W. LIEBECK, J. SAXL, and G. M. SEITZ, 'The local maximal subgroups of exceptional groups of Lie type, finite and algebraic', *Proc. London Math. Soc.* (3) 64 (1992) 21–48.
17. A. M. COHEN and G. M. SEITZ, 'The r -rank of groups of exceptional Lie type', *Indag. Math.* 90 (1987) 251–259.
18. A. M. COHEN and D. B. WALES, 'Finite subgroups of $G_2(\mathbb{C})$ ', *Comm. Algebra* 11 (1983) 441–459.
19. A. M. COHEN and D. B. WALES, 'Embeddings of the group $L(2, 13)$ in groups of Lie type E_6 ', *Israel J. Math.* 82 (1993) 45–86.
20. A. M. COHEN and D. B. WALES, 'Finite simple subgroups of semisimple complex Lie groups—a survey', *Groups of Lie type and their geometries* (eds W. M. Kantor and L. Di Martino), London Mathematical Society Lecture Note Series 207 (Cambridge University Press, 1995), pp. 77–96.
21. A. M. COHEN and D. B. WALES, ' $\text{SL}(3, 3)$ is not maximal in the Lie group of type F_4 ', *Linear Algebra Appl.* 226–228 (1995) 287–296.
22. J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. P. PARKER, and R. A. WILSON, *Atlas of finite groups* (Clarendon Press, Oxford, 1985).
23. H. S. M. COXETER, 'Integral Cayley numbers', *Duke Math. J.* 13 (1946) 561–578.

24. K. H. DAR, 'Maximal subgroups of the Tits simple group', *J. Natur. Sci. Math.* 9 (1979) 45–55.
25. L. DICKSON, 'A class of groups in the arbitrary realm connected with the configuration of the 27 lines on a cubic surface', *Quart. J. Pure Appl. Math.* 33 (1901) 145–173.
26. E. B. DYNKIN, 'Semisimple subalgebras of semisimple Lie algebras', *Amer. Math. Soc. Transl.* 6 (1957) 111–244.
27. W. FEIT, 'The current situation in the theory of finite simple groups', *Actes du Congrès International des Mathématiciens 1970*, vol. 1 (Gauthier-Villars, Paris, 1971), pp. 447–465.
28. W. FEIT, 'On finite linear groups in dimension at most 10', *Proceedings of the Conference on Finite Groups, Park City, Utah, 1975* (eds W.R. Scott and F. Gross, Academic Press, New York, 1976), pp. 397–407.
29. H. FREUDENTHAL, 'Beziehungen der E_7 und E_8 zur Oktavenebene, VIII', *Indag. Math.* 62 (1959) 447–465.
30. L. GEURTS, L. MEERTENS, and S. PEMBERTON, *ABC programmers handbook* (Prentice Hall, London, 1990).
31. D. GORENSTEIN, *Finite simple groups* (Plenum Press, New York, 1982).
32. R. L. GRIESS JR, 'Elementary abelian p -subgroups of algebraic groups', *Geom. Dedicata* 39 (1991) 253–305.
33. J. E. HUMPHREYS, *Introduction to Lie algebras and representation theory* (Springer, New York, 1972).
34. M. W. LIEBECK and G. M. SEITZ, 'Maximal subgroups of exceptional groups of Lie type', *Geom. Dedicata* 36 (1990) 353–387.
35. K. MAGAARD, 'The maximal subgroups of the Chevalley groups $F_4(F)$ where F is a finite or algebraically closed field of characteristic $\neq 2, 3$ ', Ph.D. thesis, Caltech, 1990.
36. J. G. M. MARS, 'Les nombres de Tamagawa de certains groupes exceptionnels', *Bull. Soc. Math. France* 94 (1966) 97–140.
37. W. G. MCKAY and J. PATERA, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras* (Marcel Dekker, New York, 1981).
38. R. V. MOODY and J. PATERA, 'Characters of elements of finite order in Lie groups', *SIAM J. Algebraic Discrete Methods* 5 (1984) 359–383.
39. S. P. NORTON, 'On the group Fi_{24} ', *Geom. Dedicata* 25 (1988) 483–501.
40. G. M. SEITZ, 'Maximal subgroups of exceptional algebraic groups', *Mem. Amer. Math. Soc.* 441 (1991).
41. T. A. SPRINGER and R. STEINBERG, 'Conjugacy classes in classical groups', *Seminar on algebraic groups and related finite groups* (ed. A. Borel), Lecture Notes in Mathematics 131 (Springer, Berlin, 1970), pp. E82–E100.
42. R. STEINBERG, 'Torsion in reductive groups', *Adv. in Math.* 15 (1975) 63–92.
43. R. A. WILSON, 'The geometry and maximal subgroups of the simple groups of A. Rudvalis and J. Tits', *Proc. London Math. Soc.* (3) 48 (1984) 533–563.
44. K. C. YOUNG, 'Some simple subgroups of the Tits simple group', *Notices Amer. Math. Soc.* 19 (1972) Abstract 72T–B263, A–697.

Technische Universiteit Eindhoven
 Faculteit voor Wiskunde en Informatica
 P.O. Box 513
 5600 MB Eindhoven
 The Netherlands
 E-mail: amc@win.tue.nl

Sloan Laboratory
 California Institute of Technology
 Pasadena
 California 91125
 U.S.A.
 E-mail: dbw@caltech.edu